

A CAVITY PROBLEM FOR MAXWELL'S EQUATIONS *

HABIB AMMARI[†], GANG BAO[‡], AND AIHUA W. WOOD[§]

Abstract. Consider a time-harmonic electromagnetic plane wave incident on a cavity in a ground plane. Inside the cavity, the medium may be *nonhomogeneous*. In this paper, a variational formulation is studied. Existence and uniqueness of the solutions for the model problem are established by a variational approach and the Hodge decomposition. The variational approach also forms a basis for numerical solution of the model problems.

1. Introduction. As a measure of the detectability of a target by radar systems, *radar cross section* (RCS) has always been an important subject of study in electromagnetics. Mathematical and computational methods to accurately predict the RCS of complex objects such as aircraft are of great interest to designers. Of particular importance is the prediction of the RCS of cavities due to its dominance to the target's overall RCS. Examples of cavities include jet engine inlet ducts, exhaust nozzles, and cavity-backed antennas.

We consider a time-harmonic electromagnetic plane wave incident on an arbitrarily shaped open cavity embedded in an infinite ground plane. The ground plane and the walls of the open cavity are perfect electric conductors (PEC), and the interior of the open cavity is filled with a nonmagnetic material which may be nonhomogeneous. The half-space above the ground plane is filled with a homogeneous, linear, isotropic medium characterized by its permittivity ϵ_0 and permeability μ_0 . This paper is devoted to a study of the propagation of the scattered waves from the cavity, and hence its RCS.

There is a large engineering literature available on computation and design of cavities. See, for example, [4], [6], [9], and references cited therein. However, little is known about the analysis of the problem. In the general two-dimensional setting, we have recently established in [3] the well-posedness of the scattering problem by a variational approach. This paper is concerned with the three-dimensional scattering problem from a cavity. We extend the variational approach of [3] to solve the scattering problem. However, the situation here is more complicated because:

- (i) In this work, we study directly the vector form of Maxwell's equations. The problem geometry is three dimensional.
- (ii) A Hodge decomposition is needed here due to a lack of compactness of the solution functional space. In comparison, the compactness is trivial in the two-dimensional case.

Our main result for the cavity problem indicates that the scattering problem attains a unique weak solution for a general cavity medium. An important step of our approach is to reduce the infinite nature of the scattering problem into a bounded domain (the cavity) by introducing a transparent boundary condition. The proof is given by a combination of a variational approach, the Hodge decomposition, and unique continuation. Computationally, the variational approach reported here leads naturally

*Received November 6, 2000; accepted for publication October 25, 2002.

[†]Centre de Mathématiques Appliquées, CNRS UMR 7641, Ecole Polytechnique, 91128 Palaiseau, France (ammari@cmapx.polytechnique.fr).

[‡]Department of Mathematics, Michigan State University, East Lansing, MI 48824-1027, USA.

[§]Department of Mathematics and Statistics, Air Force Institute of Technology, Wright-Patterson AFB, OH 45433, USA.

to a class of finite element methods. Analysis and computation of the finite element methods for the scattering problem will be studied and reported elsewhere.

The paper is outlined as follows. In Section 2, the Maxwell equations are presented and simplified. The scattering problem is then reduced into a bounded domain one by introducing the transparent boundary conditions in Section 3. The variational formulations are also derived. Section 4 is devoted to a well-posedness study of the model problem. It is shown that the problem attains a unique weak solution.

2. Maxwell's equations. The media are assumed to be nonmagnetic, and we assume that a constant magnetic permeability $\mu = \mu_0$ exists everywhere. We also assume that no currents are present and the fields are source free. Then the electromagnetic wave propagation is governed by the following time harmonic Maxwell equations (time dependence $e^{-i\omega t}$):

$$\nabla \times E - i\omega\mu H = 0, \quad (2.1)$$

$$\nabla \times H + i\omega\varepsilon E = 0, \quad (2.2)$$

where E and H denote the electric field and the magnetic field, respectively. The dielectric coefficient $\varepsilon(x)$ can be quite general. In addition, standard field continuity conditions are also satisfied across an interface.

Let a plane wave (E^i, H^i) be incident on an electromagnetic cavity.

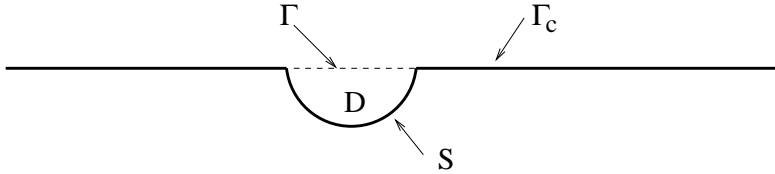


FIGURE 1. The problem geometry

As shown in Figure 1, a cavity D is placed on a perfectly conducting medium (ground). Above the flat surface $\{x_3 = 0\}$ ($\Gamma \cup \Gamma^C$), the medium is assumed to be homogeneous with a positive dielectric coefficient ε_0 . The medium inside D is non-homogeneous with a dielectric coefficient $\varepsilon(x)$. Assume further that $\varepsilon(x) \in L^\infty(D)$, $Re \varepsilon(x) \geq \varepsilon_1 > 0$, and $Im \varepsilon(x) \geq 0$ in D .

On the surface of the perfectly conducting medium, the following boundary condition is satisfied

$$\mathbf{n} \times E = 0 \text{ on } \Gamma^C \cup S, \quad (2.3)$$

where \mathbf{n} is the unit outward normal.

In addition, we require that the scattered fields satisfy the Silver-Müller radiation condition:

$$\begin{aligned} \lim_{r \rightarrow \infty} r E^s &= \lim_{r \rightarrow \infty} r H^s = 0, \\ \lim_{r \rightarrow \infty} r \{E^s - Z_0 H^s \times \mathbf{r}\} &= 0, \end{aligned} \quad (2.4)$$

where $r = |x|$, $\mathbf{r} = x/|x|$, $Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}}$.

The scattering problem is the following: For a given incoming plane wave, determine the scattered fields in the cavity and the upper-half space.

3. Transparent boundary conditions. We solve the scattering problem by a variational approach. The fundamental step of the approach is to reduce the problem into a bounded domain. This is done by introducing transparent boundary conditions on Γ .

Denote $k^2 = \omega^2 \mu \varepsilon$. Let a plane wave, $E^i = p e^{ik_0 q \cdot x}$ ($k_0 = \sqrt{\mu_0 \varepsilon_0}$), be incident on the cavity from the above, where $p, q \in \mathbf{R}^3$, $q_3 < 0$, $p \cdot q = 0$, and $|p|^2 = |q|^2 = 1$. Let $q^* = (q_1, q_2, -q_3)$. Then we may decompose the total fields as follows:

$$\begin{aligned} E^t &= E^i - p e^{ik_0 q^* \cdot x} + E^s, \\ H^t &= H^i - Z_0 p \times q^* e^{ik_0 q^* \cdot x} + H^s. \end{aligned}$$

It is easy to see that the scattered fields also satisfy the Maxwell equations (2.1), (2.2) in the upper half space, the boundary condition (2.3), and the radiation condition (2.4). For convenience, we shall, for the rest of this paper, suppress the superscript “s” for the scattered fields. Also, we shall denote $\frac{\partial A}{\partial \alpha}$ and $\frac{\partial^2 A}{\partial \alpha^2}$ by $\partial_\alpha A$ and $\partial_\alpha^2 A$, respectively.

REMARK 3.1. Since the scattering object is unbounded, the total fields are composed of three parts: incident, reflected, and scattered fields. In contrast, in the case of a bounded scattering object, the total fields only consist of incident and scattered fields.

Next, we introduce the following functional framework. Let $\tilde{\Gamma}$ be a closed smooth surface such that $\bar{\Gamma} \subset \tilde{\Gamma}$. We define div_Γ and $curl_\Gamma$ as the the surface divergence and the scalar surface rotational, respectively. Introduce the Sobolev spaces

$$H^{1/2}(\Gamma) = \{\varphi|_\Gamma, \varphi \in H^{1/2}(\tilde{\Gamma})\},$$

$$\tilde{H}^{1/2}(\Gamma) = \{\varphi|_\Gamma, \varphi \in H^{1/2}(\tilde{\Gamma}), \text{supp}(\varphi) \subset \bar{\Gamma}\},$$

$$H^{-1/2}(\Gamma) = (\tilde{H}^{1/2}(\Gamma))', \tilde{H}^{-1/2}(\Gamma) = (H^{1/2}(\Gamma))',$$

$$H_{div}^{-1/2}(\Gamma) = \{\Phi \in (H^{-1/2}(\Gamma))^3, \Phi \cdot \mathbf{n} = 0, \text{div}_\Gamma \Phi \in H^{-1/2}(\Gamma)\},$$

$$H_{curl}^{-1/2}(\Gamma) = \{\Phi \in (H^{-1/2}(\Gamma))^3, \Phi \cdot \mathbf{n} = 0, \text{curl}_\Gamma \Phi \in H^{-1/2}(\Gamma)\}.$$

The following result holds.

LEMMA 3.1. *There exists a linear continuous operator $P : (H_{div}^{-1/2}(\Gamma))' \mapsto H_{curl}^{-1/2}(\Gamma)$ such that*

$$-\mathbf{n} \times (\mathbf{n} \times H)|_\Gamma = P(-\mathbf{n} \times (\mathbf{n} \times E))|_\Gamma.$$

Proof. Observe that the medium is homogeneous in the upper-half space. The (scattered) fields satisfy

$$\begin{aligned} \nabla \times E &= i\omega \mu_0 H & \text{in } \{x_3 > 0\}, \\ \nabla \times H &= -i\omega \varepsilon_0 E & \text{in } \{x_3 > 0\} \end{aligned}$$

and

$$\lim_{r \rightarrow \infty} rE = \lim_{r \rightarrow \infty} rH = 0 ,$$

$$\lim_{r \rightarrow \infty} r\{E - Z_0H \times \mathbf{r}\} = 0 ,$$

where $E = E_1\hat{x}_1 + E_2\hat{x}_2 + E_3\hat{x}_3$, $g_j(x_1, x_2) = E_j(x_1, x_2, 0)$, $j = 1, 2, 3$.

Denote for $j = 1, 2, 3$

$$g_j(x_1, x_2) = E_j(x_1, x_2, 0) .$$

Then

$$-\mathbf{n} \times (\mathbf{n} \times E) = (g_1, g_2, 0) \text{ on } \{x_3 = 0\} ,$$

where $\text{supp}(g_1, g_2) = \Gamma$.

Thus

$$-\mathbf{n} \times (\mathbf{n} \times H)|_{x_3=0} = \frac{1}{i\omega\mu_0}(\partial_{x_2}E_3 - \partial_{x_3}E_2, \partial_{x_3}E_1 - \partial_{x_1}E_3, 0)|_{x_3=0} .$$

Now, for $j = 1, 2, 3$, we have

$$\begin{aligned} \Delta E_j + k_0^2 E_j &= 0 && \text{in } \{x_3 > 0\} , \\ E_j &= g_j && \text{on } \{x_3 = 0\} \end{aligned} \tag{3.1}$$

and

$$\lim_{r \rightarrow \infty} rE_j = 0 ,$$

$$\lim_{r \rightarrow \infty} r(\partial_r E_j - ik_0 E_j) = 0 .$$

Taking the Fourier transform with respect to (x_1, x_2) of (3.1) yields

$$\begin{aligned} [\partial_{x_3}^2 + (k_0^2 - \xi_1^2 - \xi_2^2)]\hat{E}_j(\xi, x_3) &= 0 && \text{in } \{x_3 > 0\} , \\ \hat{E}_j(\xi, 0) &= \hat{g}_j && \text{on } \{x_3 = 0\} . \end{aligned}$$

Using the radiation condition to eliminate the incoming wave term, we obtain

$$\hat{E}_j(\xi, x_3) = \hat{g}_j e^{i\sqrt{k_0^2 - \xi_1^2 - \xi_2^2}x_3} . \tag{3.2}$$

Taking the inverse Fourier transform yields, for $j = 1, 2$,

$$E_j = \frac{1}{(2\pi)^2} \int_{\mathbf{R}^2} e^{i\sqrt{k_0^2 - \xi_1^2 - \xi_2^2}x_3} \hat{g}_j e^{i\xi \cdot x} d\xi .$$

Hence

$$\partial_{x_3} E_j|_{x_3=0} = \frac{i}{(2\pi)^2} \int_{\mathbf{R}^2} \sqrt{k_0^2 - \xi_1^2 - \xi_2^2} \hat{g}_j e^{i\xi \cdot x} d\xi .$$

For E_3 , we obtain, by taking the partial derivative with respect to x_3 of (3.2)

$$\partial_{x_3} \hat{E}_3 = \hat{g}_3 i \sqrt{k_0^2 - \xi_1^2 - \xi_2^2} e^{i\sqrt{k_0^2 - \xi_1^2 - \xi_2^2}x_3} .$$

This implies that

$$\hat{E}_3(\xi, x_3) = \frac{1}{i\sqrt{k_0^2 - \xi_1^2 - \xi_2^2}} \partial_{x_3} \hat{E}_3(\xi, x_3) .$$

Note that $\nabla \cdot E = 0$ in $\{x_3 \geq 0\}$. We deduce

$$\begin{aligned} E_3|_{x_3=0} &= \frac{1}{(2\pi)^2 i} \int_{\mathbf{R}^2} \frac{1}{\sqrt{k_0^2 - \xi_1^2 - \xi_2^2}} \partial_{x_3} \hat{E}_3(\xi, 0) e^{i\xi \cdot x} d\xi \\ &= \frac{1}{(2\pi)^2 i} \int_{\mathbf{R}^2} \frac{1}{\sqrt{k_0^2 - \xi_1^2 - \xi_2^2}} [-(\partial_{x_1} \hat{E}_1 + \partial_{x_2} \hat{E}_2)(\xi, 0)] e^{i\xi \cdot x} d\xi \\ &= -\frac{1}{(2\pi)^2} \int_{\mathbf{R}^2} \frac{1}{\sqrt{k_0^2 - \xi_1^2 - \xi_2^2}} (\xi_1 \hat{g}_1 + \xi_2 \hat{g}_2) e^{i\xi \cdot x} d\xi . \end{aligned}$$

Taking the partial derivatives with respect to x_1 and x_2 and letting $x_3 = 0$ respectively gives

$$\begin{aligned} \partial_{x_2} E_3|_{x_3=0} &= -\frac{i}{(2\pi)^2} \int_{\mathbf{R}^2} \frac{1}{\sqrt{k_0^2 - \xi_1^2 - \xi_2^2}} (\xi_1 \hat{g}_1 + \xi_2 \hat{g}_2) \xi_2 e^{i\xi \cdot x} d\xi , \\ \partial_{x_1} E_3|_{x_3=0} &= -\frac{i}{(2\pi)^2} \int_{\mathbf{R}^2} \frac{1}{\sqrt{k_0^2 - \xi_1^2 - \xi_2^2}} (\xi_1 \hat{g}_1 + \xi_2 \hat{g}_2) \xi_1 e^{i\xi \cdot x} d\xi . \end{aligned}$$

Hence

$$\begin{aligned} -\mathbf{n} \times (\mathbf{n} \times H)|_{\Gamma} &= \frac{1}{(2\pi)^2 \omega \mu_0} \int_{\mathbf{R}^2} \left\{ \left[-\frac{1}{\sqrt{k_0^2 - \xi_1^2 - \xi_2^2}} (\xi_1 \hat{g}_1 + \xi_2 \hat{g}_2) \xi_2 - \sqrt{k_0^2 - \xi_1^2 - \xi_2^2} \hat{g}_2 \right], \right. \\ &\quad \left. \left[\frac{1}{\sqrt{k_0^2 - \xi_1^2 - \xi_2^2}} (\xi_1 \hat{g}_1 + \xi_2 \hat{g}_2) \xi_1 + \sqrt{k_0^2 - \xi_1^2 - \xi_2^2} \hat{g}_1 \right], 0 \right\} e^{i\xi \cdot x} d\xi \\ &:= P(-\mathbf{n} \times (\mathbf{n} \times E))|_{\Gamma} . \quad \square \end{aligned}$$

It is easy to see that P is continuous. Denote by P^* the adjoint of the operator P , that is,

$$\int_{\Gamma} P(f) \bar{g} = \int_{\Gamma} f \overline{P^*(g)} .$$

It is easily seen that the operator P^* is given by

$$\begin{aligned} P^*(g) &= \frac{1}{(2\pi)^2 \omega \mu_0} \int_{\mathbf{R}^2} \left\{ \left[-\frac{1}{\sqrt{k_0^2 - \xi_1^2 - \xi_2^2}} (\xi_1 \hat{g}_2 - \xi_2 \hat{g}_1) \xi_1 + \sqrt{k_0^2 - \xi_1^2 - \xi_2^2} \hat{g}_2 \right], \right. \\ &\quad \left. \left[\frac{1}{\sqrt{k_0^2 - \xi_1^2 - \xi_2^2}} (\xi_1 \hat{g}_2 - \xi_2 \hat{g}_1) \xi_1 - \sqrt{k_0^2 - \xi_1^2 - \xi_2^2} \hat{g}_1 \right], 0 \right\} e^{i\xi \cdot x} d\xi . \end{aligned}$$

LEMMA 3.2. *Let g be in $(H_{div}^{-1/2}(\Gamma))'$. Then*

$$Re \int_{\Gamma} P(g) \cdot \bar{g} \times \mathbf{n} \, dx_1 dx_2 = 0$$

implies $g \equiv 0$.

Proof. Clearly

$$\int_{\Gamma} P(g) \cdot \bar{g} \times \mathbf{n} \, dx_1 dx_2 = \frac{1}{\omega\mu_0} \int_{\mathbf{R}^2} [\sqrt{k_0^2 - \xi_1^2 - \xi_2^2} (|\hat{g}_1|^2 + |\hat{g}_2|^2) + \frac{1}{\sqrt{k_0^2 - \xi_1^2 - \xi_2^2}} |\xi_1 \hat{g}_1 + \xi_2 \hat{g}_2|^2] \, d\xi_1 d\xi_2 .$$

Hence $\operatorname{Re} \int_{\Gamma} P(g) \cdot \bar{g} \times \mathbf{n} \, dx_1 dx_2 = 0$ implies

$$|\hat{g}_1|^2 + |\hat{g}_2|^2 = 0 \quad \text{for } |\xi| \leq k_0 .$$

But \hat{g} is analytic with respect to ξ . Thus $\hat{g} \equiv 0$ for all ξ . Hence $g \equiv 0$. \square

REMARK 3.2. The integral $\int_{\Gamma} P(g) \cdot \bar{g} \times \mathbf{n} \, dx_1 dx_2$ is taken in the sense of duality $H_{curl}^{-1/2}(\Gamma), (H_{curl}^{-1/2}(\Gamma))'$ if g is not smooth enough.

LEMMA 3.3. *Let g be in $(H_{div}^{-1/2}(\Gamma))'$. Then there are two positive constants C_1 and C_2 , such that*

$$\operatorname{Im} \int_{\Gamma} P(g) \cdot \bar{g} \times \mathbf{n} \, dx_1 dx_2 \geq C_1 \|\operatorname{div}_{\Gamma}(g \times \mathbf{n})\|_{H^{-1/2}(\Gamma)}^2 - C_2 \|g \times \mathbf{n}\|_{H^{-1/2}(\Gamma)}^2 .$$

Proof. By some simple calculations, we obtain

$$\int_{\Gamma} P(g) \cdot \bar{g} \times \mathbf{n} \, dx_1 dx_2 = \frac{1}{\omega\mu_0} \int_{\mathbf{R}^2} [\frac{k_0^2}{\sqrt{k_0^2 - \xi_1^2 - \xi_2^2}} (|\hat{g}_1|^2 + |\hat{g}_2|^2) - \frac{1}{\sqrt{k_0^2 - \xi_1^2 - \xi_2^2}} |\xi_1 \hat{g}_1 - \xi_2 \hat{g}_2|^2] \, d\xi_1 d\xi_2 .$$

The conclusion follows immediately. \square

We now prove a uniqueness result for the cavity problem. First, introduce the Sobolev space

$$H(\operatorname{curl}, D) := \{v \in L^2(D), \nabla \times v \in L^2(D)\} .$$

Our proof is based on a unique continuation result for solutions in $H(\operatorname{curl}, D)$ to Maxwell's equations. The reader is referred to [5] for a proof of the unique continuation result. See [8] for the related Carleman estimates for Maxwell's equations.

THEOREM 3.1. *Assume that $\varepsilon \in \mathcal{C}^2(D)$. Let $E, H \in H(\operatorname{curl}, D)$ satisfy*

$$\begin{aligned} \nabla \times E &= i\omega\mu H \text{ in } D , \\ \nabla \times H &= -i\omega\varepsilon E \text{ in } D , \\ E \times \mathbf{n} &= 0 \text{ on } S , \\ \mathbf{n} \times (\mathbf{n} \times E)|_{\Gamma} &\in (H_{div}^{-1/2}(\Gamma))' , \\ -\mathbf{n} \times (\mathbf{n} \times H)|_{\Gamma} &= P(-\mathbf{n} \times (\mathbf{n} \times E)|_{\Gamma}) . \end{aligned}$$

Then

$$E = H \equiv 0 \text{ in } D .$$

Proof. If E and H in $H(\text{curl}, D)$ satisfy the homogeneous boundary value problem in D , then

$$\frac{1}{\mu_0} \int_D |\nabla \times E|^2 - \omega^2 \int_D \varepsilon |E|^2 - i\omega \int_\Gamma P(\mathbf{n} \times (\mathbf{n} \times E)) \cdot \bar{E} \times \mathbf{n} = 0 .$$

Taking the imaginary part we get

$$\text{Re} \int_\Gamma P(-\mathbf{n} \times (\mathbf{n} \times E)) \cdot \bar{E} \times \mathbf{n} = 0 .$$

Hence $E \times \mathbf{n} = 0$ on Γ by Lemma 3.2. From the boundary condition, it is immediate that $H \times \mathbf{n} = 0$ on Γ . Finally, by unique continuation for solutions in $H(\text{curl}, D)$ to Maxwell's equations [5], $E = H \equiv 0$ in D . \square

REMARK 3.3. From [5], unique continuation holds for $H^1(D)$ solutions of Maxwell's equations. Moreover, if the boundary ∂D is of class $C^{1,\alpha}$ and the coefficient ε is in $W^{1,\infty}(D) = \{\varphi \in L^\infty(D), \nabla \varphi \in L^\infty(D)\}$ then the solution of the Maxwell equations is in $H^1(D)$. Therefore, by unique continuation [5], $E = H \equiv 0$ in D follows from $E \times \mathbf{n} = H \times \mathbf{n} = 0$ on Γ .

4. Variational Formulation. With the transparent boundary condition defined by the operator P , we observe that the total fields satisfy the following boundary value problem:

$$\begin{aligned} \nabla \times E^t &= i\omega\mu H^t \text{ in } D , \\ \nabla \times H^t &= -i\omega\varepsilon E^t \text{ in } D , \\ E^t \times \mathbf{n} &= 0 \text{ on } S , \\ -\mathbf{n} \times (\mathbf{n} \times H^t)|_\Gamma &= P(-\mathbf{n} \times (\mathbf{n} \times E^t)) + g^i|_\Gamma , \end{aligned}$$

where

$$-g^i = -\mathbf{n} \times (\mathbf{n} \times (E^i - pe^{ik_0q \cdot x})) + P(-\mathbf{n} \times (\mathbf{n} \times (E^i - pe^{ik_0q^* \cdot x}))) .$$

Define

$$H_0(\text{curl}, D) := \{v \in H(\text{curl}, D), v \times \mathbf{n} = 0 \text{ on } S\} .$$

The variational formulation of the scattering problem in D is then to find $E^t \in H_0(\text{curl}, D)$, such that

$$a(E^t, v) = (g^i, v) := i\omega \int_\Gamma g^i \cdot \bar{v} \times \mathbf{n} \, dx_1 dx_2 \quad \forall v \in H_0(\text{curl}, D) , \tag{4.3}$$

where

$$a(E^t, v) = \frac{1}{\mu_0} \int_D \nabla \times E^t \cdot \nabla \times \bar{v} - \omega^2 \int_D \varepsilon E^t \cdot \bar{v} - i\omega \int_\Gamma P(\mathbf{n} \times (\mathbf{n} \times E^t)) \cdot \bar{v} \times \mathbf{n} .$$

We note that the boundary integrals are taken in the sense of duality $(H_{\text{curl}}^{-1/2}(\Gamma))'$, $H_{\text{curl}}^{-1/2}(\Gamma)$.

In order to analyze this variational problem we need the following Hodge decomposition result.

THEOREM 4.1. (*The Hodge decomposition*) For each $E \in H_0(\text{curl}, D)$, $\nabla \cdot (\varepsilon E) \in L^2(D)$, there exists a function $u \in H(\text{curl}, D)$ satisfying

$$\begin{aligned} \nabla \cdot (\varepsilon u) &= 0 \text{ in } D, \\ u \times \mathbf{n} &= 0 \text{ on } S, \\ -i\omega\varepsilon u \cdot \mathbf{n}|_\Gamma &= \text{curl}_\Gamma(P(-\mathbf{n} \times (\mathbf{n} \times u))|_\Gamma) \end{aligned}$$

and $p \in H^1(D)$, $p = 0$ on S , such that

$$E = u + \nabla p.$$

REMARK 4.1. Recall the standard notation: $\text{curl}_\Gamma(v)$ is the scalar surface rotational of the tangential vector v and

$$\text{curl}_\Gamma(v) = \text{div}_\Gamma(v \times n).$$

In order to prove the theorem, we need the following technical results.

PROPOSITION 4.1. Let $u \in H(\text{curl}, \Omega)$ where Ω is a bounded domain. Then for any positive constant η

$$\|u \times \mathbf{n}\|_{H^{-1/2}(\partial\Omega)}^2 \leq \eta \|\nabla \times u\|_{L^2(\Omega)}^2 + \frac{1}{\eta} \|u\|_{L^2(\Omega)}^2.$$

LEMMA 4.1. For any function $f \in L^2(D)$, the boundary value problem

$$\begin{aligned} \nabla \cdot (\varepsilon \nabla p) &= f \text{ in } D, \\ p &= 0 \text{ on } S, \\ -i\omega\varepsilon \partial_n p|_\Gamma &= \text{curl}_\Gamma(P(-\mathbf{n} \times (\mathbf{n} \times \nabla p))|_\Gamma) \end{aligned}$$

has a unique solution in $H_a^1(D) = \{q : q \in H^1(D), q = 0 \text{ on } S, q|_\Gamma \in \tilde{H}^{1/2}(\Gamma)\}$.

Proof. We examine the weak form of the boundary value problem. For any $q \in H_a^1(D)$, multiplying both sides of Equation (4.4) by \bar{q} and integrating over D yield

$$\int_D \text{div}(\varepsilon \nabla p) \cdot \bar{q} = \int_D f \cdot \bar{q}.$$

Integration by parts gives after using the boundary conditions that

$$\int_D \varepsilon \nabla p \cdot \overline{\nabla q} - \frac{i}{\omega} \int_\Gamma \text{curl}_\Gamma(P(-\mathbf{n} \times (\mathbf{n} \times \nabla p))) \cdot \bar{q} = - \int_D f \cdot \bar{q}, \tag{4.4}$$

where the integral

$$\int_\Gamma \text{curl}_\Gamma(P(-\mathbf{n} \times (\mathbf{n} \times \nabla p))) \cdot \bar{q}$$

is taken in the sense of duality $H^{-1/2}(\Gamma), \tilde{H}^{1/2}(\Gamma)$.

Denote the left hand side of (4.4) by $b(p, q)$. From integration by parts on the boundary, we obtain

$$b(p, q) = \int_D \varepsilon \nabla p \cdot \overline{\nabla q} - \frac{i}{\omega} \int_{\Gamma} P(-\mathbf{n} \times (\mathbf{n} \times \nabla p)) \cdot (\nabla \overline{q} \times \mathbf{n}) .$$

The variational problem takes the form: to find $p \in H_a^1(D)$, such that

$$b(p, q) = - \int_D f \cdot \overline{q}, \quad \forall q \in H_a^1(D) .$$

It is now obvious from Lemma 3.3 that there are positive constants C_1, C_2 , such that

$$\begin{aligned} \operatorname{Re} b(p, p) &= \int_D \varepsilon |\nabla p|^2 + \frac{1}{\omega} \operatorname{Im} \left\{ \int_{\Gamma} P(-\mathbf{n} \times (\mathbf{n} \times \nabla p)) \cdot (\overline{\nabla p} \times \mathbf{n}) \right\} \\ &\geq C \|\nabla p\|_{L^2(D)}^2 - C_2 \|\nabla p \times \mathbf{n}\|_{H^{-1/2}(\Gamma)}^2 . \end{aligned}$$

A simple application of Proposition 4.1 leads to

$$\operatorname{Re} b(p, p) \geq C_0 \|\nabla p\|_{L^2(D)}^2 .$$

Finally, by Poincaré's inequality, we obtain

$$\operatorname{Re} b(p, p) \geq C \|p\|_{H^1(D)}^2 .$$

The proof is complete by a direct application of the Lax-Milgram Lemma. \square

We are now ready to prove Theorem 4.1.

Proof. By Lemma 4.1, the problem

$$\begin{aligned} \nabla \cdot (\varepsilon \nabla p) &= \nabla \cdot (\varepsilon E) \text{ in } L^2(D) , \\ p &= 0 \text{ on } S , \\ -i\omega \varepsilon \partial_n p|_{\Gamma} &= \operatorname{curl}_{\Gamma}(P(-\mathbf{n} \times (\mathbf{n} \times \nabla p))|_{\Gamma}) \end{aligned}$$

is well-posed. \square

Now we define a new functional space

$$\begin{aligned} V := \{ u \in H(\operatorname{curl}, D), u \times \mathbf{n} = 0 \text{ on } S, \nabla \cdot (\varepsilon u) = 0 \text{ in } D, \\ \mathbf{n} \times (\mathbf{n} \times u) \in (H_{\operatorname{div}}^{-1/2}(\Gamma))', -i\omega \varepsilon u \cdot \mathbf{n} = \operatorname{curl}_{\Gamma}(P(-\mathbf{n} \times (\mathbf{n} \times u))) \text{ on } \Gamma \} . \end{aligned}$$

We have the following compactness result.

LEMMA 4.2. *The mapping $V \hookrightarrow L^2(D)^3$ is compact.*

Proof. Let $\tilde{u} = u$ in D , $\tilde{u} = u_1$ in $\{x_3 > 0\}$, where u_1 is the unique solution of

$$\begin{aligned} \nabla \times (\nabla \times u_1) - \omega^2 \varepsilon_0 \mu_0 u_1 &= 0 \text{ in } \{x_3 > 0\} , \\ u_1 \times \mathbf{n}|_{\Gamma} &= u \times \mathbf{n}|_{\Gamma} , \\ u_1 \times \mathbf{n}|_{\mathbf{R}^2 \setminus \overline{\Gamma}} &= 0|_{\mathbf{R}^2 \setminus \overline{\Gamma}} \end{aligned}$$

and

$$\lim_{r \rightarrow \infty} r|u_1 - Z_0(\frac{1}{i\omega\mu_0}\nabla u_1) \times \mathbf{r}| = 0.$$

Then, from

$$-i\omega\varepsilon u_1 \cdot \mathbf{n} = \text{curl}_\Gamma(P(-\mathbf{n} \times (\mathbf{n} \times u_1))) \text{ on } \Gamma,$$

we deduce that $[\varepsilon\tilde{u} \cdot \mathbf{n}] = 0$ on Γ . Further, $\tilde{u} \in H(\text{curl})$. The proof is complete by an application of Weber’s imbedding result [7]. \square

THEOREM 4.2. *The variational formulation is of Fredholm type.*

Proof. The result follows by using the decomposition

$$E^t = u + \nabla p.$$

After some simple calculations, we obtain that u and p are solutions of the following variational system:

$$\begin{aligned} \frac{1}{\mu_0} \int_D \nabla \times u \cdot \nabla \times \bar{v} - \omega^2 \int_D \varepsilon u \cdot \bar{v} - i\omega \int_\Gamma P(\mathbf{n} \times (\mathbf{n} \times u)) \cdot \bar{v} \times \mathbf{n} \\ - i\omega \int_\Gamma (P - P^*)(\mathbf{n} \times (\mathbf{n} \times \nabla p)) \cdot \bar{v} \times \mathbf{n} = i\omega \int_\Gamma g^i \cdot \bar{v} \times \mathbf{n} \quad \forall v \in V, \end{aligned}$$

$$-\omega^2 \int_D \varepsilon \nabla p \cdot \nabla \bar{q} - i\omega \int_\Gamma P(\mathbf{n} \times (\mathbf{n} \times \nabla p)) \cdot \nabla \bar{q} \times \mathbf{n} = i\omega \int_\Gamma g^i \cdot \nabla \bar{q} \times \mathbf{n} \quad \forall q \in H_a^1(D).$$

It is easily seen that the operator $P - P^* : (H_{div}^{-1/2}(\Gamma))' \mapsto H_{curl}^{-1/2}(\Gamma)$ is compact.

Let the bilinear form a_1 on $V \times V$ be defined by

$$a_1(w, v) = \frac{1}{\mu_0} \int_D \nabla \times w \cdot \nabla \times \bar{v} - \omega^2 \int_D \varepsilon w \cdot \bar{v} - i\omega \int_\Gamma P(\mathbf{n} \times (\mathbf{n} \times w)) \cdot \bar{v} \times \mathbf{n}.$$

Define the bilinear form a_2 on $H_a^1(D) \times H_a^1(D)$ by

$$a_2(\varphi, q) = -\omega^2 \int_D \varepsilon \nabla \varphi \cdot \nabla \bar{q} - i\omega \int_\Gamma P(\mathbf{n} \times (\mathbf{n} \times \nabla \varphi)) \cdot \nabla \bar{q} \times \mathbf{n}.$$

From Lemma 4.1, it follows that there exists a positive constant C_1 such that

$$|a_2(q, q)| \geq C_1 \|\nabla q\|_{L^2(D)}^2 \quad \forall q \in H_a^1(D).$$

Further, an application of Lemma 3.3 along with Proposition 4.1 yields

$$a_1(v, v) \geq C_2 \|v\|_{H(\text{curl}, D)}^2 - C_3 \|v\|_{L^2(D)}^2,$$

where C_2 and C_3 are two positive constants.

The result follows then from the compact imbedding of V into $L^2(D)$. \square

The well-posedness of the variational formulation, (4.3), of the model problem then follows immediately from Theorems 3.1 and 4.2.

REMARK 4.2. Our Hodge decomposition argument here is similar in spirit to the one used in [1] for a well-posedness study of chiral grating problems.

5. Concluding remarks. We have established well-posedness results for the scattering by a three-dimensional cavity. It is shown that a unique weak solution exists. For an open cavity, the results indicate that no eigenvalue may be present. Thus any spurious mode in computation must result from numerical artifacts.

A crucial step in our analysis is to formulate the scattering problem in the cavity. The variational approach here leads to a method for solving the problem in the cavity. This is sufficient for solving the scattering problem in the whole space. In fact, because of the homogeneous medium, the half space problem in $\{x_3 > 0\}$ can be solved easily, say by the Fourier transform method, once the cavity problem is solved.

Acknowledgments. We thank Major William D. Wood of the Air Force Research Lab for insightful discussions. The research of the first author was partially supported by ACI Jeunes Chercheurs (0693) from the Ministry of Education and Scientific Research, France. The research of the second author was partially supported by the NSF grant DMS 98-03604, the NSF University-Industry Cooperative Research Programs grants DMS 98-03809, DMS 99-72292, and the NSF Western Europe Programs grant INT 98-15798. The third author was supported in part by the Air Force Office of Scientific Research grant AFOSR-PO-990025.

REFERENCES

- [1] H. AMMARI AND G. BAO, *Electromagnetic diffraction by periodic chiral structures*, to appear in Math. Nach.
- [2] H. AMMARI AND C. LATIRI-GROUZ, *An integral equation method for the electromagnetic scattering from a scatterer on an absorbing plane*, Integral Equat. Operator Th., 39 (2001), pp. 159–181.
- [3] H. AMMARI, G. BAO, AND A. W. WOOD, *An integral equation method for the electromagnetic scattering from cavities*, Math. Meth. Appl. Sci., 23 (2000), pp. 1057–1072.
- [4] J. S. ASVESTAS AND R. E. KLEINMAN, *Electromagnetic scattering by indented screens*, IEEE Transactions on Antennas and Propagation, 42 (1994), pp. 22–30.
- [5] M. ELLER, V. ISAKOV, G. NAKAMURA, AND D. TATARU, *Uniqueness and stability in the Cauchy problem for Maxwell's and elasticity systems*, preprint.
- [6] P. M. GOGGANS AND T. H. SHUMPERT, *Backscatter RCS for TE and TM excitations of dielectric-filled cavity-backed apertures in two-dimensional bodies*, IEEE Transactions on Antenna and Propagation, Vol. 39 (1991), pp. 1224–1227.
- [7] CH. WEBER, *A local compactness theorem for Maxwell's equations*, Math. Meth. in Appl. Sci., 2 (1980), pp. 12–25.
- [8] V. VOGELANG, *On the strong unique continuation principle for inequalities of Maxwell type*, Math. Ann., 289 (1991), pp. 285–295.
- [9] W. D. WOOD, JR. AND A. W. WOOD, *Development and numerical solution of integral equations for electromagnetic scattering from a trough in a ground plane*, to appear in IEEE Transactions on Antennas and Propagation.

