METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 10, No. 1, pp. 067–080, March 2003

004

ASYMPTOTIC ANALYSIS OF THE RADIAL MINIMIZERS OF AN ENERGY FUNCTIONAL *

YUTIAN LEI†

Abstract. The author proves the $W^{1,p}$ convergence of the radial minimizers $u_\varepsilon = (u_{\varepsilon 1}, u_{\varepsilon 2}, u_{\varepsilon 3})$ of an energy functional as $\varepsilon \to 0$, and the zeros of $u_{\varepsilon 1}^2 + u_{\varepsilon 2}^2$ are located roughly. In addition, the estimates of the convergent rate of $u_{\varepsilon 3}^2$ are presented.

1. Introduction. Let $B = \{x \in R^2; x_1^2 + x_2^2 < 1\}$. Denote $S^1 = \{x \in R^3; x_1^2 + x_2^2 = 1, x_3 = 0\}$ and $S^2 = \{x \in R^3; x_1^2 + x_2^2 + x_3^2 = 1\}$. Let $g(x) = (e^{id\theta}, 0)$ where $x = (\cos\theta, \sin\theta)$ on ∂B, $d \in N$. We concern with the minimizer of the energy functional

$$E_\varepsilon(u, B) = \frac{1}{p}\int_B |\nabla u|^p dx + \frac{1}{2\varepsilon^p}\int_B u_3^2 dx \quad (p > 2)$$

in the function class

$$W = \{u(x) = (\sin f(r)e^{id\theta}, \cos f(r)) \in W^{1,p}(B, S^2); u|_{\partial B} = g\},$$

which is named the radial minimizer of $E_\varepsilon(u, B)$.

When $p = 2$, the functional $E_\varepsilon(u, B)$ was introduced in the study of some simplified model of high-energy physics, which controls the statics of planar ferromagnets and antiferromagnets (see [3][4]). The asymptotic behavior of minimizers of $E_\varepsilon(u, B)$ has been considered by Fengbo Hang and Fanghua Lin in [2]. (In particular, they discussed the asymptotic behavior of the radial minimizer of $E_\varepsilon(u, B)$ in §5.)

In this paper, we always assume $p > 2$. As in [2][3] and [4], we are interested in the behavior of minimizers of $E_\varepsilon(u, B)$ as $\varepsilon \to 0$. We will prove the $W^{1,p}_{loc}$ convergence of the radial minimizers. In addition, some estimates of the convergent rate of the radial minimizer will be presented and we will discuss the location of the points where $u_3^2 = 1$.

In polar coordinates, for $u(x) = (\sin f(r)e^{id\theta}, \cos f(r))$, we have

$$|\nabla u| = (f_r^2 + d^2 r^{-2}\sin^2 f)^{1/2},$$

$$\int_B |\nabla u|^p dx = 2\pi\int_0^1 r(f_r^2 + d^2 r^{-2}\sin^2 f)^{p/2} dr.$$

If we denote

$$V = \{f \in W^{1,p}_{loc}(0,1]; r^{1/p} f_r, r^{(1-p)/p}\sin f \in L^p(0,1), f(r) \geq 0, f(1) = \frac{\pi}{2}\},$$

then $V = \{f(r); u(x) = (\sin f(r)e^{id\theta}, \cos f(r)) \in W\}$.

*Received September 16, 2002; accepted for publication June 19, 2003.
†Department of Mathematics, Nanjing Normal University, Nanjing, Jiangsu, 210097, P.R.China (lythxl@163.com).

REMARK. $V \subset \{f \in C[0,1]; f(0)=0\}$.
In fact, if $f \in V$, denote $h(r) = f(r^{1+\frac{1}{p-2}})$, then

$$\begin{aligned}\int_0^1 |h'(r)|^p dr &= (1+\tfrac{1}{p-2})^p \int_0^1 |f'(r^{1+\frac{1}{p-2}})|^p r^{\frac{p}{p-2}} dr \\ &= (1+\tfrac{1}{p-2})^p (1-\tfrac{1}{p-1}) \int_0^1 s|f'(s)|^p ds < \infty\end{aligned}$$

which implies that $h(r) \in C[0,1]$ and hence $f(r) \in C[0,1]$.

Suppose $f(0) > 0$, then $f(r) \geq s > 0$ for $r \in [0,t)$ with $t > 0$ sufficiently small. This implies that $\sin f(r) \geq \sin s > 0$. *Noting $p > 2$, we have*

$$\int_0^1 r^{1-p} \sin^p f dr \geq \sin^p s \int_0^t r^{1-p} dr = \infty$$

which contradicts $r^{1/p-1} \sin f \in L^p(0,1)$. Therefore $f(0) = 0$.

Substituting $u(x) = (\sin f(r) e^{id\theta}, \cos f(r)) \in W$ into $E_\varepsilon(u,B)$ we obtain

$$E_\varepsilon(u,B) = 2\pi E_\varepsilon(f,(0,1)),$$

where

$$E_\varepsilon(f,(0,1)) = \int_0^1 [\frac{1}{p}(f_r^2 + d^2 r^{-2} \sin^2 f)^{p/2} + \frac{1}{2\varepsilon^p} \cos^2 f] r dr.$$

This shows that $u = (\sin f(r) e^{id\theta}, \cos f(r)) \in W$ is the minimizer of $E_\varepsilon(u,B)$ if and only if $f(r) \in V$ is the minimizer of $E_\varepsilon(f,(0,1))$. Applying the direct method in the calculus of variations we can see that the functional $E_\varepsilon(u,B)$ achieves its minimum on W by a function $u_\varepsilon(x) = (\sin f_\varepsilon(r) e^{id\theta}, \cos f_\varepsilon(r))$, hence $f_\varepsilon(r)$ is the minimizer of $E_\varepsilon(f,(0,1))$.

We will prove the following

THEOREM 1.1. *Let u_ε be a radial minimizer of $E_\varepsilon(u,B)$ on W. Then for any $\gamma \in (0,1)$, there exists a constant $h = h(\gamma)$ which is independent of $\varepsilon \in (0,1)$ such that $Z_\varepsilon = \{x \in B; |u_{\varepsilon 3}| > \gamma\} \subset B(0, h\varepsilon)$.*

This theorem shows that all the points where $u_{\varepsilon 3}^2 = 1$ are contained in $B(0,h\varepsilon)$. Hence as $\varepsilon \to 0$, these points converge to 0.

THEOREM 1.2. *Let $u_\varepsilon(x) = (\sin f_\varepsilon(r) e^{id\theta}, \cos f_\varepsilon(r))$ be a radial minimizer of $E_\varepsilon(u,B)$ on W. Then*

$$\lim_{\varepsilon \to 0} u_\varepsilon = (e^{id\theta}, 0), \quad in \quad W^{1,p}(K, R^3)$$

for any compact subset $K \subset \overline{B} \setminus \{0\}$.

THEOREM 1.3. *(convergent rate) Let $u_\varepsilon(x) = (\sin f_\varepsilon(r) e^{id\theta}, \cos f_\varepsilon(r))$ be a radial minimizer of $E_\varepsilon(u,B)$ on W. Then for any $\eta \in (0,1)$, there exist $C, \varepsilon_0 > 0$ such that as $\varepsilon \in (0,\varepsilon_0)$,*

$$\int_\eta^1 r[(f_\varepsilon')^p + \frac{1}{\varepsilon^p} \cos^2 f_\varepsilon] dr \leq C\varepsilon^p.$$

$$\sup_{x\in K} |u_{\varepsilon 3}(x)| \leq C\varepsilon^{\frac{p-2}{2}}.$$

Here $K = \overline{B} \setminus B(0,\eta)$.

The proof of Theorem 1.1 will be given in §2. In §3,we will set up the uniform estimate of $E_\varepsilon(u_\varepsilon, K)$ which implies the conclusion of Theorem 1.2 (see §4). By virtue of the uniform estimate we can also derive the proof of Theorem 1.3 in §5.

2. Proof of Theorem 1.1.

PROPOSITION 2.1. *Let* f_ε *be a minimizer of* $E_\varepsilon(f,(0,1))$. *Then*

$$E_\varepsilon(f_\varepsilon,(0,1)) \leq C\varepsilon^{2-p}$$

with a constant C *independent of* $\varepsilon \in (0,1)$.

Proof. Denote

$$I(\varepsilon,R) = Min\{\int_0^R [\frac{1}{p}(f_r^2 + \frac{d^2}{r^2}\sin^2 f)^{\frac{p}{2}} + \frac{1}{2\varepsilon^p}\cos^2 f]rdr; f \in V_R\}$$

where

$$\begin{aligned} V_R = \{f(r) \quad & \in W^{1,p}_{loc}(0,R]; f(R) = \frac{\pi}{2}, \\ & \sin f(r) r^{\frac{1}{p}-1}, f'(r) r^{\frac{1}{p}} \in L^p(0,R)\}. \end{aligned}$$

Then

$$\begin{aligned} & I(\varepsilon,1) = E_\varepsilon(f_\varepsilon,(0,1)) \\ = & \frac{1}{p}\int_0^1 r((f_\varepsilon)_r^2 + d^2 r^{-2}(\sin f_\varepsilon)^2)^{p/2} dr + \frac{1}{2\varepsilon^p}\int_0^1 r\cos^2 f_\varepsilon dr \\ = & \frac{1}{p}\int_0^{1/\varepsilon} \varepsilon^{2-p} s((f_\varepsilon)_s^2 + d^2 s^{-2}\sin^2 f_\varepsilon)^{p/2} ds + \frac{1}{2\varepsilon^p}\int_0^{\varepsilon^{-1}} \varepsilon^2 s\cos^2 f_\varepsilon ds \\ = & \varepsilon^{2-p} I(1,\varepsilon^{-1}). \end{aligned} \tag{2.1}$$

Let f_1 be the minimizer for $I(1,1)$ and define

$$f_2 = f_1, \quad as \quad 0 < s < 1; \quad f_2 = \frac{\pi}{2}, \quad as \quad 1 \leq s \leq \varepsilon^{-1}.$$

We have

$$\begin{aligned} I(1,\varepsilon^{-1}) \quad & \leq \frac{1}{p}\int_0^{\varepsilon^{-1}} s[(f_2')^2 + d^2 s^{-2}\sin^2 f_2]^{p/2} ds + \frac{1}{2}\int_0^{\varepsilon^{-1}} s\cos^2 f_2 ds \\ & \leq \frac{1}{p}\int_1^{\varepsilon^{-1}} s^{1-p} d^p ds + \frac{1}{p}\int_0^1 s((f_1')^2 + d^2 s^{-2}\sin^2 f_1)^{p/2} ds \\ & \quad + \frac{1}{2}\int_0^1 s\cos^2 f_1 ds \\ & = \frac{d^p}{p(p-2)}(1-\varepsilon^{p-2}) + I(1,1) \leq \frac{d^p}{p(p-2)} + I(1,1) = C. \end{aligned}$$

Substituting into (2.1) follows the conclusion of Proposition 2.1.

By the embedding theorem we first derive from $|u_\varepsilon| = 1$ and proposition 2.1 the following

PROPOSITION 2.2. *Let u_ε be a radial minimizer of $E_\varepsilon(u, B)$. Then there exists a constant C independent of $\varepsilon \in (0,1)$ such that*

$$|u_\varepsilon(x) - u_\varepsilon(x_0)| \leq C\varepsilon^{(2-p)/p}|x - x_0|^{1-2/p}, \qquad \forall x, x_0 \in B.$$

As a corollary of Proposition 2.1 we have

PROPOSITION 2.3. *Let u_ε be a radial minimizer of $E_\varepsilon(u, B)$. Then for some constant C independent of $\varepsilon \in (0,1]$ such that*

$$\frac{1}{\varepsilon^2}\int_B u_{\varepsilon 3}^2 dx \leq C.$$

Based on Proposition 2.2, we have the interesting result:

PROPOSITION 2.4. *Let u_ε be a radial minimizer of $E_\varepsilon(u, B)$. Then for any $\gamma \in (0,1)$, there exist positive constants λ, μ independent of $\varepsilon \in (0,1)$ such that if*

$$\frac{1}{\varepsilon^2}\int_{B\cap B^{2l\varepsilon}} u_{\varepsilon 3}^2 dx \leq \mu \tag{2.2}$$

where $B^{2l\varepsilon}$ is some disc of radius $2l\varepsilon$ with $l \geq \lambda$, then

$$|u_{\varepsilon 3}(x)| \leq \gamma, \qquad \forall x \in B \cap B^{l\varepsilon}. \tag{2.3}$$

Proof. First we observe that there exists a constant $\beta > 0$ such that for any $x \in B$ and $0 < \rho \leq 1$,

$$mes(B \cap B(x, \rho)) \geq \beta\rho^2.$$

To prove the proposition, we choose

$$\lambda = (\frac{\gamma}{2C})^{\frac{p}{p-2}}, \mu = \frac{\beta}{4}(\frac{1}{2C})^{\frac{2p}{p-2}}\gamma^{2+\frac{2p}{p-2}}$$

where C is the constant in Proposition 2.2.

Suppose that there is a point $x_0 \in B \cap B^{l\varepsilon}$ such that (2.3) is not true, i.e.

$$|u_{\varepsilon 3}(x_0)| > \gamma. \tag{2.4}$$

Then applying Proposition 2.2 we have

$$\begin{aligned}|u_\varepsilon(x) - u_\varepsilon(x_0)| &\leq C\varepsilon^{(2-p)/p}|x - x_0|^{1-2/p} \leq C\varepsilon^{(2-p)/p}(\lambda\varepsilon)^{1-2/p} \\ &= C\lambda^{1-2/p} = \frac{\gamma}{2}, \qquad \forall x \in B(x_0, \lambda\varepsilon)\end{aligned}$$

which implies that

$$|u_{\varepsilon 3}(x) - u_{\varepsilon 3}(x_0)| \leq \frac{\gamma}{2}.$$

Noticing (2.4), we obtain

$$|u_{\varepsilon 3}(x)|^2 \geq [|u_{\varepsilon 3}(x_0)| - \frac{\gamma}{2}]^2 > \frac{\gamma^2}{4}, \qquad \forall x \in B(x_0, \lambda\varepsilon).$$

Hence

$$\begin{aligned}\int_{B(x_0,\lambda\varepsilon)\cap B} u_{\varepsilon 3}^2 dx &> \frac{\gamma^2}{4} mes(B \cap B(x_0, \lambda\varepsilon)) \\ &\geq \beta\frac{\gamma^2}{4}(\lambda\varepsilon)^2 = \beta\frac{\gamma^2}{4}(\frac{\gamma}{2C})^{\frac{2p}{p-2}}\varepsilon^2 = \mu\varepsilon^2.\end{aligned} \tag{2.5}$$

Since $x_0 \in B^{l\varepsilon} \cap B$, and $(B(x_0, \lambda\varepsilon) \cap B) \subset (B^{2l\varepsilon} \cap B)$, (2.5) implies

$$\int_{B^{2l\varepsilon}\cap B} u_{\varepsilon 3}^2 dx > \mu\varepsilon^2,$$

which contradicts (2.2) and thus the proposition is proved.

To find the points where $u_{\varepsilon 3}^2 = 1$ based on Proposition 2.4, we may take (2.2) as the ruler to distinguish the discs of radius $\lambda\varepsilon$ which contain these points.

Let u_ε be a radial minimizer of $E_\varepsilon(u, B)$. Given $\gamma \in (0, 1)$. Let λ, μ be constants in Proposition 2.4 corresponding to γ. If

$$\frac{1}{\varepsilon^2}\int_{B(x^\varepsilon,2\lambda\varepsilon)\cap B} u_{\varepsilon 3}^2 dx \leq \mu,$$

then $B(x^\varepsilon, \lambda\varepsilon)$ is called $\gamma-$ good disc, or simply good disc. Otherwise $B(x^\varepsilon, \lambda\varepsilon)$ is called $\gamma-$ bad disc or simply bad disc.

Now suppose that $\{B(x_i^\varepsilon, \lambda\varepsilon), i \in I\}$ is a family of discs satisfying

$$(i) : x_i^\varepsilon \in B, i \in I; \qquad (ii) : B \subset \cup_{i\in I} B(x_i^\varepsilon, \lambda\varepsilon);$$

$$(iii) : B(x_i^\varepsilon, \lambda\varepsilon/4) \cap B(x_j^\varepsilon, \lambda\varepsilon/4) = \emptyset, i \neq j. \tag{2.6}$$

Denote

$$J_\varepsilon = \{i \in I; B(x_i^\varepsilon, \lambda\varepsilon) \ \ is \ \ a \ \ bad \ \ disc\}.$$

Proposition 2.5. *There exists a positive integer N such that the number of bad discs $Card\ J_\varepsilon \leq N$.*

Proof. Since (2.6) implies that every point in B can be covered by finite, say m (independent of ε) discs, from Proposition 2.3 and the definition of bad discs,we have

$$\begin{aligned}\mu\varepsilon^2 Card J_\varepsilon &\leq \textstyle\sum_{i\in J_\varepsilon}\int_{B(x_i^\varepsilon,2\lambda\varepsilon)\cap B} u_{\varepsilon 3}^2 dx \\ &\leq m\int_{\cup_{i\in J_\varepsilon}B(x_i^\varepsilon,2\lambda\varepsilon)\cap B} u_{\varepsilon 3}^2 dx \leq m\int_B u_{\varepsilon 3}^2 dx \leq mC\varepsilon^2\end{aligned}$$

and hence $Card \ \ J_\varepsilon \leq \frac{mC}{\mu} \leq N$.

Applying TheoremIV.1 in [1], we may modify the family of bad discs such that the new one, denoted by $\{B(x_i^\varepsilon, h\varepsilon); i \in J\}$, satisfies

$$\cup_{i\in J_\varepsilon} B(x_i^\varepsilon, \lambda\varepsilon) \subset \cup_{i\in J} B(x_i^\varepsilon, h\varepsilon), \quad \lambda \leq h; \quad Card\ J \leq Card\ J_\varepsilon,$$

$$|x_i^\varepsilon - x_j^\varepsilon| > 8h\varepsilon, i, j \in J, i \neq j.$$

The last condition implies that every two discs in the new family are not intersected. From Proposition 2.4 it is deduced that all the points where $|u_{\varepsilon 3}| = 1$ are contained in these finite, disintersected bad discs.

Now we prove our main result of this section.

THEOREM 2.6. *Let u_ε be a radial minimizer of $E_\varepsilon(u, B)$. Then for any $\gamma \in (0,1)$, there exists a constant $h = h(\gamma)$ independent of $\varepsilon \in (0,1)$ such that $Z_\varepsilon = \{x \in B; |u_{\varepsilon 3}(x)| > \gamma\} \subset B(0, h\varepsilon)$.*

Proof. Suppose there exists a point $x_0 \in Z_\varepsilon$ such that $x_0 \overline{\in} B(0, h\varepsilon)$. Then all points on the circle $S_0 = \{x \in B;\ |x| = |x_0|\}$ satisfy

$$u_{\varepsilon 3}^2(x) = \cos^2 f_\varepsilon(|x|) = \cos^2 f_\varepsilon(|x_0|) = u_{\varepsilon 3}^2(x_0) > \gamma^2.$$

By virtue of Proposition 2.4 we can see that all points on S_0 are contained in bad discs. However, since $|x_0| \geq h\varepsilon, S_0$ can not be covered by a single bad disc. As a result, S_0 has to be covered by at least two bad disintersected discs. This is impossible.

This theorem implies that all the zeros of $u_{\varepsilon 1}^2 + u_{\varepsilon 2}^2$ are contained in $B(0, h\varepsilon)$. In particular the zeros are converge to 0 as $\varepsilon \to 0$.

3. Uniform estimate. Let $u_\varepsilon(x) = (\sin f_\varepsilon(r) e^{id\theta}, \cos f_\varepsilon(r))$ be a radial minimizer of $E_\varepsilon(u, B)$, namely f_ε be a minimizer of

$$E_\varepsilon(f, (0,1)) = \frac{1}{p}\int_0^1 (f_r^2 + d^2 r^{-2} \sin^2 f)^{p/2} r dr + \frac{1}{2\varepsilon^p}\int_0^1 \cos^2 f r dr$$

in V. From Proposition 2.1, we have

$$E_\varepsilon(f_\varepsilon, (0,1)) \leq C\varepsilon^{2-p} \tag{3.1}$$

for some constant C independent of $\varepsilon \in (0,1)$.

In this section we further prove that for any $\eta \in (0,1)$, there exists a constant $C(\eta)$ such that

$$E_\varepsilon(f_\varepsilon; \eta) := E_\varepsilon(f_\varepsilon, (\eta, 1)) \leq C(\eta) \tag{3.2}$$

for $\varepsilon \in (0, \varepsilon_0)$ with small $\varepsilon_0 > 0$. Based on the estimate (3.2) and Theorem 2.6, we may obtain better convergence for minimizers, namely the $W_{loc}^{1,p}$ convergence.

To establish (3.2) we first prove

PROPOSITION 3.1. *Given $\eta \in (0,1)$. There exist constants*

$$\eta_j \in [\frac{(j-1)\eta}{N+1}, \frac{j\eta}{N+1}], (N = [p])$$

and C_j, such that

$$E_\varepsilon(f_\varepsilon, \eta_j) \leq C_j \varepsilon^{j-p} \tag{3.3}$$

for $j = 2, ..., N$, where $\varepsilon \in (0, \varepsilon_0)$.

Proof. For $j=2$, the inequality (3.3) is just the one in Proposition 2.1.

Suppose that (3.3) holds for all $j \le n$. Then we have, in particular

$$E_\varepsilon(f_\varepsilon;\eta_n) \le C_n\varepsilon^{n-p}. \tag{3.4}$$

If $n=N$ then we are done. Suppose $n<N$. We want to prove (3.3) for $j=n+1$.

Obviously (3.4) implies

$$\frac{1}{p}\int_{\frac{n\eta}{N+1}}^{\frac{(n+1)\eta}{N+1}} [(f_\varepsilon)_r^2 + d^2r^{-2}\sin^2 f_\varepsilon]^{p/2} r dr + \frac{1}{4\varepsilon^p}\int_{\frac{n\eta}{N+1}}^{\frac{(n+1)\eta}{N+1}} \cos^2 f_\varepsilon r dr \le C_n\varepsilon^{n-p}$$

from which we see by integral mean value theorem that there exists

$$\eta_{n+1} \in [\frac{n\eta}{N+1}, \frac{(n+1)\eta}{N+1}]$$

such that

$$[(f_\varepsilon)_r^2 + d^2r^{-2}\sin^2 f_\varepsilon]^{p/2}_{r=\eta_{n+1}} \le C_n\varepsilon^{n-p}, \tag{3.5}$$

$$[\frac{1}{\varepsilon^p}\cos^2 f_\varepsilon]_{r=\eta_{n+1}} \le C_n\varepsilon^{n-p}. \tag{3.6}$$

Consider the functional

$$E(\rho,\eta_{n+1}) = \frac{1}{p}\int_{\eta_{n+1}}^1 (\rho_r^2+1)^{p/2}dr + \frac{1}{\varepsilon^p}\int_{\eta_{n+1}}^1 \cos^2\rho dr.$$

It is easy to prove that the minimizer ρ_1 of $E(\rho,\eta_{n+1})$ in $W^{1,p}_{f_\varepsilon}((\eta_{n+1},1),R^+)$ exists and satisfies

$$-\varepsilon^p(v^{(p-2)/2}\rho_r)_r = \sin 2\rho, \quad in \ \ (\eta_{n+1},1) \tag{3.7}$$

$$\rho|_{r=\eta_{n+1}} = f_\varepsilon, \quad \rho|_{r=1} = f_\varepsilon(1) = \frac{\pi}{2} \tag{3.8}$$

where $v=\rho_r^2+1$. It follows from the maximum principle that $\rho_1 \le \pi/2$ and

$$\sin^2\rho(r) \ge \sin^2\rho(\eta_{n+1}) = \sin^2 f_\varepsilon(\eta_{n+1}) = 1-\cos^2 f_\varepsilon(\eta_{n+1}) \ge 1-\gamma^2, \tag{3.9}$$

the last inequality of which is implied by Theorem 2.6.

Applying (3.4) we see easily that

$$E(\rho_1;\eta_{n+1}) \le E(f_\varepsilon;\eta_{n+1}) \le C_nE_\varepsilon(f_\varepsilon;\eta_{n+1}) \le C_n\varepsilon^{n-p} \tag{3.10}$$

for $\varepsilon \in (0,\varepsilon_0)$ with $\varepsilon_0>0$ sufficiently small.

Now choosing a smooth function $\zeta(r)$ such that $\zeta=1$ on $(0,\eta), \zeta=0$ near $r=1$, multiplying (3.7) by $\zeta\rho_r(\rho=\rho_1)$ and integrating over $(\eta_{n+1},1)$ we obtain

$$v^{(p-2)/2}\rho_r^2|_{r=\eta_{n+1}} + \int_{\eta_{n+1}}^1 v^{(p-2)/2}\rho_r(\zeta_r\rho_r+\zeta\rho_{rr})dr = \frac{1}{\varepsilon^p}\int_{\eta_{n+1}}^1 \sin 2\rho\zeta\rho_r dr. \tag{3.11}$$

Using (3.10) we have

$$\begin{aligned} &|\int_{\eta_{n+1}}^{1} v^{(p-2)/2}\rho_r(\zeta_r\rho_r+\zeta\rho_{rr})dr| \\ &\leq \int_{\eta_{n+1}}^{1} v^{(p-2)/2}|\zeta_r|\rho_r^2 dr+\frac{1}{p}|\int_{\eta_{n+1}}^{1}(v^{p/2}\zeta)_r dr-\int_{\eta_{n+1}}^{1} v^{p/2}\zeta_r dr| \\ &\leq C\int_{\eta_{n+1}}^{1} v^{p/2}dr+\frac{1}{p}v^{p/2}|_{r=\eta_{n+1}}+\frac{C}{p}\int_{\eta_{n+1}}^{1} v^{p/2}dr \\ &\leq C\int_{\eta_{n+1}}^{1} v^{p/2}dr+\frac{1}{p}v^{p/2}|_{r=\eta_{n+1}}\leq C_n\varepsilon^{n-p}+\frac{1}{p}v^{p/2}|_{r=\eta_{n+1}} \end{aligned} \tag{3.12}$$

and using (3.6)(3.10) we have

$$\begin{aligned} &|\frac{1}{\varepsilon^p}\int_{\eta_{n+1}}^{1}\zeta\rho_r\sin 2\rho dr|=\frac{1}{\varepsilon^p}|\int_{\eta_{n+1}}^{1}\zeta_r\cos^2\rho dr-\int_{\eta_{n+1}}^{1}(\zeta\cos^2\rho)_r dr| \\ &\leq \frac{1}{\varepsilon^p}\cos^2\rho|_{r=\eta_{n+1}}+\frac{C}{\varepsilon^p}\int_{\eta_{n+1}}^{1}\cos^2\rho dr\leq C_n\varepsilon^{n-p}. \end{aligned} \tag{3.13}$$

Combining (3.11) with (3.12)(3.13) yields

$$v^{(p-2)/2}\rho_r^2|_{r=\eta_{n+1}}\leq C_n\varepsilon^{n-p}+\frac{1}{p}v^{p/2}|_{r=\eta_{n+1}}.$$

Hence

$$\begin{aligned} v^{p/2}|_{r=\eta_{n+1}} &= v^{(p-2)/2}(\rho_r^2+1)|_{r=\eta_{n+1}}=v^{(p-2)/2}\rho_r^2|_{r=\eta_{n+1}}+v^{(p-2)/2}|_{r=\eta_{n+1}} \\ &\leq C_n\varepsilon^{n-p}+\frac{1}{p}v^{p/2}|_{r=\eta_{n+1}}+v^{(p-2)/2}|_{r=\eta_{n+1}} \\ &\leq C_n\varepsilon^{n-p}+\frac{1}{p}v^{p/2}|_{r=\eta_{n+1}}+\delta v^{p/2}|_{r=\eta_{n+1}}+C(\delta) \\ &= C_n\varepsilon^{n-p}+(\frac{1}{p}+\delta)v^{p/2}|_{r=\eta_{n+1}}+C(\delta) \end{aligned}$$

from which it follows by choosing $\delta>0$ small enough that

$$v^{p/2}|_{r=\eta_{n+1}}\leq C_n\varepsilon^{n-p}. \tag{3.14}$$

Noting (3.9), we can see $\sin\rho>0$. Multiply both sides of (3.7) by $\cot\rho=\frac{\cos\rho}{\sin\rho}$ and integrate. Then

$$-\varepsilon^p v^{(p-2)/2}\rho_r\cot\rho|_{\eta_{n+1}}^{1}=\varepsilon^p\int_{\eta_{n+1}}^{1} v^{(p-2)/2}\rho_r^2\frac{1}{\sin^2\rho}dr+2\int_{\eta_{n+1}}^{1}\cos^2\rho dr.$$

Noting $\cot\rho(1)=0$ (which is implied by (3.8)) and $\frac{1}{\sin^2\rho}\geq 1$, we have

$$\begin{aligned} &E(\rho_1;\eta_{n+1})=\frac{1}{p}\int_{\eta_{n+1}}^{1} v^{p/2}dr+\frac{1}{\varepsilon^p}\int_{\eta_{n+1}}^{1}\cos^2\rho dr \\ &\leq C[\int_{\eta_{n+1}}^{1} v^{(p-2)/2}\rho_r^2 dr+\frac{1}{\varepsilon^p}\int_{\eta_{n+1}}^{1}\cos^2\rho dr]\leq Cv^{(p-2)/2}\rho_r\cot\rho|_{r=\eta_{n+1}}. \end{aligned}$$

From this, using(3.14)(3.6) and noticing that $n < p$, we obtain

$$\begin{aligned} &E(\rho_1;\eta_{n+1}) \le Cv^{(p-2)/2}\rho_r \cot\rho|_{r=\eta_{n+1}} \\ &\le Cv^{(p-1)/2}\cot\rho|_{r=\eta_{n+1}} \le (C_n\varepsilon^{n-p})^{(p-1)/p}(\frac{C_n\varepsilon^n}{1-C_n\varepsilon^n})^{1/2} \\ &\le C_{n+1}\varepsilon^{n+1-p+(n/2-n/p)} \le C_{n+1}\varepsilon^{n+1-p}. \end{aligned} \tag{3.15}$$

Define

$$w_\varepsilon = f_\varepsilon,\ for\ r\in(0,\eta_{n+1});\qquad w_\varepsilon=\rho_1,\ for\ r\in[\eta_{n+1},1].$$

Since f_ε is a minimizer of $E_\varepsilon(f)$, we have

$$E_\varepsilon(f_\varepsilon)\le E_\varepsilon(w_\varepsilon),$$

namely,

$$\begin{aligned} &E_\varepsilon(f_\varepsilon;\eta_{n+1}) \\ &\le \frac{1}{p}\int_{\eta_{n+1}}^1 (\rho_r^2+d^2r^{-2}\sin^2\rho)^{p/2}rdr + \frac{1}{\varepsilon^p}\int_{\eta_{n+1}}^1 \cos^2\rho r dr \\ &\le \frac{C}{p}\int_{\eta_{n+1}}^1 (\rho_r^2+1)^{p/2}dr + \frac{C}{2\varepsilon^p}\int_{\eta_{n+1}}^1 \cos^2\rho dr + C \\ &= CE(\rho_1;\eta_{n+1})+C. \end{aligned}$$

Thus, using (3.15) yields

$$E_\varepsilon(f_\varepsilon;\eta_{n+1})\le C_{n+1}\varepsilon^{n-p+1}$$

for $\varepsilon\in(0,\varepsilon_0)$. This is just (3.3) for $j=n+1$.

Proposition 3.2. *Given $\eta\in(0,1)$. There exist constants $\eta_{N+1}\in[\frac{N\eta}{N+1},\eta]$ and C_{N+1} such that*

$$E_\varepsilon(f_\varepsilon;\eta_{N+1})\le C_{N+1}\varepsilon^{N-p+1}+\frac{1}{p}\int_{\eta_{N+1}}^1 \frac{d^p}{r^{p-1}}dr \tag{3.16}$$

where $N=[p]$.

Proof. Similar to the derivation of (3.6) we may obtain from Proposition 3.1 for $j=N$ that there exists $\eta_{N+1}\in[\frac{N\eta}{N+1},\frac{(N+1)\eta}{N+1}]$, such that

$$\frac{1}{\varepsilon^p}\cos^2 f_\varepsilon|_{r=\eta_{N+1}}\le C_N\varepsilon^{N-p}. \tag{3.17}$$

Also similarly, consider the functional

$$E(\rho,\eta_{N+1})=\frac{1}{p}\int_{\eta_{N+1}}^1(\rho_r^2+1)^{p/2}dr+\frac{1}{\varepsilon^p}\int_{\eta_{N+1}}^1\cos^2\rho dr$$

whose minimizer ρ_2 in $W^{1,p}_{f_\varepsilon}((\eta_{N+1},1),R^+)$ exists and satisfies

$$-\varepsilon^p(v^{(p-2)/2}\rho_r)_r=\sin 2\rho,\qquad in\ \ (\eta_{N+1},1) \tag{3.18}$$

$$\rho|_{r=\eta_{N+1}} = f_\varepsilon, \quad \rho|_{r=1} = f_\varepsilon(1) = \frac{\pi}{2}$$

where $v = \rho_r^2 + 1$. From (3.4) for $n = N$ it follows immediately that

$$\begin{aligned} E(\rho_2; \eta_{N+1}) &\le E(f_\varepsilon; \eta_{N+1}) \le C_N E_\varepsilon(f_\varepsilon; \eta_{N+1}) \\ &\le C_N E_\varepsilon(f_\varepsilon; \eta_N) \le C_N \varepsilon^{N-p}. \end{aligned}$$

Similar to the proof of (3.14) and (3.15), we get from (3.17) that

$$v^{p/2}|_{r=\eta_{N+1}} \le C_N \varepsilon^{N-p},$$

$$E(\rho_2; \eta_{N+1}) \le C_{N+1} \varepsilon^{N+1-p}. \tag{3.19}$$

Now we define

$$w_\varepsilon = f_\varepsilon, \; for \; r \in (0, \eta_{N+1}); \qquad w_\varepsilon = \rho_2, \; for \; r \in [\eta_{N+1}, 1]$$

and then we have

$$E_\varepsilon(f_\varepsilon) \le E_\varepsilon(w_\varepsilon).$$

Notice that

$$\begin{aligned} &\int_{\eta_{N+1}}^1 (\rho_r^2 + d^2 r^{-2} \sin^2 \rho)^{p/2} r dr - \int_{\eta_{N+1}}^1 (d^2 r^{-2} \sin^2 \rho)^{p/2} r dr \\ &= \frac{p}{2} \int_{\eta_{N+1}}^1 \int_0^1 [(\rho_r^2 + d^2 r^{-2} \sin^2 \rho) s + (d^2 r^{-2} \sin^2 \rho)(1-s)]^{(p-2)/2}] ds \rho_r^2 r dr \\ &\le C \int_{\eta_{N+1}}^1 \int_0^1 [(\rho_r^2 + d^2 r^{-2} \sin^2 \rho)^{(p-2)/2} s^{(p-2)/2} \\ &\quad + (d^2 r^{-2} \sin^2 \rho)^{(p-2)/2} (1-s)^{(p-2)/2}] ds \rho_r^2 r dr \\ &= C \int_{\eta_{N+1}}^1 (\rho_r^2 + d^2 r^{-2} \sin^2 \rho)^{(p-2)/2} \rho_r^2 r dr \int_0^1 s^{(p-2)/2} ds \\ &\quad + C \int_{\eta_{N+1}}^1 (d^2 r^{-2} \sin^2 \rho)^{(p-2)/2} \rho_r^2 r dr \int_0^1 (1-s)^{(p-2)/2} ds \\ &\le C(\int_{\eta_{N+1}}^1 \rho_r^p dr + \int_{\eta_{N+1}}^1 \rho_r^2 dr) \le C \int_{\eta_{N+1}}^1 (\rho_r^2 + 1)^{p/2} dr. \end{aligned}$$

Hence

$$\begin{aligned} &E_\varepsilon(f_\varepsilon; \eta_{N+1}) \le \frac{1}{p} \int_{\eta_{N+1}}^1 ((\rho_2)_r^2 + d^2 r^{-2} (\sin \rho_2)^2)^{p/2} r dr \\ &\quad + \frac{1}{2\varepsilon^p} \int_{\eta_{N+1}}^1 (\cos \rho_2)^2 r dr \\ &\le \frac{1}{p} \int_{\eta_{N+1}}^1 (d^2 r^{-2} \sin^2 \rho)^{p/2} r dr + \frac{C}{2\varepsilon^p} \int_{\eta_{N+1}}^1 (\cos \rho_2)^2 dr \\ &\quad + C \int_{\eta_{N+1}}^1 ((\rho_2)_r^2 + 1)^{p/2} dr \\ &\le \frac{1}{p} \int_{\eta_{N+1}}^1 r (d^2 r^{-2})^{p/2} dr + C E(\rho_2; \eta_{N+1}). \end{aligned}$$

Using (3.19) we have

$$E_\varepsilon(f_\varepsilon;\eta_{N+1}) \le \frac{1}{p}\int_{\eta_{N+1}}^{1} r(d^2r^{-2})^{p/2}dr + C_{N+1}\varepsilon^{N-p+1}.$$

This is my conclusion.

4. Proof of Theorem 1.2. THEOREM 4.1. *Let* $u_\varepsilon = (\sin f_\varepsilon(r)e^{id\theta}, \cos f_\varepsilon(r))$ *be a minimizer of* $E_\varepsilon(u,B)$ *in* W. *Then*

$$\lim_{\varepsilon\to 0} f_\varepsilon = \frac{\pi}{2}, \qquad in \;\; W^{1,p}((\eta,1],R) \tag{4.1}$$

$$\lim_{\varepsilon\to 0} u_\varepsilon = (e^{id\theta},0), \qquad in \;\; W^{1,p}(K,R^3) \tag{4.2}$$

for any $\eta\in(0,1)$ *and compact subset* $K\subset \overline{B}\setminus\{0\}$.

Proof. It suffices to prove (4.2), since (4.2) implies (4.1). Without loss of generality, we may assume $K = B\setminus B(0,\eta_{N+1})$. From Proposition 3.2, We have

$$E_\varepsilon(u_\varepsilon,K) = 2\pi E_\varepsilon(f_\varepsilon,\eta_{N+1}) \le C$$

where C is independent of ε, namely

$$\int_K |\nabla u_\varepsilon|^p dx \le C, \tag{4.3}$$

$$\int_K |u_{\varepsilon 3}|^2 dx \le C\varepsilon^p. \tag{4.4}$$

(4.3) and $|u_\varepsilon| = 1$ imply the existence of a subsequence u_{ε_k} of u_ε and a function $u_*\in W^{1,p}(K,R^3)$, such that

$$\lim_{\varepsilon_k\to 0} u_{\varepsilon_k} = u_*, \qquad weakly \;\; in \;\; W^{1,p}(K,R^3)$$

$$\lim_{\varepsilon_k\to 0} u_{\varepsilon_k} = u_*, \qquad in \;\; C^\alpha(K,R^3), \alpha\in(0,1-\frac{2}{p}). \tag{4.5}$$

(4.4) and (4.5) imply $u_* = (e^{id\theta},0)$. Noticing that any subsequence of u_ε has a convergence subsequence and the limit is always $(e^{id\theta},0)$, we can assert

$$\lim_{\varepsilon\to 0} u_\varepsilon = (e^{id\theta},0), \qquad weakly \;\; in \;\; W^{1,p}(K,R^3). \tag{4.6}$$

From this and the weakly lower semicontinuity of $\int_K |\nabla u|^p$, using Proposition 3.2, we have

$$\begin{aligned}\int_K |\nabla e^{id\theta}|^p dx &\le \underline{\lim}_{\varepsilon_k\to 0}\int_K |\nabla u_\varepsilon|^p dx \le \overline{\lim}_{\varepsilon_k\to 0}\int_K |\nabla u_\varepsilon|^p dx \\ &\le C\lim_{\varepsilon\to 0}\varepsilon^{N+1-p} + 2\pi\int_{\eta_{N+1}}^1 (d^2r^{-2})^{p/2} r dr\end{aligned}$$

and hence

$$\lim_{\varepsilon\to 0}\int_K |\nabla u_\varepsilon|^p dx = \int_K |\nabla e^{id\theta}|^p dx$$

since $\int_K |\nabla e^{id\theta}|^p dx = 2\pi\int_{\eta_{N+1}}^1 (d^2r^{-2})^{p/2} r dr$. Combining this with (4.6)(4.5) complete the proof of (4.2).

5. Estimates of convergent rate. Assume $K = \overline{B} \setminus B_R$, where $R \in (0, 1/2)$. We will prove the following

THEOREM 5.1. *Let $u_\varepsilon(x) = (\sin f_\varepsilon(r)e^{id\theta}, \cos f_\varepsilon(r))$ be a radial minimizer of $E_\varepsilon(u, B)$ on W. Then there exist $C, \varepsilon_0 > 0$ such that as $\varepsilon \in (0, 1)$,*

$$\int_R^1 r[(f'_\varepsilon)^p + \frac{1}{\varepsilon^p}\cos^2 f_\varepsilon]dr \le C\varepsilon^p. \tag{5.1}$$

$$\sup_{x\in K} |u_{\varepsilon 3}(x)| \le C\varepsilon^{\frac{p-2}{2}}. \tag{5.2}$$

(5.1) gives the estimate of the rate of f_ε's convergence to $\pi/2$ in $W^{1,p}(\eta, 1]$ sense, and that of convergence of $|u_{\varepsilon 3}(x)|$ to 0 in $C^0(K)$ sense is showed by (5.2).

Proof. First, it follows from Jensen's inequality that

$$\begin{aligned} 2\pi E_\varepsilon(f_\varepsilon;\eta) &= \frac{2\pi}{p}\int_\eta^1 [(f'_\varepsilon)^2 + \frac{d^2}{r^2}\sin^2 f_\varepsilon]^{p/2} r dr + \frac{\pi}{\varepsilon^p}\int_\eta^1 \cos^2 f_\varepsilon r dr \\ &\ge \frac{2\pi}{p}\int_\eta^1 (f'_\varepsilon)^p r dr + \frac{\pi}{\varepsilon^p}\int_\eta^1 \cos^2 f_\varepsilon r dr + \frac{2\pi}{p}\int_\eta^1 \frac{d^p}{r^p}\sin^p f_\varepsilon r dr. \end{aligned}$$

Combining this with (3.16) yields

$$\begin{aligned} &\frac{2\pi}{p}\int_\eta^1 (f'_\varepsilon)^p r dr + \frac{\pi}{\varepsilon^p}\int_\eta^1 \cos^2 f_\varepsilon r dr \\ \le\ &\frac{2\pi}{p}\int_\eta^1 \frac{d^p}{r^p}(1 - \sin^p f_\varepsilon) r dr + C\varepsilon^{[p]+1-p}. \end{aligned}$$

Noticing that

$$1 - \sin^p f_\varepsilon \le C(1 - \sin^2 f_\varepsilon) = C\cos^2 f_\varepsilon,$$

and (4.4), we obtain

$$\begin{aligned} &\frac{2\pi}{p}\int_\eta^1 (f'_\varepsilon)^p r dr + \frac{\pi}{\varepsilon^p}\int_\eta^1 \cos^2 f_\varepsilon r dr \\ \le\ &C\int_\eta^1 \frac{d^p}{r^p}\cos^2 f_\varepsilon r dr + C\varepsilon^{[p]+1-p} \\ \le\ &C\varepsilon^p + C\varepsilon^{[p]+1-p} \le C\varepsilon^{[p]+1-p}. \end{aligned} \tag{5.3}$$

Using (3.16), (5.3) and the integral mean value theorem we can see that there exists $\eta_1 \in [\eta, \eta(1 + 1/2)] \subset [R/2, R]$ such that

$$[(f_\varepsilon)_r^2 + d^2 r^{-2}\sin^2 f_\varepsilon]_{r=\eta_1} \le C_1, \tag{5.4}$$

$$[\frac{1}{\varepsilon^p}\cos^2 f_\varepsilon]_{r=\eta_1} \le C_1\varepsilon^{[p]-p+1}. \tag{5.5}$$

Consider the functional

$$E(\rho, \eta_1) = \frac{1}{p}\int_{\eta_1}^1 (\rho_r^2 + 1)^{p/2} dr + \frac{1}{2\varepsilon^p}\int_{\eta_1}^1 \cos^2\rho dr.$$

It is easy to prove that the minimizer ρ_3 of $E(\rho,\eta_1)$ in $W^{1,p}_{f_\varepsilon}((\eta_1,1),R^+)$ exists.

By the same way to proof of (3.15), using (5.4) and (5.5) we have

$$E(\rho_3,\eta_1)\le v^{\frac{p-2}{2}}\rho_{3r}cot\rho_3|_{r=\eta_1}$$
$$\le C_1 cot\rho_3(\eta_1)\le C\varepsilon^{\frac{[p]+1-p}{2}+\frac{p}{2}}.$$

Hence, similar to the derivation of (3.16), we obtain

$$E_\varepsilon(f_\varepsilon;\eta_1)\le C\varepsilon^{\frac{[p]-p+1}{2}+\frac{p}{2}}+\frac{1}{p}\int_{\eta_1}^1\frac{d^p}{r^{p-1}}dr.$$

Thus (5.3) may be rewritten as

$$\int_{\eta_1}^1(f'_\varepsilon)^p rdr+\frac{1}{\varepsilon^p}\int_{\eta_1}^1\cos^2 f_\varepsilon rdr$$
$$\le C\varepsilon^{\frac{[p]+1-p}{2}+\frac{p}{2}}+C\varepsilon^p\le C_2\varepsilon^{\frac{[p]+1-p}{2}+\frac{p}{2}}.$$

Let $\eta_m=R(1-\frac{1}{2^m})$ where $R<1$. Proceeding in the way above (whose idea is improving the exponent of ε from $\frac{[p]+1-p}{2^k}+\frac{(2^k-1)p}{2^k}$ to $\frac{[p]+1-p}{2^{k+1}}+\frac{(2^{k+1}-1)p}{2^{k+1}}$ step by step), we can get that for any $m\in N$,

$$\int_{\eta_m}^1(f'_\varepsilon)^p rdr+\frac{1}{\varepsilon^p}\int_{\eta_m}^1\cos^2 f_\varepsilon rdr\le C\varepsilon^{\frac{[p]+1-p}{2^m}+\frac{(2^m-1)p}{2^m}}+C\varepsilon^p.$$

Letting $m\to\infty$, we derive

$$\int_R^1(f'_\varepsilon)^p rdr+\frac{1}{\varepsilon^p}\int_R^1\cos^2 f_\varepsilon rdr\le C\varepsilon^p.$$

This is (5.1).

From (5.1) we can see that

$$\int_K u_{\varepsilon 3}^2dx\le C\varepsilon^{2p}. \tag{5.6}$$

On the other hand, for any $x_0\in K$, we have

$$|u_{\varepsilon 3}(x)-u_{\varepsilon 3}(x_0)|\le C\varepsilon^{(2-p)/p}|x-x_0|^{1-2/p},\qquad \forall x\in B(x_0,\alpha\varepsilon),$$

by applying Proposition 2.2, where $\alpha=(\frac{|u_{\varepsilon 3}(x_0)|}{2C})^{\frac{p}{p-2}}$. Thus

$$|u_{\varepsilon 3}(x)|\ge|u_{\varepsilon 3}(x_0)|-C\alpha^{1-2/p}\ge\frac{1}{2}|u_{\varepsilon 3}(x_0)|.$$

Substituting this into (5.6) we obtain

$$C\varepsilon^{2p}\ge\int_K u_{\varepsilon 3}^2dx\ge\int_{B(x_0,\alpha\varepsilon)}u_{\varepsilon 3}^2dx$$
$$\ge\frac{\pi}{4}|u_{\varepsilon 3}(x_0)|^2(\alpha\varepsilon)^2=\frac{\pi}{4}(\frac{1}{2C})^{\frac{2p}{p-2}}|u_{\varepsilon 3}(x_0)|^{2+\frac{2p}{p-2}}\varepsilon^2,$$

which implies

$$|u_{\varepsilon 3}(x_0)| \leq C\varepsilon^{\frac{p-2}{2}}.$$

Noting x_0 is an arbitrary point in K, we have

$$\sup_{x\in K} |u_{\varepsilon 3}(x)| \leq C\varepsilon^{\frac{p-2}{2}}.$$

Thus (5.2) is derived and the proof of Theorem is complete.

REFERENCES

[1] F.Bethuel, H.Brezis, F.Helein, *Ginzburg-Landau Vortices,* Birkhauser, 1994.
[2] F.B.Hang, F.H.Lin, *Static theory for planar ferromagnets and antiferromagnets,* Acta. Math. Sinica, English Series, 17(2001), pp. 541–580.
[3] S.Komineas, N.Papanicolaou, *Vortex dynamics in two-dimensional antiferromagnets,* Nonlinearity, 11(1998), pp. 265–290.
[4] N.Papanicolaou, P.N.Spathis, *Semitopological solutions in planar ferromagnets,* Nonlinearity, 12(1999), pp. 285–302.