

## REGULARITY OF THE MINIMIZER FOR THE $D$ -WAVE GINZBURG-LANDAU ENERGY \*

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**Abstract.** We study the minimizer of the  $d$ -wave Ginzburg-Landau energy in a specific class of functions. We show that the minimizer having distinct degree-one vortices is Hölder continuous. Away from vortex cores, the minimizer converges uniformly to a canonical harmonic map. For a single vortex in the vortex core, we obtain the  $C^{\frac{1}{2}}$ -norm estimate of the fourfold symmetric vortex solution. Furthermore, we prove the convergence of the fourfold symmetric vortex solution under different scales of  $\delta$ .

**1. Introduction.** In this paper, we investigate the minimizer of the  $d$ -wave Ginzburg-Landau energy

$$E_{\epsilon, \delta}(u) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 + \frac{1}{4\epsilon^2} (1 - |u|^2)^2 + \frac{1}{2} \delta |\partial_x \partial_y u|^2 dx dy, \quad (1.1)$$

defined on a class of functions

$$V_g = \{u = u(x, y) : u \in H_g^1(\Omega; \mathbb{C}), \partial_x \partial_y u = h \in L^2(\Omega) \text{ in distribution sense}\} \quad (1.2)$$

with the norm  $\|\cdot\|$  defined by  $\|u\|^2 = \|u\|_{H^1}^2 + \|\partial_x \partial_y u\|_{L^2}^2$ . Hereafter,  $\Omega$  is a bounded smooth domain in  $\mathbb{R}^2$ ,  $g : \partial\Omega \rightarrow S^1$  is a smooth map with degree  $d \in \mathbb{N}$ ,  $0 < \epsilon, \delta \ll 1$  are small parameters, and

$$H_g^1(\Omega; \mathbb{C}) = \{u \in H^1(\Omega; \mathbb{C}) : u|_{\partial\Omega} = g\}.$$

The  $d$ -wave Ginzburg-Landau energy describes high-temperature superconductors. From [2], [7] and [8], we learned the  $d$ -wave Ginzburg-Landau energy without the magnetic field given by

$$G(u) = \int_{\mathbb{R}^2} \frac{1}{2} |\nabla u|^2 + \frac{1}{4} (1 - |u|^2)^2 + \frac{1}{2} \beta |(\partial_x^2 - \partial_y^2) u|^2 dx dy, \quad (1.3)$$

where  $\beta$  is a positive constant. Rotating the coordinates by  $45^\circ$ , we may obtain

$$G(u) = \int_{\mathbb{R}^2} \frac{1}{2} |\nabla u|^2 + \frac{1}{4} (1 - |u|^2)^2 + 2\beta |\partial_x \partial_y u|^2 dx dy, \quad (1.4)$$

Hereafter, we assume that  $|u| \rightarrow 1$  and all the derivatives of  $u$  decay fast as  $|(x, y)| \rightarrow \infty$ . Such an assumption is consistent with the results in [9] and [19]. Then we may transform (1.4) into (1.1) up to some constants.

For the minimization of (1.1), we may use the standard direct method to obtain the energy minimizer  $u_\epsilon^\delta$  of  $E_{\epsilon, \delta}(\cdot)$  over the function class  $V_g$ . Here we have used the fact that both  $H^1(\Omega)$  and  $L^2(\Omega)$  are reflexive Banach spaces. The minimizer  $u_\epsilon^\delta$  is a weak solution of

$$\Delta u + \frac{1}{\epsilon^2} (1 - |u|^2) u - \delta \partial_x^2 \partial_y^2 u = 0 \quad \text{on } \Omega, \quad (1.5)$$

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The highest derivative term is  $\delta \partial_x^2 \partial_y^2$  which is a degenerate elliptic operator. Such a term has a small divisor  $\delta$  and may lose derivatives by the standard bootstrap argument for (1.5). This may cause the main difficulty to get the regularity of  $u_\epsilon^\delta$ . Until now, there is not any regularity theorem for (1.5). Now we state a general regularity theorem of  $u_\epsilon^\delta$  as follows:

**THEOREM I.** *Suppose  $\delta = \delta_\epsilon$  is a positive constant which may depend on  $\epsilon$ . Then there exists a minimizer  $u_\epsilon^\delta$  of (1.1) over  $V_g$  such that*

$$\|u_\epsilon^\delta\|_{C^{\frac{1}{2}}(\bar{\Omega})} = O(\epsilon^{-1} \sqrt{1 + \delta}). \quad (1.6)$$

For (1.5), we may rescale the spatial variables by  $\epsilon$ , and obtain

$$\Delta u + (1 - |u|^2)u - \delta \epsilon^{-2} \partial_x^2 \partial_y^2 u = 0 \quad \text{on } \frac{1}{\epsilon} \Omega, \quad (1.7)$$

where  $\frac{1}{\epsilon} \Omega \equiv \{(x, y) : (\epsilon x, \epsilon y) \in \Omega\}$ . From physical literature (cf. [2] and [7]), the coefficient  $\delta \epsilon^{-2}$  of  $\partial_x^2 \partial_y^2 u$  is positive and bounded i.e.  $0 < \delta = O(\epsilon^2)$  as  $\epsilon \rightarrow 0+$ . Hereafter, we only consider such a quantity for  $\delta$ . By the same argument of [16], we have

**THEOREM A.** *Suppose  $0 < \delta = O(\epsilon^2)$  as  $\epsilon \rightarrow 0+$ . Then there exists a minimizer  $u_\epsilon$  of (1.1) over  $V_g$  such that*

- (i)  $u_\epsilon$  has  $d$  degree-one vortices in  $\Omega$ ,
- (ii)  $E_{\epsilon, \delta}(u_\epsilon) = \pi d \log \frac{1}{\epsilon} + O(1)$  as  $\epsilon \rightarrow 0+$ ,
- (iii)  $u_\epsilon$  converges to  $u_*$  (up to a subsequence) strongly in  $L^2(\Omega)$  and weakly in  $H_{loc}^1(\Omega \setminus \{a_1, \dots, a_d\})$ ,
- (iv)  $(a_1, \dots, a_d) \in \Omega^d$  is a global minimizer of the renormalized energy  $W_g$  in [3].

Here  $u_*$  is a canonical harmonic map defined by

$$u_*(z) = \prod_{j=1}^d \frac{z - a_j}{|z - a_j|} e^{i h(z)}, \quad \forall z \in \Omega, \quad (1.8)$$

and  $h$  is a real-valued harmonic function. For the product in (1.8), we have used the fact that  $\mathbb{R}^2$  is equivalent to  $\mathbb{C}$ . Actually, the vortices of  $u_\epsilon$  may arbitrarily tend to  $a_j$ 's (up to a subsequence) as  $\epsilon$  goes to zero. Away from the vortex cores  $B_\rho(a_j)$ 's, we obtain a uniform convergence of  $u_\epsilon$  as follows:

**THEOREM II.** *Suppose  $0 < \delta = O(\epsilon^2)$  as  $\epsilon \rightarrow 0+$ . Then for  $\rho > 0$ , the minimizer  $u_\epsilon$  converges to  $u_*$  (up to a subsequence) uniformly on  $\Omega \setminus \cup_{j=1}^d B_\rho(a_j)$  as  $\epsilon$  goes to zero, where  $u_*$  and  $(a_1, \dots, a_d) \in \Omega^d$  are defined in Theorem A. Hereafter,  $B_\rho(a_j)$  is the disk in  $\mathbb{R}^2$  with radius  $\rho$  and center at  $a_j$ .*

To estimate  $u_\epsilon$  in the vortex cores  $B_\rho(a_j)$ 's, we may simplify the minimization problem by setting  $\Omega = B_1(0)$ , where  $B_1(0)$  is the unit disk in  $\mathbb{R}^2$  with center at the origin. Moreover, we consider a modified minimization problem given by

$$\text{Minimize } E_{\epsilon, \delta} \text{ over } W_0 = V_{g_0} \cap W, \quad (1.9)$$

where  $g_0 \equiv e^{i\theta}$  on  $\partial\Omega$  and

$$W = \left\{ u = \sum_{k \in \mathbb{Z}} a_{1+4k}(r) e^{i(1+4k)\theta} \text{ on } \Omega, a_{1+4k}(r) \in \mathbb{R}, \forall r \in [0, 1], k \in \mathbb{Z} \right\}.$$

Hereafter,  $(r, \theta)$  denotes the polar coordinates in  $\mathbb{R}^2$ . The function space  $W$  provides fourfold symmetry for the minimizer. Actually, fourfold symmetry is a characteristic of vortex states in  $d$ -wave superconductors (cf. [5], [7], [9]). Please note that the function space  $W_0$  is a subspace of  $V_{g_0}$ . Hence we cannot assure that the energy minimizer  $u_\epsilon$  of  $E_{\epsilon, \delta}$  on  $W_0$  is a weak solution of (1.5). In [11], we prove that  $u_\epsilon$  is a weak solution of (1.5) and we obtain the  $H^1$ -norm estimate as follows:

**THEOREM B.** *Assume that  $0 < \delta = O(\epsilon^N)$ , where  $N$  is a positive constant independent of  $\epsilon$ . Then there exists a minimizer  $u_\epsilon$  of (1.9) such that  $u_\epsilon$  is a weak solution of (1.5) and*

$$u_\epsilon = u_0^\epsilon + v_\epsilon, \quad (1.10)$$

where  $u_0^\epsilon \equiv f_0(\frac{r}{\epsilon}) e^{i\theta}$  is the unique energy minimizer of  $E_{\epsilon, 0}$  over  $H_{g_0}^1(\Omega)$  (cf. [?]),  $v_\epsilon \in W \cap H_0^1(\Omega)$  satisfies

$$\|v_\epsilon\|_{L^2(\Omega)} = O(\sqrt{\delta}\epsilon^{-1}), \quad \|v_\epsilon\|_{H^1(\Omega)} = O(\sqrt{\delta}\epsilon^{-2}). \quad (1.11)$$

Hereafter,  $u_\epsilon$  is called the fourfold symmetric vortex solution of (1.5).

From [6], [10], [12], one may know qualitative theorems of  $u_0^\epsilon$ . Then Theorem I implies that

$$\|v_\epsilon\|_{C^{\frac{1}{2}}(\bar{\Omega})} = O(1/\epsilon). \quad (1.12)$$

The upper bound of (1.12) may tend to infinity as  $\epsilon$  goes to zero. By the fourfold symmetry of  $u_\epsilon$ , we may improve the estimate (1.12) by

**THEOREM III.** *Assume that  $0 < \delta = O(\epsilon^N)$  as  $\epsilon$  goes to zero, where  $N \geq 4$  is a constant independent of  $\epsilon$ . Then*

$$\|v_\epsilon\|_{C^{\frac{1}{2}}(\bar{\Omega})} = O(\sqrt{\delta}\epsilon^{-2}), \quad (1.13)$$

Theorem III is essential to prove the stability of the fourfold symmetric vortex solution  $u_\epsilon$ . Actually,  $H^1$ -norm estimate (cf. Theorem B) cannot assure the stability of  $u_\epsilon$ . We may consider the associated quadratic form given by

$$Q_\epsilon(w) = \int_{\Omega} \frac{1}{2} |\nabla w|^2 - \frac{1}{2\epsilon^2} (1 - |u_\epsilon|^2) |w|^2 + \frac{1}{\epsilon^2} (u_\epsilon \cdot w)^2 + \frac{1}{2} \delta |w_{xy}|^2 dx dy, \quad (1.14)$$

for  $w \in V_0$ , where

$$V_0 = \{ u = u(x, y) : u \in H_0^1(\Omega; \mathbb{C}), \partial_x \partial_y u = h \in L^2(\Omega) \text{ in distribution sense} \},$$

and  $Q_\epsilon(w) = \frac{1}{2} \frac{d^2}{dt^2} E_{\epsilon, \delta}(u_\epsilon + tw)|_{t=0}$  is the associated second variational form. In Corollary I, we will use (1.13) to prove  $Q_\epsilon(w) > 0$  for  $w \in V_0, \|w\|_{L^2} \neq 0$ , provided

the parameter  $\delta$  is sufficiently small. Therefore the fourfold symmetric vortex solution  $u_\epsilon$  is stable if the parameter  $\delta$  is sufficiently small.

From (1.13), it is remarkable that if  $N > 4$ , then  $\|v_\epsilon\|_{C^{\frac{1}{2}}(\bar{\Omega})} \rightarrow 0$  as  $\epsilon$  goes to zero.

Theorem III gives us the  $C^{\frac{1}{2}}$ -norm estimate of the perturbation term  $v_\epsilon$  in  $\Omega$ . Can we have  $C^\alpha$ -norm,  $\frac{1}{2} < \alpha < 1$ , estimate of  $v_\epsilon$ ? To answer this question, we state another result for the estimate of  $v_\epsilon$  in  $C^\alpha$ -norm,  $\frac{1}{2} < \alpha < 1$ , as follows:

**THEOREM IV.** *Assume that  $0 < \delta = O(\epsilon^N)$  as  $\epsilon$  goes to zero, where  $N \geq 6$  is a constant independent of  $\epsilon$ . Suppose*

$$\liminf_{\eta \rightarrow 0^+} \eta^{-2} \int_{\{1-\eta \leq x^2+y^2 \leq 1\}} |\partial_x^2 v_\epsilon|^2 + |\partial_y^2 v_\epsilon|^2 dx dy = O(\epsilon^{-6}), \quad (1.15)$$

for  $0 < \epsilon < \epsilon_0$ , where  $\epsilon_0$  is a positive constant. Then for  $\frac{1}{2} < \alpha < 1$ ,

$$\|v_\epsilon\|_{C^\alpha(\bar{\Omega})} = O(\sqrt{\delta} \epsilon^{-3}). \quad (1.16)$$

Theorem IV implies that the  $C^\alpha$ -norm,  $\frac{1}{2} < \alpha < 1$ , estimate of  $v_\epsilon$  may depend on the behavior of  $\partial_x^2 v_\epsilon$  and  $\partial_y^2 v_\epsilon$  near the boundary. It is remarkable that the boundary condition of  $u_\epsilon$  is

$$\begin{cases} u_\epsilon & = g_0 & \text{on } \partial\Omega, \\ \partial_x \partial_y u_\epsilon & = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.17)$$

Hence the boundary condition of  $v_\epsilon$  is

$$\begin{cases} v_\epsilon & = 0 & \text{on } \partial\Omega, \\ \partial_x \partial_y v_\epsilon & = -\partial_x \partial_y u_0^\epsilon & \text{on } \partial\Omega. \end{cases} \quad (1.18)$$

As  $\epsilon$  goes to zero,  $\partial_x \partial_y v_\epsilon|_{\partial\Omega} = -\partial_x \partial_y u_0^\epsilon|_{\partial\Omega} \sim -\partial_x \partial_y e^{i\theta} \neq 0$ . Thus  $\nabla^2 v_\epsilon$  may not tend to zero on  $\partial\Omega$ , and it is possible that  $\partial_x^2 v_\epsilon$  and  $\partial_y^2 v_\epsilon$  may have boundary layer on  $\partial\Omega$ . Therefore (1.15) is necessary to Theorem IV.

To understand more on the structure of a single vortex, we may rescale the spatial variables by  $\epsilon$  i.e. we set  $\tilde{u}_\epsilon(x, y) = u_\epsilon(\epsilon x, \epsilon y)$ , for  $(x, y) \in \frac{1}{\epsilon} \Omega$ . Then  $\tilde{u}_\epsilon$  is a weak solution of (1.7). For the convergence of  $\tilde{u}_\epsilon$ , we have

**THEOREM V.** *Let  $\tilde{u}_\epsilon(x, y) = u_\epsilon(\epsilon x, \epsilon y)$ , for  $(x, y) \in \frac{1}{\epsilon} \Omega$ . Then  $\tilde{u}_\epsilon$  converges weakly (up to a subsequence) to  $\tilde{u}$  in  $H_{loc}^1(\mathbb{R}^2)$ . Furthermore,  $\tilde{u}$  is a weak solution of*

$$\Delta u + (1 - |u|^2)u = 0 \quad \text{on } \mathbb{R}^2, \quad \text{if } \delta = o(\epsilon^2), \quad (1.19)$$

$$\Delta u + (1 - |u|^2)u - \lambda \partial_x^2 \partial_y^2 u = 0 \quad \text{on } \mathbb{R}^2, \quad \text{if } \delta = \lambda \epsilon^2, \quad (1.20)$$

where  $\lambda$  is a positive constant independent of  $\epsilon$ . In particular, if  $\delta = O(\epsilon^N)$ ,  $N > 4$ , then  $\tilde{u} = f(r) e^{i\theta}$ , where  $f$  is the unique solution of

$$\begin{aligned} f'' + \frac{1}{r} f' - \frac{1}{r^2} f + (1 - f^2) f &= 0, \quad \forall r > 0, \\ f(0) &= 0, f(+\infty) = 1. \end{aligned} \quad (1.21)$$

The equation (1.19) has been investigated to find solutions with vortex structures (cf. [3], [4], [6], [12]). However, the uniqueness of (1.19) with  $\lim_{|(x,y)| \rightarrow \infty} |u(x,y) - e^{i\theta}| = 0$  has not yet been proved. Hence it is still open that  $\tilde{u} = f(r) e^{i\theta}$  if  $0 < \delta = O(\epsilon^N)$ ,  $2 < N \leq 4$ .

In the rest of this paper, we will prove Theorem I and introduce a general regularity theorem in Section 2. In Section 3, we will prove Theorem II. In Section 4 and 5, we will complete the proof of Theorem III, IV and V, respectively.

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**2. General Regularity Theorem.** In this section, we will provide a proof of Theorem I. To prove Theorem I, we need a crucial Lemma given by

LEMMA I. *Assume  $\Omega$  is a bounded smooth domain in  $\mathbb{R}^2$ . Let  $u \in V_g$  satisfy*

$$\int_{\Omega} |\partial_x \partial_y u|^2 \leq A, \quad (2.1)$$

$$\|u\|_{H^1(\Omega)}^2 \leq B, \quad (2.2)$$

where  $A, B$  are positive constant and  $V_g$  is defined in (1.2). Then

$$u \text{ is of } C^{\frac{1}{2}}(\bar{\Omega}) \quad \text{and} \quad \|u\|_{C^{\frac{1}{2}}(\bar{\Omega})} \leq C \sqrt{A+B}, \quad (2.3)$$

where  $C$  is a positive constant depending only on  $\Omega$ .

It is remarkable that Lemma I is a general regularity theorem for functions satisfying (2.1) and (2.2). Now we prove Lemma I as follows. From extension theorem (cf. [1]), we may extend the function  $u$  on a cube  $Q = [a, b] \times [\alpha, \beta]$  such that

$$\int_Q |\partial_x \partial_y u|^2 \leq C_0 A, \quad (2.4)$$

$$\|u\|_{H^1(Q)}^2 \leq C_0 B, \quad (2.5)$$

where  $C_0$  is a positive constant depending on  $\Omega$ ,  $a < b$ ;  $\alpha < \beta$  are constants. By (2.5) and Fubini Theorem, there exists  $x_0 \in [a, b]$  such that

$$\|u(x_0, \cdot)\|_{H^1([\alpha, \beta])} \leq B_0, \quad (2.6)$$

where  $B_0 = 2C_0 B/|b-a|$ . Hence by Sobolev embedding,  $u(x_0, \cdot) \in C^{\frac{1}{2}}([\alpha, \beta])$ . Fix  $x \in [a, b]$  arbitrarily. Without loss of generality, we may assume  $u$  is smooth on  $Q$ .

Then

$$\begin{aligned}
& \int_{\alpha}^{\beta} u^2(x, y) + u_y^2(x, y) dy \\
& \leq 2 \int_{\alpha}^{\beta} u^2(x_0, y) dy + 2 \int_{\alpha}^{\beta} |u(x, y) - u(x_0, y)|^2 dy \\
& \quad + 2 \int_{\alpha}^{\beta} u_y^2(x_0, y) dy + 2 \int_{\alpha}^{\beta} |u_y(x, y) - u_y(x_0, y)|^2 dy \quad (\text{by Triangle inequality}) \\
& = 2 \int_{\alpha}^{\beta} u^2(x_0, y) + u_y^2(x_0, y) dy + 2 \int_{\alpha}^{\beta} \left| \int_{x_0}^x u_x(t, y) dt \right|^2 dy \\
& \quad + 2 \int_{\alpha}^{\beta} \left| \int_{x_0}^x u_{xy}(t, y) dt \right|^2 dy \\
& \leq 2B_0 + 2 \int_{\alpha}^{\beta} |x - x_0| \int_{x_0}^x |u_x(t, y)|^2 + |u_{xy}(t, y)|^2 dt dy \quad (\text{by Holder inequality}) \\
& \leq 2[B_0 + |b - a| C_0 (A + B)] \quad (\because (2.4), (2.5)),
\end{aligned}$$

where  $u_y = \partial_y u$  and  $u_{xy} = \partial_x \partial_y u$ . Hence

$$\|u(x, \cdot)\|_{H^1([\alpha, \beta])}^2 \leq C_1 (A + B),$$

for  $x \in [a, b]$ , where  $C_1$  is a positive constant depending only on  $\Omega$  and  $|b - a|$ . Thus by Sobolev embedding,

$$\|u(x, \cdot)\|_{C^{\frac{1}{2}}([\alpha, \beta])} \leq C_2 \sqrt{A + B}, \quad (2.7)$$

for  $x \in [a, b]$ , where  $C_2$  is a positive constant depending only on  $\Omega$ ,  $|b - a|$  and  $|\beta - \alpha|$ . Similarly, we may obtain

$$\|u(\cdot, y)\|_{C^{\frac{1}{2}}([a, b])} \leq C_3 \sqrt{A + B}, \quad (2.8)$$

for  $y \in [\alpha, \beta]$ , where  $C_3$  is a positive constant depending only on  $\Omega$ ,  $|b - a|$  and  $|\beta - \alpha|$ . Therefore by (2.7) and (2.8), we may complete the proof of Lemma I.

Now we want to prove Theorem I. From the standard direct method, it is easy to obtain a minimizer  $u_{\epsilon}^{\delta} \in V_g$  of (1.1). Let  $u_{\epsilon}^0$  be a minimizer of the energy functional

$$E_{\epsilon, 0}(u) = \int_{\Omega} \frac{1}{2} |\nabla u|^2 + \frac{1}{4\epsilon^2} (1 - |u|^2)^2$$

on  $H_g^1(\Omega)$ . From [3], we learned the quantitative properties of  $u_{\epsilon}^0$ . Then it is easy to check that

$$E_{\epsilon, 0}(u_{\epsilon}^0) = \pi d \log \frac{1}{\epsilon} + O(1), \quad (2.9)$$

$$\int_{\Omega} |\partial_x \partial_y u_{\epsilon}^0|^2 = O(\epsilon^{-2}). \quad (2.10)$$

Hence

$$\begin{aligned}
E_{\epsilon, 0}(u_{\epsilon}^0) + \frac{1}{2} \delta \int_{\Omega} |\partial_x \partial_y u_{\epsilon}^{\delta}|^2 & \leq E_{\epsilon, 0}(u_{\epsilon}^{\delta}) + \frac{1}{2} \delta \int_{\Omega} |\partial_x \partial_y u_{\epsilon}^{\delta}|^2 \\
& = E_{\epsilon, \delta}(u_{\epsilon}^{\delta}) \\
& \leq E_{\epsilon, \delta}(u_{\epsilon}^0) \\
& = E_{\epsilon, 0}(u_{\epsilon}^0) + \frac{1}{2} \delta \int_{\Omega} |\partial_x \partial_y u_{\epsilon}^0|^2 \\
& = E_{\epsilon, 0}(u_{\epsilon}^0) + \delta O(\epsilon^{-2}) \quad (\because (2.10)).
\end{aligned}$$

Hence

$$\int_{\Omega} |\partial_x \partial_y u_{\epsilon}^{\delta}|^2 = O(\epsilon^{-2}). \quad (2.11)$$

Moreover,

$$\begin{aligned} \frac{1}{2} \int_{\Omega} |\nabla u_{\epsilon}^{\delta}|^2 &\leq E_{\epsilon, \delta}(u_{\epsilon}^{\delta}) \\ &\leq E_{\epsilon, \delta}(u_{\epsilon}^0) \\ &= \pi d \log \frac{1}{\epsilon} + O(\delta \epsilon^{-2}) \quad (\because (2.9), (2.10)). \end{aligned}$$

Thus

$$\|u_{\epsilon}^{\delta}\|_{H^1(\Omega)}^2 = O\left(\log \frac{1}{\epsilon} + \delta \epsilon^{-2}\right). \quad (2.12)$$

Therefore by (2.11), (2.12) and Lemma I, we may complete the proof of Theorem I.

**3. Minimizer with Multiple Vortices.** In this section, we assume  $0 < \delta = O(\epsilon^2)$  as  $\epsilon$  goes to zero. From Theorem A in Section 1, we obtain a minimizer  $u_{\epsilon}$  having  $d$  degree-one vortices near  $a_j, j = 1, \dots, d$ . By Theorem I, the minimizer  $u_{\epsilon}$  is of  $C^{\frac{1}{2}}(\bar{\Omega})$  and  $\|u_{\epsilon}\|_{C^{\frac{1}{2}}(\bar{\Omega})} = O(\epsilon^{-1})$ . Such an upper bound is unbounded as  $\epsilon$  tends to zero, and cannot assure any strong convergence of  $u_{\epsilon}$ . The main purpose of this section is to prove Theorem II and get a bounded estimate for the  $C^{\frac{1}{2}}$ -norm of  $u_{\epsilon}$  away from the vortex cores.

Now we want to prove Theorem II. Let  $\rho > 0$  be a small constant and  $\Omega_{\rho} \equiv \Omega \setminus \cup_{j=1}^d B_{\rho}(a_j)$ , where  $a_j$ 's are defined in Theorem A. We may define a comparison map  $v_{\epsilon}$  given by

$$v_{\epsilon} = \begin{cases} u_{\epsilon} & \text{in } B_{\rho}(a_j), j = 1, \dots, d, \\ w_{\epsilon} & \text{in } \Omega_{\rho}, \end{cases} \quad (3.1)$$

where  $w_{\epsilon}$  is the minimizer of the energy functional

$$E_{\epsilon, 0}(w; \Omega_{\rho}) = \int_{\Omega_{\rho}} \frac{1}{2} |\nabla w|^2 + \frac{1}{4\epsilon^2} (1 - |w|^2)^2$$

over the function class  $H_g^1(\Omega_{\rho})$ . Here the boundary condition  $\tilde{g}$  is defined by

$$\tilde{g} = \begin{cases} g & \text{on } \partial\Omega, \\ u_{\epsilon} & \text{on } \partial B_{\rho}(a_j), j = 1, \dots, d. \end{cases}$$

From Theorem A and [3], we may obtain quantitative properties of  $w_{\epsilon}$ . Now we define the following energy functionals:

$$\begin{aligned} &E_{\epsilon, \delta}(u_{\epsilon}; \cup_{j=1}^d B_{\rho}(a_j)) \\ &= \sum_{j=1}^d \int_{B_{\rho}(a_j)} \frac{1}{2} |\nabla u_{\epsilon}|^2 + \frac{1}{4\epsilon^2} (1 - |u_{\epsilon}|^2)^2 + \frac{1}{2} \delta |\partial_x \partial_y u_{\epsilon}|^2, \end{aligned} \quad (3.2)$$

$$E_{\epsilon, \delta}(w_{\epsilon}; \Omega_{\rho}) = \int_{\Omega_{\rho}} \frac{1}{2} |\nabla w_{\epsilon}|^2 + \frac{1}{4\epsilon^2} (1 - |w_{\epsilon}|^2)^2 + \frac{1}{2} \delta |\partial_x \partial_y w_{\epsilon}|^2. \quad (3.3)$$

Then

$$\begin{aligned}
& E_{\epsilon,\delta}(u_\epsilon; \cup_{j=1}^d B_\rho(a_j)) + E_{\epsilon,0}(w_\epsilon; \Omega_\rho) + \frac{1}{2}\delta \int_{\Omega_\rho} |\partial_x \partial_y u_\epsilon|^2 \\
& \leq E_{\epsilon,\delta}(u_\epsilon; \cup_{j=1}^d B_\rho(a_j)) + E_{\epsilon,0}(u_\epsilon; \Omega_\rho) + \frac{1}{2}\delta \int_{\Omega_\rho} |\partial_x \partial_y u_\epsilon|^2 \\
& = E_{\epsilon,\delta}(u_\epsilon) \\
& \leq E_{\epsilon,\delta}(v_\epsilon) \\
& = E_{\epsilon,\delta}(u_\epsilon; \cup_{j=1}^d B_\rho(a_j)) + E_{\epsilon,\delta}(w_\epsilon; \Omega_\rho) \\
& = E_{\epsilon,\delta}(u_\epsilon; \cup_{j=1}^d B_\rho(a_j)) + E_{\epsilon,0}(w_\epsilon; \Omega_\rho) + \frac{1}{2}\delta \int_{\Omega_\rho} |\partial_x \partial_y w_\epsilon|^2,
\end{aligned}$$

i.e.

$$\int_{\Omega_\rho} |\partial_x \partial_y u_\epsilon|^2 \leq \int_{\Omega_\rho} |\partial_x \partial_y w_\epsilon|^2. \quad (3.4)$$

Similarly,

$$\begin{aligned}
& E_{\epsilon,\delta}(u_\epsilon; \cup_{j=1}^d B_\rho(a_j)) + \frac{1}{2} \int_{\Omega_\rho} |\nabla u_\epsilon|^2 \\
& \leq E_{\epsilon,\delta}(u_\epsilon; \cup_{j=1}^d B_\rho(a_j)) + E_{\epsilon,0}(u_\epsilon; \Omega_\rho) + \frac{1}{2}\delta \int_{\Omega_\rho} |\partial_x \partial_y u_\epsilon|^2 \\
& = E_{\epsilon,\delta}(u_\epsilon) \\
& \leq E_{\epsilon,\delta}(v_\epsilon) \\
& = E_{\epsilon,\delta}(u_\epsilon; \cup_{j=1}^d B_\rho(a_j)) + E_{\epsilon,\delta}(w_\epsilon; \Omega_\rho),
\end{aligned}$$

i.e.

$$\frac{1}{2} \int_{\Omega_\rho} |\nabla u_\epsilon|^2 \leq E_{\epsilon,\delta}(w_\epsilon; \Omega_\rho). \quad (3.5)$$

Hence by (3.4), (3.5) and [3], we obtain

$$\int_{\Omega_\rho} |\partial_x \partial_y u_\epsilon|^2 \leq K_0, \quad (3.6)$$

$$\int_{\Omega_\rho} |\nabla u_\epsilon|^2 \leq K_0, \quad (3.7)$$

where  $K_0$  is a positive constant depending on  $\Omega_\rho$ . Thus (3.6), (3.7) and Lemma I imply that

$$\|u_\epsilon\|_{C^{\frac{1}{2}}(\bar{\Omega}_\rho)} \leq K_1, \quad (3.8)$$

where  $K_1$  is a positive constant depending on  $\Omega_\rho$ . Therefore by (3.8) and Arzela-Ascoli Theorem, we may complete the proof of Theorem II. Note that (3.8) provides a bounded  $C^{\frac{1}{2}}$ -norm estimate of  $u_\epsilon$  on  $\Omega_\rho$ .

**4. Estimate of a Single Vortex.** In this section, we assume  $\Omega = B_1(0)$  is a unit disk with center at the origin, and  $0 < \delta = O(\epsilon^N)$  as  $\epsilon$  goes to zero, where  $N \geq 4$

is a constant independent of  $\epsilon$ . From Theorem B and using energy comparison, we have

$$\begin{aligned}
& E_{\epsilon,0}(u_0^\epsilon) + \frac{1}{2}\delta \int_{\Omega} |\partial_x \partial_y u_\epsilon|^2 \\
& \leq E_{\epsilon,0}(u_\epsilon) + \frac{1}{2}\delta \int_{\Omega} |\partial_x \partial_y u_\epsilon|^2 \\
& = E_{\epsilon,\delta}(u_\epsilon) \\
& \leq E_{\epsilon,\delta}(u_0^\epsilon) \\
& = E_{\epsilon,0}(u_0^\epsilon) + \frac{1}{2}\delta \int_{\Omega} |\partial_x \partial_y u_0^\epsilon|^2,
\end{aligned}$$

i.e.

$$\int_{\Omega} |\partial_x \partial_y u_\epsilon|^2 \leq \int_{\Omega} |\partial_x \partial_y u_0^\epsilon|^2. \quad (4.1)$$

Then by (4.1), (1.10) and Holder inequality, we obtain

$$\int_{\Omega} |\partial_x \partial_y v_\epsilon|^2 = O(\epsilon^{-2}). \quad (4.2)$$

Here we have used some properties of  $u_0^\epsilon$  (cf. [6], [12]).

From (1.11), (4.2) and extension theorem (cf. [1]), we may extend  $v_\epsilon$  to a cube  $Q_1 = [-1, 1] \times [-1, 1]$  such that

$$\|v_\epsilon\|_{H^1(Q_1)}^2 = O(\sqrt{\delta} \epsilon^{-2}), \quad (4.3)$$

$$\int_{Q_1} |\partial_x \partial_y v_\epsilon|^2 = O(\epsilon^{-2}). \quad (4.4)$$

Here we have used the fact that  $\delta = O(\epsilon^N)$ ,  $N \geq 4$  as  $\epsilon$  goes to zero.

Now we want to prove Theorem III. Without loss of generality, we may assume  $v_\epsilon$  is smooth on  $Q_1$  and satisfies

$$\int_{Q_1} |\partial_x v_\epsilon|^2 \leq \sqrt{\delta} \epsilon^{-2}, \quad (4.5)$$

$$\int_{Q_1} |\partial_y v_\epsilon|^2 \leq \sqrt{\delta} \epsilon^{-2}, \quad (4.6)$$

$$\int_{Q_1} |\partial_x \partial_y v_\epsilon|^2 \leq \epsilon^{-2}. \quad (4.7)$$

By (4.6) and Fubini Theorem, there exists  $x_0 \in [-1, 1]$  such that

$$\int_{-1}^1 |\partial_y v_\epsilon(x_0, y)|^2 dy \leq 2\sqrt{\delta} \epsilon^{-2}. \quad (4.8)$$

From Theorem B,  $u_\epsilon$  is of fourfold symmetry on  $\Omega$  i.e.  $v_\epsilon$  is of fourfold symmetry on

$Q_1$ . Hence we may set  $x_0 \in [0, 1]$ . Moreover,

$$\begin{aligned}
& \int_{-1}^1 |\partial_y v_\epsilon(x, y)|^2 dy \\
& \leq 2 \int_{-1}^1 |\partial_y v_\epsilon(x, y) - \partial_y v_\epsilon(x_0, y)|^2 dy \\
& \quad + 2 \int_{-1}^1 |\partial_y v_\epsilon(x_0, y)|^2 dy \quad (\text{by triangle inequality}) \\
& = 2 \int_{-1}^1 \left| \int_{x_0}^x \partial_x \partial_y v_\epsilon(s, y) ds \right|^2 dy + 2 \int_{-1}^1 |\partial_y v_\epsilon(x_0, y)|^2 dy \\
& \leq 2|x - x_0| \int_{Q_1} |\partial_x \partial_y v_\epsilon|^2 + 2 \int_{-1}^1 |\partial_y v_\epsilon(x_0, y)|^2 dy \quad (\text{by Holder inequality}) \\
& \leq 6\sqrt{\delta} \epsilon^{-2} \quad (\cdot (4.7), (4.8)),
\end{aligned}$$

for  $|x - x_0| \leq \sqrt{\delta}$ , i.e.

$$\int_{-1}^1 |\partial_y v_\epsilon(x, y)|^2 dy \leq 6\sqrt{\delta} \epsilon^{-2} \quad \text{for } x \in I_1 \equiv [x_0 - \sqrt{\delta}, x_0 + \sqrt{\delta}] \cap [0, 1]. \quad (4.9)$$

Let  $Q_2 = ([-1, 1] \setminus I_1) \times [-1, 1]$ . Then (4.6) implies that

$$\int_{Q_2} |\partial_y v_\epsilon|^2 dx dy \leq \sqrt{\delta} \epsilon^{-2}. \quad (4.10)$$

Hence by Fubini Theorem and the fourfold symmetry of  $v_\epsilon$ , there exists  $x_1 \in [0, 1] \setminus I_1$  such that

$$\int_{-1}^1 |\partial_y v_\epsilon(x_1, y)|^2 dy \leq 2\sqrt{\delta} \epsilon^{-2}. \quad (4.11)$$

As for (4.9), we obtain

$$\int_{-1}^1 |\partial_y v_\epsilon(x, y)|^2 dy \leq 6\sqrt{\delta} \epsilon^{-2} \quad \text{for } x \in I_2 \equiv [x_1 - \sqrt{\delta}, x_1 + \sqrt{\delta}] \cap [0, 1] \setminus I_1. \quad (4.12)$$

By induction, there exist  $x_k \in [0, 1] \setminus \cup_{j=1}^k I_j$  such that

$$\int_{-1}^1 |\partial_y v_\epsilon(x_k, y)|^2 dy \leq 2\sqrt{\delta} \epsilon^{-2}, \quad (4.13)$$

$$\int_{-1}^1 |\partial_y v_\epsilon(x, y)|^2 dy \leq 6\sqrt{\delta} \epsilon^{-2}, \quad \text{for } x \in I_{k+1}, \quad (4.14)$$

for  $k = 0, 1, 2, \dots, N$ , where

$$I_{k+1} \equiv [x_k - \sqrt{\delta}, x_k + \sqrt{\delta}] \cap [0, 1] \setminus \cup_{j=1}^k I_j.$$

Here  $N = O(1/\sqrt{\delta})$  is a positive integer such that

$$\cup_{k=1}^N I_k = [0, 1]. \quad (4.15)$$

Note that the interiors of  $I_k$ 's are disjoint to each other. Similarly, by (4.5) and the same argument as (4.13)-(4.15), we may obtain  $y_l \in [0, 1] \setminus \cup_{k=1}^l J_k$  such that

$$\int_{-1}^1 |\partial_x v_\epsilon(x, y_l)|^2 dx \leq 2\sqrt{\delta} \epsilon^{-2}, \quad (4.16)$$

$$\int_{-1}^1 |\partial_x v_\epsilon(x, y)|^2 dx \leq 6\sqrt{\delta} \epsilon^{-2}, \quad \text{for } y \in J_{l+1}, \quad (4.17)$$

for  $l = 0, 1, 2, \dots, M$ , where

$$J_{l+1} \equiv [y_l - \sqrt{\delta}, y_l + \sqrt{\delta}] \cap [0, 1] \setminus \cup_{k=1}^l J_k.$$

Here  $M = O(1/\sqrt{\delta})$  is a positive integer such that

$$\cup_{l=1}^M J_l = [0, 1]. \quad (4.18)$$

Note that the interiors of  $J_l$ 's are disjoint to each other. From (4.14), (4.15) and Sobolev embedding,

$$\|v_\epsilon(x, \cdot)\|_{C^{\frac{1}{2}}([-1, 1])} \leq C_0 \sqrt{\delta} \epsilon^{-2}, \quad (4.19)$$

for  $x \in [0, 1]$ , where  $C_0$  is a positive constant independent of  $x, y, \epsilon$  and  $\delta$ . Similarly, by (4.17), (4.18) and Sobolev embedding,

$$\|v_\epsilon(\cdot, y)\|_{C^{\frac{1}{2}}([-1, 1])} \leq C_0 \sqrt{\delta} \epsilon^{-2}, \quad (4.20)$$

for  $y \in [0, 1]$ . Thus (4.19) and (4.20) imply that

$$\|v_\epsilon\|_{C^{\frac{1}{2}}([0, 1] \times [0, 1])} \leq C_1 \sqrt{\delta} \epsilon^{-2}, \quad (4.21)$$

where  $C_1$  is a positive constant independent of  $\epsilon$  and  $\delta$ . Therefore by (4.21) and the fourfold symmetry of  $v_\epsilon$ , we may complete the proof of Theorem III.

Now we want to prove Theorem IV. From (1.5) and (1.10), we have

$$\Delta v_\epsilon - \delta \partial_x^2 \partial_y^2 v_\epsilon = \delta \partial_x^2 \partial_y^2 u_0^\epsilon + N_\epsilon(v_\epsilon), \quad (4.22)$$

where

$$N_\epsilon(v_\epsilon) = -\epsilon^{-2} [-(1 - |u_0^\epsilon|^2)v_\epsilon + 2(u_0^\epsilon \cdot v_\epsilon)u_0^\epsilon + 2(u_0^\epsilon \cdot v_\epsilon)v_\epsilon + |v_\epsilon|^2 u_0^\epsilon + |v_\epsilon|^2 v_\epsilon]. \quad (4.23)$$

By (1.11) and (4.23), it is easy to check that

$$\|N_\epsilon(v_\epsilon)\|_{L^2(\Omega)} = O(\sqrt{\delta} \epsilon^{-3}). \quad (4.24)$$

Here we have used the assumption that  $0 < \delta = O(\epsilon^N)$ ,  $N \geq 6$  as  $\epsilon$  goes to zero. Hence by (4.22), (4.24) and [3], we obtain

$$\|\Delta v_\epsilon - \delta \partial_x^2 \partial_y^2 v_\epsilon\|_{L^2(\Omega)} = O(\sqrt{\delta} \epsilon^{-3}). \quad (4.25)$$

Let  $P \in C_0^\infty(\Omega)$  be a real-valued test function defined later. Then using integration by parts, we have

$$\begin{aligned} \int_{\Omega} P \Delta v_\epsilon \cdot (\Delta v_\epsilon - \delta \partial_x^2 \partial_y^2 v_\epsilon) &= \int_{\Omega} P |\Delta v_\epsilon|^2 - \delta \int_{\Omega} P \Delta v_\epsilon \cdot \partial_x^2 \partial_y^2 v_\epsilon \\ &= \int_{\Omega} P |\Delta v_\epsilon|^2 - \delta \int_{\Omega} P \partial_x^2 v_\epsilon \cdot \partial_x^2 \partial_y^2 v_\epsilon - \delta \int_{\Omega} P \partial_y^2 v_\epsilon \cdot \partial_x^2 \partial_y^2 v_\epsilon \\ &= \int_{\Omega} P |\Delta v_\epsilon|^2 + \delta \int_{\Omega} P |\partial_x^2 \partial_y v_\epsilon|^2 - \frac{1}{2} \partial_y^2 P |\partial_x^2 v_\epsilon|^2 \\ &\quad + \delta \int_{\Omega} P |\partial_y^2 \partial_x v_\epsilon|^2 - \frac{1}{2} \partial_x^2 P |\partial_y^2 v_\epsilon|^2, \end{aligned}$$

i.e.

$$\begin{aligned} \int_{\Omega} P \Delta v_\epsilon \cdot (\Delta v_\epsilon - \delta \partial_x^2 \partial_y^2 v_\epsilon) &= \int_{\Omega} P |\Delta v_\epsilon|^2 + \delta \int_{\Omega} P (|\partial_x^2 \partial_y v_\epsilon|^2 + |\partial_y^2 \partial_x v_\epsilon|^2) \\ &\quad - \frac{1}{2} \delta \int_{\Omega} \partial_y^2 P |\partial_x^2 v_\epsilon|^2 + \partial_x^2 P |\partial_y^2 v_\epsilon|^2. \end{aligned} \quad (4.26)$$

Now we define the test function  $P$  by

$$P(x, y) = e^{-\frac{2\eta}{1-x^2-y^2}}, \quad \text{for } (x, y) \in \Omega, \quad (4.27)$$

where  $\eta > 0$  is a small parameter. Then it is easy to check that

$$\partial_x^2 P, \partial_y^2 P \leq 0 \quad \text{for } x^2 + y^2 \leq 1 - \eta, \quad (4.28)$$

$$\partial_x^2 P, \partial_y^2 P \leq \alpha_0 \eta^{-2} \quad \text{for } 1 - \eta \leq x^2 + y^2 \leq 1, \quad (4.29)$$

where  $\alpha_0 > 0$  is a universal constant independent of  $\eta$ . Hence (4.26), (4.28) and (4.29) imply that

$$\int_{\Omega} P \Delta v_\epsilon \cdot (\Delta v_\epsilon - \delta \partial_x^2 \partial_y^2 v_\epsilon) \geq \int_{\Omega} P |\Delta v_\epsilon|^2 - \frac{1}{2} \alpha_0 \delta \eta^{-2} \int_{1-\eta \leq x^2+y^2 \leq 1} |\partial_x^2 v_\epsilon|^2 + |\partial_y^2 v_\epsilon|^2. \quad (4.30)$$

From (4.24), (4.27) and Holder inequality, we have

$$\begin{aligned} \int_{\Omega} P \Delta v_\epsilon \cdot (\Delta v_\epsilon - \delta \partial_x^2 \partial_y^2 v_\epsilon) &\leq \|P \Delta v_\epsilon\|_{L^2(\Omega)} \|\Delta v_\epsilon - \delta \partial_x^2 \partial_y^2 v_\epsilon\|_{L^2(\Omega)} \\ &\leq \frac{1}{2} \|P \Delta v_\epsilon\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\Delta v_\epsilon - \delta \partial_x^2 \partial_y^2 v_\epsilon\|_{L^2(\Omega)}^2 \\ &\leq \frac{1}{2} \int_{\Omega} P |\Delta v_\epsilon|^2 dx dy + O(\delta \epsilon^{-6}) \quad (\because (4.24), (4.27)), \end{aligned}$$

i.e.

$$\int_{\Omega} P \Delta v_\epsilon \cdot (\Delta v_\epsilon - \delta \partial_x^2 \partial_y^2 v_\epsilon) \leq \frac{1}{2} \int_{\Omega} P |\Delta v_\epsilon|^2 dx dy + O(\delta \epsilon^{-6}). \quad (4.31)$$

Hence (4.30) and (4.31) imply

$$\int_{\Omega} P |\Delta v_\epsilon|^2 \leq \alpha_0 \delta \eta^{-2} \int_{1-\eta \leq x^2+y^2 \leq 1} |\partial_x^2 v_\epsilon|^2 + |\partial_y^2 v_\epsilon|^2 dx dy + O(\delta \epsilon^{-6}). \quad (4.32)$$

Hence by (1.15), (4.32) and Fatou's Lemma,

$$\int_{\Omega} |\Delta v_{\epsilon}|^2 = O(\delta \epsilon^{-6}). \quad (4.33)$$

Therefore by (4.33) and standard theorems for  $W^{2,p}$  estimate and Sobolev embedding, we may complete the proof of Theorem IV.

REMARK. From [11], the minimizer  $u_{\epsilon}$  of (1.9) is a weak solution of (1.5). However,  $u_{\epsilon}$  may not be a global minimizer of  $E_{\epsilon,\delta}$  on  $V_{g_0}$ . We will prove that  $u_{\epsilon}$  is a local minimizer of  $E_{\epsilon,\delta}$  on  $V_{g_0}$  if  $\delta$  is sufficiently small. From Theorem B and Theorem III, (1.14) becomes

$$\begin{aligned} Q_{\epsilon}(w) &= \int_{\Omega} \frac{1}{2} |\nabla w|^2 - \frac{1}{2\epsilon^2} (1 - |u_{\epsilon}^0|^2) |w|^2 + \frac{1}{\epsilon^2} (u_{\epsilon}^0 \cdot w)^2 + \frac{1}{2} \delta |w_{xy}|^2 \, dx \, dy \\ &\quad + \epsilon^{-2} O(\|v_{\epsilon}\|_{L^{\infty}} + \|v_{\epsilon}\|_{L^{\infty}}^2) \|w\|_{L^2}^2 \quad (\because (1.10)) \\ &\geq \int_{\Omega} \frac{1}{2} |\nabla w|^2 - \frac{1}{2\epsilon^2} (1 - |u_{\epsilon}^0|^2) |w|^2 + \frac{1}{\epsilon^2} (u_{\epsilon}^0 \cdot w)^2 \, dx \, dy \\ &\quad + O(\sqrt{\delta} \epsilon^{-4}) \|w\|_{L^2}^2, \quad (\because (1.13)) \end{aligned}$$

i.e.

$$Q_{\epsilon}(w) \geq Q_{\epsilon}^0(w) + O(\sqrt{\delta} \epsilon^{-4}) \|w\|_{L^2}^2, \quad (4.34)$$

for  $w \in V_0$ , where

$$Q_{\epsilon}^0(w) \equiv \int_{\Omega} \frac{1}{2} |\nabla w|^2 - \frac{1}{2\epsilon^2} (1 - |u_{\epsilon}^0|^2) |w|^2 + \frac{1}{\epsilon^2} (u_{\epsilon}^0 \cdot w)^2 \, dx \, dy.$$

By (4.34) and [17], we have

$$Q_{\epsilon}(w) > 0 \quad \text{for } w \in V_0, \|w\|_{L^2} \neq 0, \quad (4.35)$$

provided

$$0 < \delta = o(\lambda_{1,\epsilon}^2 \epsilon^8), \quad 0 < \lambda_{1,\epsilon} = o(1), \quad (4.36)$$

where  $o(1)$  is a small quantity which tends to zero as  $\epsilon$  goes to zero. Actually,  $\lambda_{1,\epsilon}$  is the minimization of  $Q_{\epsilon}^0(w)$  for  $w \in H_0^1(\Omega)$ ,  $\|w\|_{L^2} = 1$  (cf. [11], [17], [18]). Therefore we have

COROLLARY I. *Under the same assumptions as Theorem III and (4.36), the fourfold symmetric vortex solution  $u_{\epsilon}$  is stable.*

**5. Proof of Theorem V.** From Theorem A (ii), we obtain

$$\int_{\Omega} e_{\epsilon}(u_{\epsilon}) = \pi \log \frac{1}{\epsilon} + O(1), \quad (5.1)$$

where  $e_{\epsilon}$  is the energy density of  $E_{\epsilon,0}$  and is defined by

$$e_{\epsilon}(u) \equiv \frac{1}{2} |\nabla u|^2 + \frac{1}{4\epsilon^2} (1 - |u|^2)^2.$$

We let

$$\hat{u}_\epsilon = \begin{cases} u_\epsilon & \text{if } |u_\epsilon| \leq 1, \\ u_\epsilon/|u_\epsilon| & \text{if } |u_\epsilon| > 1. \end{cases}$$

Then  $\hat{u}_\epsilon$  satisfies

$$|\hat{u}_\epsilon| \leq 1 \quad \text{in } \Omega, \quad (5.2)$$

and

$$\int_{\Omega} e_\epsilon(\hat{u}_\epsilon) \leq \int_{\Omega} e_\epsilon(u_\epsilon) \leq \pi \log \frac{1}{\epsilon} + M_0, \quad (5.3)$$

where  $M_0$  is a positive constant independent of  $\epsilon$ . By (5.2), (5.3) and the proof of Proposition 1.1 in [15],  $\hat{u}_\epsilon$  has only one essential zero  $a^\epsilon$  in  $\Omega$  and  $\deg(\frac{\hat{u}_\epsilon}{|\hat{u}_\epsilon|}, \partial B) = 1$ , where  $B = B_{\epsilon^\alpha}(a^\epsilon)$  and  $\alpha \in (0, 1)$ . Without loss of generality, we may assume that  $a^\epsilon = 0$ . Now we want to prove

LEMMA II. *For each  $R_0 > 1$ , there exists a positive constant  $C$  depending only on  $R_0$  such that*

$$\int_{\Omega \setminus B_{\epsilon R_0}(0)} e_\epsilon(\hat{u}_\epsilon) \geq \pi \log \frac{1}{\epsilon} - C(R_0), \quad \text{as } \epsilon \rightarrow 0+. \quad (5.4)$$

*Proof of Lemma II.* Fix  $R_0 > 1$  arbitrarily. Let  $\tilde{\epsilon} = \epsilon R_0$ . Then (5.3) implies

$$\int_{\Omega} e_{\tilde{\epsilon}}(\hat{u}_\epsilon) \leq \int_{\Omega} e_\epsilon(\hat{u}_\epsilon) \leq \pi \log \frac{1}{\tilde{\epsilon}} + M'_0, \quad (5.5)$$

where  $M'_0 = M_0 + \pi \log R_0$ . Again, by (5.2), (5.5) and the proof of Proposition 1.1 in [15],  $\hat{u}_\epsilon$  has only one essential zero  $a^\epsilon$  in  $\Omega$  and  $\deg(\frac{\hat{u}_\epsilon}{|\hat{u}_\epsilon|}, \partial B') = 1$ , where  $B' = B_{\tilde{\epsilon}\tilde{\alpha}}(a^\epsilon)$  and  $\tilde{\alpha} \in (0, 1)$ . Moreover, by the same argument of Lemma 2.2 in [14] and [15], we obtain

$$\int_{\Omega \setminus B'} e_{\tilde{\epsilon}}(\hat{u}_\epsilon) \geq \pi \tilde{\alpha} \log \frac{1}{\tilde{\epsilon}} - M_1, \quad (5.6)$$

and

$$\int_{B'} e_{\tilde{\epsilon}}(\hat{u}_\epsilon) \leq \pi (1 - \tilde{\alpha}) \log \frac{1}{\tilde{\epsilon}} + M_1, \quad (5.7)$$

where  $M_1$  is a positive constant independent of  $\epsilon$ .

Now we claim that

$$\int_{B' \setminus B_{\tilde{\epsilon}}(0)} e_{\tilde{\epsilon}}(\hat{u}_\epsilon) \geq \pi (1 - \tilde{\alpha}) \log \frac{1}{\tilde{\epsilon}} - K, \quad (5.8)$$

where  $K$  is a positive constant independent of  $\epsilon$ . Without loss of generality, we may assume that  $B' = B_{\theta_0}(0)$ ,  $\theta_0 = \tilde{\epsilon}^{\tilde{\alpha}}$ . Then (5.7) implies that

$$\int_{B_{\theta_0}(0)} e_{\tilde{\epsilon}}(\hat{u}_\epsilon) \leq \pi \log \frac{\theta_0}{\tilde{\epsilon}} + M_1. \quad (5.9)$$

Moreover, we may rescale the spatial variable and rewrite (5.7) as

$$\int_{B_1(0)} e_{\epsilon_1}(\hat{u}_\epsilon) \leq \pi(1 - \tilde{\alpha}) \log \frac{1}{\tilde{\epsilon}} + M_1, \quad (5.10)$$

where  $\epsilon_1 = \tilde{\epsilon}^{1-\tilde{\alpha}}$ . By (5.10) and Fubini theorem (cf. [14]), there exists  $\theta_1 \in (\tilde{\epsilon}^{2\tilde{\alpha}}, \tilde{\epsilon}^{\tilde{\alpha}})$  such that

$$\theta_0 \theta_1 \int_{\partial B_{\theta_0 \theta_1}(0)} e_{\tilde{\epsilon}}(\hat{u}_\epsilon) \leq C(\tilde{\alpha}, M_1),$$

and that

$$\deg\left(\frac{\hat{u}_\epsilon}{|\hat{u}_\epsilon|}, \partial B_{\theta_0 \theta_1}(0)\right) = 1.$$

Hence by the same argument of Lemma 2.2 in [14] and [15], we have

$$\int_{B_{\theta_0}(0) \setminus B_{\theta_0 \theta_1}(0)} e_{\tilde{\epsilon}}(\hat{u}_\epsilon) \geq \pi \log \frac{1}{\theta_1} - M_2, \quad (5.11)$$

and

$$\int_{B_{\theta_0 \theta_1}(0)} e_{\tilde{\epsilon}}(\hat{u}_\epsilon) \leq \pi \log \frac{\theta_0 \theta_1}{\tilde{\epsilon}} + M_1 + M_2, \quad (5.12)$$

where  $M_2$  is a positive constant satisfying  $M_2 \leq C_0 \theta_0$ . Here  $C_0$  is a positive constant independent of  $\epsilon$ . Thus by induction, we may obtain  $\theta_1, \dots, \theta_m \in (\tilde{\epsilon}^{2\tilde{\alpha}}, \tilde{\epsilon}^{\tilde{\alpha}})$  such that  $\tilde{\epsilon} = \theta_0 \theta_1 \cdots \theta_m$  and

$$\int_{B_{\theta_0 \cdots \theta_{k-1}}(0) \setminus B_{\theta_0 \cdots \theta_k}(0)} e_{\tilde{\epsilon}}(\hat{u}_\epsilon) \geq \pi \log \frac{1}{\theta_k} - M_{k+1}, \quad (5.13)$$

and

$$\int_{B_{\theta_0 \cdots \theta_k}(0)} e_{\tilde{\epsilon}}(\hat{u}_\epsilon) \leq \pi \log \frac{\theta_0 \cdots \theta_k}{\epsilon} + \sum_{j=1}^{k+1} M_j, \quad (5.14)$$

for  $k = 1, \dots, m$ , where  $M_j$ 's are positive constants satisfying  $M_{k+1} \leq C_0 \theta_0^k$  for  $k \geq 0$ . Note that  $\sum_{j=1}^{k+1} M_j \leq M_1 + C_0 \sum_{j=1}^{\infty} \theta_0^j \leq C_1$ , where  $C_1$  is a positive constant independent of  $\epsilon$  and  $k$ . Therefore by (5.13), we may obtain (5.8). By (5.6) and (5.8), we obtain (5.4) and complete the proof of Lemma II. Here we have used the fact that  $\int_{\Omega \setminus B_{\tilde{\epsilon}}(0)} e_\epsilon(\hat{u}_\epsilon) \geq \int_{\Omega \setminus B_{\tilde{\epsilon}}(0)} e_{\tilde{\epsilon}}(\hat{u}_\epsilon)$  because of  $R_0 > 1$ .

From Lemma II, it is obvious that

$$\int_{\Omega \setminus B_{\epsilon R_0}(0)} e_\epsilon(u_\epsilon) \geq \pi \log \frac{1}{\epsilon} - C(R_0), \quad (5.15)$$

Hence (5.1) and (5.15) imply that

$$\int_{B_{\epsilon R_0}(0)} e_\epsilon(u_\epsilon) \leq K(R_0), \quad \forall R_0 > 1, \quad (5.16)$$

where  $K$  is a positive constant depending only on  $R_0$ . Since  $\tilde{u}_\epsilon(x, y) = u_\epsilon(\epsilon x, \epsilon y)$ , for  $(x, y) \in \frac{1}{\epsilon}\Omega$ , then (5.16) implies that

$$\int_{B_{R_0}(0)} e_1(\tilde{u}_\epsilon) \leq K(R_0), \quad \forall R_0 > 1. \quad (5.17)$$

Thus  $\|\tilde{u}_\epsilon\|_{H^1(B_{R_0}(0))} \leq K'(R_0)$  for all  $R_0 > 1$ , where  $K'$  is a positive constant independent of  $\epsilon$ . Therefore we may obtain that  $\tilde{u}_\epsilon$  converges to  $\tilde{u}$  weakly in  $H_{loc}^1(\mathbb{R}^2)$  as  $\epsilon$  goes to zero (up to a subsequence). Moreover, by (1.7), it is obvious that  $\tilde{u}$  is a weak solution of (1.19) if  $\delta = o(\epsilon^2)$ , and  $\tilde{u}$  is a weak solution of (1.20) if  $\delta = \lambda\epsilon^2$ , where  $\lambda$  is a positive constant independent of  $\epsilon$ . By Theorem III, [6] and [12], we may obtain  $\tilde{u} = f(r)e^{i\theta}$  if  $\delta = O(\epsilon^N)$ ,  $N > 4$ , where  $f$  satisfies (1.21), and we complete the proof of Theorem V.

## REFERENCES

- [1] R. A. ADAMS, *Sobolev spaces*, Academic Press, INC., New York, 1975.
- [2] I. AFFLECK, M. FRANZ, M. H. S. AMIN, *Generalized London free energy for high- $T_c$  vortex lattices*, Phys. Rev. B, 55, R704 (1997).
- [3] F. BETHUEL, H. BREZIS, F. HELEIN, *Ginzburg-Landau vortices*, Birkhauser, 1994.
- [4] H. BREZIS, F. MERLE, T. RIVIERE, *Quantization effects for  $-\Delta u = u(1 - |u|^2)$  in  $\mathbb{R}^2$* , Arch. Rational Mech. Anal., 126:1(1994), pp. 35–58.
- [5] A. J. BERLINSKY, A. L. FETTER, M. FRANZ, C. KALLIN, P. I. SOININEN, *Ginzburg-Landau theory of vortices in d-wave superconductors*, Phys. Rev. Lett., 75(1995), pp. 2200.
- [6] X. CHEN, C. M. ELLIOTT, T. QI, *Shooting method for vortex solutions of a complex-valued Ginzburg-Landau equation*, Proc. Roy. Soc. Edinburgh, 124A(1994), pp. 1075–1088.
- [7] D. CHANG, C.Y. MU, B. ROSENSTEIN, C. L. WU, *Static and dynamical anisotropy effects in mixed state of d-wave superconductors*, Phys. Rev. B, 57:13(1998), pp. 7955–7969.
- [8] Q. DU, *Studies of a Ginzburg-Landau model for d-wave superconductors*, SIAM J. Appl. Math., 59:4(1999), pp. 1225–1250.
- [9] M. FRANZ, C. KALLIN, P. I. SOININEN, A. J. BERLINSKY, A. L. FETTER, *Vortex state in a d-wave superconductor*, Phys. Rev. B, 53:9(1996), pp. 5795.
- [10] P. HAGAN, *Spiral Waves in Reaction Diffusion Equations*, SIAM J. Appl. Math., 42:4(1982), pp. 762–786.
- [11] Q. HAN, T. C. LIN, *Fourfold symmetric vortex solutions of the d-wave Ginzburg-Landau equation*, Nonlinearity, 15(2002), pp. 257–269.
- [12] R. M. HERVÉ, M. HERVÉ, *Étude qualitative des solutions réelles d'une équation différentielle liée à l'équation de Ginzburg-Landau*, Ann. Inst. Henri Poincaré, 11:4(1994), pp. 427–440.
- [13] F. H. LIN, *Static and moving vortices in Ginzburg-Landau theories*, Progress in nonlinear differential equations and their applications, 29(1997), Birkhäuser Verlag Basel/ Switzerland.
- [14] F. H. LIN, *Complex Ginzburg-Landau equations and dynamics of vortices, filaments, and codimension-2 submanifolds*, Comm. Pure Appl. Math., 51:4(1998), pp. 385–441.
- [15] F. H. LIN, *Vortex dynamics for the nonlinear wave equation*, Comm. Pure Appl. Math., LII:6(1999), pp. 737–761.
- [16] F. H. LIN, T. C. LIN, *Vortex State of d-Wave Superconductors in the Ginzburg-Landau Energy*, SIAM J. Math. Anal., 32:3(2000), pp. 493–503.
- [17] T. C. LIN, *The stability of the radial solution to the Ginzburg-Landau equation*, Comm. in PDE, 22(1997), pp. 619–632.
- [18] P. MIRONESCU, *On the stability of radial solutions of the Ginzburg-Landau equation*, J. Funct. Anal., 130:2(1995), pp. 334–344.
- [19] J. H. XU, Y. REN, C. S. TING, *Structures of single vortex and vortex lattice in a d-wave superconductor*, Phys. Rev. B, 53:6(1996), pp. 2991.