

## MINIMAL VARIANCE HEDGING FOR FRACTIONAL BROWNIAN MOTION \*

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**Abstract.** We discuss the extension to the multi-dimensional case of the Wick-Itô integral with respect to fractional Brownian motion, introduced by [6] in the 1-dimensional case. We prove a multi-dimensional Itô type isometry for such integrals, which is used in the proof of the multi-dimensional Itô formula. The results are applied to study the problem of minimal variance hedging in a market driven by fractional Brownian motions.

**1. Introduction.** In the following we let  $H = (H_1, H_2, \dots, H_m)$  be an  $m$ -dimensional Hurst vector with components  $H_i \in (\frac{1}{2}, 1)$  for  $i = 1, 2, \dots, m$ , and we let  $B^{(H)}(t) = (B_1^{(H)}(t), \dots, B_m^{(H)}(t))$  be an  $m$ -dimensional *fractional Brownian motion* ( $fBm$ ) with Hurst parameter  $H$ . This means that  $B^{(H)}(t) = B^{(H)}(t, \omega)$ ;  $t \in \mathbb{R}$ ,  $\omega \in \Omega$  is a continuous Gaussian stochastic process on a filtered probability space  $(\Omega, \mathcal{F}_t^{(H)}, \mu)$  with mean

$$\mathbb{E}[B^{(H)}(t)] = 0 = B^{(H)}(0) \quad \text{for all } t \quad (1.1)$$

and covariance

$$\mathbb{E}[B_i^{(H)}(s)B_j^{(H)}(t)] = \frac{1}{2}\{|s|^{2H_i} + |t|^{2H_i} - |s - t|^{2H_i}\}\delta_{ij} \quad (1.2)$$

where

$$\delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j; \quad i \leq i, j \leq m, \end{cases}$$

where  $\mathbb{E} = \mathbb{E}_\mu$  denotes the expectation with respect to the probability law  $\mu$  of  $B^{(H)}(\cdot)$ .

In other words,  $B^{(H)}(t)$  consists of  $m$  independent 1-dimensional fractional Brownian motions with Hurst parameters  $H_1, \dots, H_m$ , respectively. If  $H_i = \frac{1}{2}$  for all  $i$ , then  $B^{(H)}(t)$  coincides with classical Brownian motion  $B(t)$ . We refer to [11], [13] and [18] for more information about 1-dimensional  $fBm$ . Because of its properties (persistence/antipersistence and self-similarity)  $fBm$  has been suggested as a useful mathematical tool in many applications, including finance [10]. For example, these features of  $fBm$  seem to appear in the log-returns of stocks [18], in weather derivative models [3] and in electricity prices in a liberated electricity market [20].

In view of this it is of interest to develop a powerful calculus for  $fBm$ . Unfortunately,  $fBm$  is not a semimartingale nor a Markov process (unless  $H_i = \frac{1}{2}$  for all  $i$ ), so these theories cannot be applied to  $fBm$ . However, if  $H_i > \frac{1}{2}$  then the paths have zero quadratic variation and it is therefore possible to define a *pathwise integral*, denoted by

$$\int_{\mathbb{R}} f(t, \omega) \delta B^{(H)}(t),$$

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by a classical result of Young from 1936. See [12] and the references therein. This integral will obey Stratonovich type (i.e. “deterministic”) integration rules. Typically the expectation of such integrals is not 0 and it is known ([12], [15], [16], [19]) that the use of these integrals in finance will give markets with *arbitrage*, even in the most basic cases. In fact, this unpleasant situation (from a modelling point of view) occurs whenever we use an integration theory with Stratonovich integration rules in the generation of wealth from a portfolio. See e.g. the simple examples of [4] and [19].

Because of this – and for several other reasons – it is natural to try other types of integration with respect to *fBm*. Let  $\mathcal{L}_\phi^{1,2}$  be the set of (measurable) processes  $f(\cdot, \cdot) : \mathbb{R} \times \Omega \rightarrow \mathbb{R}$  such that  $\|f\|_{\mathcal{L}_\phi^{1,2}} < \infty$ , where

$$\|f\|_{\mathcal{L}_\phi^{1,2}}^2 := \mathbb{E} \left[ \int_{\mathbb{R}} \int_{\mathbb{R}} f(s)f(t)\phi(s,t)ds dt + \left( \int_{\mathbb{R}} D_t^\phi f(t)dt \right)^2 \right]. \tag{1.3}$$

In [6] a Wick-Itô type of integral is constructed, denoted by

$$\int_{\mathbb{R}} f(t, \omega)dB^{(H)}(t),$$

where  $B^{(H)}(t)$  is a 1-dimensional *fBm* with  $H \in (\frac{1}{2}, 1)$ . This integral exists as an element of  $L^2(\mu)$  for all (measurable) processes  $f(t, \omega)$  such that  $\|f\|_{\mathcal{L}_\phi^{1,2}} < \infty$ . Here, and in the following,

$$\phi(s, t) = \phi_H(s, t) = H(2H - 1)|s - t|^{2H-2}; \quad (s, t) \in \mathbb{R}^2, \quad \frac{1}{2} < H < 1 \tag{1.4}$$

and

$$D_t^\phi F = \int_{\mathbb{R}} \phi(s, t)D_s F ds \tag{1.5}$$

denotes the Malliavin  $\phi$ -derivative of  $F$  (see [6, Definition 3.4]). If  $f(t, \omega)$  is a step process of the form

$$f(t, \omega) = \sum_{i=1}^n f_i(\omega)\mathcal{X}_{[t_i, t_{i+1})}(t), \quad \text{where } t_1 < t_2 < \dots < t_{n+1}, \tag{1.6}$$

and  $\|f\|_{\mathcal{L}_\phi^{1,2}} < \infty$ , then the integral is defined by

$$\int_{\mathbb{R}} f(t, \omega)dB^{(H)}(t) = \sum_{i=1}^n f_i(\omega) \diamond (B^{(H)}(t_{i+1}) - B^{(H)}(t_i)), \tag{1.7}$$

where  $\diamond$  denotes the Wick product. We have the following basic properties of the Wick-Itô integral:

$$\mathbb{E} \left[ \int_{\mathbb{R}} f(t, \omega)dB^{(H)}(t) \right] = 0 \quad \text{for all } f \in \mathcal{L}_\phi^{1,2} \tag{1.8}$$

$$\mathbb{E} \left[ \left( \int_{\mathbb{R}} f(t, \omega)dB^{(H)}(t) \right) \left( \int_{\mathbb{R}} g(t, \omega)dB^{(H)}(t) \right) \right] = (f, g)_{\mathcal{L}_\phi^{1,2}} \quad \text{for all } f, g \in \mathcal{L}_\phi^{1,2} \text{ where} \tag{1.9}$$

$$(f, g)_{\mathcal{L}_\phi^{1,2}} = \mathbb{E} \left[ \int_{\mathbb{R}} \int_{\mathbb{R}} f(s)g(t)\phi(s,t)ds dt + \left( \int_{\mathbb{R}} D_t^\phi f(t)dt \right) \cdot \left( \int_{\mathbb{R}} D_t^\phi g(t)dt \right) \right]. \tag{1.10}$$

See [6] for details and proofs.

This Wick-Itô fractional calculus was subsequently extended to a white noise setting and applied to finance in [9]. Later this white noise theory was generalized to all  $H \in (0, 1)$  by [7].

All the above papers [6], [9] and [7] only deal with the 1-dimensional case. In Section 2 of this paper we discuss the extension of this integral to the  $m$ -dimensional case, i.e. we discuss the integral

$$\int_{\mathbb{R}} f(t, \omega) dB^{(H)}(t) = \sum_{i=1}^m \int_{\mathbb{R}} f_i(t, \omega) dB_i^{(H)}(t) \quad \text{for } f = (f_1, \dots, f_m) \in \mathcal{L}_{\phi}^{1,2}(m)$$

where  $B^{(H)}(t) = (B_1^{(H)}(t), \dots, B_m^{(H)}(t))$  is  $m$ -dimensional  $fBm$ ,  $\phi = (\phi_{H_1}, \dots, \phi_{H_m})$  and  $\mathcal{L}_{\phi}^{1,2}(m)$  is the corresponding class of integrands (see (2.5) below). We prove the  $m$ -dimensional analogue of the isometry (1.9), which turns out to have some unexpected features (see Theorem 2.1). By combining the multi-dimensional fractional Itô formula (Theorem 2.6) with Theorem 2.1 we obtain another fractional Itô isometry (Theorem 2.7). Finally, we end Section 2 by proving a fractional integration by parts formula (Theorem 2.9 and Theorem 2.10).

In Section 3 we apply the above results to study the problem of minimal variance hedging in a (possibly incomplete) market driven by  $m$ -dimensional  $fBm$ . Here we use fractional mathematical market model introduced by [9] and by [7]. For classical Brownian motions (and semimartingales) this problem has been studied by many researchers. See for example the survey [17] and the references therein. It turns out that for  $fBm$  this problem is even harder than in the classical case and in this paper we concentrate on a special case in order to get more specific results.

**2. Multi-dimensional Wick-Itô integration with respect to  $fBm$ .** Let  $B^{(H)}(t) = (B_1^{(H)}(t), \dots, B_m^{(H)}(t)); t \in \mathbb{R}, \omega \in \Omega$  be  $m$ -dimensional  $fBm$  with Hurst vector  $H = (H_1, \dots, H_m) \in (\frac{1}{2}, 1)^m$ , as in Section 1. Since the  $B_k^{(H)}(\cdot)$  are independent, we may regard  $\Omega$  as a product  $\Omega = \Omega_1 \times \Omega_2 \times \dots \times \Omega_m$  of identical copies  $\Omega_k$  of some  $\bar{\Omega}$  and write  $\omega = (\omega_1, \dots, \omega_m) \in \Omega$ .

Let  $\mathcal{F} = \mathcal{F}_{\infty}^{(m,H)}$  be the  $\sigma$ -algebra generated by  $\{B_k^{(H)}(s, \cdot); s \in \mathbb{R}, k = 1, 2, \dots, m\}$  and let  $\mathcal{F}_t = \mathcal{F}_t^{(m,H)}$  be the  $\sigma$ -algebra generated by  $\{B_k^{(H)}(s, \cdot); 0 \leq s \leq t, k = 1, 2, \dots, m\}$ . If  $F : \Omega \rightarrow \mathbb{R}$  is  $\mathcal{F}$ -measurable,  $1 \leq k \leq m$ , we set

$$D_{k,t}^{\phi} F = \int_{\mathbb{R}} \phi_k(s, t) D_{k,t} F dt \quad (\text{if the integral converges}) \tag{2.1}$$

where

$$\phi = (\phi_1, \dots, \phi_m) \tag{2.2}$$

$$\phi_k(s, t) = \phi_{H_k}(s, t) = H_k(2H_k - 1) |s - t|^{2H_k - 2}; \quad (s, t) \in \mathbb{R}^3, \quad k = 1, 2, \dots, m \tag{2.3}$$

and  $D_{k,t} F = \frac{\partial F}{\partial \omega_k}(t, \omega)$  is the Malliavin derivative of  $F$  with respect to  $\omega_k$ , at  $(t, \omega)$  (if it exists).

Let  $\mathcal{B} = \mathcal{B}(\mathbb{R})$  denote the Borel  $\sigma$ -algebra on  $\mathbb{R}$ . Similarly to the 1-dimensional case we can define the multi-dimensional fractional Wick-Itô integral

$$\int_{\mathbb{R}} f(t, \omega) dB^{(H)}(t) = \sum_{k=1}^m \int_{\mathbb{R}} f_k(t, \omega) dB_k^{(H)}(t) \in L^2(\mu) \tag{2.4}$$

for all  $\mathcal{B} \times \mathcal{F}$ -measurable processes  $f(t, \omega) = (f_1(t, \omega), \dots, f_m(t, \omega)) \in \mathbb{R}^m$  such that

$$\begin{aligned} & \|f_k\|_{\mathcal{L}^1_{\phi_k}} < \infty \quad \text{for all } k = 1, 2, \dots, m, \text{ where} \\ & \|f_k\|_{\mathcal{L}^1_{\phi_k}} := \mathbb{E} \left[ \int_{\mathbb{R}} \int_{\mathbb{R}} f_k(s) f_k(t) \phi_k(s, t) ds dt + \left( \int_{\mathbb{R}} D_{k,t}^{\phi} f_k(t) dt \right)^2 \right]. \end{aligned} \tag{2.5}$$

Denote the set of all such  $m$ -dimensional processes  $f$  by  $\mathcal{L}^1_{\phi}(m)$ . As in the 1-dimensional case we obtain the isometries

$$\mathbb{E} \left[ \left( \int_{\mathbb{R}} f_k dB_k^{(H)} \right)^2 \right] = \|f_k\|_{\mathcal{L}^1_{\phi_k}}^2; \quad k = 1, 2, \dots, m. \tag{2.6}$$

This is intuitively clear, since we (by independence of  $B_1^{(H)}, \dots, B_m^{(H)}$ ) can treat the remaining stochastic variables  $\omega_1, \dots, \omega_{k-1}, \omega_{k+1}, \dots, \omega_m$  as parameters and repeat the 1-dimensional approach in the  $\omega_k$  variable. It is also easy to prove (2.6) rigorously by writing  $f_k(t, \omega_1, \omega_2, \dots, \omega_m)$  as a limit of sums of products of functions depending only on  $(t, \omega_k)$  and only on  $(\omega_1, \dots, \omega_{k-1}, \omega_{k+1}, \dots, \omega_m)$ , respectively.

In view of this it is clear that if  $f = (f_1, \dots, f_m) \in \mathcal{L}^1_{\phi}(m)$ , then the Wick-Itô integral (2.4) is well-defined as an element of  $L^2(\mu)$  and by (2.6) we have

$$\left\| \int_{\mathbb{R}} f dB^{(H)} \right\|_{L^2(\mu)} \leq \sum_{k=1}^m \|f_k\|_{\mathcal{L}^1_{\phi_k}}. \tag{2.7}$$

It is useful to have an explicit expression for the norm on the left hand side of (2.7). The following formula is our main result of this section:

**THEOREM 2.1** (Multi-dimensional fractional Wick-Itô Isometry I). *Let  $f, g \in \mathcal{L}^1_{\phi}(m)$ . Then*

$$\mathbb{E} \left[ \left( \int_{\mathbb{R}} f dB^{(H)} \right) \cdot \left( \int_{\mathbb{R}} g dB^{(H)} \right) \right] = (f, g)_{\mathcal{L}^1_{\phi}(m)} \tag{2.8}$$

where

$$\begin{aligned} & (f, g)_{\mathcal{L}^1_{\phi}(m)} \\ &= \mathbb{E} \left[ \sum_{k=1}^m \int_{\mathbb{R}} \int_{\mathbb{R}} f_k(s) g_k(t) \phi_k(s, t) ds dt + \sum_{k,\ell=1}^m \left( \int_{\mathbb{R}} D_{\ell,t}^{\phi} f_k(t) dt \right) \cdot \left( \int_{\mathbb{R}} D_{k,t}^{\phi} g_{\ell}(t) dt \right) \right]. \end{aligned} \tag{2.9}$$

**REMARK.** Note the crossing of the indices  $\ell, k$  of the derivatives and the components  $f_k, g_{\ell}$  in the last terms of the right hand side of (2.9).

To prove Theorem 2.1 we proceed as in [6], but with the appropriate modifications:

In the 1-dimensional case, let  $L^2_{\phi_k}$  be the set of deterministic functions  $\alpha : \mathbb{R} \rightarrow \mathbb{R}$  such that

$$(\alpha, \alpha)_{\phi_k} := |\alpha|_{\phi_k}^2 := \int_{\mathbb{R}} \int_{\mathbb{R}} \alpha(s) \alpha(t) \phi_k(s, t) ds dt < \infty. \tag{2.10}$$

If  $\alpha \in L^2_{\phi_k}$  then clearly  $\alpha \in \mathcal{L}^{1,2}_{\phi_k}$ . Hence we can define the *Wick* (or Doleans-Dale) *exponential*

$$\mathcal{E}(\alpha) = \exp^\diamond \left( \int_{\mathbb{R}} \alpha(t) dB_k^{(H)}(t) \right) = \exp \left( \int_{\mathbb{R}} \alpha(t) dB_k^{(H)}(t) - \frac{1}{2} |\alpha|_{\phi_k}^2 \right). \tag{2.11}$$

See e.g. [6, (3.1)] or [9, Example 3.10].

Similarly, in the multidimensional case we put  $\phi = (\phi_1, \dots, \phi_m)$  and we let  $L^2_\phi$  be the set of all deterministic functions  $\alpha = (\alpha_1, \dots, \alpha_m) : \mathbb{R} \rightarrow \mathbb{R}^m$  such that  $\alpha_k \in L^2_{\phi_k}$  for  $k = 1, \dots, m$ . If  $\alpha \in L^2_\phi$  we define the corresponding Wick exponential

$$\begin{aligned} \mathcal{E}(\alpha) &= \exp^\diamond \left( \int_{\mathbb{R}} \alpha(t) dB^{(H)}(t) \right) = \exp^\diamond \left( \sum_{k=1}^m \int_{\mathbb{R}} \alpha_k(t) dB_k^{(H)}(t) \right) \\ &= \exp \left( \sum_{k=1}^m \int_{\mathbb{R}} \alpha_k(t) dB_k^{(H)}(t) - \frac{1}{2} |\alpha|_\phi^2 \right), \end{aligned} \tag{2.12}$$

where

$$|\alpha|_\phi^2 = \sum_{k=1}^m \int_{\mathbb{R}} \int_{\mathbb{R}} \alpha_k(s) \alpha_k(t) \phi_k(s, t) ds dt = \sum_{k=1}^m |\alpha|_{\phi_k}^2. \tag{2.13}$$

Let  $\mathcal{E}$  be the linear span of all  $\mathcal{E}(\alpha)$ ;  $\alpha \in L^2_\phi$ . Then we have

**THEOREM 2.2.** ([6, Theorem 3.1])  *$\mathcal{E}$  is a dense subset of  $L^p(\mathcal{F}, \mu)$ , for all  $p \geq 1$ .*

and

**THEOREM 2.3.** ([6, Theorem 3.2]) *Let  $g_i = (g_{i1}, \dots, g_{im}) \in L^2_\phi$  for  $i = 1, 2, \dots, n$  such that*

$$|g_{ik} - g_{jk}|_{\phi_k} \neq 0 \quad \text{if } i \neq j, \quad k = 1, \dots, m. \tag{2.14}$$

*Then  $\mathcal{E}(g_1), \dots, \mathcal{E}(g_n)$  are linearly independent in  $L^2(\mathcal{F}, \mu)$ .*

If  $F \in L^2(\mathcal{F}, \mu)$  and  $g_k \in L^2_{\phi_k}$  we put, as in [6],

$$D_{k, \Phi(g_k)} F = \int_{\mathbb{R}} D_{k,t}^\phi F \cdot g_k(t) dt. \tag{2.15}$$

We list some useful differentiation and Wick product rules. The proofs are similar to the 1-dimensional case and are omitted.

**LEMMA 2.4.** *Let  $f = (f_1, \dots, f_m) \in L^2_\phi$ ,  $g = (g_1, \dots, g_m) \in L^2_\phi$ . Then*

(i)  $D_{k, \Phi(g_k)} \left( \sum_{i=1}^m \int_{\mathbb{R}} f_i dB_i^{(H)} \right) = (f_k, g_k)_{\phi_k}, \quad k = 1, \dots, m,$

where

$$(f_k, g_k)_{\phi_k} = \int_{\mathbb{R}} \int_{\mathbb{R}} f_k(s) g_k(t) \phi_k(s, t) ds dt; \quad k = 1, \dots, m, \tag{2.16}$$

(ii)  $D_{k,s}^\phi \left( \sum_{i=1}^m \int_{\mathbb{R}} f_i dB_i^{(H)} \right) = \int_{\mathbb{R}} f_k(u) \phi_k(s, u) du; \quad k = 1, \dots, m,$

- (iii)  $D_{k, \Phi(g_k)} \mathcal{E}(f) = \mathcal{E}(f) \cdot (f_k, g_k)_{\phi_k}$  ;  $k = 1, \dots, m$  ,
- (iv)  $D_{k,s}^{\phi} \mathcal{E}(f) = \mathcal{E}(f) \cdot \int_{\mathbb{R}} f_k(u) \phi_k(s, u) du$  ;  $k = 1, \dots, m$  ,
- (v)  $\mathcal{E}(f) \diamond \mathcal{E}(g) = \mathcal{E}(f + g)$
- (vi)  $F \diamond \int_{\mathbb{R}} g_k dB_k^{(H)} = F \cdot \int_{\mathbb{R}} g_k dB_k^{(H)} - D_{k, \Phi(g_k)} F$  ,  $k = 1, \dots, m$  ,  
provided that  $F \in L^2(\mathcal{F}, \mu)$  and  $D_{k, \Phi(g_k)} F \in L^2(\mathcal{F}, \mu)$ .
- (vii)  $\mathbb{E}[\mathcal{E}(f) \cdot \mathcal{E}(g)] = \exp(f, g)_{\phi}$  .

We now turn to the multi-dimensional case. We will prove

LEMMA 2.5. *Suppose  $\alpha_k \in L^2_{\phi_k}$ ,  $\beta_{\ell} \in L^2_{\phi_{\ell}}$ ,  $D_{\ell, \Phi(\beta_{\ell})} F \in L^2(\mu)$  and  $D_{k, \Phi(\alpha_k)} G \in L^2(\mu)$ . Then*

$$\begin{aligned} \mathbb{E} \left[ \left( F \diamond \int_{\mathbb{R}} \alpha_k dB_k^{(H)} \right) \cdot \left( G \diamond \int_{\mathbb{R}} \beta_{\ell} dB_{\ell}^{(H)} \right) \right] \\ = \mathbb{E} \left[ (D_{\ell, \Phi(\beta_{\ell})} F) \cdot (D_{k, \Phi(\alpha_k)} G) + \delta_{k\ell} FG(\alpha_k, \beta_k)_{\phi_k} \right] , \end{aligned} \tag{2.17}$$

where

$$\delta_{k\ell} = \begin{cases} 1 & \text{if } k = \ell \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* We adapt the argument in [6] to the multi-dimensional case: First note that by a density argument we may assume that

$$F = \mathcal{E}(f) = \exp \left\{ \int_{\mathbb{R}} f(t) dB^{(H)}(t) - \frac{1}{2} |f|_{\phi}^2 \right\}$$

and

$$G = \mathcal{E}(g) = \exp \left\{ \int_{\mathbb{R}} g(t) dB^{(H)}(t) - \frac{1}{2} |g|_{\phi}^2 \right\} ,$$

for some  $f \in L^2_{\phi}$ ,  $g \in L^2_{\phi}$ .

Choose  $\delta = (\delta_1, \dots, \delta_m) \in \mathbb{R}^m$ ,  $\gamma = (\gamma_1, \dots, \gamma_m) \in \mathbb{R}^m$  and put  $\delta \times f = (\delta_1 f_1, \dots, \delta_m f_m)$  and  $\gamma \times g = (\gamma_1 g_1, \dots, \gamma_m g_m)$ . Then by Lemma 2.4

$$\begin{aligned} \mathbb{E}[(\mathcal{E}(f) \diamond \mathcal{E}(\delta \times \alpha)) \cdot (\mathcal{E}(g) \diamond \mathcal{E}(\gamma \times \beta))] \\ = \mathbb{E}[\mathcal{E}(f + \delta \times \alpha) \cdot \mathcal{E}(g + \gamma \times \beta)] = \exp(f + \delta \times \alpha, g + \gamma \times \beta)_{\phi} \end{aligned} \tag{2.18}$$

$$= \exp \left\{ \sum_{i=1}^m \int_{\mathbb{R}} \int_{\mathbb{R}} (f_i + \delta_i \alpha_i)(s) (g_i + \gamma_i \beta_i)(t) \phi_i(s, t) ds dt \right\}. \tag{2.19}$$

We now compute the double derivatives

$$\frac{\partial^2}{\partial \delta_k \partial \gamma_{\ell}}$$

of (2.18) and (2.19) at  $\delta = \gamma = 0$ . We distinguish between two cases:

Case 1.  $k \neq \ell$

Then if we differentiate (2.18) we get

$$\begin{aligned}
 & \frac{\partial^2}{\partial \delta_k \partial \gamma_\ell} \mathbb{E}[(\mathcal{E}(f) \diamond \mathcal{E}(\delta \times \alpha)) \cdot (\mathcal{E}(g) \diamond \mathcal{E}(\gamma \times \beta))]_{\delta=\gamma=0} \\
 &= \frac{\partial}{\partial \gamma_\ell} \mathbb{E}\left[\left(\mathcal{E}(f) \diamond \mathcal{E}(\delta \times \alpha) \diamond \int_{\mathbb{R}} \alpha_k dB_k^{(H)}\right) \cdot (\mathcal{E}(g) \diamond \mathcal{E}(\gamma \times \beta))\right]_{\delta=\gamma=0} \\
 &= \mathbb{E}\left[\left(\mathcal{E}(f) \diamond \int_{\mathbb{R}} \alpha_k dB_k^{(H)}\right) \cdot \left(\mathcal{E}(g) \diamond \int_{\mathbb{R}} \beta_\ell dB_\ell^{(H)}\right)\right]. \tag{2.20}
 \end{aligned}$$

On the other hand, if we differentiate (2.19) we get

$$\begin{aligned}
 & \frac{\partial^2}{\partial \delta_k \partial \gamma_\ell} [\exp(f + \delta \times \alpha, g + \gamma \times \beta)_\phi]_{\delta=\gamma=0} \\
 &= \frac{\partial}{\partial \gamma_\ell} \left[ \exp(f + \delta \times \alpha, g + \gamma \times \beta)_\phi \cdot \int_{\mathbb{R}} \int_{\mathbb{R}} \alpha_k(s)(g_k + \gamma_k \beta_k)(t) \phi_k(s, t) ds dt \right]_{\delta=\gamma=0} \\
 &= \exp(f, g)_\phi \cdot \int_{\mathbb{R}} \int_{\mathbb{R}} \alpha_k(s) g_k(t) \phi_k(s, t) ds dt \cdot \int_{\mathbb{R}} \int_{\mathbb{R}} \beta_\ell(s) f_\ell(t) \phi_\ell(s, t) ds dt \\
 &= \exp(f, g)_\phi \cdot (\alpha_k, g_k)_{\phi_k} \cdot (\beta_\ell, f_\ell)_{\phi_\ell} \\
 &= \mathbb{E}[\mathcal{E}(f) \cdot (\beta_\ell, f_\ell)_{\phi_\ell} \cdot \mathcal{E}(g) \cdot (\alpha_k, g_k)_{\phi_k}] \\
 &= \mathbb{E}[D_{\ell, \Phi(\beta_\ell)} \mathcal{E}(f) \cdot D_{k, \Phi(\alpha_k)} \mathcal{E}(g)]. \tag{2.21}
 \end{aligned}$$

This proves (2.17) in this case.

Case 2.  $k = \ell$ .

In this case, if we differentiate (2.18) we get

$$\begin{aligned}
 & \frac{\partial^2}{\partial \delta_k \partial \gamma_k} \mathbb{E}[(\mathcal{E}(f) \diamond \mathcal{E}(\delta \times \alpha)) \cdot (\mathcal{E}(g) \diamond \mathcal{E}(\gamma \times \beta))]_{\delta=\gamma=0} \\
 &= \frac{\partial}{\partial \gamma_k} \mathbb{E}\left[\left(\mathcal{E}(f) \diamond \mathcal{E}(\delta \times \alpha) \diamond \int_{\mathbb{R}} \alpha_k dB_k^{(H)}\right) \cdot (\mathcal{E}(g) \diamond \mathcal{E}(\gamma \times \beta))\right]_{\delta=\gamma=0} \\
 &= \mathbb{E}\left[\left(\mathcal{E}(f) \diamond \int_{\mathbb{R}} \alpha_k dB_k^{(H)}\right) \cdot \left(\mathcal{E}(g) \diamond \int_{\mathbb{R}} \beta_k dB_k^{(H)}\right)\right]. \tag{2.22}
 \end{aligned}$$

On the other hand, if we differentiate (2.19) we get

$$\begin{aligned}
 & \frac{\partial^2}{\partial \delta_k \partial \gamma_k} [\exp(f + \delta \times \alpha, g + \gamma \times \beta)_\phi]_{\delta=\gamma=0} \\
 &= \frac{\partial}{\partial \gamma_k} \left[ \exp(f + \delta \times \alpha, g + \gamma \times \beta)_\phi \cdot \int_{\mathbb{R}} \int_{\mathbb{R}} \alpha_k(s)(g_k + \gamma_k \beta_k)(t) \phi_k(s, t) ds dt \right]_{\delta=\gamma=0} \\
 &= \exp(f, g)_\phi \cdot \left[ (\alpha_k, g_k)_{\phi_k} \cdot (\beta_k, f_k)_{\phi_k} + \int_{\mathbb{R}} \int_{\mathbb{R}} \alpha_k(s) \beta_k(t) \phi_k(s, t) ds dt \right] \\
 &= \mathbb{E}[D_{k, \Phi(\beta_k)} \mathcal{E}(f) \cdot D_{k, \Phi(\alpha_k)} \mathcal{E}(g) + \mathcal{E}(f) \mathcal{E}(g) (\alpha_k, \beta_k)_{\phi_k}]. \tag{2.23}
 \end{aligned}$$

This proves (2.17) also for Case 2 and the proof of Lemma 2.5 is complete.  $\square$

We are now ready to prove Theorem 2.1:

*Proof.* We may consider  $\int_{\mathbb{R}} f_k(t)dB_k^{(H)}(t)$  as the limit of sums of the form

$$\sum_{i=1}^N f_k(t_i) \diamond (B_k^{(H)}(t_{i+1}) - B_k^{(H)}(t_i))$$

when  $\Delta t_i = t_{i+1} - t_i \rightarrow 0, t_1 < t_2 < \dots < t_N, N = 2, 3, \dots$ . Hence  $\mathbb{E} \left[ \left( \int_{\mathbb{R}} f dB^{(H)} \right)^2 \right] = \mathbb{E} \left[ \left( \sum_{k=1}^m \int_{\mathbb{R}} f_k dB_k^{(H)} \right)^2 \right]$  is the limit of sums of the form

$$\sum_{i,j,k,\ell} \mathbb{E} \left[ (f_k(t_i) \diamond (B_k^{(H)}(t_{i+1}) - B_k^{(H)}(t_i))) \cdot (f_\ell(t_j) \diamond (B_\ell^{(H)}(t_{j+1}) - B_\ell^{(H)}(t_j))) \right],$$

which by Lemma 2.5 is equal to

$$\sum_{i,j,k,\ell} \mathbb{E} \left[ \left( \int_{t_i}^{t_{i+1}} D_{\ell,t}^\phi f_k(t) dt \right) \cdot \left( \int_{t_j}^{t_{j+1}} D_{k,t}^\phi f_\ell(t) dt \right) + \delta_{k\ell} \int_{t_i}^{t_{i+1}} \int_{t_j}^{t_{j+1}} f_k(t_i) f_\ell(t_j) \phi_k(s,t) ds dt \right].$$

When  $\Delta t_i \rightarrow 0$  this converges to

$$\mathbb{E} \left[ \sum_{k,\ell=1}^m \left( \int_{\mathbb{R}} D_{\ell,t}^\phi f_k(t) dt \right) \cdot \left( \int_{\mathbb{R}} D_{k,t}^\phi f_\ell(t) dt \right) + \sum_{k=1}^m \int_{\mathbb{R}} \int_{\mathbb{R}} f_k(s) f_k(t) \phi_k(s,t) ds dt \right]. \tag{2.24}$$

This proves (2.9) when  $f = g$ . By polarization the proof of Theorem 2.1 is complete.  $\square$

Using Theorem 2.1 we can now proceed as in the 1-dimensional case ([6, Theorem 4.3]), with appropriate modifications, and obtain a fractional multi-dimensional Itô formula. We omit the proof.

**THEOREM 2.6** (The fractional multi-dimensional Itô formula). *Let  $X(t) = (X_1(t), \dots, X_n(t))$ , with*

$$dX_i(t) = \sum_{j=1}^m \sigma_{ij}(t, \omega) dB_j^{(H)}(t);$$

*where  $\sigma_i = (\sigma_{i1}, \dots, \sigma_{im}) \in \mathcal{L}_\phi^{1,2}(m); 1 \leq i \leq n$ .* (2.25)

*Suppose that for all  $j = 1, \dots, m$  there exists  $\theta_j > 1 - H_j$  such that*

$$\sup_i \mathbb{E}[(\sigma_{ij}(u) - \sigma_{ij}(v))^2] \leq C|u - v|^{\theta_j} \quad \text{if } |u - v| < \delta \tag{2.26}$$

*where  $\delta > 0$  is a constant. Moreover, suppose that*

$$\lim_{\substack{0 \leq u, v \leq t \\ |u-v| \rightarrow 0}} \left\{ \sup_{i,j,k} \mathbb{E}[(D_{k,u}^\phi \{\sigma_{ij}(u) - \sigma_{ij}(v)\})^2] \right\} = 0. \tag{2.27}$$

*Let  $f \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n)$  with bounded second order derivatives with respect to  $x$ . Then,*

for  $t > 0$ ,

$$\begin{aligned}
 f(t, X(t)) &= f(0, X(0)) + \int_0^t \frac{\partial f}{\partial s}(s, X(s)) ds + \int_0^t \sum_{i=1}^n \frac{\partial f}{\partial x_i}(s, X(s)) dX_i(s) \\
 &+ \int_0^t \left\{ \sum_{i,j=1}^m \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X(s)) \sum_{k=1}^m \sigma_{ik}(s) D_{k,s}^\phi(X_j(s)) \right\} ds \quad (2.28)
 \end{aligned}$$

$$\begin{aligned}
 &= f(0, X(0)) + \int_0^t \frac{\partial f}{\partial s}(s, X(s)) ds + \sum_{j=1}^m \int_0^t \left[ \sum_{i=1}^n \frac{\partial f}{\partial x_i}(s, X(s)) \sigma_{ij}(s, \omega) \right] dB_j^{(H)}(s) \\
 &+ \int_0^t \text{Tr} [\Lambda^T(s) f_{xx}(s, X(s))] ds. \quad (2.29)
 \end{aligned}$$

Here  $\Lambda = [\Lambda_{ij}] \in \mathbb{R}^{n \times m}$  with

$$\Lambda_{ij}(s) = \sum_{k=1}^m \sigma_{ik} D_{k,s}^\phi(X_j(s)); \quad 1 \leq i \leq n, \quad 1 \leq j \leq m, \quad (2.30)$$

$$f_{xx} = \left[ \frac{\partial^2 f}{\partial x_i \partial x_j} \right]_{1 \leq i, j \leq n} \in \mathbb{R}^{n \times n} \quad (2.31)$$

and  $(\cdot)^T$  denotes matrix transposed,  $\text{Tr}[\cdot]$  denotes matrix trace.

If we combine Theorem 2.6 with Theorem 2.1 we get the following result, which also may be regarded as a fractional Itô isometry:

**THEOREM 2.7** (Fractional Itô isometry II). *Suppose  $f = (f_1, \dots, f_m) \in \mathcal{L}_\phi^{1,2}(m)$ . Then, for  $T > 0$ ,*

$$\begin{aligned}
 &\mathbb{E} \left[ \left( \int_0^T D_{\ell,t}^\phi f_k(t) dt \right) \cdot \left( \int_0^T D_{k,t}^\phi f_\ell(t) dt \right) \right] \\
 &= \mathbb{E} \left[ \int_0^T \left\{ f_k(t) \int_0^t D_{k,t}^\phi f_\ell(s) dB_\ell^{(H)}(s) + f_\ell(t) \int_0^t D_{\ell,t}^\phi f_k(s) dB_k^{(H)}(s) \right\} dt \right] \quad (2.32)
 \end{aligned}$$

*Proof.* By the Itô formula (Theorem 2.6) we have

$$\begin{aligned}
 &\mathbb{E} \left[ \left( \int_0^T f_k dB_k^{(H)} \right) \cdot \left( \int_0^T f_\ell dB_\ell^{(H)} \right) \right] \\
 &= \mathbb{E} \left[ \int_0^T \left\{ f_k(t) D_{k,t}^\phi \left( \int_0^t f_\ell(s) dB_\ell^{(H)}(s) \right) + f_\ell(t) D_{\ell,t}^\phi \left( \int_0^t f_k(s) dB_k^{(H)}(s) \right) \right\} dt \right] \\
 &= \mathbb{E} \left[ \int_0^T \left\{ f_k(t) \int_0^t D_{k,t}^\phi f_\ell(s) dB_\ell^{(H)}(s) + f_\ell(t) \int_0^t D_{\ell,t}^\phi f_k(s) dB_k^{(H)}(s) \right\} dt \right] \\
 &+ \delta_{k\ell} \mathbb{E} \left[ \int_0^T \int_0^t \{ f_k(t) f_\ell(s) + f_\ell(t) f_k(s) \} \phi_k(s, t) ds dt \right], \quad (2.33)
 \end{aligned}$$

where we have used that, for  $u > 0$ ,

$$D_{k,t}^\phi \left( \int_0^u f_\ell(s) dB_\ell^{(H)}(s) \right) = \int_0^u D_{k,t}^\phi f_\ell(s) dB_\ell^{(H)}(s) + \delta_{k\ell} \int_0^u f_k(s) \phi_k(t, s) ds. \quad (2.34)$$

(See [6, Theorem 4.2].)

On the other hand, the Itô isometry (Theorem 2.1) gives that

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^T f_k dB_k^{(H)}\right) \cdot \left(\int_0^T f_\ell dB_\ell^{(H)}\right)\right] \\ = \mathbb{E}\left[\left(\int_0^T D_{\ell,t}^\phi f_k(t) dt\right) \cdot \left(\int_0^T D_{k,t}^\phi f_\ell(t) dt\right) + \delta_{k\ell} |f_k|_{\phi_k}^2\right]. \end{aligned} \tag{2.35}$$

Comparing (2.33) and (2.35) we get Theorem 2.7.  $\square$

We end this section by proving a fractional integration by parts formula. First we recall

**THEOREM 2.8** (Fractional Girsanov formula). *Suppose  $\gamma = (\gamma_1, \dots, \gamma_m) \in (L^2(\mathbb{R}))^m$  and  $\hat{\gamma} = (\hat{\gamma}_1, \dots, \hat{\gamma}_m) \in L_\phi^2$  are related by*

$$\gamma_k(t) = \int_{\mathbb{R}} \hat{\gamma}_k(s) \phi_k(s, t) ds; \quad t \in \mathbb{R}, \quad k = 1, \dots, m. \tag{2.36}$$

Let  $G \in L^2(\mu)$ . Then

$$\mathbb{E}[G(\omega + \gamma)] = \mathbb{E}[G(\omega) \exp^\diamond(\langle \omega, \hat{\gamma} \rangle)] = \mathbb{E}\left[G(\omega) \mathcal{E}\left(\int_{\mathbb{R}} \hat{\gamma} dB^{(H)}\right)\right]. \tag{2.37}$$

For a proof in the 1-dimensional case see e.g. [9, Theorem 3.16]. The proof in the multi-dimensional case is similar.

If  $F \in L^2(\mu)$  and  $\gamma = (\gamma_1, \dots, \gamma_m) \in (L^2(\mathbb{R}))^m$  the *directional derivative of  $F$  in the direction  $\gamma$*  is defined by

$$D_\gamma F(\omega) = \lim_{\varepsilon \rightarrow 0} \frac{F(\omega + \varepsilon \gamma) - F(\omega)}{\varepsilon}, \tag{2.38}$$

provided the limit exists in  $L^2(\mu)$ . We say that  $F$  is *differentiable* if there exists a process  $D_t F(\omega) = (D_{1,t} F(\omega), \dots, D_{m,t} F(\omega))$  such that  $D_{k,t} F(\omega) \in L^2(d\mu \otimes dt)$  for all  $k = 1, \dots, m$  and

$$D_\gamma F(\omega) = \int_{\mathbb{R}} D_t F(\omega) \cdot \gamma(t) dt \quad \text{for all } \gamma \in (L^2(\mathbb{R}))^m. \tag{2.39}$$

**THEOREM 2.9** (Fractional integration by parts I). *Let  $F, G \in L^2(\mu)$ ,  $\gamma \in (L^2(\mathbb{R}))^m$  and assume that the directional derivatives  $D_\gamma F, D_\gamma G$  exist. Then*

$$\mathbb{E}[D_\gamma F \cdot G] = \mathbb{E}\left[F \cdot G \cdot \int_{\mathbb{R}} \hat{\gamma} dB^{(H)}\right] - \mathbb{E}[F \cdot D_\gamma G]. \tag{2.40}$$

*Proof.* By Theorem 2.8 we have, for all  $\varepsilon > 0$ ,

$$\mathbb{E}[F(\omega + \varepsilon \gamma) G(\omega)] = \mathbb{E}[F(\omega) G(\omega - \varepsilon \gamma) \exp^\diamond(\varepsilon \langle \omega, \hat{\gamma} \rangle)].$$

Hence

$$\begin{aligned} \mathbb{E}[D_\gamma F \cdot G] &= \mathbb{E}\left[\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \{F(\omega + \varepsilon\gamma) - F(\omega)\}G(\omega)\right] \\ &= \mathbb{E}\left[\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \{F(\omega)[G(\omega - \varepsilon\gamma) \exp^\diamond(\varepsilon\langle\omega, \hat{\gamma}\rangle) - G(\omega)]\}\right] \\ &= \mathbb{E}\left[F(\omega) \frac{d}{d\varepsilon} \left\{G(\omega - \varepsilon\gamma) \exp\left(\varepsilon \int_{\mathbb{R}} \hat{\gamma} dB^{(H)} - \frac{1}{2}\varepsilon^2 |\hat{\gamma}|_\phi^2\right)\right\}_{\varepsilon=0}\right] \\ &= \mathbb{E}\left[F(\omega)G(\omega) \int_{\mathbb{R}} \hat{\gamma} dB^{(H)}\right] - \mathbb{E}[F(\omega)D_\gamma G(\omega)] \end{aligned}$$

□

We now apply the above to the fractional gradient

$$D_t^\phi F = \int_{\mathbb{R}} D_s F \cdot \phi(s, t) ds = \sum_{k=1}^m \int_{\mathbb{R}} D_{k,s} F \cdot \phi_k(s, t) ds = D_\phi F(\omega) \tag{2.41}$$

**THEOREM 2.10** (Fractional integration by parts II). *Suppose  $F, G \in L^2(\mu)$  are differentiable, with fractional gradients  $D_t^\phi F, D_t^\phi G$ . Then for each  $t \in \mathbb{R}, k \in \{1, \dots, m\}$  we have*

$$\mathbb{E}[D_{k,t}^\phi F \cdot G] = \mathbb{E}[F \cdot G \cdot B_k^{(H)}(t)] - \mathbb{E}[F \cdot D_{k,t}^\phi G] . \tag{2.42}$$

*Proof.* Choose a sequence  $\hat{\gamma}_k^{(j)} \in L^2_{\phi_k}; j = 1, 2, \dots$ , such that  $\lim_{j \rightarrow \infty} \hat{\gamma}_k^{(j)} = \delta_t(\cdot)$  (the point mass at  $t$ ), in the sense that if we define

$$\phi_k^{(j)}(s) = \int_{\mathbb{R}} \hat{\gamma}_k^{(j)} \phi_k(s, r) dr$$

then  $\phi_k^{(j)}(\cdot) \rightarrow \phi_k(\cdot, t)$  in  $L^2(\mathbb{R})$ . Then by Theorem 2.9

$$\begin{aligned} \mathbb{E}[D_{k,t}^\phi F \cdot G] &= \mathbb{E}\left[\lim_{j \rightarrow \infty} D_{\phi_k^{(j)}} F \cdot G\right] = \lim_{j \rightarrow \infty} \mathbb{E}[D_{\phi_k^{(j)}} F \cdot G] \\ &= \lim_{j \rightarrow \infty} \mathbb{E}\left[F \cdot G \cdot \int_{\mathbb{R}} \hat{\gamma}_k^{(j)} dB^{(H)}\right] - \mathbb{E}[F \cdot D_{\phi_k^{(j)}} G] \\ &= \mathbb{E}[F \cdot G \cdot B_k^{(H)}(t)] - \mathbb{E}[F \cdot D_{k,t}^\phi G] . \end{aligned}$$

□

**3. Application to minimal variance hedging.** Consider the multidimensional version of the fractional mathematical market model introduced by [9] and by [7], consisting of  $n + 1$  independent fractional Brownian motions  $B_1^{(H)}(t), \dots, B_m^{(H)}(t)$  with Hurst coefficients  $H_1, \dots, H_m$  respectively ( $\frac{1}{2} < H_i < 1$ ), as follows:

$$\text{(bond price)} \quad dS_0(t) = r(t, \omega)dt ; \quad S_0(0) = s_0 , \quad 0 \leq t \leq T \tag{3.1}$$

$$\text{(stock prices)} \quad dS_i(t) = \mu_i(t, \omega)dt + \sum_{j=1}^m \sigma_{ij}(t, \omega)dB_j^{(H)}(t) ; \quad S_i(0) = s_i, \tag{3.2}$$

$$i = 1, \dots, n, \quad 0 \leq t \leq T .$$

Here  $r(t, \omega), \mu_i(t, \omega)$  and  $\sigma_{ij}(t, \omega)$  are  $\mathcal{F}_t^{(H)}$ -adapted processes satisfying reasonable growth conditions. We refer to [7], [9], [14] and [21] for a general discussion of such markets.

We say that  $g = (g_1, \dots, g_m)$  is an *admissible portfolio* if  $g(t)$  is  $\mathcal{F}_t^{(H)}$ -adapted,  $g\sigma \in \mathcal{L}_\phi^{1,2}(m)$  and  $\mathbb{E}\left[\int_0^T \sum_{i=1}^n |g_i(t)\mu_i(t)|dt\right] < \infty$ . Here we denote by  $\sigma$  the volatility matrix  $[\sigma]_{i,j}(\cdot) = \sigma_{ij}(\cdot)$ . Suppose we are only allowed to trade in some, say  $k$ , of the securities  $S_0, \dots, S_n$ . Let  $\mathcal{K}$  be the set of  $i \in \{1, \dots, n\}$  such that trading in  $S_i$  is allowed. Then, according to our model, the *wealth* hedged by an *initial value*  $z \in \mathbb{R}$  and an admissible portfolio  $g(t) = (g_i(t, \omega))_{i \in \mathcal{K}} \in \mathbb{R}^k$  up to time  $t$  is

$$V(t) = V_z^g(t) = z + \sum_{i \in \mathcal{K}} \int_0^t g_i(u) dS_i(u); \quad 0 \leq t \leq T. \tag{3.3}$$

Now let  $F(\omega)$  be a  $T$ -claim, i.e. an  $\mathcal{F}_T^{(H)}$ -measurable random variable in  $L^2(\mu)$ .

The *minimal variance hedging problem* is to find a  $z^* \in \mathbb{R}$  and an admissible portfolio  $g^*$  such that

$$\mathbb{E}[(F - V_{z^*}^{g^*}(T))^2] = \inf_{z, g} \mathbb{E}[(F - V_z^g(T))^2]. \tag{3.4}$$

This is a difficult problem even in the classical Brownian motion setting. See e.g. [8], [17] and the references therein. For a recent general martingale approach see [5]. For fractional Brownian motion markets a special case is solved in [1] by using optimal control theory.

Here we will discuss the two-dimensional case only, and we will simply assume that

$$dS_0(t) = 0, \quad dS_1(t) = dB_1^{(H)}(t) \quad \text{and} \quad dS_2(t) = dB_2^{(H)}(t).$$

Assume that only trading in  $S_0$  and  $S_1$  is allowed. Then the problem is to minimize

$$J(z, g_1) = \mathbb{E}\left[\left(F - \left(z + \int_0^T g_1 dS_1\right)\right)^2\right] \tag{3.5}$$

over all  $z \in \mathbb{R}$  and all admissible portfolios  $g_1$ .

By the fractional Clark-Haussmann-Ocone formula ([9, Theorem 4.15]) we can write

$$F(\omega) = \mathbb{E}[F] + \int_0^T f_1(t) dB_1^{(H)}(t) + \int_0^T f_2(t) dB_2^{(H)}(t) \tag{3.6}$$

where

$$f_i(t) = \tilde{\mathbb{E}}[D_{i,t} F | \mathcal{F}_t^{(H)}]; \quad i = 1, 2.$$

Substituting this into (3.5) we get, by (1.8),

$$\begin{aligned} J(z, g_1) &= \mathbb{E}\left[\left(\mathbb{E}[F] - z + \int_0^T (f_1 - g_1) dB_1^{(H)} + \int_0^T f_2 dB_2^{(H)}\right)^2\right] \\ &= (\mathbb{E}[F] - z)^2 + \mathbb{E}\left[\left(\int_0^T (f_1 - g_1) dB_1^{(H)} + \int_0^T f_2 dB_2^{(H)}\right)^2\right]. \end{aligned} \tag{3.7}$$

Hence it is optimal to choose  $z = z^* := \mathbb{E}[F]$ . The remaining problem is therefore to minimize

$$J_0(g_1) = \mathbb{E}\left[\left(\int_0^T (f_1 - g_1)dB_1^{(H)} + \int_0^T f_2dB_2^{(H)}\right)^2\right]. \tag{3.8}$$

From now on we assume that  $f_i \in \mathcal{L}_{\phi_i}^{1,2}$  for  $i = 1, 2$ . By a Hilbert space argument on  $L^2(\mu)$  we see that  $g_1^*$  minimizes (3.8) if and only if

$$\begin{aligned} \mathbb{E}\left[\left(\int_0^T (f_1 - g_1)dB_1^{(H)} + \int_0^T f_2dB_2^{(H)}\right) \cdot \left(\int_0^T \gamma dB_1^{(H)}\right)\right] &= 0 \\ \text{for all adapted } \gamma \in \mathcal{L}_{\phi_1}^{1,2}. \end{aligned} \tag{3.9}$$

By Theorem 2.1 (3.9) is equivalent to

$$\begin{aligned} \mathbb{E}\left[\int_0^T \int_0^T (f_1(t) - g_1(t))\gamma(s)\phi_1(s, t)ds dt + \left(\int_0^T D_{1,t}^\phi(f_1(t) - g_1(t))dt\right)\left(\int_0^T D_{1,t}^\phi \gamma(t)dt\right) \right. \\ \left. + \left(\int_0^T D_{1,t}^\phi f_2(t)dt\right) \cdot \left(\int_0^T D_{2,t}^\phi \gamma(t)dt\right)\right] \\ = 0 \quad \text{for all adapted } \gamma \in \mathcal{L}_\phi^{1,2}. \end{aligned} \tag{3.10}$$

From this we immediately deduce

PROPOSITION 3.1. *The portfolio*

$$g_1(t) = g_1^*(t) := f_1(t)$$

*minimizes (3.8) if and only if*

$$\int_0^T D_{1,t}^\phi f_2(t)dt = 0 \quad a.s. \tag{3.11}$$

This result is surprising in view of the corresponding situation for classical Brownian motion, when it is *always* optimal to choose  $g_1(t) = g_1^*(t) = f_1(t)$ .

We also get

PROPOSITION 3.2. *Suppose  $g_1^*(t)$  minimizes (3.8). Then*

$$\mathbb{E}\left[\int_0^T (f_1(t) - g_1^*(t))dt\right] = 0. \tag{3.12}$$

*Proof.* This follows by choosing  $\gamma(t)$  deterministic in (3.10).  $\square$

Now assume that  $D_{1,t}^\phi(f_1(t))$  and  $D_{1,t}^\phi(g_1(t))$  are differentiable with respect to  $D_{1,s}^\phi$  and that  $D_{1,t}^\phi f_2(t)$  is differentiable with respect to  $D_{2,s}^\phi$  for all  $s \in [0, T]$ . Then we can use integration by parts (Theorem 2.10) to rewrite equation (3.10) as follows:

$$\begin{aligned}
& \mathbb{E} \left[ \int_0^T \int_0^T \{ (f_1(t) - g_1(t)) \gamma(s) \phi_1(s, t) + D_{1,t}^\phi(f_1(t) - g_1(t)) \cdot D_{1,s}^\phi \gamma(s) \right. \\
& \quad \left. + D_{1,t}^\phi f_2(t) \cdot D_{2,s}^\phi \gamma(s) \} ds dt \right] \\
&= \int_0^T \int_0^T \mathbb{E} [ (f_1(t) - g_1(t)) \phi_1(s, t) \gamma(s) + D_{1,t}^\phi(f_1(t) - g_1(t)) \gamma(s) B_1^{(H)}(s) \\
& \quad - D_{1,s}^\phi D_{1,t}^\phi(f_1(t) - g_1(t)) \gamma(s) + D_{1,t}^\phi f_2(t) \gamma(s) B_2^{(H)}(s) \\
& \quad - D_{2,s}^\phi D_{1,t}^\phi f_2(t) \gamma(s) ] ds dt \\
&= \mathbb{E} \left[ \int_0^T K(s) \gamma(s) ds \right] = 0, \tag{3.13}
\end{aligned}$$

where

$$K(s) = \int_0^T G(s, t) dt, \tag{3.14}$$

with

$$\begin{aligned}
G(s, t) &= (f_1(t) - g_1(t)) \phi_1(s, t) + D_{1,t}^\phi(f_1(t) - g_1(t)) B_1^{(H)}(s) \\
& \quad - D_{1,s}^\phi D_{1,t}^\phi(f_1(t) - g_1(t)) + D_{1,t}^\phi f_2(t) B_2^{(H)}(s) - D_{2,s}^\phi D_{1,t}^\phi f_2(t). \tag{3.15}
\end{aligned}$$

Since  $\gamma(s)$  is  $\mathcal{F}_s^{(H)}$ -measurable we get from (3.13) that

$$\begin{aligned}
0 &= \int_0^T \mathbb{E} [K(s) \gamma(s)] ds = \int_0^T \mathbb{E} [\mathbb{E} [K(s) \gamma(s) \mid \mathcal{F}_s^{(H)}]] ds \\
&= \int_0^T \mathbb{E} [\gamma(s) \mathbb{E} [K(s) \mid \mathcal{F}_s^{(H)}]] ds = \mathbb{E} \left[ \int_0^T \mathbb{E} [K(s) \mid \mathcal{F}_s^{(H)}] \gamma(s) ds \right]. \tag{3.16}
\end{aligned}$$

Since this holds for all adapted  $\gamma \in \mathcal{L}_\phi^{1,2}$  we conclude that

$$\mathbb{E} [K(s) \mid \mathcal{F}_s^{(H)}] = 0 \quad \text{for a.a. } (s, \omega). \tag{3.17}$$

or, using (3.14),

$$\begin{aligned}
& \int_0^T \{ \mathbb{E}_s [f_1(t) - g_1(t)] \phi_1(s, t) + \mathbb{E}_s [D_{1,t}^\phi(f_1(t) - g_1(t))] B_1^{(H)}(s) \\
& \quad - \mathbb{E}_s [D_{1,s}^\phi D_{1,t}^\phi(f_1(t) - g_1(t))] + \mathbb{E}_s [D_{1,t}^\phi f_2(t)] B_2^{(H)}(s) - \mathbb{E}_s [D_{2,s}^\phi D_{1,t}^\phi f_2(t)] \} dt = 0, \tag{3.18}
\end{aligned}$$

where we have used the shorthand notation

$$\mathbb{E}_s[\cdot] = \mathbb{E}[\cdot \mid \mathcal{F}_s^{(H)}].$$

We have proved:

**THEOREM 3.3.** *Suppose the claim  $F$  represented by (3.6) is such that  $D_{1,s}^\phi D_{1,t}^\phi f_1(t)$  and  $D_{2,s}^\phi D_{1,t}^\phi f_2(t)$  exist for all  $s, t \in [0, T]$ . Suppose  $\hat{g}_1(t)$  is an adapted process in  $\mathcal{L}_\phi^{1,2}$  such that  $D_{1,t}^\phi \hat{g}_1(t)$  and  $D_{1,s}^\phi D_{1,t}^\phi \hat{g}_1(t)$  exist for all  $s, t \in [0, T]$ . Then the following are equivalent:*

- (i)  $\hat{g}_1(t)$  is a minimal variance hedging portfolio for  $F$ , i.e.  $\hat{g}_1(t)$  minimizes (3.8) over all adapted  $g_1(t) \in \mathcal{L}_\phi^{1,2}$
- (ii)  $g_1(t) = \hat{g}_1(t)$  satisfies equation (3.18).

Note that the same method also applies if we assume a fractional exponential dynamics for the asset prices, which represents a more realistic financial model. To illustrate this result we consider the following special case:

**EXAMPLE 3.4.** Suppose  $f_1(t) = 0$  and

$$D_{1,t}^\phi f_2(t) = h(t), \quad \text{a deterministic function.} \tag{3.19}$$

We seek a minimal variance hedging portfolio  $g_1^*(t)$  for the claim

$$F(\omega) = \int_0^T f_2(t) dB_2^{(H)}(t). \tag{3.20}$$

In this case (3.18) gets the form

$$\int_0^T \{-\mathbb{E}_s[g_1(t)]\phi_1(s, t) - \mathbb{E}_s[D_{1,t}^\phi g_1(t)]B_1^{(H)}(s) + \mathbb{E}_s[D_{1,s}^\phi D_{1,t}^\phi g_1(t)] + h(t)B_2^{(H)}(s)\}dt = 0 \quad \text{for a.a. } (s, \omega). \tag{3.21}$$

Let us try to choose  $g_1(t)$  such that

$$D_{1,t}^\phi g_1(t) = 0. \tag{3.22}$$

Then (3.19) reduces to

$$\int_0^T \mathbb{E}_s[g_1(t)]\phi_1(s, t)dt = B_2^{(H)}(s) \int_0^T h(t)dt \tag{3.23}$$

or, since  $g_1$  is adapted,

$$\int_0^s g_1(t)\phi_1(s, t)dt + \int_s^T \mathbb{E}_s[g_1(t)]\phi_1(s, t)dt = B_2^{(H)}(s) \int_0^T h(t)dt, \quad s \in [0, T]. \tag{3.24}$$

In particular, if we choose  $s = T$  we get the equation

$$\int_0^T g_1(t)\phi_1(T, t)dt = B_2^{(H)}(T) \int_0^T h(t)dt, \tag{3.25}$$

which clearly has no adapted solution  $g_1(t)$ . (However, it obviously has a *non-adapted* solution.) Therefore an optimal portfolio  $g_1(t) = g_1^*(t)$  for the claim (3.20), if it exists, cannot satisfy (3.22).

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