METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 10, No. 3, pp. 363–376, September 2003

002

INFINITE INTERVAL PROBLEMS ARISING IN THE MODEL OF A SLENDER DRY PATCH IN A LIQUID FILM DRAINING UNDER GRAVITY DOWN AN INCLINED PLANE *

RAVI P. AGARWAL† AND DONAL O'REGAN‡

Abstract. Existence results are established for a second order boundary value problem on the half line motivated from the model of a slender dry patch in a liquid film draining under gravity down an inclined plane.

1. Introduction. Consider a thin film of viscous liquid with constant density ρ and viscosity μ flowing down a planer substrate inclined at an angle α $(0 < \alpha \leq \frac{\pi}{2})$ to the horizontal. We adopt Cartesian coordinates (x, y, z) with the x-axis down the greatest slope and the z-axis normal to the plane. With the usual lubrication approximation the height of the free surface $z = h(x, y, z)$ satisfies [4]

$$3\,\mu\,h_t = \nabla\,.\,[h^3\,\nabla(\rho\,g\,h\,\cos\alpha - \sigma\,\nabla^2\,h)] - \rho\,g\,\sin\alpha\,[h^3]_x \tag{1.1}$$

where t denotes time, g the magnitude of acceleration due to gravity and σ the coefficient of surface tension. We are interested in solutions symmetric about $y = 0$, and we seek a steady state solution for a slender dry patch for which the length scale down the plane (i.e. in the x direction) is much greater than in the transverse direction (i.e. in the y direction), so the equation (1.1) is approximated by [4]

$$[h^3\,(\rho\,g\,h\,\cos\alpha - \sigma\,h_{y\,y})_y]_y - \rho\,g\,\sin\alpha\,[h^3]_x = 0. \tag{1.2}$$

The velocity component down the plane is $u(x, y, z) = \frac{\rho\,g\,\sin\alpha\,[2\,h\,z - z^2]}{2\,\mu}$ and so for a slender dry patch of semi-width $y_e = y_e(x)$ the average volume flux around the dry patch per unit width in the transverse direction down the plane (denoted by $Q(x)$) is approximately [4]

$$Q = \frac{\rho\,g\,\sin\alpha}{3\,\mu}\,\lim_{y\to\infty}\,y^{-1}\int_{y_e(x)}^{y} h(x, w)^3\,dw. \tag{1.3}$$

We seek a similarity solution to equation (1.2) of the form $h = f(x)\,G(\eta)$ where $\eta = \frac{y}{y_e(x)}$. Note $G(1) = 0$ and (1.2) takes the form

$$\begin{aligned} &\rho\,g\,\cos\alpha\,f^2\,y_e^2\,(G^3\,G')' - \sigma\,f^2\,(G^3\,G''')' \\ &\quad -3\,\rho\,g\,\sin\alpha\,y_e^3\,G^2\,(f'\,G\,y_e - f\,G'\,y_e'\,\eta) = 0 \end{aligned} \tag{1.4}$$

with the corresponding expression for Q being

$$Q = \frac{\rho\,g\,\sin\alpha}{3\,\mu}\,f^3\,\lim_{\eta\to\infty}\,\eta^{-1}\int_1^{\eta} G(w)^3\,dw.$$

For weak surface-tension effects the second term in (1.4) can be neglected and so the only relevant similarity solution is given (after a suitable choice of origin in x) by

$$f(x) = b\,(c\,x)^m \quad \text{and} \quad y_e(x) = (c\,x)^k$$

*Received March 31, 2003; accepted for publication August 21, 2003.
†Department of Mathematical Sciences, Florida Institute of Technology, Melbourne, Florida 32901–6975, USA (agarwal@fit.edu).
‡Department of Mathematics, National University of Ireland, Galway, Ireland.

where the coefficients b and c and the exponents m and k are constants with $m = 2k - 1$. In this case $\alpha \neq \frac{\pi}{2}$ and so we may choose without loss of generality $b = c\,k\,\tan\alpha$ and so (1.4) becomes

$$((G' + \eta)'\,G^3)' - \left(7 - \frac{3}{k}\right) G^3 = 0. \tag{1.5}$$

The unknown exponent k is determined by the requirement that the average volume flux per unit width around the dry patch, Q, is independent of x. This is possible only if $m = 0$ and $G \sim G_0 > 0$ (a constant) as $\eta \to \infty$. Thus

$$Q = \frac{\rho\,g\,\sin\alpha}{3\,\mu}\,(b\,G_0)^3 \quad \text{and so} \quad m = 0,\ k = \frac{1}{2}.$$

Setting $k = \frac{1}{2}$ in (1.5) yields

$$(G^3\,G')' + \eta\,(G^3)' = 0. \tag{1.6}$$

Also the solutions to (1.6) must satisfy the boundary condition $G(1) = 0$ and the far-field condition $\lim_{\eta\to\infty} G(\eta) = G_0$. As a result one is interested in the boundary value problem

$$\begin{cases} (G^3\,G')' + \eta\,(G^3)' = 0, & 1 < \eta < \infty \\ G(1) = 0,\ \lim_{\eta\to\infty} G(\eta) = G_0 > 0. \end{cases} \tag{1.7}$$

Keeping this problem in mind, in Section 2 we discuss the general boundary value problem

$$\begin{cases} (G'(y) + p(t)\,y^m)' + q(t)\,f(t,y) = p'(t)\,y^m, & a < t < n \\ y(a) = 0,\ y(n) = b_0 > 0, \end{cases} \tag{1.8}$$

where $n > a$, $G(z) = \int_0^z g(x)\,dx$, $G'(y) = \frac{d}{dt}\,G(y(t))$ and

$$g(x) = \begin{cases} x^m, & x \geq 0 \\ -\,x^m, & x < 0 \end{cases}$$

with $m > 0$ odd. A very general existence theory will be presented for (1.8) in Section 2. Our theory relies on the following nonlinear alternative of Leray–Schauder type [1, 2].

THEOREM 1.1. *Let U be an open subset of a Banach space E, $J : \overline{U} \to E$ a continuous compact map, $p^\star \in U$ and let $N : \overline{U} \times [0,1] \to E$ be a continuous compact map with $N_1 = J$ and $N_0 = p^\star$ (here $N_\lambda(u) = N(u,\lambda)$). Also assume*

$$u \neq N_\lambda\,(u) \quad \textit{for} \quad u \in \partial U \quad \textit{and} \quad \lambda \in (0,1]. \tag{1.9}$$

Then J has a fixed point in U.

In Section 3 we discuss the following boundary value problem on the half line

$$\begin{cases} (G'(y) + p(t)\,y^m)' + q(t)\,f(t,y) = p'(t)\,y^m, & a < t < \infty \\ y(a) = 0,\ y \text{ bounded on } [a,\infty), \end{cases}$$

and our existence theory will then be applied to (1.7).

2. Existence theory on finite intervals. In this section we first establish the existence of a solution to

$$(2.1)\qquad \begin{cases} (G'(y) + p(t)\, y^m)' + q(t)\, f(t, y) = p'(t)\, y^m, & a < t < n \\ y(a) = 0,\ y(n) = b_0 > 0 \end{cases}$$

(here $n > a$) where $G(z) = \int_0^z g(x)\, dx$ and

$$g(x) = \begin{cases} x^m, & x \geq 0 \\ -x^m = |x|^m, & x < 0 \end{cases}$$

and with $m > 0$ odd. Note $G'(y) = \frac{d}{dt} G(y(t))$ and

$$G(z) = \begin{cases} \frac{z^{m+1}}{m+1}, & z \geq 0 \\ \frac{-z^{m+1}}{m+1} = \frac{-|z|^{m+1}}{m+1}, & z < 0. \end{cases}$$

By a solution to (2.1) we mean a function $y \in C[a, n]$, with $G(y) \in C^1[a, n]$, $G'(y) + p\, y^m \in AC[a, n] \cap C^1(a, n]$ which satisfies $y(a) = 0$, $y(n) = b_0$ and the differential equation in (2.1) on (a, n).

THEOREM 2.1. *Suppose the following conditions are satisfied:*

$$(2.2)\qquad f : [a, n] \times \mathbf{R} \to \mathbf{R} \quad \textit{is continuous}$$

$$(2.3)\qquad q \in C(a, n] \cap L^1[a, n] \quad \textit{with} \quad q > 0 \quad \textit{on} \quad (a, n]$$

$$(2.4)\qquad p \in C^1[a, n] \quad \textit{with} \quad p \geq 0 \quad \textit{on} \quad [a, n]$$

$$(2.5)\qquad f(t, 0) \geq 0 \quad \textit{for} \quad t \in (a, n)$$

and

$$(2.6)\qquad f(t, b_0) \leq 0 \quad \textit{for} \quad t \in (a, n).$$

Then (2.1) *has a solution* y *with* $0 \leq y(t) \leq b_0$ *for* $t \in [a, n]$.

Proof. Consider the boundary value problem

$$(2.7)_\lambda\qquad \begin{cases} (G'(y) + \lambda\, p\, y^m)' = \lambda\, f^\star(t, y), & a < t < n \\ y(a) = 0,\ y(n) = b_0 > 0, & 0 < \lambda \leq 1 \end{cases}$$

with

$$f^\star(t, y) = \begin{cases} -\,q(t)\, f(t, 0) + y, & y < 0 \\ -\,q(t)\, f(t, y) + p'(t)\, y^m, & 0 \leq y \leq b_0 \\ -\,q(t)\, f(t, b_0) + p'(t)\, b_0^m + y - b_0, & y > b_0. \end{cases}$$

Solving $(2.7)_\lambda$ is equivalent (see [2]) to finding a $y \in C[a, n]$ which satisfies

$$(2.8)\qquad y(t) = G^{-1}\Big(A\,(t - a) - \lambda \int_a^t p(s)\, y^m(s)\, ds + \lambda \int_a^t (t - x)\, f^\star(x, y(x))\, dx\Big)$$

where

$$A = \frac{G(b_0) + \lambda \int_a^n p(s)\, y^m(s)\, ds - \lambda \int_a^n (n-x)\, f^\star(x, y(x))\, dx}{n-a}. \tag{2.9}$$

Define the operator $N_\lambda : C[a,n] \to C[a,n]$ by

$$N_\lambda\, y(t) = G^{-1}(A\,(t-a) - \lambda \int_a^t p(s)\, y^m(s)\, ds + \lambda \int_a^t (t-x)\, f^\star(x, y(x))\, dx).$$

The argument in [2] guarantees that $N_\lambda : C[a,n] \to C[a,n]$ is continuous and completely continuous. We now show any solution y to $(2.7)_\lambda$ $(0 < \lambda \le 1)$ satisfies

$$0 \le G(y(t)) \le G(b_0) \quad \text{for} \quad t \in [a,n]. \tag{2.10}$$

If (2.10) is true then

$$0 \le y(t) \le b_0 \quad \text{for} \quad t \in [a,n]. \tag{2.11}$$

Suppose $G(y(t)) < 0$ for some $t \in (a,n)$. Then $G(y)$ has a negative minimum at say $t_0 \in (a,n)$, so $G'(y(t_0)) = 0$. Also there exists $\delta_1 > 0$, $\delta_2 > 0$ with $(t_0 - \delta_1, t_0 + \delta_2) \subseteq [a,n]$ and with

$$\begin{cases} G(y(t)) < 0 \quad \text{for} \quad t \in (t_0 - \delta_1, t_0 + \delta_2) \\ \text{and} \quad G(y(t_0 - \delta_1)) = G(y(t_0 + \delta_2)) = 0. \end{cases} \tag{2.12}$$

Now for $t \in (t_0 - \delta_1, t_0 + \delta_2)$ we have

$$(G'(y(t)) + \lambda\, p(t)\, y^m(t))' = -\lambda\, q(t) f(t,0) + \lambda\, y(t) < 0,$$

so integration from t_0 to $t_0 + \delta_2$ yields

$$G'(y(t_0 + \delta_2)) + \lambda\, p(t_0 + \delta_2)\, y^m(t_0 + \delta_2) \;<\; \lambda\, p(t_0)\, y^m(t_0).$$

Now $y(t_0 + \delta_2) = 0$, so (note m is odd, $p \ge 0$ and $y(t_0) < 0$)

$$G'(y(t_0 + \delta_2)) < \lambda\, p(t_0)\, y^m(t_0) \le 0. \tag{2.13}$$

Thus there exists $\delta_3 > 0$, $\delta_3 < \delta_2$ with

$$G'(y(t)) < 0 \quad \text{for} \quad t \in (t_0 + \delta_3, t_0 + \delta_2). \tag{2.14}$$

As a result

$$0 = G(y(t_0 + \delta_2)) < G(y(t_0 + \delta_3)),$$

and this contradicts (2.12). Thus $0 \le G(y(t))$ for $t \in [a,n]$, so $0 \le y(t)$ for $t \in [a,n]$. Next suppose $G(y(t)) > G(b_0)$ for some $t \in (a,n)$. Then $G(y)$ has a positive maximum at say $t_1 \in (a,n)$, so $G'(y(t_1)) = 0$. Also there exists $\delta_4 > 0$, $\delta_5 > 0$ with $(t_1 - \delta_4, t_1 + \delta_5) \subseteq [a,n]$ and with

$$G(y(t)) > G(b_0) \quad \text{for} \quad t \in (t_1 - \delta_4, t_1 + \delta_5) \tag{2.15}$$

and

$$G(y(t_1 - \delta_4)) = G(y(t_1 + \delta_5)) = G(b_0). \tag{2.16}$$

Also for $t \in (t_1 - \delta_4, t_1 + \delta_5)$ we have

$$\begin{aligned}(G'(y(t)) + \lambda\, p(t)\, y^m(t))' &= -\lambda\, q(t) f(t, b_0) + \lambda\, p'(t)\, b_0^m + \lambda\,(y(t) - b_0) \\ &> \lambda\, p'(t)\, b_0^m,\end{aligned}$$

so integration from t_1 to $t_1 + \delta_5$ yields (note (2.16))

$$G'(y(t_1 + \delta_5)) + \lambda\, p(t_1 + \delta_5)\, b_0^m \;>\; \lambda\, p(t_1)\, y^m(t_1) + \lambda\, b_0^m\,[p(t_1 + \delta_5) - p(t_1)].$$

Thus

$$G'(y(t_1 + \delta_5)) > \lambda\, p(t_1)\,[y^m(t_1) - b_0^m] \geq 0$$

since $p \geq 0$. As a result there exists $\delta_6 > 0,\ \delta_6 < \delta_5$ with

$$G'(y(t)) \;>\; 0 \quad \text{for} \quad t \in (t_1 + \delta_6, t_1 + \delta_5),$$

so

$$G(b_0) = G(y(t_1 + \delta_5)) \;>\; G(y(t_1 + \delta_6)),$$

and this contradicts (2.15). Thus $G(y(t)) \leq G(b_0)$ for $t \in [a, n]$, so (2.11) holds.

Now Theorem 1.1 applied to N_λ with $E = C[a, n]$, $U = \{u \in E : \sup_{[a,n]} |u(t)| < b_0 + 1\}$ and $p^\star = G^{-1}\left(\frac{G(b_0)\,(t-a)}{n-a}\right)$ guarantees that N_1 has a fixed point $y \in U$. Thus y is a solution of $(2.7)_1$ and the argument above guarantees that $0 \leq y(t) \leq b_0$ for $t \in [a, n]$. As a result y is a solution of (2.1). □

REMARK 2.1. It is possible to replace $p \geq 0$ on $[a, n]$ by $p \leq 0$ on $[a, n]$ and the result in Theorem 2.1 is again true; we leave the details to the reader.

Keeping our application in Section 1 in mind we now discuss the situation when our solution to (2.1) is positive on $(a, n]$. Suppose the following conditions hold:

$$(2.17)\qquad \begin{cases} \exists\, \alpha \in C[a, n] \text{ with } G(\alpha) \in C^1[a, n],\ G'(\alpha) + p\,\alpha^m \in AC[a, n] \\ \cap\, C^1(a, n] \text{ with } b_0 \geq \alpha > 0 \text{ on } (a, n],\ \alpha(a) = 0,\ \alpha(n) \leq b_0 \\ \text{and } (G'(\alpha) + p\,\alpha^m)' + q(t)\, f(t, \alpha) \geq p'(t)\,\alpha^m(t) \text{ on } (a, n) \end{cases}$$

$$(2.18)\qquad \begin{cases} \text{for each } t \in (a, n) \text{ we have } q(t)\,[f(t, y) - f(t, \alpha(t)] \geq 0 \\ \text{for } 0 \leq y \leq \alpha(t) \end{cases}$$

and

$$(2.19)\qquad p' > 0 \text{ on } (a, n).$$

Also in this case we discuss the boundary value problem

$$(2.20)\qquad \begin{cases} (y^m\, y')' + p\,(y^m)' + q\, f(t, y) = 0,\ a < t < n \\ y(a) = 0,\ y(n) = b_0 > 0. \end{cases}$$

By a solution to (2.20) we mean a function $y \in C[a, n] \cap C^1(a, n]$ with $G(y) \in C^1[a, n]$, $y^m\, y' \in C^1(a, n]$ which satisfies $y(a) = 0$, $y(n) = b_0$ and the differential equation in (2.20) on (a, n).

THEOREM 2.2. *Suppose* (2.2)–(2.6), (2.17), (2.18) *and* (2.19) *are satisfied. Then* (2.1) *has a solution* y *with* $\alpha(t) \le y(t) \le b_0$ *for* $t \in [a,n]$. *In addition* $y \in C^1(a,n]$ *with* $G'(y) = y^m y'$ *on* (a,n) *and* y *is a solution of* (2.20).

Proof. Theorem 2.1 guarantees that (2.1) has a solution y with $0 \le y(t) \le b_0$ for $t \in [a,n]$. Next we claim that

$$y(t) \ge \alpha(t) \quad \text{for} \quad t \in [a,n]. \tag{2.21}$$

Suppose $G(\alpha(t)) > G(y(t))$ for some $t \in (a,n)$. Then $G(y) - G(\alpha)$ has a negative minimum at say $t_0 \in (a,n)$, so $G'(y(t_0)) = G'(\alpha(t_0))$. Also there exists $\delta_1 > 0$, $\delta_2 > 0$ with $(t_0 - \delta_1, t_0 + \delta_2) \subseteq [a,n]$ and with

$$G(y(t)) < G(\alpha(t)) \quad \text{for} \quad t \in (t_0 - \delta_1, t_0 + \delta_2) \tag{2.22}$$

and

$$G(y(t_0 - \delta_1)) = G(\alpha(t_0 - \delta_1)) \quad \text{and} \quad G(y(t_0 + \delta_2)) = G(\alpha(t_0 + \delta_2)). \tag{2.23}$$

Also for $t \in (t_0 - \delta_1, t_0 + \delta_2)$ we have (note $0 \le y \le b_0$ on $[a,n]$)

$$\begin{aligned}(G'(y) + p\,y^m)'(t) - (G'(\alpha) + p\,\alpha^m)'(t) &\le q(t)\,[f(t,\alpha(t)) - f(t,y(t))] \\ &\quad + p'(t)\,[y^m(t) - \alpha^m(t)] \\ &< 0,\end{aligned}$$

since $p' > 0$ on (a,n). Integrate from t_0 to $t_0 + \delta_2$ to obtain

$$\begin{aligned}&G'(y(t_0+\delta_2)) + p(t_0+\delta_2)\,y^m(t_0+\delta_2) - G'(y(t_0)) - p(t_0)\,y^m(t_0) \\ &\quad < G'(\alpha(t_0+\delta_2)) + p(t_0+\delta_2)\,\alpha^m(t_0+\delta_2) - G'(\alpha(t_0)) - p(t_0)\,\alpha^m(t_0),\end{aligned}$$

so (note (2.23))

$$G'(y(t_0+\delta_2)) - G'(\alpha(t_0+\delta_2)) < \; p(t_0)\,[y^m(t_0) - \alpha^m(t_0)] \le 0,$$

since $p \ge 0$ on $[a,n]$. Thus there exists $\delta_3 > 0$, $\delta_3 < \delta_2$ with

$$G'(y(t)) - G'(\alpha(t)) < 0 \quad \text{for} \quad t \in (t_0 + \delta_3, t_0 + \delta_2).$$

As a result

$$0 = G(y(t_0+\delta_2)) - G(\alpha(t_0+\delta_2)) < G(y(t_0+\delta_3)) - G(\alpha(t_0+\delta_3)),$$

i.e.

$$G(\alpha(t_0+\delta_3)) < G(y(t_0+\delta_3)),$$

and this contradicts (2.22). Thus $G(\alpha(t)) \le G(y(t))$ for $t \in [a,n]$, so $\alpha(t) \le y(t)$ for $t \in [a,n]$ i.e (2.21) is true.

In particular note $y(t) > 0$ for $t \in (a,n]$. Also

$$\begin{aligned}\frac{y^{m+1}(t)}{m+1} &= A\,(t-a) - \int_a^t p(s)\,y^m(s)\,ds \\ &\quad + \int_a^t (t-x)[-q(x)f(x,y(x)) + p'(x)\,y^m(x)]\,dx\end{aligned}$$

where A is given in (2.9) with $\lambda = 1$ and $f^\star(x,y(x)) = -q(x)\,f(x,y(x)) + p'(x)\,y^m(x)$. Since $y > 0$ on $(a,n]$ we have $y' \in C(a,n]$. Then the change of variables theorem [3 pp. 181] guarantees that $G'(y) = g(y)\,y' = y^m\,y'$ on (a,n). Also for $t \in (a,n)$ we have

$$g(y)\,y' = A - p\,y^m + \int_a^t [-q(x)\,f(x,y(x)) + p'(x)\,y^m(x)]\,dx,$$

so $g(y)\,y' \in C^1(a,n)$. Thus for $t \in (a,n)$ we have

$$-q\,f(t,y) + p'\,y^m = (g(y)\,y' + p\,y^m)' = (g(y)\,y')' + (p\,y^m)',$$

so y is a solution of (2.20). $\square$

Suppose the following condition is satisfied:

$$\text{(2.24)} \qquad \begin{cases} \exists\,\alpha \in C[a,n] \cap C^1(a,n] \ \text{ with } \ G(\alpha) \in C^1[a,n], \\ \alpha^m\,\alpha' \in C^1(a,n], \ b_0 \geq \alpha > 0 \ \text{ on } \ (a,n], \ \alpha(a) = 0, \ \alpha(n) \leq b_0 \\ \text{and } \ (\alpha^m\,\alpha')' + p\,(\alpha^m)' + q(t)\,f(t,\alpha) \geq 0 \ \text{ on } \ (a,n). \end{cases}$$

Then we have the following theorem.

THEOREM 2.3. *Suppose* (2.2)–(2.6), (2.18), (2.19) *and* (2.24) *are satisfied. Then* (2.20) *has a solution* y *with* $\alpha(t) \leq y(t) \leq b_0$ *for* $t \in [a,n]$.

Proof. Now the change of variables theorem [3 pp. 181] guarantees that $G'(\alpha) = g(\alpha)\,\alpha' = \alpha^m\,\alpha'$ on (a,n), so for $t \in (a,n)$ we have

$$\begin{aligned}(G'(\alpha) + p\,\alpha^m)' + q\,f(t,\alpha) &= (\alpha^m\,\alpha' + p\,\alpha^m)' + q\,f(t,\alpha) \\ &= (\alpha^m\,\alpha')' + (p\,\alpha^m)' + q\,f(t,\alpha) \\ &\geq (p\,\alpha^m)' - p\,(\alpha^m)' = p'\,\alpha^m.\end{aligned}$$

Thus (2.17) holds and the result follows from Theorem 2.2. $\square$

3. Existence theory on infinite intervals. In this section we first establish the existence of a solution to

$$\text{(3.1)} \qquad \begin{cases} (G'(y) + p(t)\,y^m)' + q(t)\,f(t,y) = p'(t)\,y^m, \ \ a < t < \infty \\ y(a) = 0, \ y \ \text{ bounded on } \ [a,\infty) \end{cases}$$

where g and G are as in Section 2 and $m > 0$ is odd. By a solution to (3.1) we mean a function $y \in BC[a,\infty)$ (bounded continuous functions on $[0,\infty)$) with $G(y) \in C^1[a,\infty)$, $G'(y) + p\,y^m \in AC_{loc}[a,\infty) \cap C^1(a,\infty)$ which satisfies $y(a) = 0$ and the differential equation in (3.1) on (a,∞).

THEOREM 3.1. *Suppose the following conditions are satisfied:*

$$\text{(3.2)} \qquad f : [a,\infty) \times \mathbf{R} \to \mathbf{R} \ \ \textit{is continuous}$$

$$\text{(3.3)} \qquad q \in C(a,\infty) \cap L^1_{loc}[a,\infty) \ \ \textit{with} \ \ q > 0 \ \ \textit{on} \ \ (a,\infty)$$

$$\text{(3.4)} \qquad p \in C^1[a,\infty) \ \ \textit{with} \ \ p \geq 0 \ \ \textit{on} \ \ [a,\infty)$$

$$f(t,0) \geq 0 \quad for \quad t \in (a,\infty) \tag{3.5}$$

$$\exists\, b_0 > 0 \quad with \quad f(t,b_0) \leq 0 \quad for \quad t \in (a,\infty) \tag{3.6}$$

and

$$\begin{cases} \exists\, \mu \in L^1_{loc}[a,\infty) \quad with \quad |f(t,u)| \leq \mu(t) \\ for\ a.e. \quad t \in [a,\infty) \quad and \quad u \in [0,b_0]. \end{cases} \tag{3.7}$$

Then (3.1) *has a solution* y *with* $0 \leq y(t) \leq b_0$ *for* $t \in [a,\infty)$.

Proof. Fix $n \in N = \{1,2,...\}$ with $n \geq a+1$ and consider the boundary value problem

$$\begin{cases} (G'(y) + p(t)\, y^m)' + q(t)\, f(t,y) = p'(t)\, y^m, \quad a < t < n \\ y(a) = 0,\ y(n) = b_0 > 0. \end{cases} \tag{3.8}$$

Theorem 3.1 guarantees that there exists a solution y_n to (3.8) (i.e. $y_n \in C[a,n]$, with $G(y_n) \in C^1[a,n],\ G'(y_n) + p\, y_n^m \in AC[a,n] \cap C^1(a,n]$) with $0 \leq y_n(t) \leq b_0$ for $t \in [a,n]$. We now claim that there exist constants A_1 and A_2 (independent of n) with

$$|G'(y_n(t))| \leq A_1 + A_2 \int_a^t |p'(s)|\, ds + \int_a^t \mu(s)\, ds \quad \text{for} \quad t \in [a,n]. \tag{3.9}$$

The mean value theorem guarantees that there exists $\xi \in (a,a+1)$ with $G'(y_n(\xi)) = G(y_n(a+1)) - G(0)$, and so

$$|G'(y_n(\xi))| \leq G(b_0) \equiv K_0.$$

To prove (3.9) we consider first the case when $t \in [a,n]$ and $t > \xi$. Integrate (3.8) from ξ to t to obtain (note (3.7)),

$$\begin{aligned} |G'(y_n(t))| &\leq |G'(y_n(\xi))| + |p(t)\, y^m(t) - p(\xi)\, y^m(\xi)| \\ &\quad + b_0^m \int_\xi^t |p'(s)|\, ds + \int_\xi^t \mu(s)\, ds \\ &\leq K_0 + |p(\xi)|\, |y^m(t) - y^m(\xi)| + |p(t) - p(\xi)|\, y^m(t) \\ &\quad + b_0^m \int_a^t |p'(s)|\, ds + \int_a^t \mu(s)\, ds \\ &\leq K_0 + 2\, b_0^m \sup_{s\in[a,a+1]} p(s) + b_0^m \left| \int_\xi^t p'(s)\, ds \right| \\ &\quad + b_0^m \int_a^t |p'(s)|\, ds + \int_a^t \mu(s)\, ds \\ &\leq K_0 + 2\, b_0^m \sup_{s\in[a,a+1]} p(s) + 2\, b_0^m \int_a^t |p'(s)|\, ds \\ &\quad + \int_a^t \mu(s)\, ds, \end{aligned}$$

so (3.9) is true in this case. Next consider the case when $t < \xi$. Note in particular that $t < a+1$. Integrate the differential equation in (3.8) from t to ξ to obtain

$$\begin{aligned}|G'(y_n(t))| &\leq K_0 + |p(\xi)|\,|y^m(t) - y^m(\xi)| + |p(t) - p(\xi)|\,y^m(t) \\ &\quad + b_0^m \int_t^\xi |p'(s)|\,ds + \int_t^\xi \mu(s)\,ds \\ &\leq K_0 + 2\,b_0^m \sup_{s\in[a,a+1]} p(s) + 2\,b_0^m \int_a^{a+1} |p'(s)|\,ds \\ &\quad + \int_a^{a+1} \mu(s)\,ds,\end{aligned}$$

so (3.9) is again true.

Thus (3.9) is true in all cases, so for $t,\ s \in [a,n]$ with $s < t$ we have

$$\begin{aligned}|G(y_n(s)) - G(y_n(t))| &= \left|\int_s^t G'(y_n(x))\,dx\right| \leq A_1\,|t-s| \\ &\quad + A_2 \int_s^t \int_a^x |p'(z)|\,dz\,dx + \int_s^t \int_a^x \mu(x)\,dz\,dx.\end{aligned}$$

We can do this argument for each $k \in N$ with $k \geq n$. Define for $k \geq n$ an integer

$$u_k(x) = \begin{cases} y_k(x), & x \in [a,k] \\ b_0, & x \in [k,\infty), \end{cases}$$

so

$$G(u_k(x)) = \begin{cases} G(y_k(x)), & x \in [a,k] \\ G(b_0), & x \in [k,\infty). \end{cases}$$

It is easy to see that

$$\begin{aligned}|G(u_k(s)) - G(u_k(t))| &\leq A_1\,|t-s| + A_2 \left|\int_s^t \int_a^x |p'(z)|\,dz\,dx\right| \\ &\quad + \left|\int_s^t \int_a^x \mu(x)\,dz\,dx\right| \quad \text{for } t,\ s \in [a,\infty).\end{aligned}$$

Consider $\{u_k\}_{k=n}^{\infty}$. The Arzela–Ascoli theorem guarantees that there is a subsequence $N_n^\star$ of $\{n, n+1, ...\}$ and a function $G(z_n) \in C[a,n]$ with $G(u_k)$ converging uniformly on $[a,n]$ to $G(z_n)$ as $k \to \infty$ through $N_n^\star$. This together with the fact that G^{-1} is continuous and $G(u_k(t)) \in [0,b_0]$ for $t \in [a,n]$ implies u_k converges uniformly on $[a,n]$ to z_n as $k \to \infty$ through $N_n^\star$. Note $0 \leq z_n(t) \leq b_0$ for $t \in [a,n]$. Let $N_n = N_n^\star \setminus \{n\}$. Also the Arzela–Ascoli theorem guarantees the existence of a subsequence $N_{n+1}^\star$ of N_n and a function $G(z_{n+1}) \in C[a,n+1]$ with $G(u_k)$ converging uniformly on $[a,n+1]$ to $G(z_{n+1})$ as $k \to \infty$ through $N_{n+1}^\star$, and so u_k converges uniformly on $[a,n+1]$ to z_{n+1} as $k \to \infty$ through $N_{n+1}^\star$. Note $0 \leq z_{n+1}(t) \leq b_0$ for $t \in [a,n+1]$ and $z_{n+1} = z_n$ on $[a,n]$ since $N_{n+1}^\star \subseteq N_n$. Let $N_{n+1} = N_{n+1}^\star \setminus \{n+1\}$. Proceed inductively to obtain for $m \in \{n+2, n+3, ...\}$ a subsequence $N_m^\star$ of N_{m-1} and a function $z_m \in C[a,m]$ with u_k converges uniformly on $[a,m]$ to z_m as $k \to \infty$ through $N_m^\star$. Note $0 \leq z_m(t) \leq b_0$ for $t \in [a,m]$ and $z_m = z_{m-1}$ on $[a,m-1]$. Let $N_m = N_m^\star \setminus \{m\}$.

Define a function y as follows. Fix $x \in (a,\infty)$ and let $l \in \{n, n+1, ...\}$ with $x \le l$. Then define $y(x) = z_l(x)$ so $y \in C[a,\infty)$ and $0 \le y(t) \le b_0$ on $[a,\infty)$. Also for $n \in N_l$ we have

$$G(u_n(x)) = A_l\,(x-a) - \int_a^x p(s)\,u_n^m(s)\,ds + \int_a^x (x-s)\,[-q(s)\,f(s,u_n(s)) + p'(s)\,u_n^m(s)]\,ds$$

where

$$A_l\,(l-a) = G(u_n(l)) + \int_a^l p(s)\,u_n^m(s)\,ds - \int_a^l (l-s)\,[-q(s)\,f(s,u_n(s)) + p'(s)\,u_n^m(s)]\,ds.$$

Let $n \to \infty$ through N_l to obtain

$$G(z_l(x)) = A_l^\star\,(x-a) - \int_a^x p(s)\,z_l^m(s)\,ds + \int_a^x (x-s)\,[-q(s)\,f(s,z_l(s)) + p'(s)\,z_l^m(s)]\,ds$$

where

$$A_l^\star\,(l-a) = G(z_l(l)) + \int_a^l p(s)\,z_l^m(s)\,ds - \int_a^l (l-s)\,[-q(s)\,f(s,z_l(s)) + p'(s)\,z_l^m(s)]\,ds.$$

Thus

$$G(y(x)) = A_l^\star\,(x-a) - \int_a^x p(s)\,y^m(s)\,ds + \int_a^x (x-s)\,[-q(s)\,f(s,y(s)) + p'(s)\,y^m(s)]\,ds$$

where

$$A_l^\star\,(l-a) = G(y(l)) + \int_a^l p(s)\,y^m(s)\,ds - \int_a^l (l-s)\,[-q(s)\,f(s,y(s)) + p'(s)\,y^m(s)]\,ds.$$

We can do this for each $x > a$ and so the above integral equation yields for each $l \in N$ and $t \in [a,l]$ that

$$G'(y(t)) = -p(t)\,y^m(t) + A_l^\star + \int_a^t [-q(s)\,f(s,y(s)) + p'(s)\,y^m(s)]\,ds,$$

so $G' \in C^1[a,l]$, $G'(y) + p\,y^m \in AC[a,l] \cap C^1(a,l]$ and

$$(G'(y) + p\,y^m)'(t) = -q(t)\,f(t,y(t)) + p'(t)\,y^m(t) \quad \text{for} \quad t \in [a,l].$$

$\square$

Keeping the application in section 1 in mind it is important to discuss the situation when our solution to (3.1) is positive on (a,∞). Suppose the following conditions hold:

$$\text{(3.10)}\qquad \begin{cases} \exists\,\alpha\in BC[a,\infty) \text{ with } G(\alpha)\in C^1[a,\infty),\ G'(\alpha)+p\,\alpha^m \\ \in AC_{loc}[a,\infty)\cap C^1(a,\infty) \text{ with } b_0\geq\alpha>0 \text{ on } (a,\infty), \\ \alpha(a)=0 \text{ and } (G'(\alpha)+p\,\alpha^m)'(t)+q(t)\,f(t,\alpha)\geq p'(t)\,\alpha^m(t) \\ \text{on } (a,\infty) \end{cases}$$

$$\text{(3.11)}\qquad \begin{cases} \text{for each } t\in[a,\infty) \text{ we have } q(t)\,[f(t,y)-f(t,\alpha(t)]\geq 0 \\ \text{for } 0\leq y\leq\alpha(t) \end{cases}$$

and

$$\text{(3.12)}\qquad p'>0 \text{ on } (a,\infty).$$

Also in this case we discuss the boundary value problem

$$\text{(3.13)}\qquad \begin{cases} (g(y)\,y')'+p\,(y^m)'+q\,f(t,y)=0,\ a<t<\infty \\ y(a)=0,\ y \text{ bounded on } [a,\infty). \end{cases}$$

By a solution to (3.13) we mean a function $y\in BC[a,\infty)\cap C^1(a,\infty)$ with $y^m\,y'\in C^1(a,\infty)$ which satisfies $y(a)=0$ and the differential equation in (3.13) on (a,∞).

THEOREM 3.2. *Suppose* (3.2)–(3.7), (3.10), (3.11) *and* (3.12) *hold. Then* (3.1) *has a solution* y *with* $0\leq y(t)\leq b_0$ *for* $t\in[a,\infty)$. *In addition* $y\in C^1(a,\infty)$ *with* $G'(y)=y^m\,y'$ *on* (a,∞) *and* y *is a solution of* (3.13).

Proof. Fix $n\in N=\{1,2,...\}$ with $n\geq a+1$ and consider (3.8). Theorem 2.2 guarantees that there exists a solution y_n to (3.8) with $\alpha(t)\leq y_n(t)\leq b_0$ for $t\in[a,n]$. Essentially the same reasoning as in Theorem 3.1 guarantees that (3.1) has a solution $y\in BC[a,\infty)$ with $G(y)\in C^1[a,\infty)$, $G'(y)+p\,y^m\in AC_{loc}[a,\infty)\cap C^1(a,\infty)$ and with $\alpha(t)\leq y(t)\leq b_0$ for $t\in[a,\infty)$. In particular note $y>0$ on (a,∞). Fix $l\in\{n,n+1,...\}$ and consider $t\in[a,l]$. We know (see Theorem 3.1) that

$$\frac{y^{m+1}(t)}{m+1}=A_l^\star\,(t-a)-\int_a^t p(s)\,y^m(s)\,ds + \int_a^t (t-s)\,[-q(s)\,f(s,y(s))+p'(s)\,y^m(s)]\,ds$$

where

$$A_l^\star\,(l-a)=G(y(l))+\int_a^l p(s)\,y^m(s)\,ds - \int_a^l (l-s)\,[-q(s)\,f(s,y(s))+p'(s)\,y^m(s)]\,ds,$$

and since $y>0$ on $(a,l]$ we have $y'\in C^1(a,l)$. Then [3 pp. 181] guarantees that $G'(y)=g(y)\,y'=y^m\,y'$ on (a,l). Also for $t\in(a,l)$ we have

$$g(y)\,y'=A_l^\star-p\,y^m+\int_a^t[-q(s)\,f(s,y(s))+p'(s)\,y^m(s)]\,ds,$$

so $g(y)\,y' \in C^1(a,l)$. In addition for $t \in (a,l)$ we have

$$-q\,f(t,y) + p'\,y^m = (g(y)\,y' + p\,y^m)' = (g(y)\,y')' + (p\,y^m)'.$$

We can do this for each $l \in N$, so y is a solution of (3.13). ◻

Remark 3.1. If $\lim_{t\to\infty} \alpha(t) = b_0$ (here b_0 is as in (3.6)) then the solution y to (3.1) (guaranteed from Theorem 3.2) is a solution of the boundary value problem

$$\begin{cases} (g(y)\,y')' + p\,(y^m)' + q\,f(t,y) = 0, & a < t < \infty \\ y(a) = 0, \ \lim_{t\to\infty} y(t) = b_0. \end{cases} \tag{3.14}$$

Suppose the following condition is satisfied:

$$\begin{cases} \exists\,\alpha \in BC[a,\infty) \cap C^1(a,\infty) \ \text{ with } \ G(\alpha) \in C^1[a,\infty), \\ \alpha^m\,\alpha' \in C^1(a,\infty), \ b_0 \geq \alpha > 0 \ \text{ on } \ (a,\infty), \ \alpha(a) = 0 \\ \text{and } \ (\alpha^m\,\alpha')' + p\,(\alpha^m)' + q(t)\,f(t,\alpha) \geq 0 \ \text{ on } \ (a,\infty). \end{cases} \tag{3.15}$$

Then we have the following theorem.

Theorem 3.3. *Suppose* (3.2)–(3.7), (3.11), (3.12) *and* (3.15) *hold. Then* (3.13) *has a solution y with $\alpha(t) \leq y(t) \leq b_0$ for $t \in [a,\infty)$.*

Proof. Now [3 pp. 181] guarantees that $G'(\alpha) = g(\alpha)\,\alpha' = \alpha^m\,\alpha'$ on (a,l) for each $l \in N$, so for $t \in (a,l)$ we have

$$\begin{aligned} (G'(\alpha) + p\,\alpha^m)' + q\,f(t,\alpha) &= (\alpha^m\,\alpha' + p\,\alpha^m)' + q\,f(t,\alpha) \\ &= (\alpha^m\,\alpha')' + (p\,\alpha^m)' + q\,f(t,\alpha) \\ &\geq (p\,\alpha^m)' - p\,(\alpha^m)' = p'\,\alpha^m. \end{aligned}$$

Thus (3.10) holds and the result follows from Theorem 3.2. ◻

Remark 3.2. If $\lim_{t\to\infty} \alpha(t) = b_0$ (here b_0 is as in (3.6)) then the solution y to (3.13) (guaranteed from Theorem 3.3) is a solution of (3.14).

Example. (Slender dry patch in a liquid film).

From Section 1 consider the boundary value problem

$$\begin{cases} (y^3\,y')' + t\,(y^3)' = 0, & 1 < t < \infty \\ y(1) = 0, \ \lim_{t\to\infty} y(t) = G_0 > 0. \end{cases} \tag{3.16}$$

We will now use Theorem 3.3 (with Remark 3.2) to show that (3.16) has a solution. To see this consider

$$\begin{cases} (y^3\,y' + t\,y^3)' = y^3, & 1 < t < \infty \\ y(1) = 0, \ y \ \text{ bounded on } \ [1,\infty). \end{cases} \tag{3.17}$$

Remark 3.3. Notice $y \equiv 0$ is a solution of (3.17).

Let $m = 3$, $a = 1$, $p = t$, $q \equiv 0$, $f(t,y) \equiv 0$, $b_0 = G_0$ and

$$g(z) = \begin{cases} z^3, & z \geq 0 \\ -\,z^3 = |z|^3, & z < 0. \end{cases}$$

Clearly (3.2)–(3.7), (3.11) and (3.12) hold. Let

$$\alpha(t) = A \int_1^t \exp\left(-\frac{3\,s^2}{2\,G_0}\right)\,ds$$

where

$$A = \frac{G_0}{\int_1^\infty \exp\left(-\frac{3\,s^2}{2\,G_0}\right)\,ds}.$$

Note $\alpha(1) = 0$ and $\alpha' = A\,\exp\left(-\frac{3\,t^2}{2\,G_0}\right)$. Also for $t \in (1,\infty)$ we have

$$\begin{aligned}
(\alpha^3\,\alpha')' + t\,(\alpha^3)' &= A^4 \left(\int_1^t \exp\left(-\frac{3\,s^2}{2\,G_0}\right)\,ds\right)^2 \Big[3\,\exp\left(-\frac{3\,t^2}{G_0}\right) \\
&\quad - \frac{3\,t}{G_0}\exp\left(-\frac{3\,t^2}{2\,G_0}\right)\int_1^t \exp\left(-\frac{3\,s^2}{2\,G_0}\right)\,ds\Big] \\
&\quad + 3\,t\,A^3 \left(\int_1^t \exp\left(-\frac{3\,s^2}{2\,G_0}\right)\,ds\right)^2 \exp\left(-\frac{3\,t^2}{2\,G_0}\right) \\
&= 3\,t\,A^3 \left(\int_1^t \exp\left(-\frac{3\,s^2}{2\,G_0}\right)\,ds\right)^2 \exp\left(-\frac{3\,t^2}{2\,G_0}\right) \\
&\quad \times \left[1 - \frac{A}{G_0}\int_1^t \exp\left(-\frac{3\,s^2}{2\,G_0}\right)\,ds\right] \\
&\quad + 3\,A^4 \left(\int_1^t \exp\left(-\frac{3\,s^2}{2\,G_0}\right)\,ds\right)^2 \exp\left(-\frac{3\,t^2}{G_0}\right) \\
&\geq 0,
\end{aligned}$$

since

$$\frac{A}{G_0}\int_1^t \exp\left(-\frac{3\,s^2}{2\,G_0}\right)\,ds = \frac{\int_1^t \exp\left(-\frac{3\,s^2}{2\,G_0}\right)\,ds}{\int_1^\infty \exp\left(-\frac{3\,s^2}{2\,G_0}\right)\,ds} \leq 1.$$

Thus (3.15) holds so Theorem 3.3 guarantees that (3.17) has a solution y with $\alpha(t) \leq y(t) \leq G_0$ for $t \in [1,\infty)$. Also since $\lim_{t\to\infty} \alpha(t) = G_0$ then y is a solution of (3.16).

REFERENCES

[1] D. O'Regan, *Existence theory for the equations* $(G'(y))' = qf(t,y,y')$ *and* $(G'(y) - pH(y))] = -p'H(y) + qf(t,y)$, J. Math. Anal. Appl., 183 (1994), pp. 263–284.

[2] D. O'Regan, *Boundary value problems on noncompact intervals*, Proc. Edinburgh Math. Soc., 125A (1995), pp. 777–799.

[3] A. Torchinsky, *Real Variables*, Addison-Wesley, Redwood City, 1988.

[4] S.K. Wilson, B.R. Duffy and S.H. Davis, *On a slender dry patch in a liquid film draining under gravity down an inclined plane*, Euro. Jnl. of Applied Mathematics, 12 (2001), pp. 233–252.