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ON SOME SPECTRAL PROPERTIES OF OPERATORS GENERATED BY QUASI-DIFFERENTIAL MULTI-INTERVAL SYSTEMS *

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Abstract. We construct the common and the ordered spectral representation for operators, generated as direct sums of self-adjoint extensions of quasi-differential minimal operators on a multi-interval set (self-adjoint vector-operators), acting in a Hilbert space. The structure of the ordered representation is investigated for the case of differential coordinate operators. Results, connected with other spectral properties of such vector-operators, such as the introduction of the identity resolution and the spectral multiplicity have also been obtained.

Vector-operators have been mainly studied by W.N. Everitt, L. Markus and A. Zettl. Being a natural continuation of Everitt-Markus-Zettl theory, the presented results reveal the internal structure of self-adjoint differential vector-operators and are essential for the further study of their spectral properties.

1. Preliminaries.

1.1. Problem overview. In 1985, F. Gesztesy and W. Kirsch published their work [1], where they considered an example of a Schrödinger operator generated by the Hamiltonian

$$H = -\frac{d^2}{dx^2} + \left(s^2 - \frac{1}{4}\right)\frac{1}{\cos^2 x}, \;\; s > 0. \tag{1}$$

Since the potential of (1) has a countable number of singularities on $\mathbb{R}$ which spoil the local integrability, they constructed operators T_i, generated by (1) in the spaces

$$L^2\left(-\frac{\pi}{2} + i\pi, \frac{\pi}{2} + i\pi\right), \; i \in \mathbb{Z},$$

and then considered the direct sum operator $\oplus_{i\in\mathbb{Z}} T_i$ in the space

$$\oplus_{i\in\mathbb{Z}} L^2\left(-\frac{\pi}{2} + i\pi, \frac{\pi}{2} + i\pi\right).$$

The work [1] stimulated other researchers to generalize the problem. In [2], W.N. Everitt and the coauthors considered quasi-differential direct sum operators, generated by a countable multi-interval system, with intervals being subsets of one copy of the real line. In 1992, W.N. Everitt and A. Zettl [3] studied direct sums of minimal and maximal operators generated by arbitrary formally self-adjoint expressions in Hilbert spaces considered on arbitrary intervals (maximal and minimal vector-operators). Later in 2000, vector-operators were also considered in complete locally convex spaces by R.R. Ashurov and W.N. Everitt [4], which was a natural generalization of their work [5]. Since 1992, quasi-differential vector-operators have mostly been investigated in connection with their non-spectral theory, such as the introduction of minimal and maximal vector-operators and their relationship (it was shown that the adjoint of a minimal vector-operator is maximal in a Hilbert space [3], and the analogous result with the modification for Frechet spaces was obtained in [4]). A lot of work has been carried out by W.N. Everitt and L. Markus in order to develop the theory of self-adjoint extensions for vector-operators with the employment

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of symplectic geometry. In connection with this, see their recent memoirs [6] and [7]. Another group of scientists studied differential operators on graphs. In some cases, such theory has a close connection with that developed by W.N. Everitt, L. Markus and A. Zettl since certain boundary conditions may lead to the consideration of a differential operator on a graph as a direct sum operator. Some most modern results in connection with the spectral theory of differential operators on graphs belong to R. Carlson [8, 9] and P. Kurasov, F. Stenberg [10].

The theory of operators generated by multi-interval systems finds its applications in many problems of quantum mechanics, theory of semiconductors and theoretical computer science; good bibliographical references for these subjects may be found in [7].

Since the theory of quasi-differential vector-operators in a Hilbert space is quite young and the most recent studies have concerned mostly problems connected with their common theory, small attention was given to its spectral aspects. Some results, describing the position of spectra of vector-operators were presented in 1985 in [1] and the most recent results belong to Sobhy El-Sayed Ibrahim [11, 12]. Some spectral properties of self-adjoint vector-operators were presented by M.S. Sokolov in [13] and R.R. Ashurov, M.S. Sokolov in [14, 15]. Nevertheless, a rigorous structural spectral theory for such operators has not been developed yet. The present work is designed to make essential steps in this direction. It completely covers abstract results, briefly described in [13, 14] with some modifications. It also presents the new results, describing the structure of the ordered spectral representation and eigenfunction expansions.

1.2. Quasi-differential operators and vector-operators. Basic concepts of quasi-differential operators are described in [3, 6]. A good reference for operators with real coefficients is the book of M.A. Naimark [16].

Let us have a number $n \in \mathbb{N}, n \geqslant 2$, and an arbitrary interval $I \subseteq \mathbb{R}$. Let $Z_n(I)$ be a set of Shin-Zettl matrices. These are matrices $A = \{a_{rs}\},\ a_{rs} : I \to \mathbb{C}$ of the order $n \times n$, such that for almost all $x \in I$:

$$\begin{cases} (i) & a_{rs} \in L_{loc}(I), \quad r, s = \overline{1,n}; \\ (ii) & a_{r,r+1}(x) \neq 0, \quad r = \overline{1,n-1}; \\ (iii) & a_{rs} = 0, \qquad s = \overline{r+2,n};\ r = \overline{1,n-2}. \end{cases}$$

Consider a function $f : I \to \mathbb{C}$; its quasi-derivatives relatively to a Shin-Zettl matrix A are defined by

$$\begin{cases} (i) & f_A^{[0]} := f; \\ (ii) & f_A^{[r]} := \frac{1}{a_{r,r+1}} \left[\frac{d}{dx} f_A^{[r-1]} - \sum_{s=1}^{r} a_{rs} f_A^{[s-1]} \right], \quad r = \overline{1,n-1}; \\ (iii) & f_A^{[n]} := \frac{d}{dx} f_A^{[n-1]} - \sum_{s=1}^{n} a_{ns} f_A^{[s-1]}. \end{cases}$$

Let us introduce a linear manifold $D(A) \subset AC_{loc}(I)$:

$$D_A(I) := \{ f : I \to \mathbb{C} |\ f_A^{[r-1]} \in AC_{loc}(I) \quad (r = \overline{1,n}) \}.$$

It is possible to see, that $f \in D_A(I)$ implies $f_A^{[n]} \in L_{loc}(I)$, and it is possible to prove that $D_A(I)$ is dense in $L_{loc}(I)$.

Relative to a matrix $A \in Z_n(I)$, we have the quasi-differential expression $M_A[f] = i^n f_A^{[n]}$, $f \in D_A(I)$.

The matrix $A^+ \in Z_n(I)$ designates the Lagrange adjoint matrix to A if $A^+ := -L_n^{-1} A^* L_n$, where A^* is the adjoint matrix, and $L_n = \{l_{rs}\}$ is the $(n \times n)$-matrix, defined as:

$$l_{r,n+1-r} = \begin{cases} (-1)^{r-1}, & r = \overline{1,n}; \\ 0, & \text{for other } r, s. \end{cases}$$

Using this notation we suppose that in this work we deal only with Lagrange symmetric (formally self-adjoint) expressions, that is $M_{A^+}[f] = M_A[f] = \tau(f)$, where τ is an alternative denotation for a Lagrange symmetric expression.

For a quasi-differential expression $M_A[f]$, the Lagrange formula is known ($[\alpha, \beta] \subseteq I$ - an arbitrary compact subinterval of I):

$$\int_\alpha^\beta \{\overline{g(x)} M_A[f](x) - f(x)\overline{M_{A^+}[g(x)]}\}\, dx = [f,g]_A(\beta) - [f,g]_A(\alpha), \tag{2}$$

where $f \in D_A,\ g \in D_{A^+},\ [f,g]_A(\beta)$ and $[f,g]_A(\alpha)$ may be derived from:

$$[f,g]_A(x) = i^n \sum_{i=1}^{n} (-1)^{i-1} f_A^{[i-1]}(x) \overline{g_{A^+}^{[n-i]}(x)}, \quad x \in I.$$

Let $\omega > 0$ be a weight function from $L_{loc}(I)$, $\omega : I \to \mathbb{R}$; the Hilbert space $L^2(I : \omega)$ is formed as usual.

We define maximal and minimal operators as follows:

DEFINITION 1.1. Operators T_{max} and T_{min} are called respectively *maximal* and *minimal* operators if they are generated by $\tau(f)$ on the domains $D(T_{max})$ and $D(T_{min})$:

$$D(T_{max}) = \{f : I \to \mathbb{C} |\ f \in D_A(I);\ \omega^{-1}\tau(f) \in L^2(I : \omega)\},$$

$$T_{max} f = \omega^{-1}\tau(f),\ (f \in D(T_{max}));$$

$$D(T_{min}) = \{f |\ f \in D(T_{max});\ [f,g]_A(b) - [f,g]_A(a) = 0\ (g \in D(T_{max}))\},$$

$$T_{min} f = \omega^{-1}\tau(f),\ (f \in D(T_{min})),$$

where $[f,g]_A(b)$ and $[f,g]_A(a)$ are the limits (which necessarily exist) of the bilinear forms from (2), that is $\lim_{\beta \to b}[f,g]_A(\beta) = [f,g]_A(b)$ and $\lim_{\alpha \to a}[f,g]_A(\alpha) = [f,g]_A(a)$.

The following general theorem is known for the operators T_{max} and T_{min} :

THEOREM 1.2. *For the operators T_{max} and T_{min} and their domains the following facts are valid :*

(*a*) $D(T_{min}) \subseteq D(T_{max})$. *Domains $D(T_{min})$ and $D(T_{max})$ are dense in $L^2(I : \omega)$;*

(*b*) *The operator T_{min} is closed and symmetric, the operator T_{max} is closed in*

$L^2(I:\omega)$;
(c) $T^*_{min} = T_{max}$ *and* $T^*_{max} = T_{min}$.

All self-adjoint extensions of T_{min} appear to be the contractions of T_{max}.

Let Ω be a finite or a countable set of indices. On Ω, we have an Everitt-Markus-Zettl multi-interval quasi-differential system $\{I_i, \tau_i; \omega_i\}_{i\in\Omega}$. This EMZ system generates a family of the weighted Hilbert spaces $\{L^2(I_i : \omega_i) = L^2_i\}_{i\in\Omega}$ and families of minimal $\{T_{min,i}\}_{i\in\Omega}$ and maximal $\{T_{max,i}\}_{i\in\Omega}$ operators. Consider a respective family $\{T_i\}_{i\in\Omega}$ of self-adjoint extensions.

We introduce the system Hilbert space $\mathbf{L}^2 = \oplus_{i\in\Omega} L^2_i$ consisting of vectors $\mathbf{f} = \oplus_{i\in\Omega} f_i$, such that $f_i \in L^2_i$ and

$$\|\mathbf{f}\|^2 = \sum_{i\in\Omega} \|f_i\|^2_i = \sum_{i\in\Omega} \int_{I_i} |f_i|^2 \omega_i \, dx < \infty,$$

where $\|\cdot\|^2_i$ are the norms in L^2_i. In the space $\mathbf{L}^2$ consider the operator $T : D(T) \subseteq \mathbf{L}^2 \to \mathbf{L}^2$, defined on the domain

$$D(T) = \left\{ \mathbf{f} \in \oplus_{i\in\Omega} D(T_i) \subseteq \mathbf{L}^2 : \sum_{i\in\Omega} \|T_i f_i\|^2_i < \infty \right\}$$

by $T\mathbf{f} = \oplus_{i\in\Omega} T_i f_i$.

Definition 1.3. The operator $T = \oplus_{i\in\Omega} T_i$ is called a *differential vector-operator* generated by the self-adjoint extensions T_i on an *EMZ* system, or simply a vector-operator. If Ω is infinite, the vector-operator T is called *infinite*. The operators T_i are called *coordinate* operators. For $\Omega' \subset \Omega$, the operator $\oplus_{k\in\Omega'} T_k$ is called a *sub-vector-operator* of the vector-operator $\oplus_{i\in\Omega} T_i$.

The following abstract preliminaries may be found, for instance, in books [17, 18].

Fix $i \in \Omega$. For each T_i there exists a unique resolution of the identity E^i_λ and a unitary operator U_i, making the isometrically isomorphic mapping of the Hilbert space L^2_i onto the space $L^2(M_i, \mu_i)$, where the operator T_i is represented as a multiplication operator. Below, we remind the structure of the mapping U_i.

We call $\phi \in L^2_i$ a *cyclic vector* if for each $z \in L^2_i$ there exists a Borel function f, such that $z = f(T_i)\phi$. Generally, there is no a cyclic vector in L^2_i but there is a collection $\{\phi^k\}$ of them in L^2_i, such that $L^2_i = \oplus^k L^2_i(\phi^k)$, where $L^2_i(\phi^k)$ are T_i-invariant subspaces in L^2_i generated by the cyclic vectors ϕ^k. That is

$$L^2_i(\phi^k) = \overline{\{f(T_i)\phi^k\}},$$

for a varying Borel function f, such that $\phi^k \in D(f(T_i))$. There exist unitary operators

$$U^k : L^2_i(\phi^k) \to L^2(\mathbb{R}, \mu^k),$$

where $\mu^k(\Delta) = \|E^i(\Delta)\phi^k\|^2_i$ for any Borel set Δ. In $L^2(\mathbb{R}, \mu^k)$, the operator T_i has the form of multiplication by λ, i.e.

$$\left(U^k T_i|_{L^2_i(\phi^k)} {U^k}^{-1} z \right)(\lambda) = \lambda z(\lambda).$$

Then the operator

$$U_i = \oplus^k U^k : \oplus^k L^2_i(\phi^k) \to \oplus^k L^2(\mathbb{R}, \mu^k)$$

makes the spectral representation of the space L_i^2 onto the space $L^2(M_i, \mu_i)$, where M_i is a union of nonintersecting copies of the real line (*a sliced union*) and $\mu_i = \sum_k \mu^k$. That is $(U_i T_i U_i^{-1} z)(\lambda) = f(\lambda) z(\lambda)$, where $z \in U[D(T_i)]$ and f is a Borel function defined almost everywhere according to the measure μ_i.

A vector $\phi \in L_i^2$ is called *maximal* relative to the operator T_i, if each measure $(E^i(\cdot)x, x)_i$, $x \in L_i^2$, is absolutely continuous relative to the measure $(E^i(\cdot)\phi, \phi)_i$.

For each Hilbert space L_i^2, there exist a unique (up to unitary equivalence) decomposition $L_i^2 = \oplus_k L_i^2(\varphi_i^k)$, where φ_i^1 is maximal in L_i^2 relative to T_i, and a decreasing set of multiplicity sets e_k^i, where e_1^i is the whole line, such that $\oplus_k L_i^2(\varphi_i^k)$ is equivalent with $\oplus_k L^2(e_k^i, \mu_i)$, where the measure of the ordered representation is defined as $\mu_i(\cdot) = (E^i(\cdot)\varphi_i^1, \varphi_i^1)_i$. A spectral representation of T_i in $\oplus_k L^2(e_k^i, \mu_i)$ is called the *ordered representation* and it is unique, up to a unitary equivalence. Two operators are called *equivalent*, if they create the same ordered representation of their spaces.

2. The spectral representation for the vector-operator T. In this section we show, how the common spectral representation of the vector-operator T depends on the common spectral representations of the given operators T_i. For this purpose, we first prove some auxiliary results.

Definition 2.1. For $i \in \Omega$, we introduce a *sliced union* of sets M_i (see also preliminaries) as a set M, containing all M_i on different copies of $\cup_{i\in\Omega} M_i$. The sets M_i do not intersect in M, but they can *superpose*, i.e. two sets M_i and M_j superpose, if their projections in the set $\cup_{i\in\Omega} M_i$ intersect.

Separate arguments show, that the following auxiliary proposition is true.

Proposition 2.2. *Let us have a set of measures μ_i, $i \in \Omega$, defined on nonintersecting supports. If*

$$\sum_{i\in\Omega} \int_{-\infty}^{\infty} f(\lambda)\, d\mu_i(\lambda) < \infty,$$

for any Borel function $f(\lambda)$, then the following equality is true:

$$\sum_{i\in\Omega} \int_{-\infty}^{\infty} f(\lambda)\, d\mu_i(\lambda) = \int_{-\infty}^{\infty} f(\lambda)\, d\sum_{i\in\Omega} \mu_i(\lambda).$$

Lemma 2.3. *The identity resolution $\{E_\lambda\}$ of the vector-operator T equals to the direct sum of the coordinate identity resolutions $\{E_\lambda^i\}$, that is:*

$$\{E_\lambda\} = \oplus_{i\in\Omega} \{E_\lambda^i\}$$

Proof. A vector $\mathbf{x}$ belongs to $D(T)$ if and only if

$$\|T\mathbf{x}\|^2 = \sum_{i\in\Omega} \|T_i x_i\|_i^2 = \sum_{i\in\Omega} \int_{-\infty}^{\infty} \lambda^2\, d\|E_\lambda^i x_i\|_i^2 < \infty.$$

Then, using Proposition 2.2 we find out that:

$$\sum_{i\in\Omega} \int_{-\infty}^{\infty} \lambda^2\, d\|E_\lambda^i x_i\|_i^2 = \int_{-\infty}^{\infty} \lambda^2\, d\sum_{i\in\Omega} \|E_\lambda^i x_i\|_i^2.$$

This means, that $\mathbf{x} \in D(T)$, if and only if

$$\int_{-\infty}^{\infty} \lambda^2 \, d \sum_{i \in \Omega} \|E_\lambda^i x_i\|_i^2 < \infty$$

and

$$\|T\mathbf{x}\|^2 = \int_{-\infty}^{\infty} \lambda^2 \, d \sum_{i \in \Omega} \|E_\lambda^i x_i\|_i^2.$$

Using the uniqueness property of an identity resolution, the last two equations show that $\oplus_{i\in\Omega}\{E_\lambda^i\}$ is the identity resolution of the vector-operator T. That is, according to our notations $\{E_\lambda\} = \oplus_{i\in\Omega}\{E_\lambda^i\}$. The lemma is proved. □

LEMMA 2.4. *For any Borel function f and any vector $\mathbf{x} \in D(f(T))$, the following equality holds: $f(T)\mathbf{x} = [\oplus_{i\in\Omega} f(T_i)]\mathbf{x}$.*

Proof. Let $\mathbf{x} \in D(f(T))$. Then, paying attention to Proposition 2.2 and Lemma 2.3, for any $\mathbf{y} \in \mathbf{L}^2$, we obtain:

$$(f(T)\mathbf{x}, \mathbf{y}) = \int_{-\infty}^{\infty} f(\lambda) \, d(E_\lambda \mathbf{x}, \mathbf{y}) = \int_{-\infty}^{\infty} f(\lambda) \, d \sum_{i\in\Omega} (E_\lambda^i x_i, y_i)_i =$$
$$= \sum_{i\in\Omega} \int_{-\infty}^{\infty} f(\lambda) \, d(E_\lambda^i x_i, y_i)_i = \sum_{i\in\Omega} (f(T_i) x_i, y_i)_i = ([\oplus_{i\in\Omega} f(T)]\mathbf{x}, \mathbf{y}).$$

Since $\mathbf{y}$ is arbitrary, we have $f(T)\mathbf{x} = [\oplus_{i\in\Omega} f(T_i)]\mathbf{x}$. This completes the proof of the lemma. □

For $z_i \in L_i^2$, $i \in \Omega$, define $\overline{\mathbf{z_i}} = \{0, ..., 0, z_i, 0, ..., 0\} \in \mathbf{L}^2$, where z_i is on the i-th place.

For each $i \in \Omega$, let $\epsilon(T_i)$ denote the *subspectrum* of the operator T_i, i.e. the set where the spectral measures of T_i are concentrated. Note, that $\overline{\epsilon(T_i)} = \sigma(T_i)$. For instance, the subspectrum of an operator having the complete system of eigenfunctions with eigenvalues being the rational numbers of $[0, 1]$ equals to $\mathbb{Q} \cap [0, 1]$; the subspectrum of an operator having the continuous spectrum [0,1] is assumed to equal to (0,1) without loss of generality.

Consider a projecting mapping $P : M \to \cup_{i\in\Omega} M_i$ (see Definition 2.1), such that $P(\epsilon(T_i)) = \epsilon(T_i)$.

DEFINITION 2.5. Let $\Omega = \cup_{k=1}^K A_k$, $A_k \cap A_s = \emptyset$ for $k \neq s$ and

$$A_k = \{s \in \Omega : \forall s, l \in A_k, s \neq l, P(\epsilon(T_s)) \cap P(\epsilon(T_l)) = B_{sl},$$
$$\text{where } \|E^t(B_{sl})\varphi_t\|_t^2 = 0 \text{ for any cyclic } \varphi_t \in L_t^2, \, t = s, l\}.$$

From all such divisions of Ω we choose and fix the one, which contains the minimal number of A_k. In case all the coordinate spectra $\sigma(T_i)$ are simple, we define the number $\Lambda = \min\{K\}$ as the *spectral index* of the vector-operator T.

THEOREM 2.6. *Let each T_i have a cyclic vector a_i in L_i^2. Then the vector-operator T has Λ cyclic vectors $\{\mathbf{a}_k\}_{k=1}^{\Lambda}$, having the form $\mathbf{a}_k = \sum_{i\in A_k} \overline{\mathbf{a_i}}$.*

Proof. First we consider the case of two coordinate operators. Let $s, l \in \Omega$. Then, in order to obtain one cyclic vector in $L^2_s \oplus L^2_l$ having the form $a_s \oplus a_l$, for any $\mathbf{x} = x_s \oplus x_l \in L^2_s \oplus L^2_l$ we have to find a Borel function f, such that

$$\mathbf{x} = f(T_s \oplus T_l)[a_s \oplus a_l].$$

From Lemma 2.4 it follows that

$$\mathbf{x} = [f(T_s) \oplus f(T_l)][a_s \oplus a_l].$$

On the other hand we must obtain each space $L^2_p\,(p = s, l)$ by closing the set $\{f_p(T_p)a_p\}$, letting f_p vary over all the Borel functions such that $a_p \in D(f_p(T_p))$. If $s, l \in A_k$, then supposing that $f = f_p$ on $P(\epsilon(T_p))$, we obtain the required function f, since any functions in the isomorphic space L^2 are considered equal on the set of measure zero. Hence, it is clear that for all $i \in A_k$, we may build a single cyclic vector of the form

$$\mathbf{a}_k = \oplus_{i \in A_k} a_i = \sum_{i \in A_k} \overline{\mathbf{a_i}},$$

using the process described above, each time operating with a pair of operators.

We recall, that we have the minimal number of A_k. Consider the Hilbert space

$$[\oplus_{i \in A_k} L^2_i] \oplus [\oplus_{j \in A_q} L^2_j], k \neq q. \tag{3}$$

We know, that then

$$[\cup_{i \in A_k} P(\epsilon(T_i))] \cap [\cup_{j \in A_q} P(\epsilon(T_j))] = B_{kq}$$

has a non-zero spectral measure. From the reasonings described in the beginning of this proof we see, that for joining the cyclic vectors $\mathbf{a}_k = \oplus_{i \in A_k} a_i$ and $\mathbf{a}_q = \oplus_{j \in A_q} a_j$ into the one

$$\mathbf{a}_k + \mathbf{a}_q = \sum_{i \in A_k} \overline{\mathbf{a_i}} + \sum_{j \in A_q} \overline{\mathbf{a_j}},$$

we would have to derive the Hilbert space (3) by closing the set

$$\{f_k(\oplus_{i \in A_k} T_i)\mathbf{a}_k\} \oplus \{f_q(\oplus_{j \in A_q} T_j)\mathbf{a}_q\},$$

with varying the Borel functions f_k and f_q, which coincide on B_{kq}. This is not possible, since the set of such functions is not dense in the isomorphic space L^2 (the isomorphism is understood as in the spectral representation of the space (3)). Hence, we have obtained Λ cyclic vectors

$$\mathbf{a}_k = \sum_{i \in A_k} \overline{\mathbf{a_i}} \in \mathbf{L}^2,\ k = \overline{1, \Lambda}$$

and have proved the theorem. □

Corollary 2.7. *Let each T_i have a single cyclic vector. Then*
1. $\Lambda = 1$ *if and only if the coordinate operators $T_i, i \in \Omega$, have almost everywhere (relatively to the spectral measure) pairwise non-superposing subspectra.*
2. a) $\mathrm{card}(\Omega) < \aleph_0$. $\Lambda = \mathrm{card}(\Omega)$, *if and only if all the coordinate operators T_i*

have pairwise superposing subspectra; b) $\mathrm{card}(\Omega) = \aleph_0$. $\Lambda = \infty$, *if and only if* T *has an infinite sub-vector-operator, the coordinate operators of which have pairwise superposing subspectra.*

Proof. The proof directly follows from the reasonings of the proof of Theorem 2.6. □

In the next section we will rigorously show what the spectral multiplicity of a vector-operator is. Nevertheless, this notation is intuitively clear. Running ahead, let us present here an example, which will show the difference between the spectral index and the spectral multiplicity of the vector-operator T.

Example 1. We have a three-interval EMZ system $\{I_i, \tau_i\}_{i=1}^3$ (a kinetic energy, a mirror kinetic energy, an impulse):

$$I_1 = [0, +\infty), \ \ \tau_1 = -\left(\frac{d}{dt}\right)^2, \qquad D(T_1) = \{f \in D(T_{max,1}) : f(0) + kf'(0) = 0, -\infty < k \leqslant \infty\};$$

$$I_2 = [0, +\infty), \ \ \tau_2 = \left(\frac{d}{dt}\right)^2, \qquad D(T_2) = \{f \in D(T_{max,2}) : f(0) + sf'(0) = 0, -\infty < s \leqslant \infty\};$$

$$I_3 = [0, 1], \ \ \tau_3 = \frac{1}{i}\frac{d}{dt}, \ \ D(T_3) = \{f \in D(T_{max,3}) : f(0) = e^{i\alpha} f(1),\ \alpha \in [0, 2\pi]\}.$$

If $k, s \in (-\infty, 0] \cup \{+\infty\}$ then

$$\epsilon(T_1) = (0, +\infty), \ \ \epsilon(T_2) = (-\infty, 0), \ \ \epsilon(T_3) = \bigcup_{n=-\infty}^{\infty} (2\pi n - \alpha).$$

For this system we have: $\{1, 2, 3\} = \cup_{k=1}^2 A_k$ and $A_1 = \{1, 2\}$, $A_2 = \{3\}$. Thus, here the spectral index does not coincide with the spectral multiplicity (which equals to 1) and equals to 2.

The case $0 < k, s < +\infty$ leads to the following

$$\epsilon(T_1) = \left\{-\frac{1}{k^2}\right\} \cup (0, +\infty), \ \ \epsilon(T_2) = (-\infty, 0) \cup \left\{\frac{1}{s^2}\right\}, \ \ \epsilon(T_3) = \bigcup_{n=-\infty}^{\infty} (2\pi n - \alpha).$$

If

$$\alpha \notin \left[\bigcup_{n=-\infty}^{\infty} \left(2\pi n + \frac{1}{k^2}\right)\right] \bigcup \left[\bigcup_{n=-\infty}^{\infty} \left(2\pi n - \frac{1}{s^2}\right)\right],$$

we have $A_1 = \{1\}$, $A_2 = \{2\}$, $A_3 = \{3\}$. That is $\Lambda = 3$ but $\oplus_{i=1}^3 T_i$ has a simple spectrum.

Example 2. Let us have a vector-operator $\oplus_{i=1}^3 T_i$ with

$$\epsilon(T_1) = \bigcup_{n \in \mathbb{Z}, n \geqslant 0} \pi n, \ \ \epsilon(T_2) = \bigcup_{n \in \mathbb{Z}, n \leqslant 0} \pi n, \ \ \epsilon(T_3) = \bigcup_{n \in \mathbb{Z}, n \neq 0} \pi n.$$

Spectral index equals to 3 but spectral multiplicity equals to 2.

DEFINITION 2.8. A vector-operator $T = \oplus_{i\in\Omega} T_i$ with simple coordinate spectra $\sigma(T_i)$ is called *distorted* if its spectral index does not equal its spectral multiplicity.

Note that in the above example 1, it is possible to unite the cyclic vectors into one just taking their direct sum (as it is shown in the proof of Theorem 2.6). But nevertheless, it is convenient to consider such operators as distorted satisfying Definition 2.8. The distorted vector-operator from Example 2 is 'completely' distorted and it is not possible to unite the coordinate cyclic vectors into a cyclic direct sum.

With some loss of technical value but more clearly for applications, Theorem 2.6 may be reformulated as

COROLLARY 2.9. *Let each T_i have a simple spectrum. Then undistorted vector-operator T has Λ-multiple spectrum.*

Let us pass to the general case when each operator T_i has m_i cyclic vectors. There exists a decomposition

$$T = \oplus_{i\in\Omega} T_i = \oplus_{i\in\Omega} \oplus_{k=i}^{m_i} T_i^k = \oplus_s T_s,$$

where each T_s has a single cyclic vector. For the vector-operator T decomposed as above, we apply Theorem 2.6 and find the spectral index Λ. It is clear, that in this case for the spectral index there exists the estimate

$$\Lambda \geqslant \max\{m_i\}. \tag{4}$$

As it has been stated in the preliminaries, for each operator T_i there exists the unitary operator U_i, such that $U_i : L_i^2 \to L^2(M_i, \mu_i)$. Hence

$$\oplus_{i\in\Omega} U_i : \oplus_{i\in\Omega} L_i^2 \to \oplus_{i\in\Omega} L^2(M_i, \mu_i).$$

Or, in the general case (i.e. when there are T_i with more then one cyclic vector),

$$\oplus_{i\in\Omega} U_i : \oplus_{i\in\Omega} \oplus_{k=1}^{m_i} L_{i,k}^2 \to \oplus_{i\in\Omega} \oplus_{k=1}^{m_i} L^2(\mathbb{R}, \mu_i^k).$$

From Theorem 2.6 it follows that there exists a unitary operator

$$V : \oplus_{i\in\Omega} \oplus_{k=1}^{m_i} L^2(\mathbb{R}, \mu_i^k) = \oplus_s L^2(\mathbb{R}, \mu_s) \to \oplus_{q=1}^{\Lambda} L^2 \left(\mathbb{R}, \sum_{j\in A_q} \mu_j\right). \tag{5}$$

This means that for any vector-operator T there exists the unitary operator $V \oplus_{i\in\Omega} U_i$, which represents the space $\mathbf{L}^2$ on the space $L_2(N, \mu)$:

$$V \oplus_{i\in\Omega} U_i : \mathbf{L}^2 \to L^2(N, \mu),$$

where N is the sliced union of Λ copies of $\mathbb{R}$ and

$$\mu = \sum_{q=1}^{\Lambda} \sum_{j\in A_q} \mu_j,$$

according to the symbols in (5). We finally obtain

THEOREM 2.10. *Let the vector-operator $T = \oplus_{i\in\Omega} T_i$ be undistorted and let the unitary operator V be defined as in (5). If unitary operators U_i give spectral representations of the Hilbert spaces L_i^2 on the spaces $L^2(M_i,\mu_i)$, then the unitary operator*

$$W = V \oplus_{i\in\Omega} U_i$$

gives a spectral representation of the space $\mathbf{L}^2$ on the space $L^2(N,\mu)$.

Directly from the definition of a distorted vector-operator, it follows that only for undistorted vector-operators, the transform V does reduce the quantity of cyclic vectors to the minimal possible. Note, that distorted differential vector-operators appear to be frequent objects if vector-operators are considered on a set of closed bounded intervals, and on the other hand quite rare, if coordinate operators have continuous spectra. For them Theorem 2.10 is not efficient and needs to be strengthened. Such strengthening is the construction of an ordered representation for arbitrary (distorted or not) differential vector-operators, the process which seems to be essential for further development of spectral theory of vector-operators.

3. The ordered spectral representation for the vector-operator T.

THEOREM 3.1. *If θ_i and $\{e_n^i\}_{n=1}^{m_i}$ are measures and multiplicity sets of ordered representations for coordinate operators T_i, $i\in\Omega$, then there exist processes Pr_1 and Pr_2, such that the measure*

$$\theta = Pr_1(\{\theta_i\}_{i\in\Omega})$$

is the measure of an ordered representation and the sets

$$s_n = Pr_2\left(\{e_k^i\}_{i\in\Omega;\,k=\overline{1,m_i}}\right)$$

are the canonical multiplicity sets of the ordered representation of the operator T. Thus, the unitary representation of the space $\mathbf{L}^2$ on the space $\oplus_n L^2(s_n,\theta)$ is the ordered representation and it is unique up to unitary equivalence.

Proof. We divide the proof into units for convenience. Units **(A)** and **(B)** represent the process, which we call 'the process of division on subspectra'.

(A) Let a_i be maximal vectors relative to the operators T_i in L_i^2. We want to find a maximal vector relative to the vector-operator T. We know, that the vector $\oplus_{i\in\Omega} a_i$ does not give a single measure, if a set $P(\epsilon(T_i))\cap P(\epsilon(T_j))$ has a non-zero spectral measure for $i\neq j$. Consider restrictions $T_i|_{L_i^2(a_i)} = T_i'$. Since all the operators T_i' have single cyclic vectors a_i, we can divide Ω into A_k, $k=\overline{1,\Lambda}$ (see Definition 2.5) and apply Theorem 2.6 for the operator $\oplus_{i\in\Omega} T_i'$. Thus, we have derived Λ vectors $\mathbf{a}^k = \oplus_{j\in A_k} a_j$, which are maximal in the respective spaces $\mathbf{L}^2(\mathbf{a}^k) = \oplus_{j\in A_k} L_j^2(a_j)$. Indeed, this is obvious for the case $\mathrm{card}(A_k) < \aleph_0$. For the infinite case, if arbitrary $\mathbf{y} = \oplus_{j\in A_k} y_j \in \mathbf{L}^2(\mathbf{a}^k)$ and if

$$\left([\oplus_{j\in A_k} E^j](\cdot)\mathbf{a}^k, \mathbf{a}^k\right) = \sum_{j\in A_k} \left(E^j(\cdot)a_j, a_j\right)_j = 0, \tag{6}$$

then from the maximality of the vectors a_j for all $j\in A_k$, and since $P(\epsilon(T_j'))\cap P(\epsilon(T_k'))$ has zero spectral measures for $j\neq k$, we obtain

$$\sum_{j\in A_k} \left(E^j(\cdot)y_j, y_j\right)_j = \left([\oplus_{j\in A_k} E^j](\cdot)\mathbf{y}, \mathbf{y}\right) = 0,$$

which follows from the convergence to zero of the series with the positive maximal elements (6). Thus, in particular, we have constructed a maximal vector in $\mathbf{L}^2$ for the case $\Lambda = 1$.

(B) Let now $1 < \Lambda < \infty$. Define $T^k = \oplus_{j \in A_k} T'_j$. For any two operators T^k and T^s, $k \neq s$, let us introduce the sets $\epsilon_{k,s} = P(\epsilon(T^k)) \cap P(\epsilon(T^s))$ and $\epsilon_k = P(\epsilon(T^k)) \setminus \epsilon_{k,s}$. There exist unitary representations $U^k : \mathbf{L}^2(\mathbf{a}^k) \to L^2(\mathbb{R}, \mu_{\mathbf{a}^k})$ (see formula (5) supposing there $\Lambda = 1$). Consider measures μ_k and $\mu_{k,s}$, defined as

$$\mu_{k,s}(e) = \mu_{\mathbf{a}^k}(e \cap \epsilon_{k,s}) \text{ and } \mu_k(e) = \mu_{\mathbf{a}^k}(e \cap \epsilon_k),$$

for any measurable set e. For any operator T^k (with respect to T^s), measures μ_k and $\mu_{k,s}$ are mutually singular and $\mu_k + \mu_{k,s} = \mu_{\mathbf{a}^k}$; therefore

$$L^2(\mathbb{R}, \mu_{\mathbf{a}^k}) = L^2(\mathbb{R}, \mu_k) \oplus L^2(\mathbb{R}, \mu_{k,s}).$$

This means that (according to our designations):

$${U^k}^{-1} : L^2(\mathbb{R}, \mu_{\mathbf{a}^k}) \longrightarrow \mathbf{L}^2(\mathbf{a}^k_k) \oplus \mathbf{L}^2(\mathbf{a}^k_{k,s})$$

and $\mathbf{a}^k = \mathbf{a}^k_k \oplus \mathbf{a}^k_{k,s}$, where $\mathbf{a}^k_k$ and $\mathbf{a}^k_{k,s}$ form the measures μ_k and $\mu_{k,s}$ respectively. Define also as $\max\{w, \psi\}$ the vector, which is maximal of the two vectors in the brackets (Note, that this designation is valid only for vectors, considered on the same set. In order not to complicate the investigation we assume here that any two vectors are comparable in this sense. In order to achieve this, it is enough to decompose each coordinate operator T_i into the direct sum $T_i^{pp} \oplus T_i^{cont}$, where the operators have respectively pure point and continuous spectra. Then after redesignation we obtain the equivalent vector-operator to the initial vector-operator $\oplus T_i$).

Consider first two operators T^1 and T^2. It is clear, that the vector

$$\mathbf{a}^{1\oplus 2} = \mathbf{a}^1_1 \oplus \mathbf{a}^2_2 \oplus \max\left\{\mathbf{a}^1_{1,2}, \mathbf{a}^2_{2,1}\right\}$$

is maximal in $\mathbf{L}^2(\mathbf{a}^1) \oplus \mathbf{L}^2(\mathbf{a}^2)$. Note, that $\mathbf{a}^1_1$ and $\mathbf{a}^2_2$ and they both may equal zero. The maximal vector in $\mathbf{L}^2(\mathbf{a}^1) \oplus \mathbf{L}^2(\mathbf{a}^2) \oplus \mathbf{L}^2(\mathbf{a}^3)$ will have the form:

$$\mathbf{a}^{1\oplus 2\oplus 3} = \mathbf{a}^{1\oplus 2}_{1\oplus 2} \oplus \mathbf{a}^3_3 \oplus \max\left\{\mathbf{a}^{1\oplus 2}_{1\oplus 2,3}, \mathbf{a}^3_{3,1\oplus 2}\right\}.$$

Continuing this process, we obtain a maximal vector in the main space $\mathbf{L}^2$:

$$\mathbf{a}^{1\oplus\cdots\oplus\Lambda} = \mathbf{a}^{1\oplus\cdots\oplus\Lambda-1}_{1\oplus\cdots\oplus\Lambda-1} \oplus \mathbf{a}^\Lambda_\Lambda \oplus \max\left\{\mathbf{a}^{1\oplus\cdots\oplus\Lambda-1}_{1\oplus\cdots\oplus\Lambda-1,\Lambda}, \mathbf{a}^\Lambda_{\Lambda,1\oplus\cdots\oplus\Lambda-1}\right\}. \tag{7}$$

Formula 7 may be simplified, if we divide the measures $\mu_{\mathbf{a}^k}$ into continuous and pure point components, that is $\mu_{\mathbf{a}^k} = \mu^{cont}_{\mathbf{a}^k} + \mu^{pp}_{\mathbf{a}^k}$. Then $\mathbf{a}^k = \mathbf{a}^{k,cont} \oplus \mathbf{a}^{k,pp}$. Relatively to any operator T^s, $k \neq s$, we have

$$\mathbf{a}^{k,cont} = \mathbf{a}^{k,cont}_k \oplus \mathbf{a}^{k,cont}_{k,s} \text{ and } \mathbf{a}^{k,pp} = \mathbf{a}^{k,pp}_k \oplus \mathbf{a}^{k,pp}_{k,s}.$$

Now we can repeat the process described above in **(B)**, separately for the continuous and the pure point parts. Since measures with the same null set may be considered equivalent, we have

$$\max\{w^{cont}, \psi^{cont}\} = \text{ either } w^{cont} \text{ or } \psi^{cont},$$

$$\max\{w^{pp}, \psi^{pp}\} = \text{ either } w^{pp} \text{ or } \psi^{pp},$$

for any two vectors w and ψ. Thus we obtain

$$\mathbf{a}^{1\oplus\cdots\oplus\Lambda,cont} = \mathbf{a}^{1,cont} \oplus \left[\oplus_{j=2}^{\Lambda}\mathbf{a}_j^{j,cont}\right].$$

Similarly,

$$\mathbf{a}^{1\oplus\cdots\oplus\Lambda,pp} = \mathbf{a}^{1,pp} \oplus \left[\oplus_{j=2}^{\Lambda}\mathbf{a}_j^{j,pp}\right].$$

Since $\max\{w^{cont}, \psi^{pp}\} = \psi^{pp}$, we finally derive

$$\mathbf{a}^{1\oplus\cdots\oplus\Lambda} = \mathbf{a}^{1\oplus\cdots\oplus\Lambda,pp} \oplus \mathbf{a}_{1\oplus\cdots\oplus\Lambda}^{1\oplus\cdots\oplus\Lambda,cont}.$$

Let $\Lambda = \infty$. We obtain $\mathbf{a}^{1\oplus\cdots\oplus\Lambda}$ as a vector which satisfies the following equality:

$$\left\|\left[\oplus_{i\in\Omega}E^i(\cdot)\right]\mathbf{a}^{1\oplus\cdots\oplus\Lambda}\right\|^2 = \lim_{L\to\infty}\left\|\left[\oplus_{j=1}^{L}E^j(\cdot)\right]\mathbf{a}^{1\oplus\cdots\oplus L}\right\|^2, \tag{8}$$

since the limit on the right side exists. Indeed $\lim_{L\to\infty}\left\|\left[\oplus_{j=1}^{L}E^j(\cdot)\right]\mathbf{a}^{1\oplus\cdots\oplus L}\right\|^2$ can be rewritten as $\sum_{j=1}^{\infty}\left\|E^j(\cdot)\,\widehat{a_j}\right\|_j^2$, where $\widehat{a_j}$ are the restricted a_j. Noticing that

$$\sum_{j=1}^{\infty}\left\|E^j(\cdot)\,\widehat{a_j}\right\|_j^2 \leqslant \sum_{j=1}^{\infty}\left\|E^i(\cdot)\,a_j\right\|_j^2 < \infty,$$

we prove the convergence (without loss of generality, the vectors a_i can be always chosen such, that $\sum_{i=1}^{\infty}\|a_i\|_i^2 < \infty$).

(C) The next step is to build the measure of the ordered representation for the vector-operator. From Lemma 2.3 and the reasonings above, it follows that such a measure will be

$$\theta(\cdot) = \left(\left[\oplus_{i\in\Omega}E^i(\cdot)\right]\mathbf{a}^{1\oplus\cdots\oplus\Lambda}, \mathbf{a}^{1\oplus\cdots\oplus\Lambda}\right).$$

Thus we have constructed the process Pr_1.

(D) The final step is to construct the canonical multiplicity sets s_n of the vector-operator; s_1 is the whole line; s_2 must contain all the spectrum the multiplicity of which exceeds or equals to 2. For this purpose, we are primarily to unite all e_2^i. But, nevertheless, $\cup_i e_2^i$ will not include all the sets of multiplicity $\geqslant 2$, since we know that if $P(e_1^i \setminus e_2^i) \cap P(e_1^j \setminus e_2^j)$ has a non-zero spectral measure, all the intersections of this sort will represent the multiplicity 2 and should be included into s_2 (since then it is not possible to construct a single cyclic vector). That is $s_2 = \left(\cup_i P(e_2^i)\right) \cup \left(\cup \cap \left(P(e_1^i \backslash e_2^i)\right)\right)$. Using this idea and the fact that an infinite intersection of measurable sets is a measurable set, by induction we may finally build s_n:

$$s_n = \left[\bigcup_i P(e_n^i)\right] \bigcup \left[\bigcup_{\sum m_i \geqslant n} \bigcap P\left(e_{m_i}^i \backslash e_{m_i+1}^i\right)\right]. \tag{9}$$

We have constructed the process Pr_2. □

The constructed measure and the multiplicity sets induce the ordered representation. It is known that such a representation is unique up to unitary equivalence.

Let us return to the example 2. For the distorted vector-operator $T_1 \oplus T_2 \oplus T_3$, two spectral measures will be constructed on vectors $\mathbf{a}^{1\oplus 2\oplus 3}$ and

$$\min\{a^1_{1,2}, a^2_{2,1}\} \oplus \min\{a^2_{2,3}, a^3_{3,2}\} \oplus \min\{a^3_{3,1}, a^1_{1,3}\},$$

where the sense of the minimums is clear.

Here the term 'distorted vector-operator' is clearly explained by the form of its cyclic vectors. The multiplicity set e_2 will be

$$[P(\epsilon(T_1)) \cap P(\epsilon(T_2))] \cup [P(\epsilon(T_1)) \cap P(\epsilon(T_3))] \cup [P(\epsilon(T_2)) \cap P(\epsilon(T_3))].$$

Using the obtained spectral representation we can construct equivalence classes in families of self-adjoint operators:

Definition 3.2. Two families of self-adjoint extensions $\{T_i\}_{i=1}^N$ and $\{S_j\}_{j=1}^L$ are called equivalent, if respective vector-operators $\oplus_{i=1}^N T_i$ and $\oplus_{j=1}^L S_j$ are equivalent.

Note, that if two families $\{T_i\}_{i=1}^N$ and $\{S_j\}_{j=1}^L$ are equivalent, it is not necessarily the case that $N = L$ and T_i is equivalent with S_i.

4. The structure of the ordered spectral representation. Up to now, we have not used the structure of the coordinate operators as differential operators. In this section we make precise the ordered representation obtained in the previous section.

Let $I = \bigvee_{i\in\Omega} I_i$ denote the sliced union of intervals I_i. Similarly, $I^k = \bigvee_{j\in A_k} I_j$. If x_i are variables on I_i, then $\vee x_i$ will designate a variable either on I or I^k depending on the context. This notation shows, that a vector-function

$$z = \{z_1(x_1), \dots, z_n(x_n), \dots\}$$

on I or I^k may be written as $z(\vee x_i)$. In particular, we may also write $\mathbf{z}(\vee x_i)$ instead of $\mathbf{z} = \oplus_{i\in\Omega} z_i$.

Let us introduce the space $\oplus_{i\in\Omega} L^\infty(I_i^n)$. Here, $\mathbf{z}(\vee x_i) \in \oplus_{i\in\Omega} L^\infty(I_i^n)$ means that

$$\sup_{i\in\Omega} \left\{ \operatorname{ess}\sup_{x_i\in I_i^n} |z_i(x_i)\chi_{I_i^n}(x_i)| \right\} < \infty,$$

where for each i, families $\{I_i^n\}_{n=1}^\infty$ represent compact subintervals of I_i, such that $\cup_{n=1}^\infty I_i^n = I_i$ and χ is the characteristic function. In [4, Lemma 2.1], it was shown that $\oplus_{i\in\Omega} L^\infty(I_i^n) = (\oplus_{i\in\Omega} L^1(I_i^n))^*$, where the space of Lebesgue-integrable vector-functions $\oplus_{i\in\Omega} L^1(I_i^n)$ is defined analogously to $\mathbf{L}^2$.

We also need to introduce a symbolic integral $\int_{\bigvee J_i} f(\vee x_i)\, d(\vee x_i)$ defined by:

$$\int_{\bigvee J_i} f(\vee x_i)\, d(\vee x_i) = \oplus_i \int_{J_i} f_i(x_i)\, dx_i,$$

where $f(\vee x_i)$ is understood to be measurable relatively to $d(\vee x_i)$, if and only if $f_i(x_i)$ are measurable relatively to Lebesgue measures dx_i. Then

$$\int_{\bigvee J_i} f(\vee x_i)\, d(\vee x_i) < \infty$$

if and only if $\sup_i \int_{J_i} f_i(x_i)\,dx_i < \infty$.

The main result of the current work is to prove the following theorem:

THEOREM 4.1. *Let T be a self-adjoint vector-operator, generated by an EMZ system $\{I_i, \tau_i\}_{i\in\Omega}$. Let U be an ordered representation of the space $\mathbf{L}^2 = \oplus_{i\in\Omega} L^2(I_i)$ relative to T with the measure θ and the multiplicity sets s_k, $k = \overline{1,m}$. Then there exist kernels $\Theta_k(\vee x_i, \lambda)$, measurable relative to $d(\vee x_i)\times\theta$, such that $\Theta_k(\vee x_i,\lambda) = 0$ for $\lambda\in\mathbb{R}\setminus s_k$ and $(\oplus_{i\in\Omega}\tau_i - \lambda)\Theta_k(\vee x_i,\lambda) = 0$ for each fixed λ. Moreover,*

$$\int_\Delta |\Theta_k(\vee x_i,\lambda)|^2\,d\theta(\lambda) \in \oplus_{i\in\Omega} L^\infty(I_i^n)\ \ \forall n\in\mathbb{N}. \tag{10}$$

$$(U\mathbf{w})^k(\lambda) = \lim_{n\to\infty}\int_{I^n} \mathbf{w}(\vee x_i)\,\overline{\Theta_k(\vee x_i,\lambda)}\,d(\vee x_i), \quad \mathbf{w}\in\mathbf{L}^2, \tag{11}$$

where the limit exists in $L^2(s_k,\theta)$. The kernels $\{\Theta_k(\vee x_i,\lambda)\}_{k=1}^n$, $n\leqslant m$, are linearly independent as vector-functions of the first variable almost everywhere relatively to the measure θ on s_n.

Proof. Fix i. If θ_i and $\{e_p^i\}_{p=1}^{m_i}$ are respectively the measure and the multiplicity sets of an ordered representation for T_i, then there exists the decomposition $L_i^2 = \oplus_{p=1}^{m_i} L^2(e_p^i,\theta_i)$, which implies $T_i = \oplus_{p=1}^{m_i} T_i^p$ and $L^2(e_p^i,\theta_i)$ are T_i^p-invariant. For vector-operator $(\oplus_{i\in\Omega}\oplus_{p=1}^{m_i} T_i^p) \to$ redesignate $\to \oplus_s T_s$, $s = \{i,p\}\in\Omega_1$, we may write $\Omega_1 = \cup_{k=1}^{\Lambda} A_k$.

Let us separate the proof into units for convenience.

(A) For each T_j, $j\in A_k\subset\Omega_1$ and $k=\overline{1,\Lambda}$, there exists a single cyclic vector $a_j\in L_j^2$ and [18, XII.3, Lemma 9 and XIII.5, Theorem 1(I)] a function $W_j(x_j,\lambda)$ defined on $I_j\times e_j$ (note, that for a fixed $i\in\Omega$, $I_j = I_i$ for all $p=\overline{1,m_i}$) and measurable relatively to $dx_j\times\mu_{a_j}$, such that $W_j(x_j,\lambda) = 0$, $\lambda\in\mathbb{R}\setminus e_j$ and for any bounded $\Delta\subset e_j$:

$$\int_\Delta |W_j(x_j,\lambda)|^2\,d\mu_{a_j}(\lambda) \in L^\infty(I_j^n),\ \ n\in\mathbb{N}.$$

Also

$$\left(E^j(\Delta)F_j(T_j)a_j\right)(x_j) = \int_\Delta W_j(x_j,\lambda)F_j(\lambda)\,d\mu_{a_j}(\lambda), \tag{12}$$

for any $F_j\in L^2(e_j,\mu_{a_j})$. On $I^k = \bigvee_{j\in A_k} I_j$, we construct the vector-function

$$W^k(\vee x_j,\lambda) = \{W_1(x_1,\lambda),\dots,W_n(x_n,\lambda),\dots\},$$

which is obviously measurable relative to $d(\vee x_j)\times\sum\mu_{a_j}$. Since $W_j(\cdot,\lambda)\in L^2(\Delta,\mu_{a_j})$, then substituting $\overline{W_j(\cdot,\lambda)} = \overline{W_j(\lambda)}$ in (12) in place of F_j, we obtain

$$\left(E^j(\Delta)\overline{W_j(T_j)}a_j\right)(x_j) = \int_\Delta |W_j(x_j,\lambda)|^2\,d\mu_{a_j}(\lambda).$$

Remembering, that $P(\epsilon(T_s))\cap P(\epsilon(T_j))$ has zero measure, for $s\neq j$ and $s,j\in A_k$, we obtain

$$\left(\left[\oplus_{j\in A_k}E^j\right](\Delta)\overline{W^k(\oplus_{j\in A_k}T_j)}\mathbf{a}^k\right)(\vee x_j) = \int_\Delta |W^k(\vee x_j,\lambda)|^2\,d\mu_{\mathbf{a}^k}(\lambda),$$

where $\mathbf{a}^k = \oplus_{j\in A_k} a_j$.

Since elements f_j from $D(T_j)$ are continuous and thus essentially bounded on I_i^n for any $n \in \mathbb{N}$, $\oplus_{j\in A_k} f_j \in \oplus_{j\in A_k} D(T_j)$ implies that

$$\text{Range}\left[\oplus_{j\in A_k} E^j\right](\Delta) \subseteq \oplus_{j\in A_k} D(T_j) \subset \oplus_{j\in A_k} L^\infty(I_j^n)$$

and hence, we obtain

$$\int_\Delta |W^k(\vee x_j, \lambda)|^2 \, d\mu_{\mathbf{a}^k} \in \oplus_{j\in A_k} L^\infty(I_j^n) \ \ \forall n \in \mathbb{N}. \tag{13}$$

In [18, XIII.5, Theorem 1(I)] it was shown that if we have ordered representations U_i of L_i^2 relative to the operators T_i, $i \in \Omega$, the following formula is valid for $j \in \Omega_1$:

$$(U_j w_j)(\lambda) = \lim_{n\to\infty} \int_{I_j^n} w(x_j)\, \overline{W_j(x_j, \lambda)}\, dx_j, \quad w_j \in L_j^2,$$

where the limit exists in $L^2(e_j, \mu_{a_j})$. Taking direct sums in both sides of the last equality, for each system of compact subintervals we obtain

$$(U^k \oplus_{j\in A_k} w_j^n)(\lambda) = \oplus_{j\in A_k} \int_{I_j^n} w_j(x_j)\, \overline{W_j(x_j, \lambda)}\, dx_j, \quad w_j^n = w_j \chi_{I_j^n}.$$

From (13), it follows that for any bounded Borel set $\Delta \in e_j$ and $I^{k,n} = \bigvee_{j\in A_k} I_j^n$,

$$\int_{I^{k,n}} \int_\Delta |W^k(\vee x_j, \lambda)|^2 \, d\mu_{\mathbf{a}^k}\, d(\vee x_j) < \infty$$

and since $\mathbf{w}^k = \oplus_{j\in A_k} w_j(x_j)$ is assumed to belong to $\oplus_{j\in A_k} L_j^2$, we may write:

$$(U^k \mathbf{w}^{k,n})(\lambda) = \int_{I^{k,n}} \mathbf{w}^k(\vee x_j)\, \overline{W^k(\vee x_j, \lambda)}\, d(\vee x_j).$$

Taking the limit in the both sides and defining $\mathbf{w}^k = \oplus_j \lim_{n\to\infty} w_j^n$ we obtain the formula

$$(U^k \mathbf{w}^k)(\lambda) = \lim_{n\to\infty} \int_{I^{k,n}} \mathbf{w}^k(\vee x_j)\, \overline{W^k(\vee x_j, \lambda)}\, d(\vee x_j), \quad \mathbf{w}^k \in \oplus_{j\in A_k} L_j^2. \tag{14}$$

Note, that since for all $p = \overline{1, m_i}$ there exists the equality $(\tau_i - \lambda)W_i^p = 0$ (see [18, XIII.5, Theorem 1]), it is obvious that $(\oplus_{j\in A_k} \tau_j - \lambda)W^k = 0$, where $\tau_j = \tau_i$ for a fixed i and all $p = \overline{1, m_i}$. If $P(\epsilon(T_i)) \cap P(\epsilon(T_j))$ has zero spectral measures for all $i, j \in \Omega$, then $A_k : \Omega_1 = \cup_{k=1}^{\Lambda_1} A_k$ may be constructed such that A_k contains of indices $\{i, k\}$, $i \in \Omega$, $k = \overline{1, \max_i\{m_i\}}$.

(B) Consider the set of indices $\Omega_2 = \{j \in \Omega_1 : j = \{i, 1\}, i \in \Omega\}$. Construct $A_k : \Omega_2 = \cup_{k=1}^{\Lambda_2} A_k$. Apply the reasonings used in **(A)**, considering everywhere Ω_2 instead of Ω_1. Hence, for each A_k and we find a vector-function $W_1^k(\vee x_j, \lambda)$ which is the solution of the equation $(\oplus_{j\in A_k} \tau_j - \lambda)\mathbf{y} = 0$. Consider W_1^k and W_1^s for $s \neq k$. For $\mathbf{a}^k$ there exists the decomposition $\mathbf{a}^k = \mathbf{a}_k^k \oplus \mathbf{a}_{k,s}^k$ (see the proof of Theorem 3.1). This fact induces the decomposition for W_1^k: $W_1^k = W_{1,k}^k \oplus W_{1,k,s}^k$. It is clear that being the restrictions of W_1^k, the vector-functions $W_{1,k}^k$ and $W_{1,k,s}^k$ are also the solutions of the equation $(\oplus_{j\in A_k}\tau_j-\lambda)\mathbf{y}=0$. They, along with $\mathbf{a}_k^k$ and $\mathbf{a}_{k,s}^k$ define unitary transformations U_k^k and $U_{k,s}^k$ by formula (14), such that:

$$U_k^k:\mathbf{L}^2(\mathbf{a}_k^k)\to L^2(\mathbb{R},\mu_k)\text{ and }U_{k,s}^k:\mathbf{L}^2(\mathbf{a}_{k,s}^k)\to L^2(\mathbb{R},\mu_{k,s})$$

(see the definitions in the proof of Theorem 3.1). This implies, that the decomposition $W^k=W_{1,k}^k\oplus W_{1,k,s}^k$ is correct.

Define as $\max\{W_{1,k,s}^k,W_{1,s,k}^s\}$ the vector-function, which corresponds to the vector $\max\{\mathbf{a}_{k,s}^k,\mathbf{a}_{s,k}^s\}$, respectively $\min\{W_{1,k,s}^k,W_{1,s,k}^s\}$ as the vector-function which corresponds to that $\mathbf{a}_{k,s}^k$ or $\mathbf{a}_{s,k}^s$, which is not maximal of the two.

(C) Without loss of generality, suppose that $k=1$ and $s=2$. From the reasonings presented in Unit **(A)** of this proof, it follows that

$$\Theta_1^{1\oplus 2}=W_{1,1}^1\oplus W_{1,2}^2\oplus\max\left\{W_{1,1,2}^1,W_{1,2,1}^2\right\}$$

is correctly constructed vector-function satisfying the statement of the theorem for the case $T=[\oplus_{j\in A_1}T_j]\oplus[\oplus_{q\in A_2}T_q]$. Apply the above described process to $\Theta_1^{1\oplus 2}$ and W_1^3 to obtain the correctly constructed vector-function:

$$\Theta_1^{1\oplus 2\oplus 3}=\Theta_{1,1\oplus 2}^{1\oplus 2}\oplus W_{1,3}^3\oplus\max\left\{\Theta_{1,1\oplus 2,3}^{1\oplus 2},W_{1,3,1\oplus 2}^3\right\}.$$

Continuing this process, we finally obtain:

$$\begin{aligned}\Theta_1(\vee x_i,\lambda)&=\Theta_1^{1\oplus\cdots\oplus\Lambda_2}=\\&=\Theta_{1,1\oplus\cdots\oplus\Lambda_2-1}^{1\oplus\cdots\oplus\Lambda_2-1}\oplus W_{1,\Lambda_2}^{\Lambda_2}\oplus\max\left\{\Theta_{1,1\oplus\cdots\oplus\Lambda_2-1,\Lambda_2}^{1\oplus\cdots\oplus\Lambda_2-1},W_{1,\Lambda_2,1\oplus\cdots\oplus\Lambda_2-1}^{\Lambda_2}\right\},\end{aligned}$$

where in the case of $\Lambda_2=\infty$, $\Theta_1^{1\oplus\cdots\oplus\Lambda_2}$ is the function which satisfies (analogously to (8)):

$$\begin{aligned}(15)\quad&\left([\oplus_{i\in\Omega}E^i(\Delta)]\int_\Delta\Theta_1^{1\oplus\cdots\oplus\Lambda_2}d\theta(\lambda),\int_\Delta\Theta_1^{1\oplus\cdots\oplus\Lambda_2}d\theta(\lambda)\right)=\\&=\lim_{L\to\infty}\left([\oplus_{j=1}^L E^j(\Delta)]\int_\Delta\Theta_1^{1\oplus\cdots\oplus L}\,d\theta_L(\lambda),\int_\Delta\Theta_1^{1\oplus\cdots\oplus L}\,d\theta_L(\lambda)\right),\end{aligned}$$

for any bounded Borel set Δ, where $\theta_L(\cdot)=\left([\oplus_{j=1}^L E^j(\cdot)]\,\mathbf{a}^{1\oplus\cdots\oplus L},\mathbf{a}^{1\oplus\cdots\oplus L}\right)$ is the measure of the ordered representation of the space $\oplus_{j=1}^L L_j^2$. The limit on the right side exists since for any bounded Borel Δ:

$$\begin{aligned}&\left([\oplus_{j=1}^L E^j(\Delta)]\int_\Delta\Theta_1^{1\oplus\cdots\oplus L}\,d\theta_L(\lambda),\int_\Delta\Theta_1^{1\oplus\cdots\oplus L}\,d\theta_L(\lambda)\right)=\\&=\left([\oplus_{j=1}^L E^j(\Delta)]\,\mathbf{a}^{1\oplus\cdots\oplus L},\mathbf{a}^{1\oplus\cdots\oplus L}\right)\leqslant\left([\oplus_{i=1}^\infty E^i(\Delta)]\;\oplus_{i=1}^\infty a_i,\oplus_{i=1}^\infty a_i\right)<\infty,\end{aligned}$$

for all $L\in\mathbb{N}$ (Lemma 2.3). Despite seeming weak, such convergence is quite natural. Indeed, (15) implies that the cyclic subspace

$$\mathbf{L}^2\left(\int_\Delta\Theta_1^{1\oplus\cdots\oplus L}\,d\theta_L(\lambda)\right)$$

is ε-close with the cyclic subspace

$$\mathbf{L}^2\left(\int_\Delta\Theta_1^{1\oplus\cdots\oplus\Lambda_2}\,d\theta(\lambda)\right),$$

when L is sufficiently big. That is, in the topology of $\mathbf{L}^2$ for any Borel set Δ,

$$f(T)\int_{\Delta}\Theta_1^{1\oplus\cdots\oplus L}\,d\theta_L(\lambda)\to f(T)\int_{\Delta}\Theta_1^{1\oplus\cdots\oplus\Lambda_2}\,d\theta(\lambda),$$

for any Borel f as $L\to\infty$. This means that

$$\int_{\Delta}\Theta_1^{1\oplus\cdots\oplus L}\,d\theta_L(\lambda)\to\int_{\Delta}\Theta_1^{1\oplus\cdots\oplus\Lambda_2}\,d\theta(\lambda),\text{ as } L\to\infty.$$

(D) Define $\Omega_3=\{j\in\Omega_1 : j=\{i,2\}, i\in\Omega\}$. Construct $A_k:\Omega_3=\cup_{k=1}^{\Lambda_3}A_k$. Apply processes **(B)** and **(C)** of this proof, substituting everywhere Ω_3 instead of Ω_2. We obtain a vector-function $\Theta_2^{1\oplus\cdots\oplus\Lambda_3}$, which is defined on the set $\cup_i P(e_2^i)$. But, as we know (see (9)), the set s_2 also includes the sets where there are non-empty superpositions of $\epsilon(T_i)$. Therefore, designating

$$\Theta_2^1=\Theta_2^{1\oplus\cdots\oplus\Lambda_3},\ \Theta_2^2=\min\{W_{1,1,2}^1,W_{1,2,1}^2\},\ldots,$$
$$\Theta_2^{\Lambda_2+1}=\min\left\{\Theta_{1,1\oplus\cdots\oplus\Lambda_2-1,\Lambda_2}^{1\oplus\cdots\oplus\Lambda_2-1},W_{1,\Lambda_2,1\oplus\cdots\oplus\Lambda_2-1}^{\Lambda_2}\right\},$$

we may again use the process **(C)** to build the vector-function $\Theta_2(\vee x_i,\lambda)$ defined on s_2 and $\Theta_2(\vee x_i,\lambda)=0$ for $\lambda\in\mathbb{R}\setminus s_2$. Using processes **(B)**, **(C)**, **(D)** and formula (9), we finally obtain $\Theta_m(\vee x_i,\lambda)$.

(E) The above presented constructions show, that all vector-functions $\Theta_k(\vee x_i,\lambda)$, $k=\overline{1,m}$ are the solutions of the equation $(\oplus_{i\in\Omega}\tau_i-\lambda)\mathbf{y}=0$, moreover they equal zero on $\mathbb{R}\setminus s_k$ and satisfy formulas (10) and (11).

The last thing is to prove the linear independence. In order to make the reasonings more transparent, we prove the linear independence for the special case of two vector-functions

$$\Theta_1=W_{1,1}^1\oplus W_{1,2}^2\oplus\max\left\{W_{1,1,2}^1,W_{1,2,1}^2\right\}$$

and

$$\Theta_2=\min\left\{W_{1,1,2}^1,W_{1,2,1}^2\right\}.$$

Without loss of generality suppose that $\max\left\{W_{1,1,2}^1,W_{1,2,1}^2\right\}=W_{1,1,2}^1$. It is clear that

$$\begin{aligned}&\alpha\Theta_1+\beta\Theta_2=\\&=\alpha\left(\{W_{1,1}^1,0,0,0\}+\{0,0,W_{1,2}^2,0\}+\{0,W_{1,1,2}^1,0,0\}\right)+\beta\{0,0,0,W_{1,2,1}^2\}=\\&=\{\alpha W_{1,1}^1,\alpha W_{1,1,2}^1,\alpha W_{1,2}^2,\beta W_{1,2,1}^2\}=0\end{aligned}$$

implies $\alpha=\beta=0$. The linear independence in the general case is proved using the same ideas. Thus, the linear independence is proved and this finishes the proof of the theorem. ∎

Note, that the given proof introduces the general method of constructing eigenfunctions for a vector-operator. For theoretical purposes, the form of the obtained eigenfunctions could be simplified by totally ordering the set $\{T^j\}_{j=1}^{\Lambda_2}$. This is achieved by saying that $T^k\preceq T^s$ if $\max\{W_{1,k,s}^k,W_{1,s,k}^s\}=W_{1,k,s}^k$. At that, $T^k\simeq T^s$ if and only if $T^k\preceq T^s$ and $T^s\preceq T^k$. According to this, we build $\oplus_{j=1}^{\Lambda_2}T^j$, where $T^j\preceq T^{j+1}$,

$j=\overline{1,\Lambda_2-1}$ if $\Lambda_2\geqslant 2$. The obtained vector-operator is obviously equivalent to the initial vector-operator (comprising unordered operators).

As an important corollary of Theorem 4.1 we obtain

THEOREM 4.2 (Eigenfunction expansions). *For any* $\mathbf{w}\in\mathbf{L}^2$, *there exists a decomposition*

$$\mathbf{w}=\sum_{k=1}^{m}\lim_{n\to\infty}\int_{-n}^{+n}(U\mathbf{w})^k(\lambda)\Theta_k(\vee x_i,\lambda)\,d\theta(\lambda).$$

Proof. From the process of building Θ_k in the previous proof and, in particular (4), it follows that

$$\left(E(\Delta)F(T)\mathbf{a}^k\right)(\vee x_i)=\int_\Delta\Theta_k(\vee x_i,\lambda)F(\lambda)\,d\mu_{\mathbf{a}_k}(\lambda).$$

Substituting here $F=(U\mathbf{w})^k$, we obtain

$$\int_{-n}^{n}\Theta_k(\vee x_i,\lambda)(U\mathbf{w})^k(\lambda)\,d\mu_{\mathbf{a}_k}(\lambda)=E[-n,n]F(T)\mathbf{a}^k\to F(T)\mathbf{a}^k=U^{k^{-1}}F=\mathbf{w}^k.$$

Now the statement of the theorem becomes clear, since $\mathbf{w}=\oplus_{k=1}^m\mathbf{w}_k$. □

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