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EXISTENCE OF POSITIVE SOLUTIONS FOR THE ONE-DIMENSION SINGULAR $P-$LAPLACIAN EQUATION WITH SIGN CHANGING NONLINEARITIES VIA THE METHOD OF UPPER AND LOWER SOLUTION *

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Abstract. A result concerning the existence of positive solutions for the Dirichlet boundary value problem$-\left(\varphi_p\left(u'\right)\right)' = f\left(t,u\right), \ t \in (0,1), \ u(0) = c > 0$ and $u(1) = 0$, is given in this paper. Here $f(t,y)$ may change sign and may be singular at $y = 0$.

1. Introduction. This paper establishes a new result concerning the existence of nonnegative solutions for the Dirichlet boundary value problem

$$\left\{ \begin{array}{l} -\left(\varphi_p\left(u'\right)\right)' = f\left(t,u\right), \quad t \in (0,1) \\ u(0) = c > 0, \ u(1) = 0; \end{array} \right. \tag{1.1}$$

here $\varphi_p(x) = |x|^{p-2}x$, $p > 1$. For $p = 2$, the above problem models steady-state diffusion with reaction (see [1]) and many results have been obtained in the literature when $f(t,u) \le 0$ or $f(t,u) \ge 0$, (see $[2-4]$ and the references therein). However, very few results are available when $f(t,u)$ changes sign.

For $p \neq 2$, the above problem occurs in the study of the $n-$dimensional $p-$Laplace equation, non-Newtonian fluid theory and the turbulent flow of a gas in a porous medium [5].

2. Main Results. Consider the boundary value problem

$$\left\{ \begin{array}{l} -\left(\varphi_p\left(u'\right)\right)' = F\left(t,u\right) \quad \text{for all } t \in (0,1) \\ u(0) = a, \ u(1) = b \end{array} \right. \tag{2.1}$$

where $F : D \to R$ is continuous function and $D \subset (0,1) \times [0,+\infty)$.

DEFINITION 2.1. Let $\alpha \in C([0,1],R) \cap C^1((0,1),R)$ and $\varphi_p(\alpha') \in C^1((0,1),R)$. Now α is called a lower solution for problem (2.1) if $(t,\alpha(t)) \in D$ for all $t \in (0,1)$ and

$$\left\{ \begin{array}{ll} -\left(\varphi_p\left(\alpha'\right)\right)' \le F\left(t,\alpha(t)\right), & t \in (0,1) \\ \alpha(0) \le a, \ \alpha(1) \le b. \end{array} \right.$$

Let $\beta \in C([0,1],R) \cap C^1((0,1),R)$ and $\varphi_p(\beta') \in C^1((0,1),R)$. Now β is called an upper solution for problem (2.1) if $(t,\beta(t)) \in D$ for all $t \in (0,1)$ and

$$\left\{ \begin{array}{ll} -\left(\varphi_p\left(\beta'\right)\right)' \ge F\left(t,\beta(t)\right), & t \in (0,1) \\ \beta(0) \ge a, \ \beta(1) \ge b. \end{array} \right.$$

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LEMMA 2.1 [7]. Suppose α, β are lower and upper solution of problem (2.1) and assume the following conditions are satisfied:

$(H1)$ $\alpha(t) \leq \beta(t)$ for all $0 \leq t \leq 1$;

$(H2)$ $D_{a\beta} \subseteq D$, here $D_{\alpha\beta} = \{(t,y) \,|0 < t < 1,\ \alpha(t) \leq y \leq \beta(t)\}$;

$(H3)$ there exists a continuous function $q \in C(0,1)$ such that

$$|F(t,y)| \leq q(t)\,,\ \ \forall (t,y) \in D_{a\beta},$$

and

$$\int_0^1 q(t)\,dt < +\infty.$$

Then the BVP (2.1) has at least one solution $u \in C([0,1], R) \cap C^1((0,1), R)$ with $\varphi_p(u') \in C^1((0,1), R)$ such that

$$\alpha(t) \leq u(t) \leq \beta(t)\,,\ \ 0 \leq t \leq 1.$$

THEOREM 2.1. Suppose the following conditions hold:

$(H4)$ $f : [0,1] \times (0,+\infty) \to (-\infty,\infty)$ is continuous and $\lim_{y\to 0^+} f(t,y) = -\infty$ uniformly on $[0,1]$;

$(H5)$ there exist a constant $a \in (0,c]$ and a continuous function $g_1 : (0,\infty) \to (0,\infty)$ such that

$$f(t,u) \geq -g_1(u) \text{ for } t \in [0,1]\,,\ u > 0,\ \int_0^a g_1(s)\,ds < +\infty$$

$$\text{and } \int_0^a (G_1(x))^{-1/p}\,dx > q^{1/p};$$

here $G_1(x) = \int_0^x g_1(s)\,ds,\ 0 < x < a,\ q = \frac{p}{p-1}$;

$(H6)$ there exists $b_2 > c,\ b_1 \in [0,b_2)$ and a continuous function $g_2 : (0,\infty) \to (0,\infty)$ such that

$$f(t,u) \leq g_2(u) \text{ for } t \in [0,1]\,,\ u > 0 \text{ and } \int_{b_1}^{b_2} (G_2(x))^{-1/p}\,dx > q^{1/p};$$

here $G_2(x) = \int_x^{b_2} g_2(s)\,ds,\ b_1 < x < b_2$.

Then (1.1) has at least one positive solution $u \in C^1[0,1] \cap C(0,1)$.

Proof. We first prove the following four Claims.

CLAIM 1. The problem

$$\begin{cases} (\varphi_p(\phi'))' = g_1(\phi)\,,\ \ t \in (0,1) \\ \phi(1) = 0,\ \phi'(1) = 0,\ \phi(t) > 0 \text{ for } t \in [0,1) \end{cases} \tag{2.2}$$

has a unique solution $\phi \in C[0,1] \cap C^1(0,1)$ with

$$0 < \phi(t) < a,\ \ \forall t \in [0,1).$$

In addition

$$-(\varphi_p(\phi'))' \leq f(t,\phi) \ \text{ for } t \in (0,1)\,.$$

Proof of Claim 1. From [9] we know that (2.2) has a unique positive solution. On the other hand, since $(\varphi_p(\phi'))' \geq 0$ we have that ϕ' is increasing. Using $\phi'(1) = 0$ we obtain $\phi' \leq 0$.

Multiply both sides of (2.1) by ϕ' and then integrate from s to 1, to obtain

$$\int_s^1 (\varphi_p(\phi'(t)))' \phi'(t)\, dt = \int_s^1 g_1(\phi(t))\phi'(t)\, dt,$$

so

$$\int_0^{\varphi_p(\phi'(s))} \varphi_p^{-1}(x)\, dx = G_1(\phi(s)).$$

Now use

$$\int_0^u \varphi_p^{-1}(s)\, ds = \frac{1}{q}|u|^q,$$

to obtain

$$\frac{1}{q}|\phi'|^p = G_1(\phi).$$

Thus

$$\phi' = -q^{\frac{1}{p}}(G_1(\phi))^{\frac{1}{p}},$$

so

$$\int_t^1 \frac{d\phi}{(G_1(\phi))^{\frac{1}{p}}} = -q^{\frac{1}{p}}(1-t) \text{ for } t \in [0,1].$$

Consequently

$$\int_0^{\phi(t)} (G_1(x))^{\frac{-1}{p}}\, dx = q^{\frac{1}{p}}(1-t) \text{ for } t \in [0,1]. \tag{2.3}$$

Let

$$H_1(u) = \int_0^u (G_1(x))^{\frac{-1}{p}}\, dx.$$

Then

$$\phi(s) = H_1^{-1}\left(q^{\frac{1}{p}}(1-s)\right)$$

is a solution of (2.2). Now $\int_0^{\phi(0)} (G_1(x))^{\frac{-1}{p}}\, dx = q^{\frac{1}{p}}$ and $\int_0^a (G_1(x))^{\frac{-1}{p}}\, dx > q^{\frac{1}{p}}$ imply $0 < \phi(0) < a$. Also since $\phi' \leq 0$ we have

$$(\varphi_p(\phi')) + f(t,\phi) \geq (\varphi_p(\phi'))' - g_1(\phi) = 0 \text{ for } t \in (0,1).$$

Claim 2. The problem

$$\begin{cases} -(\varphi_p(\theta'))' = g_2(\theta), & t \in (0,1) \\ \theta(0) = b_2,\ \theta'(0) = 0,\ \theta(t) > 0 \text{ for } t \in (0,1] \end{cases} \tag{2.4}$$

has a unique solution $\theta \in C[0,1] \cap C^1(0,1)$ such that

$$b_1 < \theta(t) < b_2, \quad \forall t \in (0,1]. \tag{2.5}$$

In addition

$$-(\varphi_p(\theta'))' \geq f(t,\theta) \text{ for } t \in (0,1).$$

Proof of Claim 2. From [9] we know that (2.4) has a unique positive solution. On the other hand since $-(\varphi_p(\theta'))' \geq 0$ we have that θ' is decreasing. Using $\theta'(0) = 0$ we obtain $\theta' \leq 0$. Let

$$H_2(x) = \int_x^{b_2} (G_2(s))^{\frac{-1}{p}} ds, \quad 0 < x < b_2.$$

Argue as in Claim 1 to obtain

$$\int_{\theta(t)}^{b_2} (G_2(x))^{\frac{-1}{p}} dx = q^{\frac{1}{p}} t \text{ for } t \in (0,1]. \tag{2.6}$$

Then

$$\theta(t) = H_2^{-1}\left(q^{\frac{1}{p}} t\right) \text{ for } t \in (0,1]$$

is a solution of (2.4). Now since $\int_{\theta(1)}^{b_2} (G_2(x))^{\frac{-1}{p}} dx = q^{\frac{1}{p}}$ and $\int_{b_1}^{b_2} (G_2(x))^{\frac{-1}{p}} dx > q^{\frac{1}{p}}$ we have $b_1 < \theta(1) < b_2$. Moreover

$$(\varphi_p(\theta'))' + f(t,\theta) \leq (\varphi_p(\theta'))' + g_2(\theta) = 0 \text{ for } t \in (0,1).$$

Claim 3. $\phi(t) < \theta(t)$ for all $t \in (0,1)$.

Proof of Claim 3. If $a \leq b_1$ then $\phi(t) < a \leq b_1 < \theta(t)$ for $t \in (0,1)$. We now consider the case $b_1 < a$. We easily obtain that

$$G_1(u) > 0 \text{ for } u \in (0,a), \; G_2(u) > 0 \text{ for } u \in (b_1,b_2)$$

and

$$q^{\frac{1}{p}} = \int_0^{\phi(t)} (G_1(x))^{\frac{-1}{p}} dx + \int_{\theta(t)}^{b_2} (G_2(x))^{\frac{-1}{p}} dx \text{ for } t \in (0,1).$$

Let

$$\Phi(u) = \int_0^u (G_1(x))^{\frac{-1}{p}} dx + \int_u^{b_2} (G_2(x))^{\frac{-1}{p}} dx \text{ for } u \in [b_1,a].$$

It is obvious that $\Phi \in C[b_1,a] \cap C^2(b_1,a)$ with

$$\Phi'(u) = (G_1(u))^{\frac{-1}{p}} - (G_2(u))^{\frac{-1}{p}} \text{ for } u \in (b_1,a),$$

and

$$-p\Phi''(u) = (G_1(u))^{\frac{-1}{p}-1} g_1(u) + (G_2(u))^{\frac{-1}{p}-1} g_2(u) \text{ for } u \in (b_1,a).$$

If $\Phi'(u_0) = 0$ for $u_0 \in [b, a]$, then $G_1(u_0) = G_2(u_0) > 0$. On the other hand, $g_1(u_0) + g_2(u_0) > 0$, so

$$-p\Phi''(u_0) = (G_1(u_0))^{\frac{-1}{p}-1}(g_1(u_0) + g_2(u_0)) > 0.$$

Consequently, Φ has no locally minimum point in (b_1, a). Notice

$$\Phi(b_1) = \int_0^{b_1} (G_1(x))^{\frac{-1}{p}} dx + \int_{b_1}^{b_2} (G_2(x))^{\frac{-1}{p}} dx \geq \int_{b_1}^{b_2} (G_2(x))^{\frac{-1}{p}} dx > q^{\frac{1}{p}}.$$

Since $a < b_2$ we have

$$\Phi(a) = \int_0^{a} (G_1(x))^{\frac{-1}{p}} dx + \int_{a}^{b_2} (G_2(x))^{\frac{-1}{p}} dx \geq \int_0^{a} (G_1(x))^{\frac{-1}{p}} dx > q^{\frac{1}{p}}.$$

Consequently

$$\Phi(u) = \int_0^{u} (G_1(x))^{\frac{-1}{p}} dx + \int_{u}^{b_2} (G_2(x))^{\frac{-1}{p}} dx > q^{\frac{1}{p}} \text{ for } u \in [b_1, a]. \tag{2.7}$$

Suppose there exists $t_0 \in (0,1)$ such that $\theta(t_0) < \phi(t_0)$. Then $b_1 < \theta(t_0) < \phi(t_0) < a$. By (2.3) and (2.6) we have

$$\begin{aligned} q^{\frac{1}{p}} &= \int_0^{\phi(t_0)} (G_1(x))^{\frac{-1}{p}} dx + \int_{\theta(t_0)}^{\phi(t_0)} (G_2(x))^{\frac{-1}{p}} dx + \int_{\phi(t_0)}^{b_2} (G_2(x))^{\frac{-1}{p}} dx \\ &\geq \int_0^{\phi(t_0)} (G_1(x))^{\frac{-1}{p}} dx + \int_{\phi(t_0)}^{b_2} (G_2(x))^{\frac{-1}{p}} dx \\ &= \Phi(\phi(t_0)) \\ &> q^{\frac{1}{p}} \text{ (see (2.7))}, \end{aligned}$$

a contradiction.

Claim 4. There exists $\eta \in C[0,1] \cap C^1(0,1)$ such that $\phi(t) \leq \eta(t) \leq \theta(t)$, $\forall t \in (0,1)$ and

$$\begin{cases} -(\varphi_p(\eta'))' \geq f(t, \eta), & t \in (0,1) \\ \eta(0) = c, \ \eta(1) = 0. \end{cases}$$

Proof of Claim 4. Let $R = \min_{t\in[0,1]} \theta(t) > 0$ and

$$F(t,y) = \begin{cases} f(t,y), & y \geq R \\ \max\{f(t,y), f(t,R)\}, & 0 < y < R \\ f(t,R), & y = 0 \end{cases}$$

First we prove that $F : [0,1] \times [0,\infty) \to (-\infty,\infty)$ is continuous. By $(H4)$, there exist δ, $0 < \delta < R$, such that $f(t,y) < f(t,R)$ for all $(t,y) \in [0,1] \times (0,\delta]$. As a result

$$F(t,y) = f(t,R) \quad \text{for} \quad (t,y) \in [0,1] \times (0,\delta],$$

so $F : [0,1] \times [0,\infty) \to (-\infty,\infty)$ is continuous.

By Claim 1 and Claim 2, we have

$$-\left(\varphi_p\left(\theta'\left(t\right)\right)\right)' - F\left(t,\theta\left(t\right)\right) = -\left(\varphi_p\left(\theta'\left(t\right)\right)\right)' - f\left(t,\theta\left(t\right)\right) \geq 0, \quad t \in (0,1)$$

$$-\left(\varphi_p\left(\phi'\left(t\right)\right)\right)' - F\left(t,\phi\left(t\right)\right) \leq -\left(\varphi_p\left(\phi'\left(t\right)\right)\right)' - f\left(t,\phi\left(t\right)\right) \leq 0, \quad t \in (0,1)$$

and

$$0 < \phi(0) < c \leq \theta(0)\,,\ \phi(1) = 0 < \theta(1)\,,\ 0 < \phi(t) < \theta(t) \text{ for } t \in (0,1)\,.$$

From Lemma 2.1, we know the problem

$$\begin{cases} -\left(\varphi_p\left(\eta'\left(t\right)\right)\right)' = F\left(t,\eta\left(t\right)\right), & t \in (0,1) \\ \eta(0) = c, \quad \eta(1) = 0. \end{cases}$$

has a solution $\eta \in C[0,1] \cap C^1(0,1)$ with $\phi(t) \leq \eta(t) \leq \theta(t)\,,\ \forall t \in (0,1)$. Since $F(t,y) \geq f(t,y)\,,\ (t,y) \in (0,1) \times (0,\infty)$, we have $-\left(\varphi_p\left(\eta'\left(t\right)\right)\right)' \geq f\left(t,\eta\left(t\right)\right)$ for all $t \in (0,1)\,.$

Proof of Theorem 2.1. For $n \in \{3,4,\cdots\}$, consider the problem

$$\begin{cases} \left(\varphi_p\left(z'\left(t\right)\right)\right)' - f\left(t,z\left(t\right)\right) = 0, & t \in \left(0,\frac{n-1}{n}\right) \\ z(0) = c, \quad z\left(\frac{n-1}{n}\right) = \eta\left(\frac{n-1}{n}\right). \end{cases} \tag{2.8}$$

From Claim 1 and Claim 4, we have

$$\begin{cases} -\left(\varphi_p\left(\eta'\right)\right)' \geq f(t,\eta)\,, & t \in \left(0,\frac{n-1}{n}\right) \\ \eta(0) = c,\ \eta\left(\frac{n-1}{n}\right) = \eta\left(\frac{n-1}{n}\right) \end{cases}$$

and

$$\begin{cases} -\left(\varphi_p\left(\phi'\right)\right)' \leq f(t,\phi)\,, & t \in \left(0,\frac{n-1}{n}\right) \\ \phi(0) \leq c,\ \phi\left(\frac{n-1}{n}\right) \leq \eta\left(\frac{n-1}{n}\right). \end{cases}$$

Then η is an upper solution and ϕ is a lower solution of problem (2.8). On the other hand $0 < \phi(t) \leq \eta(t)\,,\ t \in \left[0,1-\frac{1}{n}\right]$ and $f : \left[0,1-\frac{1}{n}\right] \times D_{\phi\eta} \to (-\infty,\infty)$ is continuous. From Lemma 2.1, problem (2.8) has at least one solution $z_n \in C\left(\left[0,\frac{n-1}{n}\right],R\right) \cap C^1\left(\left(0,\frac{n-1}{n}\right),R\right)$ and $\varphi_p(z_n') \in C^1\left(\left(0,\frac{n-1}{n}\right),R\right)$ such that

$$\phi(t) \leq z_n(t) \leq \eta(t) \text{ for } 0 \leq t \leq \frac{n-1}{n}.$$

Fix $n_0 \in \{3,4,\cdots\}$. Now lets look at the interval $\left[0,1-\frac{1}{n_0}\right]$. Let

$$R_{n_0} = \sup\left\{|f(t,u)| : t \in \left[0,1-\frac{1}{n_0}\right] \text{ and } u \in D_{\phi\eta}\right\}.$$

The Mean Value Theorem implies that there exists $\tau \in \left(0,1-\frac{1}{n_0}\right)$ with $|z_n'(\tau)| \leq 3\sup_{[0,1]}\eta(t) \equiv L_{n_0}$. Hence for $t \in \left[0,1-\frac{1}{n_0}\right]$ we have

$$|z_n'(t)| \leq \varphi_p^{-1}\left(|z_n'(\tau)| + \left|\int_\tau^t \left(\varphi_p\left(z_n'\right)\right)' dx\right|\right) \leq \left(\varphi_p\left(L_{n_0}\right) + R_{n_0}\right)^{\frac{1}{p-1}}$$

where φ_p^{-1} is an inverse function of φ_p.

As a result

$$\{z_n\}_{n=n_0}^{\infty} \text{ is bounded, equicontinuous family on } \left[0,1-\frac{1}{n_0}\right].$$

The Arzela-Ascoli theorem guarantees the existence of a subsequence N_{n_0} of integers and a function $u_{n_0} \in C\left[0,1-\frac{1}{n_0}\right]$ with z_n converging uniformly to u_{n_0} on $\left[0,1-\frac{1}{n_0}\right]$ as $n \to \infty$ through N_{n_0}. Similarly

$$\{z_n\}_{n=n_0}^{\infty} \text{ is bounded, equicontinuous family on } \left[0,1-\frac{1}{n_0+1}\right],$$

so there is a subsequence N_{n_0+1} of N_{n_0} and a function $u_{n_0+1} \in C\left[0,1-\frac{1}{n_0+1}\right]$ with z_n converging uniformly to u_{n_0+1} on $\left[0,1-\frac{1}{n_0+1}\right]$ as $n \to \infty$ through N_{n_0+1}. Note $u_{n_0+1} = u_{n_0}$ on $\left[0,1-\frac{1}{n_0}\right]$ since $N_{n_0+1} \subseteq N_{n_0}$. Proceed inductively to obtain subsequence on integers

$$N_{n_0} \supseteq N_{n_0+1} \supseteq \cdots \supseteq N_k \supseteq \cdots$$

and functions

$$u_k \in \left[0,1-\frac{1}{k}\right]$$

with

$$z_n \text{ converging uniformly to } u_k \text{ on } \left[0,1-\frac{1}{k}\right] \text{ as } n \to \infty \text{ through } N_k$$

and

$$u_{k+1} = u_k \text{ on } \left[0,1-\frac{1}{k}\right].$$

Define a function $u : [0,1] \to [0,\infty)$ by $u(t) = u_k(t)$ on $\left[0,1-\frac{1}{k}\right]$ and $u(1) = 0$. Notice u is well defined and $\phi(t) \le u(t) \le \eta(t)$ for $t \in (0,1)$. Next fix $t \in [0,1)$ and let $m \in \{n_0, n_0+1, \cdots\}$ be such that $0 \le t < 1-\frac{1}{m}$. Let $N_m^+ = \{n \in N_m : n \ge m\}$. Let $n \in N_m^+$ and let $a = 0$, $b = 1-\frac{1}{m}$.

Define the operator, $L : C[a_0,b_0] \to C[a_0,b_0]$ by

$$(Ly)(t) = y(a_0) + \int_{a_0}^{t} \varphi_p^{-1}\left(A_y + \int_s^{b_0} q(\tau) f(\tau, y(\tau))\, d\tau\right) ds$$

where A_y satisfies

$$\int_{a_0}^{b_0} \varphi_p^{-1}\left(A_y + \int_s^{b_0} f(\tau, y(\tau))\, d\tau\right) ds = y(b_0) - y(a_0).$$

Let $y_n \to y$ uniformly on $[a_0,b_0]$. As the proof in Theorem 2.4[5], if we show $\lim_{n\to\infty} A_{y_n} = A$, then this together with φ_p^{-1} continuous, implies that

$L : C[a_0, b_0] \to C[a_0, b_0]$ is continuous. Associate A_{y_n} with y_n and notice

$$\int_{a_0}^{b_0} \left(\varphi_p^{-1} \left(A_{y_n} + \int_s^{b_0} f(\tau, y(\tau))\, d\tau \right) - \varphi_p^{-1} \left(A_y + \int_s^{b_0} f(\tau, y(\tau))\, d\tau \right) \right) ds$$
$$= y_n(b_0) - y_n(a_0) - y(b_0) + y(a_0).$$

The Mean Value Theorem for integrals implies that there exists $\eta_n \in [0,1]$ with

$$\varphi_p^{-1} \left(A_{y_n} + \int_{\eta_n}^{b_0} f(\tau, y(\tau))\, d\tau \right) - \varphi_p^{-1} \left(A_y + \int_{\eta_n}^{b_0} f(\tau, y(\tau))\, d\tau \right)$$
$$= \frac{y_n(b_0) - y_n(a_0) - y(b_0) + y(a_0)}{b_0 - a_0},$$

and since $y_n \to y$ uniformly on $[a_0, b_0]$ we have $\lim_{n\to\infty} A_{y_n} = A_y$.

Now since z_n converges uniformly on $[a_0, b_0]$ to u as $n \to \infty$ and $Lz_n = z_n$ we obtain $Lu = u$, i.e.

$$-(\varphi_p(u'(t)))' = f(t,u), \quad a_0 \le t \le b_0.$$

We can do this argument for each $t \in (0,1)$ and so $-(\varphi_p(u'(t)))' = f(t,u), \quad 0 < t < 1$. It remains to show u is continuous at 1.

Let $\varepsilon > 0$ be given. Now since $0 < \phi(t) \le \eta(t)$, $t \in (0,1)$ and $\phi(1) = \eta(1) = 0$, there exists $\delta > 0$ with

$$0 \le \phi(t) \le \eta(t) < \frac{\varepsilon}{2} \text{ for } t \in [1-\delta, 1].$$

This together with the fact that $\phi(t) \le u_n(t) \le \eta(t)$ for $t \in (0,1)$ implies that

$$\phi(t) \le u_n(t) \le \eta(t) < \frac{\varepsilon}{2} \text{ for } t \in [1-\delta, 1].$$

Consequently

$$0 \le \phi(t) \le u(t) \le \eta(t) < \frac{\varepsilon}{2} \text{ for } t \in [1-\delta, 1]$$

and so u is continuous at 1. Thus $u \in C[0,1] \cap C^1(0,1)$ and u is a positive solution of (1.1). The proof of Theorem 2.1 is complete.

Remark 2.1. The ideas in this section can be used to discuss the BVP

$$\begin{cases} -(\varphi_p(u'))' = f(t,u), & t \in (0,1) \\ u(0) = 0, \ u(1) = c > 0. \end{cases}$$

Only minor adjustments are needed, so we leave the details to the reader.

To illustrate the above ideas we consider the following problem

$$\begin{cases} -(\varphi_p(u'))' = \lambda a(t) \left(u^\beta - \frac{1}{u^\alpha} \right) \text{ for } t \in (0,1) \\ u(0) = c > 0, \ u(1) = 0; \end{cases} \tag{2.9}$$

here $\lambda > 0$, $a \in C[0,1]$ and $a(t) > 0$ for $t \in [0,1]$, $0 < \alpha < 1$ and $0 < \beta$.

Corollary 2.1. (2.9) has at least one positive solution $u \in C^1[0,1] \cap C(0,1)$ if $\lambda > 0$ is chosen sufficiently small.

Proof. To see this we will apply Theorem 2.1. Let $f(t,u) = \lambda a(t)\left(u^{\beta} - \frac{1}{u^{\alpha}}\right)$. Then $f : [0,1] \times (0,+\infty) \to (-\infty,\infty)$ is continuous and $\lim_{u\to 0^+} f(t,u) = -\infty$ uniformly on $[0,1]$.

In conditions $(H5)$ and $(H6)$ we let $a = c$, $b_2 = 2c$, $b_1 = 0$. Also we let

$$d_0 = \max_{t\in[0,1]} a(t) \quad \text{and} \quad g_1(u) = \frac{\lambda d_0}{u^{\alpha}} \text{ for } u \in (0,\infty).$$

Then $g_1 : (0,\infty) \to (0,\infty)$ is a continuous function and

$$\lambda a(t)\left(u^{\beta} - \frac{1}{u^{\alpha}}\right) \geq -g_1(u) \text{ for } t \in [0,1],\ u > 0,$$

$$\int_0^a g_1(s)\,ds = \int_0^a \frac{\lambda d_0}{s^{\alpha}} ds = \frac{\lambda d_0}{1-\alpha} a^{1-\alpha} < +\infty.$$

Also we have

$$G_1(x) = \int_0^x \frac{\lambda d_0}{u^{\alpha}} du = \frac{\lambda d_0}{1-\alpha} x^{1-\alpha}$$

and

$$\begin{aligned}\int_0^a (G_1(x))^{\frac{-1}{p}}\,dx &= \left(\frac{1-\alpha}{\lambda d_0}\right)^{\frac{1}{p}} \int_0^a x^{\frac{\alpha-1}{p}} dx \\ &= \left(\frac{1-\alpha}{\lambda d_0}\right)^{\frac{1}{p}} \cdot \frac{p}{\alpha+p+1} \cdot c^{\frac{\alpha+p+1}{p}}.\end{aligned}$$

Then $\int_0^a (G_1(x))^{\frac{-1}{p}}\,dx > q^{\frac{1}{p}}$, provided

$$\lambda^{\frac{1}{p}} < \lambda^* = \left(\frac{1-\alpha}{qd_0}\right)^{\frac{1}{p}} \cdot \frac{p}{\alpha+p+1} \cdot c^{\frac{\alpha+p+1}{p}}. \tag{2.10}$$

Next let

$$g_2(u) = \lambda d_0 u^{\beta} \text{ for } u \in (0,\infty).$$

Then $g_2 : (0,\infty) \to (0,\infty)$ is a continuous function and

$$\lambda a(t)\left(u^{\beta} - \frac{1}{u^{\alpha}}\right) \leq g_2(u) \text{ for } t \in [0,1],\ u > 0,$$

and

$$G_2(x) = \int_x^{b_2} g_2(u)\,du = \frac{\lambda d_0}{\beta+1}\left((2c)^{\beta+1} - x^{\beta+1}\right) \text{ for } x \in (0,2c).$$

Thus

$$\int_{b_1}^{b_2} (G_2(x))^{-\frac{1}{p}}\, dx = \int_0^{2c} \left[\frac{\lambda d_0}{\beta+1}\left((2c)^{\beta+1} - x^{\beta+1}\right)\right]^{-\frac{1}{p}} dx$$
$$= \left(\frac{\beta+1}{\lambda d_0}\right)^{\frac{1}{p}} \int_0^{2c} \frac{dx}{\left[(2c)^{\beta+1} - x^{\beta+1}\right]^{\frac{1}{p}}}$$
$$= \left(\frac{\beta+1}{\lambda d_0}\right)^{\frac{1}{p}} \cdot (2c)^{\frac{p-\beta-1}{p}} \int_0^1 \frac{dy}{\left[1 - y^{\beta+1}\right]^{\frac{1}{p}}}$$
$$\geq \left(\frac{\beta+1}{\lambda d_0}\right)^{\frac{1}{p}} \cdot (2c)^{\frac{p-\beta-1}{p}}\,.$$

Then $\int_{b_1}^{b_2} (G_2(x))^{\frac{-1}{p}}\, dx > q^{\frac{1}{p}}$, provided with

$$\lambda^{\frac{1}{p}} < \lambda^{**} = \left(\frac{\beta+1}{q d_0}\right)^{\frac{1}{p}} \cdot (2c)^{\frac{p-\beta-1}{p}}\,. \tag{2.11}$$

Thus if $0 < \lambda^{\frac{1}{p}} < \min\{\lambda^*, \lambda^{**}\}$, the conditions of Theorem 2.1 are satisfied. As a result problem (2.9) has at least one positive solution.

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