

BOUNDEDNESS FOR MULTILINEAR LITTLEWOOD-PALEY OPERATORS ON TRIEBEL-LIZORKIN SPACES *

LIU LANZHE†

Abstract. In this paper, we prove the boundedness in the context of Triebel-Lizorkin spaces for multilinear Littlewood-Paley operators.

1. Introduction and results. Let $\delta > 0$ and ψ be a function on R^n which satisfies the following properties:

- (1) $\int_{R^n} \psi(x) dx = 0$,
- (2) $|\psi(x)| \leq C(1 + |x|)^{-(n+1-\delta)}$,
- (3) $|\psi(x+y) - \psi(x)| \leq C|y|(1 + |x|)^{-(n+2-\delta)}$ when $2|y| < |x|$.

Fixed $\mu > 1$. Let m be a positive integer and A be a function on R^n . The multilinear Littlewood-Paley operator is defined by

$$g_\mu^A(f)(x) = \left[\int \int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu} |F_t^A(f)(x, y)|^2 \frac{dy dt}{t^{n+1}} \right]^{1/2},$$

where

$$F_t^A(f)(x, y) = \int_{R^n} \frac{R_{m+1}(A; x, z)}{|x - z|^m} f(z) \psi_t(y - z) dz,$$

$$R_{m+1}(A; x, y) = A(x) - \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha A(y) (x - y)^\alpha,$$

and $\psi_t(x) = t^{-n+\delta} \psi(x/t)$ for $t > 0$. Let $F_t(f)(y) = f * \psi_t(y)$. We also define that

$$g_\mu(f)(x) = \left(\int \int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu} |F_t(f)(y)|^2 \frac{dy dt}{t^{n+1}} \right)^{1/2},$$

which is the Littlewood-Paley operator (see [18]).

Let H be the Hilbert space $H = \left\{ h : \|h\| = \left(\int \int_{R_+^{n+1}} |h(t)|^2 dy dt / t^{n+1} \right)^{1/2} < \infty \right\}$.

Then for each fixed $x \in R^n$, $F_t^A(f)(x, y)$ may be viewed as a mapping from $(0, +\infty)$ to H , and it is clear that

$$g_\mu^A(f)(x) = \left\| \left(\frac{t}{t + |x - y|} \right)^{n\mu/2} F_t^A(f)(x, y) \right\|,$$

$$g_\mu(f)(x) = \left\| \left(\frac{t}{t + |x - y|} \right)^{n\mu/2} F_t(f)(y) \right\|.$$

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†College of Mathematics and Computer, Changsha University of Science and Technology, Changsha 410077, P.R. of China (lanzhelu@263.net).

Note that when $m = 0$, g_μ^A is just the commutator of Littlewood-Paley operator (see [1][13-16]). It is well known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors (see [3-6][9][10]). In [11][17], Janson and Paluszynski obtain the boundedness of commutators generated by the Calderón-Zygmund operator or fractional integral operator with Lipschitz functions on Triebel-Lizorkin spaces. The main purpose of this paper is to discuss the boundedness of the multilinear Littlewood-Paley operators in the context of Triebel-Lizorkin spaces. First, let us introduce some notation. We will work on R^n , $n > 1$. Throughout this paper, $M(f)$ will denote the Hardy-littlewood maximal function of f , Q will denote a cube of R^n with side parallel to the axes, and for a cube Q , let $f_Q = |Q|^{-1} \int_Q f(x)dx$ and $f^\#(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f(y) - f_Q|dy$. For $\beta > 0$ and $p > 1$, let $\dot{F}_p^{\beta, \infty}$ be the homogeneous Triebel-Lizorkin space. The Lipschitz space $\dot{\Lambda}_\beta$ is the space of functions f such that

$$\|f\|_{\dot{\Lambda}_\beta} = \sup_{\substack{x, h \in R^n \\ h \neq 0}} \left| \Delta_h^{[\beta]+1} f(x) \right| / |h|^\beta < \infty,$$

where Δ_h^k denotes the k -th difference operator (see[17]).

We shall prove the following theorems in Section 3.

THEOREM 1. Let $0 \leq \delta < n$, $0 < \beta < 1/2$, $1 < p < n/\delta$, $1/p - 1/q = \delta/n$ and $D^\alpha A \in \dot{\Lambda}_\beta$ for $|\alpha| = m$. Then g_μ^A is bounded from $L^p(R^n)$ to $\dot{F}_q^{\beta, \infty}(R^n)$.

THEOREM 2. Let $0 \leq \delta < n$, $0 < \beta < 1/2$, $1 < p < n/(\delta + \beta)$, $1/p - 1/q = (\delta + \beta)/n$ and $D^\alpha A \in \dot{\Lambda}_\beta$ for $|\alpha| = m$. Then g_μ^A is bounded from $L^p(R^n)$ to $L^q(R^n)$.

THEOREM 3. Let $0 \leq \delta < n$, $0 < \beta < 1/2$, $\delta + \beta < n$ and $D^\alpha A \in \dot{\Lambda}_\beta$ for $|\alpha| = m$. Then for any $\lambda > 0$,

$$|\{x \in R^n : g_\mu^A(f)(x) > \lambda\}| \leq C \left(\sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\Lambda}_\beta} \|f\|_{L^1} / \lambda \right)^{n/(n-\delta-\beta)}.$$

2. Some lemmas. We begin with some preliminary lemmas.

LEMMA 1 (see [17]). For $0 < \beta < 1$, $1 < p < \infty$, we have

$$\begin{aligned} \|f\|_{\dot{F}_p^{\beta, \infty}} &\approx \left\| \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q| dx \right\|_{L^p} \\ &\approx \left\| \sup_{c \in Q} \inf_c \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - c| dx \right\|_{L^p}. \end{aligned}$$

LEMMA 2 (see [17]). For $0 < \beta < 1$, $1 \leq p \leq \infty$, we have

$$\|f\|_{\dot{\Lambda}_\beta} \approx \sup_Q \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - f_Q| dx \approx \sup_Q \frac{1}{|Q|^{\beta/n}} \left(\frac{1}{|Q|} \int_Q |f(x) - f_Q|^p dx \right)^{1/p}.$$

LEMMA 3 (see [2]). For $1 \leq r < \infty$ and $0 < \delta < n$, let

$$M_{\delta, r}(f)(x) = \sup_{x \in Q} \left(\frac{1}{|Q|^{1-\delta r/n}} \int_Q |f(y)|^r dy \right)^{1/r}.$$

Suppose that $r < p < n/\delta$ and $1/q = 1/p - \delta/n$. Then $\|M_{\delta,r}(f)\|_{L^q} \leq C\|f\|_{L^p}$.

LEMMA 4 (see [8]). Let $Q_1 \subset Q_2$. Then

$$|f_{Q_1} - f_{Q_2}| \leq C\|f\|_{\dot{\lambda}_\beta}|Q_2|^{\beta/n}.$$

LEMMA 5 (see [6]). Let A be a function on R^n and $D^\alpha A \in L^q(R^n)$ for $|\alpha| = m$ and some $q > n$. Then

$$|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x, y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where $\tilde{Q}(x, y)$ is the cube centered at x and having side length $5\sqrt{n}|x - y|$.

LEMMA 6. Let $0 \leq \delta < n$, $1 < p < \infty$ and $D^\alpha A \in \dot{\lambda}_\beta$ for $|\alpha| = m$, $1 < r \leq \infty$, $1/q = 1/p + 1/r - \delta/n$. Then g_μ^A is bounded from $L^p(R^n)$ to $L^q(R^n)$, that is

$$\|g_\mu^A(f)\|_{L^q} \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \|f\|_{L^p}.$$

Proof. By the Minkowski' inequality and the condition of ψ , we have

$$\begin{aligned} &g_\mu^A(f)(x) \\ &\leq \int_{R^n} \frac{|f(z)||R_{m+1}(A; x, z)|}{|x - z|^m} \left(\int_{R_+^{n+1}} \left(\frac{t}{t + |x - y|} \right)^{n\mu} |\psi_t(y - z)|^2 \frac{dydt}{t^{1+n}} \right)^{1/2} dz \\ &\leq C \int_{R^n} \frac{|f(z)||R_{m+1}(A; x, z)|}{|x - z|^m} \\ &\quad \times \left(\int_0^\infty \int_{R^n} \left(\frac{t}{t + |x - y|} \right)^{n\mu} \frac{t^{-2n+2\delta}}{(1 + |y - z|/t)^{2n+2-2\delta}} \frac{dydt}{t^{1+n}} \right)^{1/2} dz \\ &\leq C \int_{R^n} \frac{|f(z)||R_{m+1}(A; x, z)|}{|x - z|^m} \\ &\quad \times \left[\int_0^\infty \left(t^{-n} \int_{R^n} \left(\frac{t}{t + |x - y|} \right)^{n\mu} \frac{dy}{(t + |y - z|)^{2n+2-2\delta}} \right) tdt \right]^{1/2} dz. \end{aligned}$$

Noting that

$$\begin{aligned} t^{-n} \int_{R^n} \left(\frac{t}{t + |x - y|} \right)^{n\mu} \frac{dy}{(t + |y - z|)^{2n+2-2\delta}} &\leq CM \left(\frac{1}{(t + |x - z|)^{2n+2-2\delta}} \right) \\ &\leq \frac{C}{(t + |x - z|)^{2n+2-2\delta}} \end{aligned}$$

and

$$\int_0^\infty \frac{tdt}{(t + |x - z|)^{2n+2-2\delta}} = C|x - z|^{-2n+2\delta},$$

we obtain

$$\begin{aligned} g_\mu^A(f)(x) &\leq C \int_{R^n} \frac{|f(z)||R_{m+1}(A; x, z)|}{|x - z|^m} \left(\int_0^\infty \frac{tdt}{(t + |x - z|)^{2n+2-2\delta}} \right)^{1/2} dz \\ &= C \int_{R^n} \frac{|f(z)||R_{m+1}(A; x, z)|}{|x - z|^{m+n-\delta}} dz, \end{aligned}$$

thus, the lemma follows from [3].

3. Proof of theorems.

Proof of Theorem 1. Fix a cube $Q = Q(x_0, l)$ and $\tilde{x} \in Q$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A)_{\tilde{Q}} x^\alpha$, then $R_m(A; x, y) = R_m(\tilde{A}; x, y)$ and $D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A)_{\tilde{Q}}$ for $|\alpha| = m$. We write, for $f_1 = f\chi_{\tilde{Q}}$ and $f_2 = f\chi_{R^n \setminus \tilde{Q}}$,

$$\begin{aligned} F_t^A(f)(x, y) &= \int_{R^n} \frac{R_{m+1}(\tilde{A}; x, z)}{|x-z|^m} \psi_t(y-z) f(z) dz \\ &= \int_{R^n} \frac{R_{m+1}(\tilde{A}; x, z)}{|x-z|^m} \psi_t(y-z) f_2(z) dz + \int_{R^n} \frac{R_m(\tilde{A}; x, z)}{|x-z|^m} \psi_t(y-z) f_1(z) dz \\ &\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{(x-z)^\alpha \psi_t(y-z)}{|x-z|^m} D^\alpha \tilde{A}(z) f_1(z) dz, \end{aligned}$$

then

$$\begin{aligned} & \left| g_\mu^A(f)(x) - g_\mu^{\tilde{A}}(f_2)(x_0) \right| \\ &= \left\| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t^A(f)(x, y) \right\| - \left\| \left(\frac{t}{t+|x_0-y|} \right)^{n\mu/2} F_t^{\tilde{A}}(f_2)(x_0, y) \right\| \\ &\leq \left\| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t \left(\frac{R_m(\tilde{A}; x, \cdot)}{|x-\cdot|^m} f_1 \right) (y) \right\| \\ &\quad + \sum_{|\alpha|=m} \frac{1}{\alpha!} \left\| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t \left(\frac{(x-\cdot)^\alpha}{|x-\cdot|^m} D^\alpha \tilde{A} f_1 \right) (y) \right\| \\ &\quad + \left\| \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t^{\tilde{A}}(f_2)(x, y) - \left(\frac{t}{t+|x_0-y|} \right)^{n\mu/2} F_t^{\tilde{A}}(f_2)(x_0, y) \right\| \\ &:= A(x) + B(x) + C(x), \end{aligned}$$

thus,

$$\begin{aligned} & \frac{1}{|Q|^{1+\beta/n}} \int_Q \left| g_\mu^A(f)(x) - g_\mu^{\tilde{A}}(f_2)(x_0) \right| dx \\ &\leq \frac{1}{|Q|^{1+\beta/n}} \int_Q A(x) dx + \frac{1}{|Q|^{1+\beta/n}} \int_Q B(x) dx + \frac{1}{|Q|^{1+\beta/n}} \int_Q C(x) dx \\ &:= I + II + III. \end{aligned}$$

Now, let us estimate I , II and III , respectively. First, for $x \in Q$ and $z \in \tilde{Q}$, using Lemma 2 and Lemma 5, we get

$$\begin{aligned} |R_m(\tilde{A}; x, z)| &\leq C|x-z|^m \sum_{|\alpha|=m} \sup_{x \in \tilde{Q}} |D^\alpha A(x) - (D^\alpha A)_{\tilde{Q}}| \\ &\leq C|x-z|^m |Q|^{\beta/n} \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta}, \end{aligned}$$

thus, taking r, s such that $1 < r < p$ and $1/s = 1/r - \delta/n$, by the (L^r, L^s) -boundedness of g_μ (see Lemma 6) and the Hölder inequality, we obtain

$$\begin{aligned} I &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \frac{1}{|Q|} \int_Q |g_\mu(f_1)(x)| dx \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \|g_\mu(f_1)\|_{L^s} |Q|^{-1/s} \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \|f_1\|_{L^r} |Q|^{-1/s} \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \left(\frac{1}{|\tilde{Q}|^{1-r\delta/n}} \int_{\tilde{Q}} |f(y)|^r dy \right)^{1/r} \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} M_{\delta,r}(f)(\tilde{x}). \end{aligned}$$

Secondly, using the following inequality (see [17]):

$$\|(D^\alpha A - (D^\alpha A)_{\tilde{Q}}) f \chi_{\tilde{Q}}\|_{L^r} \leq C |Q|^{1/s+\beta/n} \|D^\alpha A\|_{\dot{\lambda}_\beta} M_{\delta,r}(f),$$

and similar to the proof of I , we gain

$$\begin{aligned} II &\leq \frac{C}{|Q|^{1+\beta/n}} \sum_{|\alpha|=m} \|g_\mu((D^\alpha A - (D^\alpha A)_{\tilde{Q}}) f_1)\|_{L^s} |Q|^{1-1/s} \\ &\leq C |Q|^{-\beta/n-1/r} \sum_{|\alpha|=m} \|(D^\alpha A - (D^\alpha A)_{\tilde{Q}}) f_1\|_{L^r} \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} M_{\delta,r}(f)(\tilde{x}). \end{aligned}$$

To estimate III , we write

$$\begin{aligned} &\left(\frac{t}{t+|x-y|} \right)^{n\mu/2} F_t^{\tilde{A}}(f_2)(x, y) - \left(\frac{t}{t+|x_0-y|} \right)^{n\mu/2} F_t^{\tilde{A}}(f_2)(x_0, y) \\ &= \int_{R^n} \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} \left[\frac{1}{|x-z|^m} - \frac{1}{|x_0-z|^m} \right] R_m(\tilde{A}; x, z) \psi_t(y-z) f_2(z) dz \\ &\quad + \int_{R^n} \left(\frac{t}{t+|x-y|} \right)^{n\mu/2} \frac{\psi_t(y-z) f_2(z)}{|x_0-z|^m} [R_m(\tilde{A}; x, z) - R_m(\tilde{A}; x_0, z)] dz \\ &\quad + \int_{R^n} \left[\left(\frac{t}{t+|x-y|} \right)^{n\mu/2} - \left(\frac{t}{t+|x_0-y|} \right)^{n\mu/2} \right] \frac{R_m(\tilde{A}; x_0, z) \psi_t(y-z) f_2(z)}{|x_0-z|^m} dz \\ &\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \left[\left(\frac{t}{t+|x-y|} \right)^{n\mu/2} \frac{(x-z)^\alpha}{|x-z|^m} - \left(\frac{t}{t+|x_0-y|} \right)^{n\mu/2} \frac{(x_0-z)^\alpha}{|x_0-z|^m} \right] \\ &\quad \times D^\alpha \tilde{A}(z) \psi_t(y-z) f_2(z) dz \\ &:= III_1 + III_2 + III_3 + III_4. \end{aligned}$$

Note that $|x - z| \sim |x_0 - z|$ for $x \in Q$ and $z \in R^n \setminus \tilde{Q}$. By the condition of ψ and similar to the proof of Lemma 6, we obtain

$$\begin{aligned} & \frac{1}{|Q|^{1+\beta/n}} \int_Q ||III_1|| dx \\ \leq & \frac{C}{|Q|^{1+\beta/n}} \int_Q \left(\int_{R^n \setminus \tilde{Q}} \frac{|x - x_0|}{|x_0 - z|^{m+n+1-\delta}} |R_m(\tilde{A}; x, z)| |f(z)| dz \right) dx \\ \leq & C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} \frac{|x - x_0|}{|x_0 - z|^{n+1-\delta}} |f(z)| dz \\ \leq & C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \sum_{k=1}^\infty 2^{-k} \frac{1}{|2^k\tilde{Q}|^{1-\delta/n}} \int_{2^k\tilde{Q}} |f(z)| dz \\ \leq & C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \sum_{k=1}^\infty 2^{-k} M_{\delta,1}(f)(\tilde{x}) \\ \leq & C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} M_{\delta,1}(f)(\tilde{x}). \end{aligned}$$

For III_2 , by the formula (see [6]):

$$R_m(\tilde{A}; x, z) - R_m(\tilde{A}; x_0, z) = \sum_{|\eta| < m} \frac{1}{\eta!} R_{m-|\eta|}(D^\eta \tilde{A}; x, x_0) (x - z)^\eta$$

and Lemma 5, we get

$$|R_m(\tilde{A}; x, z) - R_m(\tilde{A}; x_0, z)| \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} |Q|^{\beta/n} |x - x_0| |x_0 - z|^{m-1},$$

thus

$$\begin{aligned} & \frac{1}{|Q|^{1+\beta/n}} \int_Q ||III_2|| dx \\ \leq & C \frac{1}{|Q|^{1+\beta/n}} \int_Q \int_{R^n \setminus \tilde{Q}} \frac{|R_m(\tilde{A}; x, z) - R_m(\tilde{A}; x_0, z)|}{|x_0 - z|^{m+n-\delta}} |f(z)| dz dx \\ \leq & C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q} \setminus 2^k\tilde{Q}} \frac{|x - x_0|}{|x_0 - y|^{n+1-\delta}} |f(z)| dz \\ \leq & C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} |Q|^{\beta/n} M_{\delta,1}(f)(\tilde{x}). \end{aligned}$$

For III_3 , by the inequality: $a^{1/2} - b^{1/2} \leq (a - b)^{1/2}$ for $a \geq b > 0$, we gain, similar to the proof of Lemma 6,

$$\begin{aligned} & \frac{1}{|Q|^{1+\beta/n}} \int_Q ||III_3|| dx \\ & \leq \frac{C}{|Q|^{1+\beta/n}} \int_Q \int_{R^n} \\ & \quad \left(\int_{R_+^{n+1}} \left[\frac{t^{n\mu/2} |x - x_0|^{1/2} |\psi_t(y - z)| |R_m(\tilde{A}; x_0, z)| |f_2(z)|}{|x - z|^m (t + |x - y|)^{(n\mu+1)/2}} \right]^2 \frac{dy dt}{t^{n+1}} \right)^{1/2} dz dx \\ & \leq \frac{C}{|Q|^{1+\beta/n}} \int_Q \int_{R^n} \frac{|f_2(z)| |R_m(\tilde{A}; x_0, z)| |x - x_0|^{1/2}}{|x - z|^m} \\ & \quad \times \left(\int_0^\infty \frac{dt}{(t + |x - z|)^{2n+2-2\delta}} \right)^{1/2} dz dx \\ & \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \int_{R^n} \frac{|x - x_0|^{1/2}}{|x_0 - z|^{n+1/2-\delta}} |f_2(z)| dz \\ & \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \sum_{k=1}^\infty 2^{-k/2} M_{\delta,1}(f)(\tilde{x}) \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} M_{\delta,1}(f)(\tilde{x}). \end{aligned}$$

For III_4 , by Lemma 4, we get

$$|D^\alpha A(z) - (D^\alpha A)_{\tilde{Q}}| \leq \|D^\alpha A\|_{\dot{\lambda}_\beta} |x_0 - z|^\beta,$$

thus, similar to the proof of III_1 and III_3 , we obtain

$$\begin{aligned} & \frac{1}{|Q|^{1+\beta/n}} \int_Q ||III_4|| dx \\ & \leq \frac{C}{|Q|^{1+\beta/n}} \int_Q \sum_{|\alpha|=m} \int_{R^n} \left(\frac{|x - x_0|}{|x_0 - z|^{n+1-\delta}} + \frac{|x - x_0|^{1/2}}{|x_0 - z|^{n+1/2-\delta}} \right) |f_2(z)| |D^\alpha \tilde{A}(z)| dz dx \\ & \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \sum_{k=1}^\infty (2^{k(\beta-1)} + 2^{k(\beta-1/2)}) M_{\delta,1}(f)(\tilde{x}) \\ & \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} M_{\delta,1}(f)(\tilde{x}). \end{aligned}$$

Thus,

$$III \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} M_{\delta,1}(f)(\tilde{x}).$$

We now put these estimates together, and taking the supremum over all Q such that $\tilde{x} \in Q$, and using Lemma 1 with Lemma 3, we obtain

$$\|g_\mu^A(f)\|_{\dot{F}_q^{\beta,\infty}} \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \|f\|_{L^p}.$$

This completes the proof of Theorem 1.

Proof of Theorem 2. By the same argument as in the proof of Theorem 1, we have, for $1 \leq s < p$ and $1/r = 1/s - \delta/n$,

$$\begin{aligned} & \frac{1}{|Q|} \int_Q |g_\mu^A(f)(x) - g_\mu^{\tilde{A}}(f_2)(x_0)| dx \\ & \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} (M_{\beta+\delta,r}(f) + M_{\beta+\delta,1}(f)), \end{aligned}$$

thus, the sharp estimate of g_μ^A is obtained as following

$$(g_\mu^A(f))^\# \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} (M_{\beta+\delta,r}(f) + M_{\beta+\delta,1}(f)).$$

Now, using Lemma 3, we gain

$$\begin{aligned} \|g_\mu^A(f)\|_{L^q} & \leq C \|(g_\mu^A(f))^\#\|_{L^q} \\ & \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} (\|M_{\beta+\delta,r}(f)\|_{L^q} + \|M_{\beta+\delta,1}(f)\|_{L^q}) \leq C \|f\|_p. \end{aligned}$$

This completes the proof of Theorem 2.

Proof of Theorem 3. We first prove the following estimate:

$$|g_\mu^A(f)(x)| \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \left(\lambda_1^{\delta+\beta} Mf(x) + \lambda_1^{\delta+\beta-n/r'} \|f\|_{L^1}^{1/r'} (Mf(x))^{1/r} \right)$$

for any $\lambda_1 > 0$ and $n/(n - \delta - \beta) < r$. In fact, fix the cube $Q = Q(x, \lambda_1)$, similar to the proof of Lemma 6, we have

$$\begin{aligned} |g_\mu^A(f)(x)| & \leq C \int_{R^n} \frac{|f(z)| |R_{m+1}(A; x, z)|}{|x - z|^{m+n-\delta}} dz \\ & = C \left(\int_Q + \int_{Q^c} \right) \frac{|f(z)| |R_{m+1}(A; x, z)|}{|x - z|^{m+n-\delta}} dz = I_1 + I_2. \end{aligned}$$

For I_1 , we let that, for $k > 0$,

$$\tilde{A}_k(y) = A(y) - \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A)_{2^{-k}Q} y^\alpha,$$

then, by Lemma 5, for $z \in 2^{-k}Q$,

$$|R_{m+1}(\tilde{A}_k; x, z)| \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} (2^{-k} \lambda_1)^\beta |x - z|^m,$$

thus, by Lemma 4 and Lemma 5,

$$\begin{aligned}
 I_1 &\leq C \sum_{k=0}^{\infty} \int_{2^{-k}Q \setminus 2^{-k-1}Q} \frac{|f(z)| |R_{m+1}(\tilde{A}_k; x, z)|}{|x-z|^{m+n-\delta}} dz \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \sum_{k=0}^{\infty} (2^{-k} \lambda_1)^\beta \int_{2^{-k}Q \setminus 2^{-k-1}Q} \frac{|f(z)|}{|x-z|^{n-\delta}} dz \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \sum_{k=0}^{\infty} (2^{-k} \lambda_1)^{\beta+\delta-n} \int_{2^{-k}Q \setminus 2^{-k-1}Q} |f(z)| dz \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \lambda_1^{\beta+\delta} Mf(x);
 \end{aligned}$$

For I_2 , taking $\varepsilon > 0$ such that $(n + \varepsilon)/(n - \delta - \beta) < r$, we write $n - \delta = (n + \varepsilon)/r + n/r' - \varepsilon/r - \delta$, then, by the Hölder's inequality,

$$\begin{aligned}
 I_2 &\leq C \left(\int_{Q^c} \frac{|f(z)| dz}{|x-z|^{n+\varepsilon}} \right)^{1/r} \left[\int_{Q^c} \frac{|f(z)|}{|x-z|^{n-(\delta+\varepsilon/r)r'}} \left(\frac{|R_{m+1}(A; x, z)|}{|x-z|^m} \right)^{r'} dz \right]^{1/r'} \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \left(\sum_{k=0}^{\infty} (2^k \lambda_1)^{-\varepsilon-n} \int_{|x-z| < 2^k \lambda_1} |f(z)| dz \right)^{1/r} \\
 &\quad \times \left(\sum_{k=0}^{\infty} (2^k \lambda_1)^{\beta r'} \int_{2^{-k}Q \setminus 2^{-k-1}Q} \frac{|f(z)| dz}{|x-z|^{n-(\delta+\varepsilon/r)r'}} \right)^{1/r'} \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \left(\sum_{k=0}^{\infty} 2^{-k\varepsilon} \lambda_1^{-\varepsilon} Mf(x) \right)^{1/r} \lambda_1^{\delta+\beta-n/r'+\varepsilon/r} \\
 &\quad \times \left(\sum_{k=0}^{\infty} 2^{k(\delta+\beta-n/r'+\varepsilon/r)r'} \int_{2^{-k}Q \setminus 2^{-k-1}Q} |f(z)| dz \right)^{1/r'} \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \lambda_1^{-\varepsilon/r} (Mf(x))^{1/r} \lambda_1^{\delta+\beta-n/r'+\varepsilon/r} \|f\|_{L^1}^{1/r'} \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \lambda_1^{\delta+\beta-n/r'} \|f\|_{L^1}^{1/r'} (Mf(x))^{1/r}.
 \end{aligned}$$

Thus, our claim holds. Now we can prove Theorem 3. For any $\lambda > 0$ and $f \in L^1(R^n)$, taking $\lambda_1 = (\sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \|f\|_{L^1} \lambda^{-1})^{1/(n-\delta-\beta)}$ in above estimate,

we gain, by the weak type boundedness of M ,

$$\begin{aligned}
& |\{x \in R^n : g_\mu^A(f)(x) > \lambda\}| \\
& \leq \left| \left\{ x \in R^n : Mf(x) > \frac{\lambda}{2C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \lambda_1^{\delta+\beta}} \right\} \right| \\
& \quad + \left| \left\{ x \in R^n : Mf(x) > \left(\frac{\lambda}{2C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \lambda_1^{\delta+\beta-n/r'} \|f\|_{L^1}^{1/r}} \right)^r \right\} \right| \\
& \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \lambda_1^{\delta+\beta} \|f\|_{L^1} / \lambda + C \left(\sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \lambda_1^{\delta+\beta-n/r'} \|f\|_{L^1}^{1/r} / \lambda \right)^r \\
& \leq C \left(\sum_{|\alpha|=m} \|D^\alpha A\|_{\dot{\lambda}_\beta} \|f\|_{L^1} / \lambda \right)^{n/(n-\delta-\beta)}.
\end{aligned}$$

This completes the proof of Theorem 3.

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