

LOCAL TIME DECAY FOR A NONLINEAR BEAM EQUATION *

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Abstract. Using the Morawetz’ Radial Identity, we show that the local energy of a solution is integrable in time and the local L^2 norm of the solution approaches zero as time approaches the infinity for a nonlinear beam equation with the spatial dimension > 5 .

1. Introduction. Consider a nonlinear beam equation

$$u_{tt} + \Delta^2 u + f(u) = 0 \tag{1}$$

where $u = u(x, t)$, $x = (x_1, x_2, \dots, x_n) \in R^n$, R^n is the n -dimensional Euclidean space, $n > 5$, $t \geq 0$, $\Delta =$ Laplacian in x , and $f(u)$ satisfies

$$c_1(uf(u) - 2F(u)) + c_0u^2 \geq F(u) \geq c_0u^2 \tag{2}$$

for some positive constants c_0 and c_1 , where $F'(u) = f(u)$ with $F(0) = 0$. As usual, the subscript in variables denotes the partial derivative, thus, $u_t = \partial u / \partial t$, etc... We also use the notations $\partial_j = \partial / \partial x_j$, and $u_r = (x/r) \cdot \nabla u$, where ∇ is the gradient in x , and $r = |x|$. Moreover, for a function of one variable $g(s)$, $g'(s) = d(g(s))/ds$ denotes the derivative of g in s . Finally, that a function is C^n means that its n^{th} partial derivatives are continuous. In this work, we show that the local energy of a solution is integrable in time and the local L^2 norm of a solution approaches zero as time approaches the infinity. Our method follows [1] in utilizing the Morawetz’ Radial Identity [5].

The global scattering problem was considered in [1] along with several inequalities. It was conjectured in [1] that the local energy is integrable in t and tends to zero as t approaches the infinity. This work proves the first part of this conjecture. The well-posedness, low-energy scattering, stability and instability of solitary and standing waves, and the time decay of solutions for the nonlinear beam equation with a slightly different $f(u)$ can be found in [2, 3]. In the one-spatial dimension, a similar equation to (1) with a different nonlinear term has been studied as a model for a suspension bridge [4].

We shall need the following result from [1]:

The energy $E[u] = \int_{R^n} [(1/2)u_t^2 + (1/2)|\Delta u|^2 + F(u)]dx$ is a constant, and for $n > 5$, assuming that u is a solution that is smooth enough and small enough at the spatial infinity, then there is a positive constant c such that

$$\int_0^\infty \int_{R^n} (1/r)[uf(u) - 2F(u)]dxdt \leq cE[u] \tag{3}$$

$$\int_0^\infty \int_{R^n} (1/r^3)|\nabla u|^2 dxdt \leq cE[u] , \text{ and} \tag{4}$$

$$\int_0^\infty \int_{R^n} (1/r^5)|u|^2 dxdt \leq cE[u] , \text{ provided } n \geq 6 \tag{5}$$

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2. The Morawetz' radial identity. Let ζ be a C^4 radially symmetric function of x . Multiplying (1) by $\zeta(u_r + ((n-1)/(2r))u)$ we get the following identity, assuming that u is C^2 in t and C^4 in x ,

$$\begin{aligned} 0 &= (u_{tt} + \Delta^2 u + f(u))\zeta(u_r + (n-1)u/(2r)) \\ &= X_t + \nabla \cdot Y + Z \end{aligned} \quad (6)$$

where

$$X = u_t \zeta(u_r + (n-1)u/(2r)),$$

$$Y = -((n-1)u^2/4)(\nabla(\Delta(\zeta/r))) + \text{a function depending on } \zeta, \zeta', \zeta'', \zeta''', x, u, F(u), \nabla u, \Delta u, \nabla(\Delta u), u_r, \nabla u_r, u_i, 1/r, 1/r^2 \text{ and } 1/r^3,$$

and

$$\begin{aligned} Z &= (\zeta'/2)(u_t)^2 + (\Delta u)^2[3\zeta'/2] + A(u_r)^2 + B(|\nabla u|^2 - (u_r)^2) + Cu^2 + (n-1)\zeta(uf(u) \\ &\quad - 2F(u))/(2r) - \zeta'F(u) + (\zeta - r\zeta')P, \end{aligned}$$

where

$$A = -7\zeta'''/2 - \zeta'(n-1)(n-3)/(2r^2) + \zeta(n-1)(n-3)/(2r^3),$$

$$B = -3\zeta'''/2 + \zeta''(n-5)/r - \zeta'(n^2 + 2n - 19)/(2r^2) + \zeta(n^2 + 2n - 19)/(2r^3),$$

$$C = ((n-1)/2)[\zeta'''/(2r) + \zeta'''(n-3)/(r^2) + \zeta''(n-3)(n-7)/(2r^3) - 3\zeta'(n-3)(n-5)/(2r^4) + 3\zeta(n-3)(n-5)/(2r^5)],$$

$$P = (2/r)[\sum_{i,j}(S_{ij}u)^2 - \sum_i(\sum_j(x_j/r)S_{ij}u)^2] \geq 0$$

$$\text{with } S_{ij}u = ((x_i)/r^3) \sum_k [x_k(x_k \partial_j - x_j \partial_k)u_r] + \partial_j \sum_k [(x_k/r^2)(x_k \partial_i - x_i \partial_k)u].$$

REMARK. If $\zeta = 1$, this identity is the identity shown in the proof of Theorem 1 of [1].

3. The integrability of the local energy. We now state the main result of this work.

THEOREM. Consider the nonlinear beam equation (1) with the condition (2) on $f(u)$. Assume that the spatial dimension n is > 5 . Assume also that u is C^2 in t and C^4 in x , u_t is bounded, and u and all its partial derivatives in x up to the 4th order approach zero as $|x|$ approaches the infinity. Then the local energy is integrable in t .

Proof. Assume that ζ and ζ' are non-negative functions and $\zeta, \zeta', \zeta'',$ and ζ''' are bounded functions. We integrate both sides of (6) with respect to x in R^n and t from 0 to T . With the assumption on the smoothness and smallness of u at the spatial infinity, we have $\int_{R^n} \nabla \cdot Y dx = 0$ for $n > 5$.

Thus

$$0 \geq \int_{R^n} X(x, T) dx - \int_{R^n} X(x, 0) dx + \int_0^T \int_{R^n} Z dx dt.$$

Now we can rewrite X into two ways:

$$\begin{aligned} X &= -(\zeta/2)(u_t^2 + |\nabla u|^2) + (\zeta/2)[u_t^2 + |W|^2 + 2((x/r) \cdot W)u_t] \\ &\quad - \nabla \cdot [(n-1)\zeta u^2 x/(4r^2)] + (n-1)(n-3)\zeta u^2/(8r^2) + (n-1)\zeta' u^2/(4r), \end{aligned} \quad (7)$$

and

$$\begin{aligned} X = & (\zeta/2)(u_t^2 + |\nabla u|^2) - (\zeta/2)[u_t^2 + |W|^2 - 2((x/r) \cdot W)u_t] \\ & + \nabla \cdot [(n-1)\zeta u^2 x/(4r^2)] - (n-1)(n-3)\zeta u^2/(8r^2) - (n-1)\zeta' u^2/(4r), \end{aligned} \quad (8)$$

where $W = \nabla u + (n-1)ux/(2r^2)$.

Using the first way (7) for $X(x, T)$ and the second way (8) for $X(x, 0)$, we get

$$\int_{R^n} (1/2)\zeta(u_t^2 + |\nabla u|^2)(x, 0)dx + \int_{R^n} (1/2)\zeta(u_t^2 + |\nabla u|^2)(x, T)dx \geq \int_0^T \int_{R^n} Z dx dt.$$

Let T approach the infinity, we get

$$\int_0^\infty \int_{R^n} Z dx dt \leq c_2 E[u], \text{ for some positive constant } c_2, \quad (9)$$

where c_2 depends on c_0, c_1 and the bound for ζ .

Now, let $\zeta(x) = \zeta(r) = 1 - 1/(r^2 + 4)^2$, where $r = |x|$. Since ζ and ζ' are non-negative and $\zeta, \zeta', \zeta'',$ and ζ''' are bounded functions, the inequality (9) holds. Substituting ζ into (9), we get

$$\begin{aligned} & \int_0^\infty \int_{R^n} [(u_t^2 + (\Delta u)^2)r/(r^2 + 4)^3 + |\nabla u|^2/(r^3(r^2 + 4)^5) + P/(r^2 + 4)^3 \\ & \quad + u^2/(r^5(r^2 + 4)^6) + (uf(u) - 2F(u))/(r(r^2 + 4)^2)] dx dt \\ & \leq \int_0^\infty \int_{R^n} 4F(u)r/(r^2 + 4)^3 dx dt + c_2 E[u] \\ & \leq \int_0^\infty \int_{R^n} [4c_1(uf(u) - 2F(u)) + 4c_0 u^2]r/(r^2 + 4)^3 dx dt + c_2 E[u] \\ & \leq \int_0^\infty \int_{R^n} [4c_1(uf(u) - 2F(u))/r + 4c_0 u^2/r^5] dx dt + c_2 E[u] \\ & \leq c_3 E[u] \end{aligned}$$

for some positive constant c_3 . Note that we have used (3) and (5) in the above inequality.

Let $h > b > 0$, we get

$$\int_0^\infty \int_{b \leq |x| \leq h} [u_t^2 + (\Delta u)^2 + |\nabla u|^2 + u^2 + (uf(u) - 2F(u))] dx dt \leq c_4 E[u]$$

for some positive constant c_4 depending on b and h .

Therefore, $\int_0^\infty \int_{b \leq |x| \leq h} [u_t^2 + (\Delta u)^2 + F(u)] dx dt \leq c_5 E[u]$, for some positive constant c_5 depending on c_0, c_1, b and h . Since the equation (1) is invariant under spatial translation, we get

$$\int_0^\infty \int_{|x| \leq h} [u_t^2 + (\Delta u)^2 + F(u)] dx dt \leq c_6 E[u], \text{ for some positive constant } c_6.$$

Therefore the local energy is integrable in time.

4. Time decay of the local L^2 norm. Now, we are going to show that the local L^2 norm of u approaches 0 as t approaches the infinity. The idea of the proof is from [5]. Let $h > 0$ and $t > t_1 > 0$, then

$$\begin{aligned} (t - t_1) \int_{|x| \leq h} u^2(x, t) dx &= \int_{t_1}^t \partial[(\tau - t_1) \int_{|x| \leq h} u^2(x, \tau) dx] / \partial \tau d\tau \\ &= \int_{t_1}^t \int_{|x| \leq h} u^2(x, \tau) dx d\tau + \int_{t_1}^t (\tau - t_1) \int_{|x| \leq h} 2uu_t(x, \tau) dx d\tau. \end{aligned}$$

Let $t_1 = t - 1$, we get

$$\int_{|x| \leq h} u^2(x, t) dx \leq \int_{t-1}^t \int_{|x| \leq h} u^2(x, \tau) dx d\tau + \int_{t-1}^t \int_{|x| \leq h} (u^2 + u_t^2)(x, \tau) dx d\tau.$$

Hence $\int_{|x| \leq h} u^2(x, t) dx$ approaches 0 as t approaches the infinity since the local energy is integrable in time.

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