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MONOTONE MAPS OF $\mathbb{R}^n$ ARE QUASICONFORMAL*

KARI ASTALA†, TADEUSZ IWANIEC‡, AND GAVEN J. MARTIN§

For Neil Trudinger

Abstract. We give a new and completely elementary proof showing that a δ–monotone mapping of $\mathbb{R}^n$, $n \geq 2$ is K–quasiconformal with linear distortion

$$K \leq \frac{1+\sqrt{1-\delta^2}}{1-\sqrt{1-\delta^2}}$$

This sharpens a result due to L. Kovalev.

Key words. Monotone mapping, quasiconformal.

AMS subject classifications. 30C60

1. Introduction. In [5] L.V. Kovalev proved the interesting fact that a δ–monotone mapping of $\mathbb{R}^n$ is K–quasiconformal for some distortion constant K depending only on δ. Here we give a new poof of this result using methods which are rather more elementary than those employed in [5], going through a compactness argument which is more or less standard in the theory of quasiconformal mappings. We are also able to give the precise estimates relating the monotonicity constant δ and the distortion constant K (these precise estimates were already given in two dimensions in our earlier work [2].) We remark that the proof given here works without modification for monotone mappings of Hilbert spaces.

Let us recall the relevant definitions. A function $h : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is called *δ–monotone*, $0 < \delta \leq 1$ if for every $z, w \in \Omega$

$$\langle h(z) - h(w), z - w \rangle \geq \delta |h(z) - h(w)||z - w| \tag{1}$$

There is no supposition of continuity here. It is obvious from the definition at (1) that the family of δ–monotone maps is invariant under rescaling and translation. Of course $\langle h(z) - h(w), z - w \rangle = |h(z) - h(w)||z - w| \cos(\alpha)$ where α is the angle between these vectors. Thus δ–monotone maps are prevented from rotating the vector formed from a pair of points more than an angle $|\arccos(\delta)| < \pi/2$. Monotone mappings have found wide application in partial differential equations for decades, particularly those second order PDEs of divergence type, because of the well known Minty-Browder theory [6, 3]. Roughly the monotonicity condition is used to bound a nonlinear operator away from a curl. See the monograph [1] for some of this theory and connections with quasiconformal mappings and second order nonlinear divergence equations in the plane. This brings us to our next definition. An orientation preserving injection

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†Department of Mathematics and Statistics, University of Helsinki, FI-00014, Finland (kari.astala@helsinki.fi).
‡Department of Mathematics, Syracuse University, New York, NY 13244-1150, USA (tiwaniec@syr.edu).
§Institute of Information and Mathematical Sciences, Massey University, Private Bag 102-904, Auckland, New Zealand (g.j.martin@massey.ac.nz).

$f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is called K–*quasiconformal* if there is $H < \infty$ so that for each $x \in \mathbb{R}^n$ the infinitesimal linear distortion

$$\limsup_{r\to 0} \frac{\max_{|\zeta|=r} |f(x+\zeta) - f(x)|}{\min_{|\zeta|=r} |f(x+\zeta) - f(x)|} \le H \tag{2}$$

The maximal linear distortion K is the essential supremum of the quantity of the left-hand side of (2). Condition (2) guarantees the map has $W^{1,n}_{loc}(\Omega)$ regularity among many other things [4].

2. The main result. Here then is the theorem we want to prove.

THEOREM 1. *Let $h : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ be δ–monotone. Then either h is constant or else a quasiconformal homeomorphism with linear distortion bounded by*

$$K = \frac{1 + \sqrt{1-\delta^2}}{1 - \sqrt{1-\delta^2}}$$

This bound on the linear distortion is sharp for every $\delta \in (0, 1]$.

Proof. Let us begin by exhibiting sharpness. It suffices to consider monotone maps of the complex plane $\mathbb{C}$. Higher dimensional examples follow by the obvious extension. Considering an arbitrary linear map $h(z) = \alpha z + \beta \bar{z}$ of the complex plane $\mathbb{C}$ we need an estimate of the monotonicity of this map. As monotonicity is invariant under adding a constant we need an estimate at 0, where the condition $\langle h(z), z\rangle \ge \delta |h(z)||z|$ can be written as

$$\Re e[(\alpha z + \beta \bar{z})\bar{z}] \ge \delta |\alpha z + \beta \bar{z}|, \quad |z| = 1$$

Assuming $\beta \ne 0$, we ask that $\Re e(\frac{\alpha}{|\beta|} + \lambda) \ge \delta \big|\frac{\alpha}{|\beta|} + \lambda\big|$ for every $|\lambda| = 1$, or that the disk with center $\alpha/|\beta|$ and radius 1 is contained in the cone

$$C(\delta) = \{z = x + iy : \ \delta|y| \le \sqrt{1-\delta^2}\, x\ \}$$

The set of such possible center points forms another cone, with same opening and direction as $C(\delta)$ but with vertex $z_0 = \frac{1}{\sqrt{1-\delta^2}}$. Hence the requirement of δ-monotonicity takes the form

$$\delta \left|\Im m(\alpha)\right| = \delta \left|\Im m\big(\alpha - \frac{|\beta|}{\sqrt{1-\delta^2}}\big)\right| \le \sqrt{1-\delta^2}\, \Re e\big(\alpha - \frac{|\beta|}{\sqrt{1-\delta^2}}\big)$$

Multiplying and reorganizing we have that the linear map $h(z) = \alpha z + \beta \bar{z}$ is δ-monotone if and only if

$$|\beta| + \delta\, |\Im m\ \alpha| \le \sqrt{1-\delta^2}\, \Re e\ \alpha \tag{3}$$

As a particular consequence, under δ-monotonicity we have $|\beta| \le \sqrt{1-\delta^2}\, |\alpha|$, so that the linear distortion of h,

$$K(h) = \frac{|\alpha| + |\beta|}{|\alpha| - |\beta|} \le \frac{1 + \sqrt{1-\delta^2}}{1 - \sqrt{1-\delta^2}} \tag{4}$$

The equality occurs for the δ-monotone mapping $h(z) = z + k\bar{z}$, where $k = \sqrt{1-\delta^2} \in [0, 1)$. Thus the result, if true, is sharp.

To study the general δ-monotone mappings we adopt the following notation. The cone

$$C_w^\delta(z) = \left\{\zeta \in \Omega : \left|\frac{\zeta - w}{|\zeta - w|} - \frac{z-w}{|z-w|}\right| < \frac{\delta}{2}\right\} \tag{5}$$

has w as its vertex and opens up in the direction $z - w$. It is the union of all rays starting at w and making an angle less than $2\arcsin(\delta/4)$ with the ray in direction $z - w$.

By definition, if h is δ–monotone we see that if $\zeta \in C_w^\delta(z)$ where $z, w \in \Omega$, then

$$|h(\zeta) - h(w)| \leq \frac{2}{\delta}\,\langle h(\zeta) - h(w), \frac{z-w}{|z-w|}\rangle \tag{6}$$

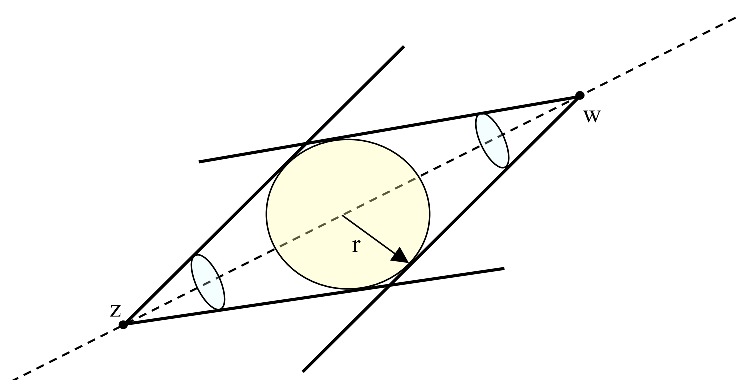
This is because

$$\begin{aligned}
&\frac{|\zeta - w|}{|z-w|}\langle h(\zeta) - h(w), z - w\rangle \\
&= \langle h(\zeta) - h(w), \zeta - w\rangle - \langle h(\zeta) - h(w), \zeta - w - \frac{|\zeta - w|}{|z-w|}(z-w)\rangle \\
&\geq (\delta - \frac{\delta}{2})|h(\zeta) - h(w)||\zeta - w|
\end{aligned}$$

and rearranging the non-zero terms gives (6). From this we deduce the following estimate for h simply by adding the relevant estimates obtained by swapping z and w.

Lemma 1. (Kovalev [5]) *If h is δ–monotone and $\zeta \in C_z^\delta(w) \cap C_w^\delta(z) =: Q_{z,w}^\delta$, then*

$$|h(\zeta) - h(w)| + |h(\zeta) - h(z)| \leq \frac{2}{\delta}\,\langle h(z) - h(w), \frac{z-w}{|z-w|}\rangle \leq \frac{2}{\delta}\,|h(z) - h(w)| \tag{7}$$



Intersection of cones $C_z^\delta(w)$ and $C_w^\delta(z)$

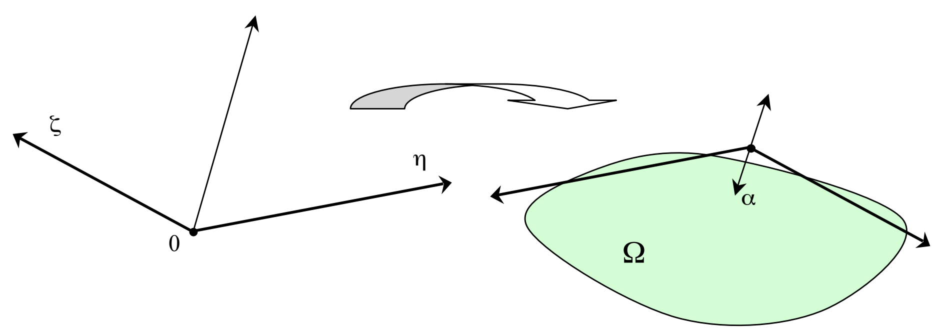
The following lemma is obvious.

Lemma 2. *Let $r \leq \frac{1}{5}\delta\,|z-w|$. Then the intersection $Q_{z,w}^\delta$ of the cones contains the ball $B\left(\frac{1}{2}(z+w), r\right) \cap \Omega$.*

The following easy lemma concerning convex sets will be useful.

LEMMA 3. *Let $L = \{t\zeta : t \geq 0\} \cup \{t\eta : t > 0\}$ with directions $\zeta, \eta \in \mathbb{S}^{n-1}$ not equal or antipodal. Let Ω be a proper convex subset of $\mathbb{R}^n$. Then there is a euclidean motion ψ of $\mathbb{R}^n$ so that $\psi(0) \notin \Omega$ yet for some s, t we have both $\psi(t\zeta), \psi(s\eta) \in \Omega$.*

Proof. It suffices to consider the two dimensional case. Then find a point $x \in \partial\Omega$ with a uniquely defined support line and inward normal α. Rotate and translate so that x is the image of 0 while the image of $\zeta + \eta$ is parallel to α. Now move the image of 0 in the direction $-\alpha$, away from Ω. For a sufficiently small move, the image of L will have the desired properties. □



REMARK. Lemma 3 is local in the sense that if Ω is a relatively convex proper subset of a domain D (the intersection of a convex subset of $\mathbb{R}^n$ with D), then we may find ψ so that $\psi(0) \notin \Omega$ yet $\psi(0) \in D$ and for some s, t we have both $\psi(t\zeta), \psi(s\eta) \in \Omega$.

2.1. Weak quasisymmetry. A mapping $h : \Omega \to \mathbb{R}^n$ is *weakly quasisymmetric* if there is a constant $H < \infty$ such that for all $z_1, z_2, w \in \Omega$,

$$|z_1 - w| \leq |z_2 - w| \quad \text{implies} \quad |h(z_1) - h(w)| \leq H|h(z_2) - h(w)| \tag{8}$$

Note this *a'priori* does not require f to be continuous. However, that readily follows.

LEMMA 4. (Tukia-Väisälä [7]) *Let h be a weakly quasisymmetric function in a domain $\Omega \subset \mathbb{R}^n$. Then f is either a homeomorphism or a constant.*

Proof. If h is not constant, to see the mapping is a homeomorphism onto its image, by (8) it is enough to establish continuity. Suppose that h is not continuous at $z_0 \in \Omega$. Then there is a sequence of points $z_j \to z_0$ such that for some $\epsilon > 0$ we have $|h(z_j) - h(z_0)| \geq \epsilon$. Passing to a subsequence we may assume that $|z_{j+1} - z_0| < \frac{1}{2}|z_j - z_0|$. This in turn implies

$$|z_{j+1} - z_0| \leq |z_{j+1} - z_j| \tag{9}$$

Now weak quasisymmetry implies the image sequence is bounded, $|h(z_j) - h(z_0)| \leq H|h(z_1) - h(z_0)|$ for all j. We may again pass to a subsequence so as to be able to assume that $h(z_j) \to a \in \mathbb{R}^n$, $a \neq h(z_0)$. But now of course we have from (9)

$$|h(z_{j+1}) - h(z_0)| \leq H\, |h(z_{j+1}) - h(z_j)|$$

which is a clear contradiction as the left hand side is bounded below by ϵ and the right hand side is tending to 0. Thus h is continuous, and hence a homeomorphism. □

2.2. Compactness. We begin with the following lemma.

Lemma 5. *Let $\Omega \subset \mathbb{R}^n$ be a domain containing the closed ball $\overline{B}(0, \frac{3}{\delta})$ and let $\alpha \in \mathbb{S}^{n-1}$. Define*

$$\mathcal{F}_\alpha = \{h : \Omega \to \mathbb{R}^n : h \text{ is } \delta\text{–monotone, } h(0) = 0 \text{ and } |h(\alpha)| = 1\}$$

Then there is $H = H(\delta) < \infty$ such that for all $|z| \leq 1$

$$\sup_{h \in \mathcal{F}_\alpha} |h(z)| < H \tag{10}$$

Proof. Let $X = \{z \in \Omega : \sup_{h\in\mathcal{F}_\alpha} |h(z)| < \infty\}$. Then X is nonempty, $\{0, \alpha\} \subset X$ and relatively convex by (7) with the choice $\zeta \in [z, w] \cap \Omega$, given $z, w \in X$. Suppose $X \neq \Omega$. Using Lemma 3 and the subsequent remark, we can find $z_0 \in \Omega \setminus X$ and two points $u, v \in X$ such that the angle $\angle(u, z_0, v)$ is as close to π as we like. As $u, v \in X$, $R = \sup_{h\in\mathcal{F}_\alpha} |h(u)| + h(v)| < \infty$. But $z_0 \notin X$ implies there are δ–monotone maps $h_j \in \mathcal{F}_\alpha$ with $|h_j(z_0)| \to \infty$. But then $\angle(h_j(u), h_j(z_0), h_j(v)) \to 0$ as the first and last points here are in the ball $B(0, R)$. Thus one of $u - z_0$ or $v - z_0$ is eventually rotated by the mappings by an angle greater than $\pi/2 - \varepsilon$, for $\varepsilon > 0$ as small as we like. This contradicts δ–monotonicity. Thus $X = \Omega$. We need uniformity in this estimate. By hypothesis $\pm w = (\pm\frac{3}{\delta}, 0 \ldots, 0) \subset \Omega$. Let $M = \sup_{h\in\mathcal{F}_\alpha} |h(w)| + |h(-w)| < \infty$. Then Lemma 2 gives $B(0, 1) \subset C^\delta_{-w}(w) \cap C^\delta_w(-w)$. Hence we can apply (7) to see that for all $z \in B(0, 1)$ we have $|h(w) - h(z)| + |h(-w) - h(z)| \leq \frac{2}{\delta} M$ whereupon

$$|h(z)| \leq \left(\frac{1}{2} + \frac{1}{\delta}\right) M = H$$

Finally to see that H does not depend on α it obviously suffices to make the following observation: if h is δ–monotone and O is an orthogonal rotation, then $O^t hO$ is δ–monotone,

$$\begin{aligned} \langle O^t hO(z) - O^t hO(w), z - w\rangle &= \langle hO(z) - hO(w), Oz - Ow\rangle \\ &\geq \delta |hO(z) - hO(w)||Oz - Ow| \\ &\geq \delta |O^t hO(z) - O^t hO(w)||z - w| \end{aligned}$$

This completes the proof of the lemma. □

2.3. Quasiconformality. We first establish quasiconformality without good estimates.

Lemma 6. *Let $h : \Omega \to \mathbb{R}^n$ be a non constant δ–monotone mapping in a domain $\Omega \subset \mathbb{R}^n$. Then h is a continuous injection whose linear distortion is bounded by $H = H(\delta)$ of Lemma 5.*

Proof. If h is not injective, $h(x) = h(y)$ for two distinct points $x, y \in \Omega$ and, arguing as in the proof of Lemma 5, we see from (7) that $X = \{z \in \Omega : h(z) = h(x)\}$ is relatively convex in Ω while using Lemma 3 we obtain $X = \Omega$. Thus h is constant.

Therefore we only need to establish the bound on the linear distortion. Let $z_0 \in \Omega$ with $\overline{B}(z_0, d) \subset \Omega$ and $r < \delta d/3$. Choose $\eta \in \mathbb{S}^{n-1}$, such that

$$\min_{|\zeta|=r} |h(z_0 + \zeta) - h(z_0)| = |h(z_0 + r\eta) - h(z_0)|$$

Then define

$$g(z) = \frac{h(z_0 + rz) - h(z_0)}{|h(z_0 + r\eta) - h(z_0)|}$$

and note that g is a δ–monotone mapping, $g(0) = 0$, $|g(\eta)| = 1$ and g is defined on a domain containing $\overline{B}(0, \frac{3}{\delta})$. Hence

$$\frac{\max_{|\zeta|=r} |h(z_0 + \zeta) - h(z_0)|}{\min_{|\zeta|=r} |h(z_0 + \zeta) - h(z_0)|} = \max_{|\xi|=1} |g(\xi)| < H$$

We see that h is weakly quasisymmetric in $B(z_0, \delta d/9)$, hence continuous, with linear distortion bounded by H. Thus Lemma 6 is completed. □

Finally, to get the sharp bound on the linear distortion we note that as a quasiconformal map any δ–monotone function is in $W^{1,n}_{loc}(\mathbb{R}^n)$ and admits a non-degenerate (invertible) derivative almost everywhere. Given $z \in \mathbb{R}^n$ we set

$$dh[z_0](z) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \ (h(z_0 + \epsilon z) - h(z_0))$$

Using the continuity of the inner product we see that $z \mapsto dh[z_0](z)$ is a δ–monotone linear map. Furthermore, the linear distortion of h is the essential supremum of the linear distortions of the maps $dh[z_0]$, $z_0 \in \Omega$. Thus it is enough to consider the linear mappings $dh[z_0]$. We restrict this to the two plane Π spanned by the directions in which the minimal and maximal stretchings occur. Let $P : \mathbb{R}^n \to \Pi$ be the projection into this plane. It is easy to see that

$$P \circ dh(x_0)|\Pi : \Pi \to \Pi$$

is δ–monotone as a map of Π (identified as $\mathbb{R}^2$) to itself - the angle between a vector and its image is only decreased under projection. The result then follows as per our very first calculation at (4). For further details and interesting connections see [2].

This finally completes the proof of Theorem 1. □

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