Statistics and Its Interface Volume 2 (2009) 101–113

Fisher information in ordered data: A review

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Fisher information is a fundamental concept of statistical inference and plays an important role in many areas of statistical analysis. Research in Fisher information in order statistics started around 1965 by John Tukey. Recently, the research in this area has been extended from classical order statistics and Type-I and Type-II censored data to other situations, including hybrid censoring, random censoring, progressive censoring, record values, and concomitants, with new applications in genetic linkage analysis. In this article, we provide a comprehensive review of various developments concerning the theory and applications of Fisher information in ordered data, show how Fisher information provides insight into properties of many statistical procedures involving order statistics.

AMS 2000 subject classifications: Primary 62F07, 62F12, 62N05; secondary 62-02.

Keywords and phrases: Asymptotics, censored data, characterizations, hazard functions, life testing, order statistics, ranked set sampling.

1. INTRODUCTION

Fisher information is a fundamental concept in statistical theory and applications. In classical inference with a random sample, for example, Fisher information appears in the Cramer-Rao lower bound for unbiased estimators [85]. In many situations with a non iid sample, Fisher information also plays an important role. For example, consider a linear estimate of location using several sample percentiles [66, 25]. The optimal percentiles that minimize the variance of the linear unbiased estimate are equivalent to ones that contain the maximal Fisher information about the location parameter. In genetic linkage analysis using sib pairs [52], statistics based on extreme discordant sib pairs are more powerful than those based on a random sample of sib pairs. The Fisher information now provides insight as the extreme discordant sib pairs contain more Fisher information about the linkage parameter, the recombination fraction, than random samples with equal sample size.

Calculation of Fisher information contained in a random sample is simple [85]. We focus on calculation of Fisher information in many situations involving ordered data. Order statistics of a random sample $(X_1, \ldots, X_n)$ from a parametric distribution $F_\theta(x)$ are denoted as $X_{1:n} \le \cdots \le X_{n:n}$, where θ is a vector of parameters. Although the definition of Fisher information in these nonstandard situations is simple, the calculation of Fisher information is not trivial, as order statistics are correlated and not identically distributed. The research of Fisher information in order statistics started with Tukey [94]. He examined the sensitivity of asymptotic efficiency in inference using consecutive order statistics $(X_{1:n}, \ldots, X_{k:n})$ when another order statistic $X_{k+1:n}$ was added or $X_{k:n}$ was deleted. The sensitivity that he derived is related to the asymptotic Fisher information in order statistics, which has important applications in asymptotic inference using L-statistics, spacings, and Type-II censored data [26, 6, 70, 30]. Mehrotra et al. [68] simplified the calculation of exact Fisher information in Type-II censored data $(X_{1:n}, \ldots, X_{k:n})$. Park [74] derived an alternative approach to simplify the calculation of Fisher information in Type-II censored data by using the Markovian property of order statistics, which can be applied to obtain Fisher information under progressive censoring [106]. The computation of Fisher information in censored and truncated data was studied by Escobar and Meeker [35] with applications to life testing. Iyengar et al. [60] studied Fisher information in a single order statistic and examined when it contains more Fisher information than a single random sample. Zheng and Gastwirth [101] generalized the results of Mehrotra et al. [68] to any collection of order statistics. The connections between the exact and asymptotic Fisher information in order statistics were obtained by Takahashi and Sugiura [93] and Zheng [97]. Asymptotic Fisher information and the recurrence relations of Fisher information were further studied by Park [75, 76]. Recently, the research in Fisher information in order statistics has been extended to other situations, including records and concomitants. Fisher information provides insight into properties of many statistical procedures in these areas and characterizes properties of the underlying distributions.

We review Fisher information in ordered statistics from a known parametric distribution with unknown parameters and illustrate how the Fisher information in ordered data provides insight into properties of genetic linkage analysis using selected sib pairs. The exact Fisher information in order statistics, ranked set samples, hybrid censored data,

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†Prepared when he was a Eugene Lukacs Visiting Professor, Department of Mathematics and Statistics, Bowling Green State University.

‡Partially supported by Yonsei University Research Fund of 2007.

randomly censored data, and other ordered data is discussed in Section 2. Asymptotic Fisher information and its relation with the exact Fisher information and applications are given in Section 3. Section 4 summarizes the characterizations of distributions using Fisher information in order statistics. Applications are given in Section 5. We finish in Section 6 with discussion and future research in this area.

2. EXACT FISHER INFORMATION IN ORDERED DATA

2.1 Definition

Let $\mathbf{X_m} = (X_{k_1:n}, X_{k_2:n}, \ldots, X_{k_m:n})$, $1 \le k_1 < \cdots < k_m \le n$, be a collection of m order statistics from a random sample of size n from a population with cumulative distribution function (cdf) $F_\theta(x)$ and continuous density function $f_\theta(x)$, where θ is a vector of parameters. The likelihood function of $\mathbf{X_m}$ is proportional to [30]

$$(1)\quad f_{k_1\cdots k_m:n}(x_{k_1}, \ldots, x_{k_m}) \propto \prod_{i=1}^{m} f_\theta(x_{k_i}) \prod_{i=1}^{m+1} \{F_\theta(x_{k_i}) - F_\theta(x_{k_{i-1}})\}^{k_i - k_{i-1} - 1},$$

with $-\infty = x_{k_0} < x_{k_1} < \cdots < x_{k_m} < x_{k_{m+1}} = \infty$. Its cdf is denoted as $F_{k_1\cdots k_m:n}$. Under the same regularity conditions of Abo-Eleneen and Nagaraja [2], the Fisher information (FIM) about the parameter θ contained in $\mathbf{X_m}$ is given by

$$(2)\quad I_{k_1k_2\cdots k_m:n}(\theta) = \int \cdots \int \Big(\frac{\partial}{\partial\theta} \log f_{k_1k_2\cdots k_m:n}\Big)^T \Big(\frac{\partial}{\partial\theta} \log f_{k_1k_2\cdots k_m:n}\Big) dF_{k_1\cdots k_m:n}.$$

To distinguish (2) from the asymptotic FIM discussed later, we refer to (2) as the exact FIM. In the following we focus on a scalar parameter θ. The results for the FIM matrix are readily obtained.

2.2 Exact Fisher information in order statistics

Directly calculating $I_{k_1k_2\cdots k_m:n}(\theta)$ from (2) is tedious and involves calculation of multiple integrals. Our approaches for calculating $I_{k_1k_2\cdots k_m:n}(\theta)$ depend on FIM decompositions and have three major advantages. Firstly, they simplify and unify the calculation that can be applied to other ordered data such as randomly censored data, concomitants and progressive censoring. Secondly, they provide new properties and applications of FIM in order statistics. Thirdly, they link the exact and asymptotic FIM which lets us define the asymptotic FIM from $I_{k_1k_2\cdots k_m:n}(\theta)$ rather than from studying asymptotic variances.

The first approach is to decompose $I_{k_1k_2\cdots k_m:n}(\theta)$ as a linear combination of $\tau_{ij} = E\{\phi_\theta(X_{i:n})\phi_\theta(X_{j:n})\}$, where $\phi_\theta(x) = \partial \log f_\theta(x)/\partial\theta$ [68, 101]. We can show that $I_{k_1k_2\cdots k_m:n}(\theta)$ involves the moments of the following three *extended* hazard functions [68]

$$h_1(x_i) = -\frac{F'_\theta(x_i)}{1 - F_\theta(x_i)},$$
$$h_2(x_j) = \frac{F'_\theta(x_j)}{F_\theta(x_j)},$$
$$h_3(x_i, x_j) = \frac{F'_\theta(x_j) - F'_\theta(x_i)}{F_\theta(x_j) - F_\theta(x_i)},$$

and that these moments can be expressed as a linear combination of τ_{ij} where $F'_\theta = \partial F/\partial\theta$. For example,

$$(n-s)E\{h_1^2(X_{s:n})\} = \frac{2}{n-s-1} \sum_{i=s+1}^{n-1} \sum_{j=i+1}^{n} \tau_{ij}.$$

A simple example is FIM in all n order statistics, $(X_{1:n}, \ldots, X_{n:n})$,

$$(3)\quad I_{1\cdots n:n}(\theta) = \sum_{i=1}^{n}\sum_{j=1}^{n} \tau_{ij} = \sum_{i=1}^{n} \tau_{ii} = n\int_{-\infty}^{\infty} \Big\{\frac{\partial}{\partial\theta} \log f_\theta(x)\Big\}^2 dF_\theta(x).$$

For the exponential family of distributions $f_\theta(x) = \alpha(x)\exp(\theta T(x) - C(\theta))$, $\phi_\theta(x) = T(x) - C'(\theta)$ and $\tau_{ij} = E\{T(X_{i:n})T(X_{j:n})\} - C'(\theta)E\{T(X_{i:n}) + T(X_{j:n})\} + \{C'(\theta)\}^2$.

In general, a matrix approach is useful for this FIM decomposition. Define an $n\times n$ symmetric matrix $\tau = (\tau_{ij})_{n\times n}$, where τ_{ij} is given before. From (3), $I_{1\cdots n:n}(\theta)$ is the trace of τ, $\mathrm{tr}(\tau)$. To illustrate the use of τ, we consider the FIM contained in two tails of order statistics (the smallest s order statistics and the largest $n-t+1$ order statistics), $(X_{1:n}, \ldots, X_{s:n}; X_{t:n}, \ldots, X_{n:n})$, denoted by $I_{1\cdots st\cdots n:n}(\theta)$, $s < t$. First, we partition the matrix τ into a 3×3 matrix $\tau = (\tau^{ij})_{3\times 3}$, where τ^{11}, τ^{22}, and τ^{33} are $s \times s$, $(t-s-1)\times(t-s-1)$, and $(n-t+1)\times(n-t+1)$ diagonal submatrices, respectively, corresponding to the three blocks of consecutive order statistics: $b_1 = (X_{1:n}, \ldots, X_{s:n})$, $b_2 = (X_{s+1:n}, \ldots, X_{t-1:n})$, and $b_3 = (X_{t:n}, \ldots, X_{n:n})$. Blocks b_1 and b_3 are two observed tails while b_2 is censored. Accordingly, $I_{1\cdots st\cdots n:n}(\theta)$ is the sum of three parts, where the first and third parts, corresponding to the observed blocks b_1 and b_2, are $\mathrm{tr}(\tau^{11}) = \sum_{i=1}^{s}\tau_{ii}$ and $\mathrm{tr}(\tau^{33}) = \sum_{i=t}^{n}\tau_{ii}$, respectively. The second part, corresponding to the censored block b_2, equals the sum of the off-diagonal elements of τ^{22} multiplied by $1/(t-s-2)$, the inverse of the size of the censored block minus one. Thus,

$$I_{1\cdots s\,t\cdots n:n}(\theta) = \sum_{i=1}^{s}\tau_{ii} + \sum_{i=t}^{n}\tau_{ii} + \frac{2}{t-s-2}\sum_{i=s+1}^{t-2}\sum_{j=i+1}^{t-1}\tau_{ij}. \tag{4}$$

In summary, when the block is observed, the trace of the corresponding diagonal submatrix is calculated, while the sum of off-diagonal elements multiplied by a constant is used when the block is censored. For example, consider a random sample of size 10 from the normal distribution with the location parameter θ. Then $\tau_{ij} = E(X_{i:10}X_{j:10})$, which can be found using Harter and Balakrishnan [51]. The FIM contained in the first three order statistics and last three order statistics can be obtained using (4) as $I_{1\,2\,3\;8\,9\,10:10} = 10 - 2(\tau_{44}+\tau_{55}) + 2(2\tau_{45}+2\tau_{46}+\tau_{47}+\tau_{56})/3 = 9.4239$. Applying the matrix approach to the Type-II censored data, $(X_{1:n},\ldots,X_{s:n})$, which are usually observed in life testing and to the order statistics in the upper tail, $(X_{t:n},\ldots,X_{n:n})$, we have

$$I_{1\cdots s:n}(\theta) = \sum_{i=1}^{s}\tau_{ii} + \frac{2}{n-s-1}\sum_{i=s+1}^{n-1}\sum_{j=i+1}^{n}\tau_{ij}, \tag{5}$$

$$I_{t\cdots n:n}(\theta) = \frac{2}{t-2}\sum_{i=1}^{t-2}\sum_{j=i+1}^{t-1}\tau_{ij} + \sum_{i=t}^{n}\tau_{ii}. \tag{6}$$

Notice that, from (4), (5) and (6), $I_{1\cdots s\,t\cdots n:n}(\theta) \neq I_{1\cdots s:n}(\theta) + I_{t\cdots n:n}(\theta)$. This property would not be easily obtained by calculating FIM using (2). The reason that the above matrix decomposition holds generally for $I_{k_1k_2\cdots k_m:n}(\theta)$ is the Markovian property of order statistics (see David and Nagaraja [30]: pp. 17-20).

The second approach is to decompose $I_{k_1k_2\cdots k_m:n}(\theta)$ as a linear combination of $I_{ij}(\theta)$ (see Park [74, 76]). Let A_1 and A_2 be any two sets such that $A_1\bigcap A_2 \neq \emptyset$ and i_A be the indicator function of the set A. Then $i_{A_1\bigcup A_2} = i_{A_1} + i_{A_2} - i_{A_1\bigcap A_2}$. If the elements of A_1 and A_2 are ordered ranks of order statistics, then the Fisher information has the same decomposition, $I_{A_1\bigcup A_2:n}(\theta) = I_{A_1:n}(\theta) + I_{A_2:n}(\theta) - I_{A_1\bigcap A_2:n}(\theta)$. For example, $A_1 = \{1,\ldots,t\}$ and $A_2 = \{s,\ldots,n\}$, $s < t$. Then $I_{1\cdots n:n}(\theta) = I_{1\cdots t:n}(\theta) + I_{s\cdots n:n}(\theta) - I_{s\cdots t:n}(\theta)$. Likewise, the FIM contained in the sample median $X_{m:n}$ $(n = 2m-1)$ may be written as

$$\begin{aligned} I_{m:n}(\theta) &= I_{1\cdots m:n}(\theta) + I_{m\cdots n:n}(\theta) - I_{1\cdots n:n}(\theta) \\ &= I_{s\cdots m:n}(\theta) + I_{m\cdots t:n}(\theta) - I_{s\cdots t:n}(\theta) \end{aligned}$$

for $1 \le s \le m \le t \le n$. In general, let A_i be a set of ordered integers such that $A_i\bigcap A_j \neq \phi$, and $A = \bigcup_{i=1}^{k} A_i$. Then the following FIM decomposition holds:

$$I_{A:n}(\theta) = \sum_{i=1}^{k} I_{A_i:n}(\theta) - \sum_{i=1}^{k-1} I_{A_i\bigcap A_{i+1}:n}(\theta). \tag{7}$$

Applying (7), $I_{k_1k_2\cdots k_m:n}(\theta)$ can be decomposed as linear combinations of double integrals $I_{ij:n}(\theta)$.

The second approach actually uses the conditional FIM in order statistics. Note that $I_{1\cdots n:n}(\theta) = I_{1\cdots s:n}(\theta) + I_{s+1\cdots n|s:n}(\theta)$, where $I_{s+1\cdots n|s:n}(\theta)$ is the conditional FIM in $X_{s+1:n},\ldots,X_{n:n}$ given $X_{s:n}$, which is written as $I_{s+1\cdots n|s:n}(\theta) = (n-s)E\{g_\theta(X_{(s)})\}$, where

$$g_\theta(x) = \int_x^\infty \left\{\frac{\partial}{\partial\theta}\log\frac{f_\theta(y)}{1-F_\theta(x)}\right\}^2 \frac{f_\theta(y)}{1-F_\theta(x)}dy.$$

This result is easy to use for life time distributions as the hazard function is used. It can also be used to obtain the asymptotic FIM in Type II censored data. For example, for the exponential distribution with a scale parameter θ, $I_{s+1\cdots n|s:n}(\theta) = (n-s)/\theta^2$ by the lack of memory property. Hence, $I_{1\cdots s:n}(\theta) = s/\theta^2$. Several applications of this result will be discussed later.

It is well-known that there are several recurrence relations and identities between cdf's of order statistics. FIM decompositions can be instantly written in terms of recurrence relationships [74]. Two examples using the recurrence of cdf's of order statistics are given here. From Cole [28], $F_{r:n-1} = \frac{n-r}{n}F_{r:n} + \frac{r}{n}F_{r+1:n}$, which yields the FIM decomposition: $nI_{1\cdots r:n-1}(\theta) = (n-r-1)I_{1\cdots r:n}(\theta) + rI_{1\cdots r+1:n}(\theta)$. Using Srikantan [89],

$$F_{r:n-1} = \sum_{i=n-r}^{n-1}(-1)^{i-n+r}C_{i-1,n-r-1}C_{n-1,i}F_{1:i},$$

where $C_{i,j} = i!/(j!(j-i)!)$, $I_{r:n}(\theta)$ can be written in terms of FIM in the smallest order statistic as $I_{r:n}(\theta) = \sum_{i=n-r+1}^{n}(-1)^{i-n+r-1}C_{i-2,n-r-1}C_{n,i}I_{1:i}(\theta)$, where $I_{1:i}(\theta)$ is calculated using Efron and Johnstone [33], who studied the FIM in a random sample in terms of the hazard function $h_\theta(x) = \frac{f_\theta(x)}{1-F_\theta(x)}$ as

$$\begin{aligned} I_{1:1}(\theta) &= \int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial\theta}\log f_\theta(x)\right\}^2 dF_\theta(x) \\ &= \int_{-\infty}^{\infty}\left\{\frac{\partial}{\partial\theta}\log h_\theta(x)\right\}^2 dF_\theta(x). \end{aligned} \tag{8}$$

As the hazard function of $X_{1:n}$ is n times that of X, $I_{1:n}(\theta) = E\{\partial\log h_\theta(X_{1:n})/\partial\theta\}^2$. The last FIM decomposition shows that the FIM in Type-II censored data, $I_{1\cdots r:n}(\theta)$, is determined by the FIM in the smallest order statistics $\{I_{1:i}(\theta), i = 1,\ldots,n\}$.

Iyengar et al. [60] studied the FIM in a random sample from weighted distributions with density function $f_\theta^w(x) = w(x)f_\theta(x)/E\{w(X)\}$, where $f_\theta(x) = \alpha(x)\times\exp(\theta T(x) -$

$C(\theta)$) belongs to the exponential family of distributions. The density of a single order statistic $X_{k:n}$ is a weighted distribution. For the exponential family of distributions $f_\theta(x)$, they proved that the FIM in $X_{k:n}$ is greater than that in a random observation X from the density $f_\theta(x)$ if and only if $(k-1)\log F_\theta(x)+(n-k)\log\{1-F_\theta(x)\}$ is a concave function of θ. Applying this result to the normal distribution with a location parameter θ, they showed that $I_{k:n}(\theta) > I_{1:1}(\theta)$ for $k = 1, \ldots, n$. It follows that $\sum_{k=1}^{n} I_{k:n}(\theta) > I_{1\cdots n:n}(\theta)$, which is the strong subadditivity property of information [38, 62]. In Section 2.3, we show that the subadditivity property of FIM holds in general and has applications to ranked set sampling.

The above FIM decompositions can be readily extended to the multiparameter case. For example, for $\theta = (\theta_1, \ldots, \theta_m)$, let $\tau_{ij} = (\tau_{ij}(u, v))_{m\times m}$ be an $m \times m$ matrix with $\tau_{ij}(u, v) = E\{\phi_{\theta_u}(X_{i:n})\phi_{\theta_v}(X_{j:n})\}$, where $\phi_{\theta_u}(x) = \partial \log f_\theta(x)/\partial\theta_u$. Then, we only need to replace all τ_{ij} in (4), (5) and (6) by $(\tau_{ij} + \tau_{ji})/2$.

2.3 Fisher information in ranked set samples

Ranked set sampling (RSS) is an efficient alternative to simple random sampling (SRS) when sampling units are difficult and/or expensive to measure but easy to rank without actual quantification; for example, the heights of trees. In destructive analysis, the sampling unit can be measured only after it is destroyed but it may be ranked visually without being destroyed. The RSS has wide applications in ecological studies, industrial statistics, biostatistics, and statistical genetics. Many statistical procedures using RSS are shown to be more efficient than those using SRS with the same sample size [23]. A RSS of size k consists of independent order statistics obtained from k independent replications. At the ith replication, a SRS of size k is identified and ranked ascending without actual measurements as $X_{(1)}, \ldots, X_{(k)}$ where only $X_{(i)}$ is retained for actual measurement. The set size k is usually small, $2 \le k \le 5$, as the ranking may be imperfect without measuring. To get a larger sample size, the previous procedure is repeated m times, so the sample size is $n = mk$. The RSS is then denoted as $X_{(1)j}, \ldots, X_{(k)j}$ for $j = 1, \ldots, m$. When the ranking is perfect, $X_{(i)j}$ has the same distribution as the ith order statistic $X_{i:k}$ for all j.

For parametric inference, FIM in $n = mk$ RSS is the sum of FIM in each of m samples, $m\sum_{i=1}^{k} I^*_{i:k}(\theta)$, as they are independent, while FIM in SRS of size n is $nI_{1:1}(\theta) = mkI_{1:1}(\theta) = mI_{1\cdots k:k}(\theta)$. Here, I^* is used to indicate that the ranking may not be perfect. Stokes [91] and Chen [22] proved that $\sum_{i=1}^{k} I^*_{i:k}(\theta) \ge I_{1\cdots k:k}(\theta)$ for any ranking model including ranking errors. Under perfect ranking, this is the subadditivity property of FIM (Section 2.2). Zheng [100] showed that the subadditivity property of FIM in RSS holds more generally $\sum_{i=s}^{t} I^*_{i:k}(\theta) \ge I_{s\cdots t:k}(\theta)$, $1 \le s \le t \le k$, for any ranking model. These results show that FIM in RSS is always more than that in SRS with the same sample size, which provide insight into the properties of parametric inference using RSS compared with SRS. Balakrishnan and Li [14, 13] considered the ordered ranked set sample (viz., the order statistics that arise from a RSS) and discussed the best linear unbiased estimate (BLUE) based on such an ordered RSS and its relative efficiency compared to those based on RSS; they also compared these two schemes in terms of Tukey's linear sensitivity measure as well as the FIM. In particular, they have shown that the ordered RSS has a higher linear sensitivity compared to the RSS which simply implies that the BLUE based on the ordered RSS are more efficient than the BLUE based on the RSS.

2.4 Fisher information in hybrid censored data

In most life-testing experiments, we can not continue the experiment until the last failure is observed. So, the experiment is usually terminated when either a pre-fixed censoring time C arrives (corresponding Type-I censoring scheme) or when the rth failure is observed (corresponding to Type-II censoring scheme).

The Fisher information about θ in the Type-I censored data, wherein the censoring time is a constant to be C, is well-known [44] to be $I_C(\theta)$ as

$$I_C(\theta) = n\int_0^C \left\{\frac{\partial}{\partial\theta}\log h_\theta(x)\right\}^2 dF_\theta(x),$$

while the Fisher information in the Type-II censored data can be obtained from the Fisher information in order statistics.

Type-I censoring scheme has the merit that the termination time of the experiment is guaranteed, but the level of efficiency may be too low or too high since we are uncertain about the number of failures. On the other hand, Type-II censoring scheme has the merit that the level of efficiency is guaranteed (since the number of failures to be observed is fixed in advance), but the experiment may be terminated too early or too late since the exact time of the rth failure is uncertain. For these reasons, we need another censoring scheme where both efficiency level and termination time can be controlled. For this specific purpose, a hybrid censoring scheme, which is a mixture of Type-I and Type-II censoring schemes, has been proposed; see, for example, Epstein [34], Chen and Bhattachayya [21], Gupta and Kundu [46] and Childs et al. [27]. In the Type-I hybrid censoring scheme, the experiment is terminated when either the rth failure or a pre-fixed censoring time C comes first. In the Type-II hybrid censoring scheme, the experiment is terminated when either the rth failure or the censoring time C comes later.

The Fisher information in the Type-I hybrid censored data, where the censoring time is $\min(X_{r:n}, C)$, can be obtained [96] as

$$I_{\min(X_{r:n},C)}(\theta) = \int_0^C \left\{\frac{\partial}{\partial\theta}\log h(x;\theta)\right\}^2 \big(f_{1:n}(x;\theta)+\cdots+f_{r:n}(x;\theta)\big)dx,$$

and the Fisher information in the Type-II hybrid censored data, where the censoring time is $\max(X_{r:n}, C)$, can be obtained [78] as

$$I_{\max(X_{r:n},C)}(\theta) = I_C(\theta) + I_{1\cdots r:n}(\theta) - I_{\min(X_{r:n},C)}(\theta).$$

However, the efficiency level of the experiment under the Type-I hybrid censoring scheme may be still too low. To overcome this, we can let the experiment continue until at least rth failure ($r < s$) comes, which is called a generalized Type-I hybrid censoring scheme. In a similar manner, the experiment under the Type-II hybrid censoring scheme may be terminated still too late. To overcome this, we can terminate the experiment if a censoring time C_2 ($C_1 < C_2$) comes, which is called a generalized Type-II hybrid censoring scheme; see Chandrasekar et al. [20].

The Fisher information in the generalized Type-I hybrid censored data where the censoring time is $\max(X_{r:n}, \min(C, X_{s:n}))$ [simply denoted by $r \vee (C \wedge s)$] can be obtained [79] as

$$\begin{aligned} &I_{\max(X_{r:n},\min(C,X_{s:n}))}(\theta) \\ &\quad = I_{1\cdots r:n}(\theta) + I_{\min(C,X_{s:n})}(\theta) - I_{\min(C,X_{r:n})}(\theta), \end{aligned}$$

and the Fisher information in the generalized Type-I hybrid censored data where the censoring time is $\min(\max(C_1, X_{r:n-1}), C_2)$ can be obtained as

$$\begin{aligned} &I_{\min(\max(C_1,X_{r:n}),C_2)}(\theta) \\ &\quad = I_{\max(C_1,X_{r:n})}(\theta) + I_{C_2}(\theta) - I_{\max(C_2,X_{r:n})}(\theta) \\ &\quad = I_{C_1}(\theta) + I_{\min(C_2,X_{r:n})}(\theta) - I_{\min(C_1,X_{r:n})}(\theta). \end{aligned}$$

2.5 Fisher information in randomly censored data

Random censoring is commonly encountered in survival analysis. Let X and Y be lifetime and censoring variables with cdf's $F_\theta(x)$ and $G_\theta(y)$ and densities $f_\theta(x)$ and $g_\theta(y)$, respectively. Note that, under noninformative censoring, the censoring distribution G is independent of θ. Informative censoring, however, may also occur in applications. For example, late-stage cancer patients may be given higher doses of drug who are more likely to be censored due to poor compliance to the treatment with high drop out rates. Under random censorship, one only observes $Z = \min(X, Y)$ and an indicator $\delta = I(X \le Y)$. Denote the likelihood function for the pair (Z, δ) as $L(\theta; z, \delta)$. Under some regularity conditions, the FIM about θ contained in (Z, δ) is defined as $I^{Z,\delta}(\theta) = E\{\partial \log L(\theta; Z, \delta)/\partial\theta\}^2$, which has been studied in parametric inference of survival data by Miller [69], Abdushukrov and Kim [1], Prakasa Rao [84], Gastwirth and Wang [43], and Zheng and Gastwirth [102].

In practice, n pairs are observed in order $(Z_{1:n}, \delta_{[1]}), \ldots, (Z_{n:n}, \delta_{[n]})$, where $Z_{1:n} \le \cdots \le Z_{n:n}$ are order statistics and $\delta_{[i]}$ is the indicator for $Z_{i:n}$. First, the FIM in n pairs $(Z_{i:n}, \delta_{[i]}), i = 1, \ldots, n$ can be written as $I^{Z,\delta}_{1\cdots n:n}(\theta) = nI^{Z,\delta}(\theta)$. The FIM decompositions still hold for FIM in randomly censored data [104]. For example, for the median survival time $Z_{m:n}$, we have

$$I^{Z,\delta}_{m:n}(\theta) = I^{Z,\delta}_{1\cdots m:n}(\theta) + I^{Z,\delta}_{m\cdots n:n}(\theta) - I^{Z,\delta}_{1\cdots n:n}(\theta).$$

In general, let $I^Z_{k_1\cdots k_m}(\theta)$ be the FIM in any censored data $(Z_{k_1:n}, \ldots, Z_{k_m:n})$ without the indicators. Then, by FIM decompositions,

$$I^{Z,\delta}_{k_1\cdots k_m}(\theta) = I^Z_{k_1\cdots k_m}(\theta) + \Delta_{k_1\cdots k_m}(\theta),$$

where $\Delta_{k_1\cdots k_m}(\theta) \ge 0$ and the equality holds if and only if the hazard functions of X and Y are proportional, i.e., $f\bar{G}/(g\bar{F})$ is independent of θ, which is referred to as the Koziol-Green random censoring model (KGM) [64, 29, 24]. Thus, under KGM, whether or not an observation is censored contains no FIM about θ.

Under KGM, Hollander et al. [57] noticed that the asymptotic variance of the nonparametric estimator of the survival function $\bar{F}$ can decrease as the degree of censoring increases. FIM provides insight into why censoring improves the efficiency of the estimate. Under KGM and informative censoring, the censored data (Z, δ) may contain more Fisher information than the uncensored data X, e.g., $I^{Z,\delta}_{1\cdots r:n}(\theta) > I^X_{1\cdots r:n}(\theta)$. The insight was given by Zheng and Gastwirth [102], who showed that $I^{Z,\delta}(\theta) > I^X(\theta)$ if and only if

$$\int\left\{\frac{\partial}{\partial\theta}\log h_\theta(x)\right\}^2 g_\theta(x)\bar{F}_\theta(x)dx > \int\left\{\frac{\partial}{\partial\theta}\log h_\theta(x)\right\}^2 f_\theta(x)G_\theta(x)dx. \tag{9}$$

The FIM in X, $I^X(\theta)$, can then be written as

$$\begin{aligned} I^X(\theta) &= \int\left\{\frac{\partial}{\partial\theta}\log h_\theta(x)\right\}^2 f_\theta(x)dx \\ &= \int\left\{\frac{\partial}{\partial\theta}\log h_\theta(x)\right\}^2 f_\theta(x)G_\theta(x)dx \\ &\quad + \int\left\{\frac{\partial}{\partial\theta}\log h_\theta(x)\right\}^2 f_\theta(x)\bar{G}_\theta(x)dx, \end{aligned} \tag{10}$$

where the first term on the right hand side of (10) is the FIM in X lost due to censoring and the second term is the FIM in X not lost due to censoring. Under KGM, the censoring variable Y contains FIM $\int\{\partial \log h_\theta(x)/\partial\theta\}^2 g_\theta(x)dx$, among which $\int\{\partial \log h_\theta(x)/\partial\theta\}^2 g_\theta(x)\bar{F}_\theta(x)dx$ is contained in (Z, δ). Note that (9) indicates that, under KGM when the FIM contributed from the censoring variable Y is more than the FIM in X lost due to censoring, the randomly censored data contain more FIM than uncensored data.

2.6 Fisher information in other ordered data

The research of FIM in order statistics has been extended to other ordered data, including record data [3, 4, 55, 92, 53], concomitants [2], and Type-II progressive censoring [106]. The FIM in these ordered data is more difficult to calculate as their likelihood functions are more complicated than that of order statistics in (1). Here, we provide a brief review of recent developments.

One example of record data is Olympic high jump records studied in Carlin and Gelfand [17]. The upper (lower) record data consist of a sequence of observed records, each of which is greater (less) than the previous record, and the record times to observe the records. Let T_i and R_i be the ith record time and upper record value. Given a random sample of $(X_1, \ldots, X_n)$, we have $T_1 = 1$ and $R_1 = X_1$, and $T_i = \min\{j > T_{i-1}; X_j > X_{T_{i-1}}\}$ and $R_i = X_{T_i}$ for $i \geq 2$. Lower records can be defined similarly. See Arnold et al. [7]. Ahmadi and Arghami [3, 4] and Hofmann and Nagaraja [55] all discussed the FIM contained in n record values and compared it with the FIM in a simple random sample of n observations. By considering discrete populations and weak record values (which are values that are at least as large as the previous record), Stepanov et al. [92] derived an exact expression for the distributions of such weak record values and the FIM contained in n weak records. Finally, Hofmann and Balakrishnan [53] discussed the k-record values, which are the kth largest (or smallest) value yet seen in the original sequence (originally introduced by Dziubdziela and Kopociński [31]), and derived the Fisher information contained in such k-record values.

Concomitants arise from ranking bivariate observations $\{(X_i, Y_i) : i = 1, \ldots, n\}$, where all X_i's are ranked. The Y value corresponding to the ith order statistic $X_{i:n}$ is a concomitant, denoted by $Y_{[i]}$. Concomitants appear in many applications. In bivariate life testing, (X, Y) are lifetimes of two components of a tested item but X is first observed. In survival analysis, Y is the indicator whether or not the observation is censored. It also has applications in ranked set sampling [90]. Abo-Eleneen and Nagaraja [2] studied the FIM in a single concomitant $Y_{[i]}$ and applied the results to various bivariate exponential distributions. In Section 2.3, FIM decompositions were applied to randomly censored data $\{(Z_{k_1:n}, \delta_{[k_1]}), \ldots, (Z_{k_m:n}, \delta_{[k_m]})\}$. Likewise, these properties of FIM can be extended to $\{(X_{k_1:n}, Y_{[k_1]}), \ldots, (X_{k_m:n}, Y_{[k_m]})\}$ and to obtain asymptotic FIM.

Type-II progressive censoring extends Type-II censoring and has many applications in life testing [10]. In Type-II censoring, the first m outcomes of $n > m$ tested items are observed. In Type-II progressive censoring, immediately after observing the ith outcome at time denoted by $X_{i:m:n}$, $i = 1, \ldots, m-1$, R_i surviving items are randomly removed from the test. When the mth outcome is observed at time $X_{m:m:n}$, the rest of the items, $R_m = n - R_1 - \cdots - R_{m-1}$, are removed from the test. The Type-II progressive censored data consist of $\mathbf{X}_{1\cdots m:m:m} = (X_{1:m:n}, \ldots, X_{m:m:n})$. When $R_i = 0$ for $i = 1, \ldots, m-1$ and $R_m = n - m$, it reduces to the usual Type-II censored data. Denote by $\mathbf{X}_{1\cdots r:m:m} = (X_{1:m:n}, \ldots, X_{r:m:n})$, the first r outcomes from Type-II progressive censoring. The decomposition of FIM using the conditional FIM can be applied to $I_{1\cdots m:m:m}(\theta)$, FIM in $\mathbf{X}_{1\cdots m:m:m}$. Zheng and Park [106] showed that $I_{1\cdots r:m:n}(\theta) = I_{1:m:n}(\theta) + I_{2|1:m:n}(\theta) + \cdots + I_{r|(r-1):m:n}(\theta)$, where, using the hazard function h, $I_{s|(s-1):m:n}(\theta) = \int\{\partial \log h_\theta(x)/\partial\theta\}^2 f_{s:m:n}(x)dx$ and the densities $f_{s:m:n}(x)$ were explicitly given in Balakrishnan et al. [11]. When applying the results to the exponential or Weibull distributions with a scale parameter, we have $I_{1\cdots r:m:n}(\theta) = rI_{1:1}(\theta)$.

3. ASYMPTOTIC FISHER INFORMATION

3.1 Asymptotic Fisher information and inference

Consider a block of consecutive order statistics $\mathbf{X}_{st} = (X_{s:n}, \ldots, X_{t:n})$ of a random sample from cdf $F_\theta(x)$. The mean vector and covariance matrix of $\mathbf{X}_{st}$ are denoted by $\mu = E(\mathbf{X}_{st})$ and $\mathbf{\Sigma} = Var(\mathbf{X}_{st})$, respectively. For a scalar parameter θ and some real valued numbers $c = (c_1, \ldots, c_{t-s+1})$, Tukey's linear sensitivity [94], $S(\mathbf{X}_{st}; \theta)$, was defined as $S(\mathbf{X}_{st}; \theta) = \sup_c\{(c^T\mu)^2/c^T\mathbf{\Sigma}c\}$. $S(\mathbf{X}_{st}; \theta)$ was introduced as a measure of efficiency for unbiased L-estimators for location-scale family distributions. In fact, for the location parameter, Nagaraja (1994) showed that, when $s/n \to p_s$ and $t/n \to p_t$ as $n \to \infty$, $S(\mathbf{X}_{st}; \theta)$ is approximately the FIM contained in all observations between the $(100p_s)$th and $(100p_t)$th sample percentiles of n observations. The multiparameter version of Tukey's linear sensitivity was studied by Chandrasekar and Balakrishnan [19].

Like Tukey's linear sensitivity, asymptotic FIM in order statistics was usually obtained from the asymptotic relative efficiency of large sample inferences. Here are some examples. For location-scale family distributions, the asymptotic FIM in consecutive order statistics was obtained as the inverse of asymptotic variances for L-statistics [26] and for asymptotically most powerful rank tests [95, 40, 88, 49, 101]. It is also obtained from inferences using Type-I censored data [36] and Type-II censored data [50, 61, 16, 98, 75] and other censored data [106].

We give a formal definition of asymptotic FIM, which is indeed equivalent to those obtained from large sample inferences mentioned above, but not limited to location-scale distributions studied, for example, by Chernoff et al. [26]. Denote $I_{[0,1]}(\theta) = I_{1:1}(\theta)$ as the FIM in a single observation.

We first consider Type-II censored data $(X_{1:n}, \ldots, X_{r:n})$ and its FIM $I_{1\cdots r:n}(\theta)$. Suppose $r/n \to p \in (0,1)$ as $n \to \infty$. The asymptotic FIM contained in the lower 100pth percentile of $F_\theta(x)$ is defined as

$$I_{[0,p]}(\theta) = \lim_{n\to\infty} \frac{1}{n} I_{1\cdots r:n}(\theta). \tag{11}$$

The proportion of FIM contained in the lower (100p)th percentile is $I_{[0,p]}(\theta)/I_{[0,1]}(\theta)$. For example, for the exponential distribution with scale parameter θ, $\theta^2 I_{1\cdots r:n}(\theta) = r$ (Section 2.2). Thus, $\theta^2 I_{[0,p]}(\theta) = p$ and $I_{[0,p]}(\theta)/I_{[0,1]}(\theta) = p$. So, the proportion of FIM contained in the lower (100p)th percentile is exactly p. In general, the asymptotic FIM contained in k blocks grouped by percentiles $[p_1, q_1], \ldots, [p_k, q_k]$, denoted by $I_{\bigcup_{i=1}^k [p_i,q_i]}(\theta)$, is defined as the limit of FIM in k blocks of consecutive order statistics standardized by n. By this definition, for example, the asymptotic FIM contained in the median of the distribution is $I_{[1/2,1/2]}(\theta) = \lim_{m\to\infty} I_{m:2m-1}(\theta)/(2m-1)$. The FIM decompositions can be directly applied to the asymptotic FIM, e.g., $I_{[0,p]}(\theta) + I_{[p,1]}(\theta) = I_{[0,1]}(\theta) + I_{[p,p]}(\theta)$, where $I_{[p,p]}(\theta)$ is the asymptotic FIM in the (100p)th percentile.

3.2 Calculating asymptotic Fisher information

To calculate the asymptotic FIM from the matrix decomposition, we only need the following results [93, 97]. Let $r/n \to p \in (0,1)$ and $s/n \to q \in (0,1)$ as $n \to \infty$ and $q > p$. Under some suitable conditions,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=r}^{s} \tau_{ii} = \int_{F^{-1}(p)}^{F^{-1}(q)} \Big\{ \frac{\partial}{\partial\theta} \log f_\theta(x) \Big\}^2 f_\theta(x)dx,$$

$$\lim_{n\to\infty} \frac{2}{n(s-r)} \sum_{i=r}^{s-1} \sum_{j=i+1}^{s} \tau_{ij} = \frac{1}{q-p} \Big\{ \int_{F^{-1}(p)}^{F^{-1}(q)} \frac{\partial}{\partial\theta} f_\theta(x)dx \Big\}^2.$$

When applying the above results to Type-II censored data, we have

$$I_{[0,p]}(\theta) = \int_{-\infty}^{F^{-1}(p)} \Big\{ \frac{\partial}{\partial\theta} \log f_\theta(x) \Big\}^2 f_\theta(x)dx + \frac{1}{1-p} \Big\{ \int_{F^{-1}(p)}^{\infty} \frac{\partial}{\partial\theta} f_\theta(x)dx \Big\}^2 \tag{12}$$

which was obtained by Halperin [50] and Bhattacharyya [16] from asymptotic inference using Type-II censored data. For location-scale distributions, it was also obtained in Tukey [94] and Nagaraja [70] from linear sensitivities, Chernoff et al. [26] from asymptotic inference for L-statistics, and Zheng and Gastwirth [101] from the correlation of the most powerful rank test for censored data. $I_{[0,p]}(\theta)$ can also be obtained from the decomposition $I_{1\cdots r:n}(\theta) = I_{1\cdots n:n}(\theta) - I_{r+1\cdots n|r:n}(\theta)$ [75] and from the FIM in Type-I censored data [36]. Expressing (12) in terms of the hazard function h_θ of X,

$$I_{[0,p]}(\theta) = \int_{-\infty}^{F^{-1}(p)} \Big\{ \frac{\partial}{\partial\theta} \log h_\theta(x) \Big\}^2 f_\theta(x)dx,$$

one extends (8) to Type-II censored data [98]. The FIM in terms of reversed hazard functions are obtained by Gupta and Kundu [48]. Takahashi and Sugiura [93] also studied the rate of convergence for (11). Suppose $r/n = p + o(n^{-1/2+\gamma})$ and $\gamma \in (0, 1/2)$ is a constant. Then $I_{1\cdots r:n}(\theta)/n - I_{[0,p]}(\theta) = O(n^{-1/2+\gamma})$. A similar convergence rate holds for $I_{\bigcup_{i=1}^k [p_i,q_i]}(\theta)$.

We mention two simple uses of asymptotic FIM. In the first example, consider Type-II censored data, $X_{1:n}, \ldots, X_{r:n}$, from cdf $F_\theta(x)$. When $r/n \to p \in (0,1)$, under some regularity conditions, the maximum likelihood estimate (MLE) of θ satisfies $n^{1/2}(\hat\theta - \theta) \xrightarrow{D} N(0, p/I_{[0,p]}(\theta))$. For example, the first 18 of 23 ordered numbers of million revolutions of ball bearings before failure are listed here: 17.88, 28.92, 33.00, 41.12, 45.60, 48.80, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84 [67]. Gupta and Kundu [47] indicated that the data can be fitted by the exponentiated exponential distribution with pdf $f_{\alpha,\beta}(x) = \alpha\beta^{-1}\{1 - \exp(-x/\beta)\}^{\alpha-1} \exp(-x/\beta)$, where $\alpha > 0, \beta > 0$ are parameters. The MLEs for the parameters using the data are $\hat\alpha = 3.1953$ and $\hat\beta = 45.5663$. Let $p = 18/23$. Thus, $I_{[0,p]}(\hat\alpha) = .0978$, $I_{[0,p]}(\hat\beta) = .001129$, and $I_{[0,p]}(\hat\alpha, \hat\beta) = .008765$ [99]. In the second example, consider the percentiles $F^{-1}(p_1) < \cdots < F^{-1}(p_k)$, which are called k optimal spacings if the variance of an asymptotically best linear unbiased estimator for location-scale distributions based on these k percentiles is minimized [73, 26, 66, 25, 12]. To find optimal k spacings is equivalent to finding k percentiles that contain most FIM. Using FIM decompositions, the asymptotic FIM contained in any k percentiles is given by

$$I_{\bigcup_{i=1}^k [p_i,p_i]}(\theta) = \sum_{i=0}^{k} \frac{1}{p_{i+1} - p_i} \Big\{ \int_{F^{-1}(p_i)}^{F^{-1}(p_{i+1})} \frac{\partial}{\partial\theta} f_\theta(x)dx \Big\}^2,$$

where $p_0 = 0$ and $p_{k+1} = 1$. Thus, the concept of optimal spacings is not limited to location-scale distributions when asymptotic FIM is used. Moreover, using the exact FIM, it is easier to obtain the asymptotic FIM in general situations than deriving the asymptotic variances.

4. CHARACTERIZATIONS BY FISHER INFORMATION IN ORDER STATISTICS

Characterization problems have received considerable attention in the literature (see Kagan et al. [63], Azlarov and Volodin [9], Prakasa Rao [83], Rao and Shanbhag [86]). The results of exact and asymptotic FIM in order statistics enable us to characterize some distributions using FIM.

Let T be the fixed censoring time. Then, Type-I censored data contain all order statistics up to time T, that is, $X_{1:n} \le \cdots \le X_{r:n} \le T$ and $X_{r+1:n} > T$. The FIM contained in Type-I censored data is equivalent to the asymptotic FIM in Type-II censoring $I_{[0,p]}(\theta)$ with $p = F^{-1}(T)$ [36]. Gertsbakh and Kagan [44] and Zheng [98] studied characterizations of lifetime distributions $F(x/\theta)$ or $F(x-\theta)$ by FIM in Type-I and Type-II censored data, respectively, and showed that the proportion of FIM in the first pth percentile is a constant p, i.e., $I_{[0,p]}(\theta) = pI_{[0,1]}(\theta)$, if and only if F is the Weibull distribution when θ is the scale parameter or the extreme value distribution when θ is the location parameter. Moreover, the hazard function can be factorized as $h_\theta(x) = u(x)v(\theta)$ if and only if $I_{1\cdots r:n}(\theta) = rI_{1:1}(\theta)$ for any $n \ge 1$ and $1 \le r \le n$. Recently, Hofmann et al. [54] improved this condition for the factorization of the hazard function as $I_{1:n}(\theta) = I_{1:1}(\theta)$ for any $n \ge 1$ and further obtained similar characterizations for the reversed hazard function and for record values. On the other hand, Zheng and Gastwirth [104] showed that results to those of Gertsbakh and Kagan [44] and Zheng [98] also hold for randomly censored data.

It is known that a parent distribution is uniquely determined by the moments of its order statistics $\{E(X_{i;n}) : i = 1, \ldots, n\}$ or its subset [18, 82, 8, 58]. The FIM in order statistics cannot uniquely determine a parent distribution. For example, order statistics from normal distributions $N(\theta, 1)$ and $N(\theta - 1, 1)$ contain the same FIM about θ. Park and Zheng [80], however, studied the condition under which the FIM in order statistics can uniquely determine the parent distribution. Let F_θ be the cdf of X. Define $\alpha^F_\theta(u) = \{\partial F_\theta(x)/\partial\theta\}|_{x=F_\theta^{-1}(u)}$ for $u \in (0, 1)$ and $\alpha^F_\theta(0) = \alpha^F_\theta(1) = 0$. For location and scale parameters, $\alpha^F_\theta(u) = -f(F^{-1}(u))$ and $\alpha^F_\theta(u) = -\theta^{-1}F^{-1}(u)f(F^{-1}(u))$, respectively, which have applications in robust linear estimators [42]. Define $\eta(F_\theta) = (\alpha^F_\theta)^2$. For the distribution function F, $I^F_{k_1\cdots k_m:n}(\theta)$ is FIM in m order statistics from cdf F. Then, using FIM decompositions, for $n \ge 1$, $I^F_{k_1\cdots k_m:n}(\theta) = I^G_{k_1\cdots k_m:n}(\theta)$ if and only if $\eta(F_\theta) = \eta(G_\theta)$. The result also holds for asymptotic FIM. The function $\eta(F_\theta)$ can be used to define an equivalence relation, which yields the following results for location-scale distributions. For the location (scale) parameter, any two subsets of order statistics with the same ranks contain the same FIM if and only if $G(x - \theta) = F(x - \theta - c)$ $(G(x/\theta) = F(x/(|c|\theta)))$ for some constant c.

5. APPLICATIONS

5.1 Location-scale distributions and BLUE

For symmetric distributions, the middle order statistics and the tails of order statistics seem to be more informative about the location and scale parameters, respectively. Zheng and Gastwirth [101, 103] studied this problem for the normal, logistic, Laplace and Cauchy distributions using asymptotic FIM, and the results were applied to selecting order statistics for BLUE for these parameters.

Which block of ordered data contain most FIM about the location parameter? Consider the FIM $I_{[p,p+q]}(\theta)$ of the four distributions, where $q > 0$ is the percentage of data and is fixed. The results show that $p = (1 - q)/2$ (the middle portion of order statistics) maximizes $I_{[p,p+q]}(\theta)$ for normal, logistic and Laplace distributions. The result does not hold for the Cauchy distribution. However, the 50% of middle order statistics contain more than 85% of total FIM for all four distributions. For the scale parameter, consider the FIM in two tails $I_{[p,p+q/2]\bigcup[1-p-q/2,1-p]}(\theta)$ with fixed $q > 0$. The results show that, using at least 30% of order statistics ($q > 0.30$), the extreme two tails ($p = 0$) from normal, logistic and Laplace distributions contain the most FIM about the scale parameter. For the Cauchy distribution, $I_{[p,p+q/2]\bigcup[1-p-q/2,1-p]}(\theta)$ is maximized when the mid-tails are used, $p = (1-q)/2$. Table 1 reports the FIM in two-tails for the scale parameter and in the middle portion of order statistics for the location parameter for Cauchy, Laplace, logistic and normal distributions. Overall, 40% of order statistics in two-tails contain more than 80% of total FIM. However, the middle 40% of order statistics only contain 28%, 37%, 40% and 80% of FIM about the scale parameter for normal, logistic, Laplace and Cauchy distributions, respectively [101]. Applying these results to BLUE for the scale parameter, for example, the extreme order statistics can be selected (the mid-tails for Cauchy). For a small sample size $n = 12$, the relative efficiencies, defined as the ratio of the variances of two estimates, one using the extreme 30% order statistics and the other using all order statistics, are over 0.85 for normal, logistic and Laplace distributions, but only 0.50 for Cauchy distribution using the 30% mid-tails. Through simulation studies, it has been observed that a large sample size of $n = 48$ is needed to have the relative efficiency over 0.85 for Cauchy distribution using only 30% mid-tails.

5.2 Robust estimates

Small sample quick robust estimators for the mean using three order statistics or percentiles, Tukey's trimean (TRI) and Gastwirth's estimate (GAS), have been studied in the literature [41, 5, 59, 56, 81]. In practice, the true distribution underlying the data is often unknown while a family of plausible distributions may be identified. A robust estimator has high efficiency for each possible distribution. The efficiency robustness is measured by the lowest efficiency of the estimator across the whole family of distributions. The higher the minimum efficiency, the more robust the estimator is.

As quick estimators, TRI and GAS use only three order statistics at the (25th, 50th and 75th) and ($33\frac{1}{3}$rd, 50th and $66\frac{1}{3}$rd) percentiles, respectively. When the sample size n is odd, they are written as TRI $= 0.25X_{[n/4]:n} +$

Table 1. Percentage of total FIM contained in two-tails (scale parameter) and middle order statistics (location parameter)

Distribution	Percentage of data	Two-tails (scale)	Middle (location)
Cauchy	30%	64.1%	86.9%
	40%	79.9%	89.4%
	50%	90.5%	91.6%
Laplace	30%	92.1%	97.1%
	40%	95.9%	99.0%
	50%	98.1%	99.7%
Logistic	30%	94.1%	89.1%
	40%	97.5%	92.7%
	50%	99.5%	95.5%
Normal	30%	96.8%	78.7%
	40%	98.8%	83.1%
	50%	99.6%	87.1%

$0.50X_{(n+1)/2:n}+0.25X_{n+1-[n/4]:n}$ and GAS $= 0.30X_{[n/3]:n}+0.40X_{(n+1)/2:n}+0.30X_{n+1-[n/3]:n}$, where $[x]$ indicates the largest integer contained in x. The three order statistics used are all within 25th and 75th percentiles, which cover more than 85% of the total FIM in data about the location parameter of normal, logistic, Laplace and Cauchy distribution from light to heavy tails, as mentioned earlier. Zheng and Gastwirth [101] calculated FIM in the location parameter in three order statistics including the median for $n = 19$ and found the minimum percentage of total FIM in three order statistics across the four distributions. We use ranks to represent the three order statistics. Then the order statistics $(X_{5:19}, X_{10:19}, X_{15:19})$ in TRI and $(X_{6:19}, X_{10:19}, X_{14:19})$ in GAS are represented by (5,10,15) and (6,10,14). The minimum percentage of FIM in three order statistics (2,10,18), (3,10,17), (4,10,16), (5,10,15), (6,10,14), and (7,10,13) are 80%, 82%, 83%, 85% (TRI), 85% (GAS), and 81%, respectively. Note that TRI and GAS contain the highest minimum percentage of FIM when the true distribution is unknown. Thus, calculating FIM in order statistics provides insight into some robustness properties of these quick estimators.

5.3 Genetic linkage analysis

Haseman and Elston [52] studied linkage analysis between genes and a quantitative trait locus (QTL) using random samples of sibling pairs. Recent research shows that the power detecting linkage is substantially enhanced when selected sib pairs are used (see, Risch and Zhang [87]). One type of selected sib pairs is discordant sib pairs (DSP), where one sib's trait value is in the top percentile of the trait distribution while the other is in the lower percentile. Zheng and Gastwirth [105] extended a parametric model of Haseman and Elston [52] for the abstract value of trait difference between two sibs and calculated the FIM about some parameters contained in DSP. However, they did not study the FIM about the linkage parameter, the recombination fraction in DSP, which will be discussed here.

Let (X_1, X_2) be quantitative traits of two sibs in a family. Assume that the trait is controlled by an unknown gene (QTL) with two alleles M and N with frequencies $p = Pr(M)$ and $q = Pr(N) = 1 - p$. Each sib has one of three genotypes MM, MN and NN. Haseman and Elston [52] proposed the regression model for the trait X as $X = \mu + g + e$, where μ is overall mean and g is the random genetic effect defined as $g = a, d, -a$ for QTL genotype MM, MN, NN, where $a > 0$ and $d \in (-\infty, \infty)$ are unknown parameters. The error e has zero mean and variance σ_e^2. We only observe sibling marker information and not their QTL. One important quantity for linkage analysis is the proportion of alleles shared identical-by-descent (IBD) by a sib pair. This proportion is denoted as π_t at the QTL and π_m at the marker. Parental genotypes at the marker are required in order to figure out the IBD proportion at the marker. We assume the complete information of sibling and their parental genotypes at the marker, denoted by I_m, as Haseman and Elston [52], are available to determine π_m. The recombination fraction $r \in [0, 0.5]$ was introduced by the joint distribution of (π_t, π_m) [52]. Linkage corresponds to $r < 0.5$. We use a 1-1 transformation $\psi = r^2 + (1-r)^2 \in [0.5, 1]$, where linkage corresponds to $\psi > 0.5$.

Zheng and Gastwirth [105] extended the parametric model of Haseman and Elston [52]. Let $Z = |X_1 - X_2|$ be the absolute value of the trait difference of two sibs. The likelihood of Z depends on I_m. For illustration, we only consider parental mating type (MT) $A_2A_2 \times A_1A_2$ and sib-pair genotypes (SP) $A_2A_2 - A_1A_2$, i.e, $I_m = \{\textit{mating type} = \text{MT} = A_2A_2 \times A_1A_2; \text{SP} = A_2A_2 - A_1A_2\}$ for all sib pairs, which is I_m type 3 given in Zheng and Gastwirth [105] (Table 2). We also consider the additive genetic model, so $d = 0$ and assume $\sigma_e^2 = 1$. Then the likelihood of a sib pair can be written as a mixture distribution

$$f(z) = f(z|p, a, \psi, I_m) = \sum_{i=1}^{3} \omega_i(p, \psi|I_m) f_i(z; a),$$

where $f_1(z; a) = \phi(z)$, $f_2(z; a) = \phi(z-a) + \phi(z+a)$, $f_3(z; a) = \phi(z-2a) + \phi(z+2a)$ and ϕ is pdf of the standard normal distribution, and $\omega_1(p, \psi|I_m) = \psi(p^4 + 4p^2q^2 + q^4)/2 + (1 - pq)$, $\omega_2(p, \psi|I_m) = 2pq(p^2 + q^2)\psi + pq$, and $\omega_3(p, \psi|I_m) = p^2q^2\psi$. A similar version of $f(z|p, a, \psi, I_m)$ was mentioned in Haseman and Elston [52]. Denote the cdf of Z by $F(z) = F(z|p, a, \psi, I_m)$. Zheng and Gastwirth [105] considered the FIM using the marginal likelihood $\sum_{I_m} f(z|p, a, \psi, I_m)Pr(I_m) = g(z|p, a)$, which was independent of the linkage parameter ψ. Thus, they only examined the FIM in DSP about other parameters (a, p). For using FIM to explain why power of linkage analysis is substantially increased when DSP is used, we need to calculate FIM in DSP about ψ. Thus, we use $f(z|p, a, \psi, I_m)$. For

Table 2. Percentage of total FIM about ψ in DSP and ARE using DSP relative to random samples of same size. The top panel has $p = 0.10$ and the bottom panel has $p = 0.30$

		% of total FIM		ARE	
a	ψ	Top 20%	Top 10%	Top 20%	Top 10%
1.0	0.68	69.45%	41.99%	347%	420%
	0.90	64.92%	35.97%	325%	360%
1.5	0.68	76.10%	47.97%	381%	480%
	0.90	71.02%	40.35%	355%	404%
1.0	0.68	88.37%	86.99%	442%	870%
	0.90	98.28%	87.06%	491%	871%
1.5	0.68	62.04%	59.22%	310%	592%
	0.90	75.03%	64.32%	375%	643%

n independent sib pairs, we calculate Z_i, $i = 1, \ldots, n$ and rank them as $Z_{1:n} \le \cdots \le Z_{n:n}$. The DSP corresponds to largest order statistics of Z_i. This selection of DSP was also considered by Fullerton et al. [39]. Let $\theta = (p, a, \psi)$ and $I_{[1-s,1]}(\theta) = (I_{uv,s})_{3\times 3}$ be the FIM matrix about θ contained in the top $(100s\%)$ percentile. From Section 3, we have

$$I_{[1-s,1]}(\theta) = \int_{\lambda_s}^{\infty} \Big\{\frac{\partial}{\partial\theta}\log f(z)\Big\}\Big\{\frac{\partial}{\partial\theta}\log f(z)\Big\}^T dF(z) + \frac{1}{1-s}\Big\{\int_0^{\lambda_s}\frac{\partial}{\partial\theta}f(z)dz\Big\}\Big\{\int_0^{\lambda_s}\frac{\partial}{\partial\theta}f(z)dz\Big\}^T,$$

where λ_s is the $100(1-s)\%$ percentile solved from $F(\lambda_s) = 1 - s$. Then the FIM matrix in all data is $I_{[0,1]}(\theta) = (I_{uv,1})_{3\times 3}$. We find the inverse of these FIM matrices, denoted as $(i_{uv,s})_{3\times 3}$ and $(i_{uv,1})_{3\times 3}$. The percentage FIM $i_{33,1}/i_{33,s}$ about ψ contained in DSP is evaluated. The ARE using DSP [the top $(100s\%)$] relative to random sib pairs with the same sample size is given by $100(i_{33,1}/i_{33,s})/s\%$. These results are presented in Table 2.

Table 2 shows that n discordant sib pairs contains at least three times more FIM about the recombination fraction than n random sib pairs. For example, when the minor allele frequency at QTL is $p = 0.30$, $a = 1$ and the recombination fraction $r = 0.2$ ($\psi = .68$), the top 10 DSP contain about 87% of the total FIM about r in 100 random sib pairs. Table 2 also shows that the more extreme sib pairs are, the higher the percentage of FIM relative to the sample size is. Thus, discordant sib pairs are more informative than random sib pairs. These results provide insight into the findings of Risch and Zhang [87] that the power of testing linkage substantially increases when extremely discordant sib pairs are used. Other methods for increasing power of linkage analysis using selected sib pairs were reviewed by Feingold [37].

6. DISCUSSION AND FUTURE RESEARCH

Fisher information plays an important role in inference with non-random samples. In this article, we have reviewed properties of exact Fisher information in order statistics based on their decompositions. The asymptotic Fisher information in order statistics is defined using the limit of decompositions of the exact Fisher information. Results have been extended to other censored data, e.g., concomitants, randomly censored data, and progressively censored data, and have been used to characterize some parametric distributions. Several applications of exact and asymptotic Fisher information have been given, which provide insight into robust or optimal properties of inference based on ordered data.

Research on Fisher information in order statistics has primarily focused on univariate order statistics. In many applications, bivariate samples are usually observed, e.g. life testing, and genetic data analysis. Fisher information has been studied for special bivariate samples $(X_{i:n}, Y_{[i]})$, $i = 1, \ldots, n$, where $Y_{[i]}$ are concomitants of order statistics $X_{i:n}$. General results are not available and ordering bivariate samples is not trivial [15]. Different methods of ordering bivariate data and inference based on them can be compared through Fisher information in the ordered bivariate data.

For random samples, Fisher information is related to Kullback-Leibler information [65]. Kullback-Leibler information in order statistics has been recently studied by Ebrahimi et al. [32] and Park [77]. The relationship of Fisher information and Kullback-Leibler information in order statistics and selected percentiles are of interest in theory and applications. In this review, we have focused on the expected Fisher information in order statistics. However, in practice, the observed Fisher information in order statistics may be used while making inference based on ordered data. So, properties of the observed Fisher information in different censored samples are worth studying. Ng et al. [71, 72] studied the inference under progressive censoring. The FIM may be used to design an optimal progressive censoring scheme where the removal may be determined to maximize the FIM in the censored data.

ACKNOWLEDGMENTS

We would like to thank Dr. Michael Proschan and Prof. Joseph Gastwirth for their careful reading and suggestions.

Received 22 September 2008

REFERENCES

[1] Abdushukrov, I. A. and Kim, L. V. (1987). Lower Cramer-Rao and Bhattacharya bounds for randomly censored observations. *J. Soviet Math.* 2171–2185.

[2] Abo-Eleneen, Z. A. and Nagaraja, H. N. (2002). Fisher information in an order statistic and its concomitant. *Ann. Inst. Statist. Math.* **54** 667–680. MR1932410

[3] Ahmadi, J. and Arghami, N. R. (2001). On the Fisher information in record values. *Metrika* **53** 195–206. MR1855494

[4] Ahmadi, J. and Arghami, N. R. (2003). Comparing the Fisher information in record values and IID observations. *Statistics* **37** 435–441. MR2006790

[5] Andrews, D. F., Bickel, P. J., Hampel, F. R., Huber, P. J., Rogers, W. H., and Tukey, J.W. (1972). *Robust Estimates of Location*, Princeton University Press, NJ. MR0331595

[6] Arnold, B. C., Balakrishnan, N., and Nagaraja, H. N. (1992). *A First Course in Order Statistics*, Wiley, NY. MR1178934

[7] Arnold, B. C., Balakrishnan, N., and Nagaraja, H. N. (1998). *Records*, Wiley, NY. MR1628157

[8] Arnold, B. C. and Meeden, G. (1975). Characterization of distributions by sets of moments of order statistics. *Ann. Statist.* **3** 754–758. MR0378264

[9] Azlarov, T. and Volodin, N. A. (1986). *Characterization Problems Associated With the Exponential Distribution*, Springer-Verlag, NY. MR0841073

[10] Balakrishnan, N. and Aggarwala, R. (2000). *Progressive Censoring: Theory, methods and Applications*, Birkhäuser, Boston, MA. MR1763020

[11] Balakrishnan, N., Childs, A., and Chandrasekar, B. (2002). An efficient computational method for moments of order statistics under progressive censoring. *Statist. Probab. Lett.* **60** 359–365. MR1947175

[12] Balakrishnan, N. and Cohen, A. C. (1991). *Order Statistics and Inference: Estimation Methods*, Academic Press, Boston, MA. MR1084812

[13] Balakrishnan, N. and Li, T. (2006). Fisher information and Tukey's linear sensitivity measure based on ordered ranked set samples. In Balakrishnan, N., Castillo, E., and Sarabia, J. M., editors, *Advances in Distribution Theory, Order Statisitcs, and Inference*, pp. 187–206. Birkhäuser, Boston, MA. MR2226222

[14] Balakrishnan, N. and Li, T. (2008). Ordered ranked set samples and applications to inference. *J. Statist. Plann. Inf.* **138** 3512–3524. MR2450092

[15] Barnett, V. (1976). The ordering of multivariate data (with discussion). *J. Roy. Statist. Soc., Ser. A* **139** 318–343. MR0445726

[16] Bhattacharyya, G. K. (1985). The asymptotics of maximum likelihood and related estimators based on Type II censored data. *J. Amer. Statist. Assoc.* **80** 398–404. MR0792740

[17] Carlin, B. P. and Gelfand, A. E. (1993). Parametric likelihood inference for record breaking problems. *Biometrika* **80** 507–515.

[18] Chan, L. K. (1967). On a characterization of distributions by expected values of extreme order statistics. *Amer. Math. Monthly.* **74** 950–951. MR0221699

[19] Chandrasekar, B. and Balakrishnan, N. (2002). On a multiparameter version of Tukey's linear sensitivity measure and its properties. *Ann. Inst. Statist. Math.* **54** 796–805. MR1954047

[20] Chandrasekar, B., Childs, A., and Balakrishnan, N. (2004). Exact likelihood inference for the exponential distribution under generalized Type-I and Type-II hybrid censoring. *Nav. Res. Log.* **51** 994–1004. MR2083561

[21] Chen, S. M. and Bhattacharyya, G. K. (1988). Exact confidence-bound for an exponential parameter under hybrid censoring. *Comm. Stat. – Theor. Meth.* **17** 1857–1870. MR0945789

[22] Chen, Z. (2000). The efficiency of ranked-set sampling relative to simple random sampling under multi-parameter families. *Statistica Sinica* **10** 247–263. MR1742111

[23] Chen, Z., Bai, Z. D., and Sinha, B. K. (2004). *Ranked Set Sampling*, Springer-Verlag, NY. MR2151099

[24] Chen, Y. Y., Hollander, M., and Lanberg, N. A. (1982). Small-sample results for the Kaplan–Meier estimator. *J. Amer. Statist. Assoc.* **77** 141–144. MR0648036

[25] Cheng, S. W. (1975). A unified approach to choosing optimum quantiles for the ABLE's. *J. Amer. Statist. Assoc.* **70** 155–159.

[26] Chernoff, H., Gastwirth, J. L., and Johns, M. V. (1967). Asymptotic distribution of linear combination of functions of order statistics with applications to estimation. *Ann. Math. Statist.* **38** 52–72. MR0203874

[27] Childs, A., Chandrasekar, B., and Balakrishnan, N. (2003). Exact likelihood inference on Type-I and Type-II hybrid censored samples from the exponential distribution. *Ann. Inst. Stat. Math.* **55** 319–330. MR2001867

[28] Cole, R. H. (1951). Relations between moments of order statistics. *Ann. Math. Statist.* **22** 308–310. MR0041390

[29] Csörgő, S. and Horváth, L. (1981). On the Koziol–Green model for random censoreship. *Biometrika* **68** 391–401. MR0626398

[30] David, H. A. and Nagaraja, H. N. (2003). *Order Statistics*, 3rd ed., Wiley, NY. MR1994955

[31] Dziubdziela, W. and Kopociński, B. (1976). Limiting properties of the kth record values. *Zastosowania Matematyki* **15** 187–190. MR0420788

[32] Ebrahimi, N., Soofi, E. S., and Zahedi, H. (2004). Information properties of order statistics and spacings. *IEEE Trans. Info. Theory* **50** 177–183. MR2051426

[33] Efron, B. and Johnstone, I. (1990). Fisher information in terms of the hazard rate. *Ann. Statist.* **18** 38–62. MR1041385

[34] Epstein, B. (1954). Truncated life tests in the exponential case. *Ann. Math. Statist.* **25** 555–564. MR0064372

[35] Escobar, L. A. and Meeker, W. Q. (1998). Fisher information matrices with censoring, truncation, and explanatory variables. *Statistica Sinica* **8** 221–237. MR1621160

[36] Escobar, L. A. and Meeker, W. Q. (2001). The asymptotic equivalence of the Fisher information matrices for type I and type II censored data from location-scale families. *Comm. Statist. – Theory Meth.* **30** 2211–2225. MR1862955

[37] Feingold, E. (2001). Methods for linkage analysis of quantitative trait loci in humans. *Theor. Popul. Biol.* **60** 167–180.

[38] Ferentinos, K. and Papaioannou, T. (1981). New parametric measures of information. *Information and Control* **51** 193–208. MR0686839

[39] Fullerton, J. et al. (2003). Linkage analysis of extremely discordant and concordant sibling pairs identifies quantitative-trait loci that influence variation in the human personality trait neuroticism. *Amer. J. Hum. Genet.* **72** 879–890.

[40] Gastwirth, J. L. (1965). Asymptotic most powerful rank tests for the two-sample problem with censored data. *Ann. Math. Statist.* **36** 1243–1247. MR0176580

[41] Gastwirth, J. L. (1966). On robust procedures. *J. Amer. Statist. Assoc.* **61** 929–948. MR0205397

[42] Gastwirth, J. L. and Rubin, H. (1969). On robust linear estimators. *Ann. Math. Statist.* **40** 24–39. MR0242329

[43] Gastwirth, J. L. and Wang, J. L. (1988). Control percentile test procedures for censored-data. *J. Statist. Plann. Inf.* **18** 267–276. MR0926631

[44] Gertsbakh, I. (1995). On the Fisher information in Type I censored and quantal response data. *Statist. Probab. Lett.* **23** 297–306. MR1342739

[45] Gertsbakh, I. and Kagan, A. M. (1999). Characterization of the Weibull distribution by properties of the Fisher information under type-I censoring. *Statist. Probab. Lett.* **42** 99–105. MR1671785

[46] Gupta, R. D. and Kundu, D. (1998). Hybrid censoring schemes with exponential failure distribution. *Comm. Statist. – Theory Meth.* **27** 3065–3083.

[47] Gupta, R. D. and Kundu, D. (2001). Exponentiated exponential family: An alternative to gamma and Weibull distributions. *Biometrical J.* **43** 117–130. MR1847768

[48] Gupta, R. D. and Kundu, D. (2006). On the comparison of Fisher information of the Weibull and GE distributions. *J. Statist. Plann. Inf.* **136** 3130–3144. MR2256220

[49] Hájek, J. and Šidák, Z. (1967). *Theory of Rank Tests*, Academic Press, NY.

[50] Halperin, M. (1952). Maximum likelihood estimation in truncated samples. *Ann. Math. Statist.* **23** 226–238. MR0048760

[51] Harter, H. L. and Balakrishnan, N. (1996). *CRC Handbook of Tables for the Use of Order Statistics in Estimation*, CRC Press, Boca Raton, FL. MR1383819

[52] Haseman, J. K. and Elston, R. C. (1972). The investigation of

linkage between a quantitative trait and a marker locus. *Behavior Genet.* **2** 3–19.

[53] Hofmann, G. and Balakrishnan, N. (2004). Fisher information in k-records. *Ann. Inst. Statist. Math.* **56** 383–396. MR2067163

[54] Hofmann, G., Balakrishnan, N., and Ahmadi, J. (2005). Characterization of hazard function factorization by Fisher information in minima and upper record values. *Statist. Probab. Lett.* **72** 51–57. MR2126293

[55] Hofmann, G. and Nagaraja, H. N. (2003). Fisher information in record data. *Metrika* **57** 177–193. MR1969251

[56] Hogg, R. V. (1974). Adaptive robust procedures: a partial review and some suggestions for future applications and theory. *J. Amer. Statist. Assoc.* **69** 909–923. MR0461779

[57] Hollander, M., Proschan, F., and Sconing, J. (1985). Efficiency loss with the Kaplan–Meier estimator. Technical Report M707, Department of Statistics, Florida State University, Tallahassee, FL.

[58] Huang, J. S. (1989). Moment problem of order statistics: A review. *Inter. Statist. Rev.* **57** 59–66.

[59] Huber, P. J. (1972). Robust statistics: A review. *Ann. Math. Statist.* **43** 1041–1067. MR0314180

[60] Iyengar, S., Kvam, P., and Singh, H. (1999). Fisher information in weighted distributions. *Canadian J. Statist.* **27** 833–841. MR1767150

[61] Johnson, R. A. (1974). Asymptotic results for inference procedures based on r smallest observations. *Ann. Statist.* **2** 1138–1151. MR0418313

[62] Kagan, A. M. and Landsman, Z. (1997). Statistical meaning of Carlen's superadditivity of the Fisher information. *Statist. Probab. Lett.* **32** 175–179. MR1436863

[63] Kagan, A. M., Linnik, Y. V., and Rao, C. R. (1973). *Characterization Problems in Mathematical Statistics*, Wiley, NY. MR0346969

[64] Koziol, J. A. and Green, S. B. (1976). A Cramer-von Mises statistic for randomly censored data. *Biometrika* **63** 465–474. MR0448695

[65] Kullback, S. (1959). *Information Theory and Statistics*, Wiley, NY. MR0103557

[66] Kulldorff, G. (1973). A note on the optimum spacing of sample quantiles from the six extreme value distributions. *Ann. Statist.* **1** 562–567. MR0356339

[67] Lawless, J. K. (1982). *Statistical Models and Methods for Lifetime Data*, Wiley, NY. MR0640866

[68] Mehrotra, K. G., Johnson, R. A., and Bhattacharyya, G. K. (1979). Exact Fisher information for censored samples and the extended hazard rate functions. *Comm. Statist. – Theory Meth.* **15** 1493–1510. MR0547411

[69] Miller, R. G. (1983). What price Kaplan–Meier? *Biometrics* **39** 1077–1081. MR0733872

[70] Nagaraja, H. N. (1994). Tukey's linear sensitivity and order statistics. *Ann. Inst. Statist. Math.* **46** 757–768. MR1325994

[71] Ng, H. K. T., Chan, P. S., and Balakrishnan, N. (2002). Estimation of parameters from progressively censored data using EM algorithm. *Comput. Statist. Data Anal.* **39** 371–386. MR1910018

[72] Ng, H. K. T., Chan, P. S., and Balakrishnan, N. (2004). Optimal progressive censoring plans for the Weibull distribution. *Technometrics* **46** 470–481. MR2101515

[73] Ogawa, J. (1951). Contributions to the theory of systematic statistics I. *Osaka Math. J.* **3** 175–213. MR0046621

[74] Park, S. (1996). Fisher information in order statistics. *J. Amer. Statist. Assoc.* **91** 385–390. MR1394095

[75] Park, S. (2003). On the asymptotic Fisher information in order statistics. *Metrika* **57** 71–80. MR1963712

[76] Park, S. (2005a). On calculating the Fisher information in order statistics. *Statist. Papers* **46** 293–301. MR2155717

[77] Park, S. (2005b). Testing exponentiality based on the Kullback–Leibler information with the type II censored data. *IEEE Trans, Reliabil.* **54** 22–26.

[78] Park, S., Balakrishnan, N., and Zheng, G. (2008). Fisher information in hybrid censored data. *Statist. Probab. Lett.* **78** 2781-2786.

[79] Park, S. and Balakrishnan, N. (2008). On simple calculation of the Fisher information in hybrid censoring schemes. Submitted.

[80] Park, S. and Zheng, G. (2004). Equal Fisher information in order statistics. *Sankhyā* **66** 20–34. MR2082905

[81] Patel, K. R., Mudholkar, G. S., and Indrasiri Fernando, J. L. (1988). Student's t approximations for three simple robust estimators. *J. Amer. Statist. Assoc.* **83** 1203–1210. MR0997604

[82] Pollak, M. (1973). On equal distribution. *Ann. Statist.* **1** 180–182. MR0331582

[83] Prakasa Rao, B. L. S. (1992). *Identifiability in Stochastic Models – Characterization of Probability Distributions*, Academic Press, Boston. MA. MR1171013

[84] Prakasa Rao, B. L. S. (1995). Remarks on Cramer–Rao type integral inequalities for randomly censored data. In *Analysis of Censored Data* (Eds., H.L. Koul and J.V. Deshpande), IMS Lecture Notes, Vol. **27**, Institute of Mathematical Statistics, Hayward, CA. MR1483346

[85] Rao, C. R. (1973). *Linear Statistical Inference and Its Applications*, 2nd ed., Wiley, NY. MR0346957

[86] Rao, C. R. and Shanbhag, D. N. (1998). Recent approaches to characterizations based on order statistics and record values. In *Order Statistics: Theory and Methods* (Eds., N. Balakrishnan and C.R. Rao), Handbook of Statistics **16**, pp. 231–256, North-Holland, Amsterdam. MR1668747

[87] Risch, N. and Zhang, H. (1995). Extreme discordant sib pairs for mapping quantitative trait loci in humans. *Science* **268** 1584–1589.

[88] Sen, P. K. (1967). Asymptotically most powerful rank order tests for grouped data. *Ann. Math. Statist.* **38** 1229–1239. MR0212955

[89] Srikantan, K. S. (1962). Recurrence relations between the pdf's of order statistics and some applications. *Ann. Math. Statist.* **33** 169–177. MR0137237

[90] Stokes, S. L. (1980). Inference on the correlation coefficient in bivariate normal population from ranked set samples. *J. Amer. Statist. Assoc.* **75** 989–995. MR0600988

[91] Stokes, S. L. (1995). Parametric ranked set sampling. *Ann. Inst. Statist. Math.* **47** 465–482. MR1364255

[92] Stepanov, A. V., Balakrishnan, N., and Hofmann, G. (2003). Exact distribution and Fisher information of weak record values. *Statist. Probab. Lett.* **64** 69–81. MR1995811

[93] Takahashi, H. and Sugiura N. (1989). The rate of convergence of Fisher information for Type II censored sample. *J. Japan. Statist. Soc.* **19** 139–144. MR1046589

[94] Tukey, J. W. (1965). Which part of the sample contains the information? *Proc. Nat. Acad. Sci., USA* **53** 127–134. MR0172387

[95] van Eeden, C. (1963). The relation between Pitman's asymptotic relative efficiency of two-tests and the correlation coefficient between their test statistics. *Ann. Math. Statist.* **34** 1442–1451. MR0157419

[96] Wang, Y. and He, S. (2005). Fisher information in censored data. *Statist. Probab. Lett.* **73** 199–206. MR2159255

[97] Zheng, G. (2000). On the rate of convergence of Fisher information in multiple type II censored data. *J. Japan. Statist. Soc.* **30** 197–204. MR1841885

[98] Zheng, G. (2001). A characterization of the factorization of hazard function by the Fisher information under Type II censoring with application to the Weibull family. *Statist. Probab. Lett.* **52** 249–253. MR1838212

[99] Zheng, G. (2002). On the Fisher information matrix in type II censored data from the exponentiated exponential family. *Biometrical J.* **44** 353–357. MR1898411

[100] Zheng, G. (2004). Some remarks on the Fisher information in ranked set samples. *Comm. Statist. – Theory. Meth.* **33** 1511–1525. MR2066365

[101] Zheng, G. and Gastwirth, J. L. (2000). Where is the Fisher

information in an ordered sample? *Statistica Sinica* **10** 1267–1280. MR1804545

[102] Zheng, G. and Gastwirth, J. L. (2001). On the Fisher information in randomly censored data. *Statist. Probab. Lett.* **52** 421–426. MR1841610

[103] Zheng, G. and Gastwirth, J. L. (2002). Do tails of symmetric distributions contain more Fisher information about the scale parameter? *Sankyhā, Ser. B* **64** 289–300. MR1993915

[104] Zheng, G. and Gastwirth, J. L. (2003). Fisher information in ordered randomly censored data with applications to characterization problems. *Statistica Sinica* **13** 507–517. MR1977740

[105] Zheng, G. and Gastwirth, J. L. (2005). Fisher information in randomly sampled sib pairs and extremely discordant sib pairs in genetic analysis for a quantitative trait locus. *J. Statist. Plann. Inf.* **130** 299–315. MR2128009

[106] Zheng, G. and Park, S. (2004). On the Fisher information in multiply censored and progressively censored data. *Comm. Statist. – Theory. Meth.* **33** 1821–1835. MR2065176

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