

Preface

This is the second and last part of an introduction to analysis which is based on my beginner's course Analysis I–III and the more advanced course Tensoranalysis that I use to give in Heidelberg from time to time.

The present volume comprises material for a four semesters course. It starts with a fairly comprehensive introduction to functional analysis, which might serve as a basis for a separate independent lecture.

In the next two chapters the theory of differentiation in Banach spaces and the fundamental existence theorems in analysis are treated in great detail and generality.

In Chapter 9 we develop the existence and regularity theory for ordinary differential equations in Banach spaces always having in mind to apply these results later to differential equations of arbitrary order in semi-Riemannian manifolds.

The last three chapters, Chapter 10–12, contain some fairly advanced topics from measure theory and differential geometry. In addition to providing the basic definitions and results of these theories we included material that is of great importance from an analytical point of view like covering theorems, Hausdorff measures and vector valued measures, or a thorough treatment of submanifolds, tubular neighbourhoods, the Riemannian and Lorentzian distance functions with respect to a hypersurface, and solving evolution equations in manifolds.

The only topic that is missing, at least from my point of view, is the treatment of partial differential equations, especially looking at partial differential equations in manifolds. However, after reading the material in Chapter 11 and 12, anyone, who knows the PDE theory in Euclidean space, should be able to apply this theory to PDE problems in semi-Riemannian manifolds.

I would like to thank Heiko Kröner for proof reading large parts of the final manuscript and Shing-Tung Yau for accepting the manuscript for the International Series in Analysis.

Heidelberg, April 2005

Claus Gerhardt

Contents

Chapter 6. Elements of functional analysis	1
6.1. Linear and multilinear mappings	1
6.2. The Hahn-Banach theorem	5
6.3. Compact operators	8
6.4. Baire's theorem and its applications	10
6.5. Reflexive Banach spaces	14
6.6. Completion of metric spaces	24
6.7. Elementary properties of Hilbert spaces	27
6.8. The Fréchet-Riesz representation theorem	31
6.9. Adjoint, selfadjoint and normal operators	37
6.10. Projectors	43
6.11. Orthonormal bases and Fourier series	49
6.12. Spectral decomposition of compact, selfadjoint operators	54
6.13. Fredholm's alternative	59
Chapter 7. Differentiation in Banach spaces	61
7.1. Differentiable functions	61
7.2. The mean value theorem and its consequences	69
7.3. Differentiation of sequences of functions	70
7.4. Partial derivatives	73
7.5. Derivatives of higher order	82
7.6. Taylor's formula	96
7.7. Local extrema	98
Chapter 8. Existence theorems	103
8.1. Banach's fixed point theorem	103
8.2. The inverse function theorem	105
8.3. The implicit function theorem	113
8.4. The rank theorem	118
8.5. Extrema with side conditions	122
Chapter 9. Ordinary differential equations	127
9.1. Local existence and uniqueness	127
9.2. Comparison theorems	133
9.3. Linear differential equations	138
9.4. Regularity results for the flow of a vector field	142

9.5.	Affine differential equations of first order	156
9.6.	Ordinary differential equations of higher order	165
9.7.	Linear differential operators with constant coefficients	167
Chapter 10.	Lebesgue's Theory of Integration	173
10.1.	Measures and measurable functions	173
10.2.	The Lebesgue Integral	183
10.3.	The Riesz representation theorem	194
10.4.	Approximation theorems	205
10.5.	Covering theorems	211
10.6.	The Lebesgue measure in \mathbb{R}^n	214
10.7.	Transformation rule for integrals	223
10.8.	The theorem of Fubini	228
10.9.	Complex measures	235
10.10.	Vector valued measures	240
10.11.	L^p -spaces	244
10.12.	Mollification of integrable functions	254
Chapter 11.	Tensor analysis	259
11.1.	Manifolds and tangential spaces	259
11.2.	The tensor product	265
11.3.	Semi-Riemannian manifolds	273
11.4.	The Riemannian curvature tensor	280
11.5.	Ordinary differential equations in manifolds	289
11.6.	Geodesics and the exponential map	299
11.7.	The Hopf-Rinow theorem	310
11.8.	Measure and integration theory in manifolds	316
11.9.	Differentiable mappings between manifolds	325
Chapter 12.	Theory of submanifolds	329
12.1.	Submanifolds	329
12.2.	Hypersurfaces in semi-Riemannian manifolds	333
12.3.	Submanifolds of higher codimension	341
12.4.	Polar coordinates in \mathbb{R}^{n+1}	343
12.5.	Gaussian coordinate systems	348
12.6.	Global Gaussian coordinate systems	363
12.7.	First and second variation of the arc length	373
	Bibliography	383
	List of Symbols	385
	Index	389