

## CONVEX HULL PROPERTIES OF HARMONIC MAPS

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### 0. Introduction

In 1975, Yau [18] proved, by way of a gradient estimate, that a complete manifold  $M$  with non-negative Ricci curvature must satisfy the strong Liouville property for harmonic functions. The strong Liouville property (Liouville property) asserts that any positive (bounded) harmonic function defined on  $M$  must be identically constant. In 1980, Cheng [4] generalized the gradient estimate to harmonic maps from a manifold  $M$  with non-negative Ricci curvature to a Cartan-Hadamard manifold  $N$ . In particular, the Liouville property for harmonic maps can be derived for this situation. The Liouville property for harmonic maps asserts that if the image of the harmonic map is contained in a bounded set, then the map must be identically constant. In fact, Cheng's gradient estimate actually yields a slightly stronger theorem. It implies that if a harmonic map from a manifold with non-negative Ricci curvature into a Cartan-Hadamard manifold is of sublinear growth, then the map must be constant. A map  $u : M \rightarrow N$  is of sublinear growth if there exist a point  $p \in M$  and a point  $o \in N$  such that the distance  $d(u(x), o)$  between the image of  $u$  to the point  $o$  satisfies

$$d(u(x), o) = o(\rho(x)),$$

with  $\rho(x)$  being the distance from  $x \in M$  to  $p$ . Later, Kendall [10] proved that if a stochastically complete manifold satisfies the Liouville

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property for harmonic functions, then it must also satisfy the Liouville property for harmonic maps into a Cartan-Hadamard manifold.

A few years ago, Grigor'yan [8] and Saloff-Coste [15] proved that if a manifold  $M$  satisfies a weak Poincaré inequality ( $\mathcal{P}$ ) and a weak volume growth property ( $\mathcal{V}$ ) (see §3 for definition), then it must satisfy a Harnack inequality for harmonic functions. A consequence of the Harnack inequality is the strong Liouville property for harmonic functions. The Harnack inequality also implies that there exists a constant  $0 < \alpha \leq 1$  such that any harmonic function  $f$ , defined on  $M$ , satisfying the growth condition

$$|f(x)| = o(\rho^\alpha(x))$$

as  $x \rightarrow \infty$ , must be identically constant. Since it is known that a manifold with non-negative Ricci curvature satisfies conditions ( $\mathcal{P}$ ) and ( $\mathcal{V}$ ) and also both of these conditions are invariant under quasi-isometries, this will include the class of manifolds which are quasi-isometric to a manifold with non-negative Ricci curvature.

Two years ago, Shen [14] formulated a version of the strong Liouville property for harmonic maps into a Cartan-Hadamard manifold  $N$  with strictly negative curvature. A manifold is said to have strictly negative curvature if its sectional curvature  $K_N$  is bounded from above by some negative constant  $-a < 0$ . Shen proved that if the image of a harmonic map from a manifold with non-negative Ricci curvature lies inside a horoball of  $N$ , then it must be a constant map. In a recent paper, Tam [16] generalized the result of Grigor'yan and Saloff-Coste and the result of Shen to harmonic maps from manifolds satisfying conditions ( $\mathcal{P}$ ) and ( $\mathcal{V}$ ). In particular, he proved that if  $M$  is a manifold satisfying conditions ( $\mathcal{P}$ ) and ( $\mathcal{V}$ ), then there exists a constant  $0 < \alpha \leq 1$  such that any harmonic map  $u$  from  $M$  into a Cartan-Hadamard manifold satisfying

$$d(u(x), o) = o(\rho^\alpha(x)),$$

as  $x \rightarrow \infty$ , must be the constant map. He also proved that if  $M$  is a manifold satisfying conditions ( $\mathcal{P}$ ) and ( $\mathcal{V}$ ), then any harmonic map whose image is in a horoball of a Cartan-Hadamard manifold with strictly negative curvature must be a constant map.

In a different direction, a recent article of Han, Tam, Treibergs, and Wan [9] discussed the structure of the image sets of harmonic diffeomorphisms from the Euclidean plane  $\mathbb{R}^2$  into the hyperbolic plane  $\mathbb{H}^2$ . In this setting, they proved that if the Hopf differential is polynomial of degree  $\ell$ , then the image of the harmonic map must be an ideal polygon

with  $\ell + 2$  vertices. Conversely, if the image of the harmonic diffeomorphism  $u$  is an ideal polygon with  $\ell + 2$  vertices, and if, in addition, the conformal metric  $||\partial u||^2|dz|^2$  is complete, then the Hopf differential must be polynomial of degree  $\ell$ . The arguments involved in proving these results utilized many key facts which are specific to dimension 2. The first fact is that one can realize such a harmonic diffeomorphism as Gauss map of some constant mean curvature, space like, complete, hypersurface in Minkowski space. Another purely 2-dimensional phenomenon used in the proof is the Gauss-Bonnet theorem. It is evident that such a clean statement is not to be expected in higher dimensions. However, it is hopeful that some form of higher dimension statement which can be viewed as a partial generalization to this result would be valid.

The purpose of this note is to study the image sets of harmonic maps  $u : M \rightarrow N$  from a manifold, whose space of bounded harmonic function  $\mathcal{H}_0(M)$  is finite dimensional, into a Cartan-Hadamard manifold. In view of the above discussion, a manifold with non-negative Ricci curvature has  $\dim \mathcal{H}_0(M) = 1$ . More generally, because of the results of Grigor'yan and Saloff-Coste, a manifold satisfying conditions  $(\mathcal{P})$  and  $(\mathcal{V})$  also has  $\dim \mathcal{H}_0(M) = 1$ . In fact, using a recent theorem of the first author [11], manifolds satisfying a mean value inequality  $(\mathcal{M})$  (see §3 for definition) and condition  $(\mathcal{V})$  will have  $\dim \mathcal{H}_0(M) < \infty$ . The interested reader should refer to [11] and the survey paper [12] for a detail comparison of the various conditions.

Let us denote  $S_\infty(N)$  to be the geometric boundary of the Cartan-Hadamard manifold  $N$ , and  $A = \overline{u(M)} \cap S_\infty(N)$  to be the image of  $u$  in the geometric boundary. Suppose  $\{A_n\}$  is any monotonically decreasing sequence of closed subsets of  $S_\infty(N)$  with the properties that  $A$  is properly contained in each  $A_n$  and  $\bigcap_{n=1}^\infty A_n = A$ . We will show in Theorem 2.1 that there exists a set of  $k$  points  $\{y_i\}_{i=1}^k$  in  $\overline{u(M)} \cap N$ , with  $k \leq \dim \mathcal{H}_0(M)$ , such that

$$u(M) \subset \bigcap_{n=1}^\infty \mathcal{C}(\{y_i\}_{i=1}^k \cup A_n),$$

where  $\mathcal{C}(\{y_i\}_{i=1}^k \cup A_n)$  denotes the convex hull over the set  $\{y_i\}_{i=1}^k \cup A_n$ . In the event when  $A = \emptyset$ , then  $u(M) \subset \mathcal{C}(\{y_i\}_{i=1}^k)$ . When  $A \neq \emptyset$ , it is tempting to conclude that  $u(M) \subset \mathcal{C}(\{y_i\}_{i=1}^k \cup A)$ . Unfortunately, we do not know if  $\bigcap_{n=1}^\infty \mathcal{C}(\{y_i\}_{i=1}^k \cup A_n) = \mathcal{C}(\{y_i\}_{i=1}^k \cup A)$  is valid for arbitrary Cartan-Hadamard manifolds. However, if we assume that  $N$  satisfies a separation property (Definition 1.4), then we prove in Lemma 1.5 that this is the case. It is easily verified that two-dimensional visibility

manifolds and hyperbolic spaces have separation property. On the other hand, for Cartan-Hadamard manifolds with strongly negative sectional curvature, i.e.,  $K_N$  is bounded between two negative constants, we show that the two sets  $\bigcap_{n=1}^{\infty} \mathcal{C}(\{y_i\}_{i=1}^k \cup A_n)$  and  $\mathcal{C}(\{y_i\}_{i=1}^k \cup A)$  are bounded distance away from each other. It turns out that this is sufficient for us to conclude that  $u(M) \subset \mathcal{C}(\{y_i\}_{i=1}^k \cup A)$ .

Observe that the convex hull theorem asserts that if  $\dim \mathcal{H}_0(M) = 1$  and  $u(M)$  is contained in a bounded set, then  $A = \emptyset$  and  $u(M)$  must lie in the convex hull of one point. This is precisely Kendall's theorem (also see [16]) without the assumption that  $M$  is stochastically complete. Another application of the convex hull theorem is given in Theorem 2.4, where we assume that  $N$  either is a two-dimensional visibility manifold or has strongly negative sectional curvature, and that all positive harmonic functions on  $M$  are bounded with  $\dim \mathcal{H}_0(M) = k_0 < \infty$ . In this case, if the image at infinity  $A = \overline{u(M)} \cap S_{\infty}(N)$  of the harmonic map has at most 1 point, then  $A$  is in fact empty and  $u(M)$  must be contained in the convex hull of at most  $k_0$  points in  $N$ . When  $k_0 = 1$ , this generalizes the horoball theorems of Shen and Tam (Corollary 2.5) for this special case of  $N$ .

In §3, we consider harmonic maps from a manifold satisfying condition  $(\mathcal{M})$  and having polynomial volume growth, into a Cartan-Hadamard manifold satisfying the separation property at infinity (Definition 1.7). We will prove that if the harmonic map is polynomial growth of at most degree  $\ell$ , then  $A = \overline{u(M)} \cap S_{\infty}(N)$  must have only finitely many points. Moreover, the number of points can be estimated by  $\ell$ . Combining with Theorem 2.1, we conclude that (Theorem 3.5)  $u(M)$  must lie in the convex hull over a finite set of interior vertices  $\{y_j\}$  union a finite set of boundary vertices  $\{a_i\}$  if the target manifold  $N$  is either a two-dimensional visibility manifold or has strongly negative sectional curvature. If we impose the stronger assumption that  $M$  satisfies condition  $(\mathcal{P})$  also, then the set of interior vertices  $\{y_j\}$  must be empty unless  $u$  is a constant map. In particular, the image must lie in a convex hull over a finite number of boundary points, which we call vertices at infinity. Moreover, the number of vertices at infinity is bounded by a constant depending on  $\ell$ . This last result can be viewed as the partial higher dimensional generalization of the theorem of Han-Tam-Treibergs-Wan.

The key ingredient in §3 is the notion of  $\ell$ -massive sets (Definition 3.1). In Grigor'yan's [7] work, he defined massive sets, which are the same as 0-massive sets, to study the space  $\mathcal{H}_0(M)$ . He showed that  $\dim \mathcal{H}_0(M)$  is given by the maximum number of disjoint massive sets in

$M$ . If we define the space  $\mathcal{H}_\ell(M)$  of harmonic functions with polynomial growth of at most degree  $\ell$ , it is not known that  $\dim \mathcal{H}_\ell(M)$  is related to the maximum number of disjoint  $\ell$ -massive sets. However, in [11] the first author estimated  $\dim \mathcal{H}_\ell(M)$  on a manifold satisfying conditions  $(\mathcal{M})$  and  $(\mathcal{V})$ . It turns out that by modifying that argument, one can also estimate the maximum number of disjoint  $\ell$ -massive sets on  $M$ . This estimate allows us to bound the number of points in  $A = \overline{u(M)} \cap S_\infty(N)$ .

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### 1. Properties of Convex Hull

Throughout this paper, we shall assume that  $N$  is a Cartan-Hadamard manifold, namely,  $N$  is simply connected and has nonpositive sectional curvature. It is well known that  $N$  can be compactified by adding a sphere at infinity  $S_\infty(N)$ . The resulting compact space  $\overline{N} = N \cup S_\infty(N)$  is homeomorphic to a closed Euclidean ball. Two geodesic rays  $\gamma_1$  and  $\gamma_2$  in  $N$  are called equivalent if their Hausdorff distance is finite. Then the geometric boundary  $S_\infty(N)$  is simply given by the equivalence classes of geodesic rays in  $N$ . A sequence of points  $\{x_n\}$  in  $\overline{N}$  converges to  $x \in \overline{N}$  if for some fixed point  $p \in N$ , the sequence of geodesic rays  $\{\overline{px_n}\}$  converges to a geodesic ray  $\gamma \in x$ . In this case, we say  $\gamma$  is the geodesic segment  $\overline{px}$  joining  $p$  to  $x$ . Recall that a subset  $C$  in  $N$  is strictly convex if any geodesic segment between any two points in  $C$  is also contained in  $C$ . For a subset  $K$  in  $N$ , the convex hull of  $K$ , denoted by  $\mathcal{C}(K)$ , is defined to be the smallest strictly convex subset  $C$  in  $N$  containing  $K$ . The convex hull can also be obtained by taking the intersection of all convex sets  $C \subset N$  containing  $K$ . When  $N$  is a Cartan-Hadamard manifold, there is only one geodesic segment joining a pair of points in  $N$ . In this case, there is only one notion of convexity, and we will simply say a set is convex when it is a strictly convex set. For the purpose of this article, we will need a notion of convexity for  $\overline{N}$ . Since a geodesic line is a geodesic segment joining the two end points in  $S_\infty(N)$ , it still makes sense to talk about geodesics joining two points in  $\overline{N}$ . However, it is not true in general that any two points in  $S_\infty(N)$  can always be joined by a geodesic segment given by a geodesic line, as indicated by two non-antipodal points in  $S_\infty(\mathbb{R}^n)$ . If every pair of points

in  $S_\infty(N)$  can be joined by a geodesic line in  $N$ , then  $N$  is said to be a visibility manifold. This class of manifolds was extensively studied in [6]. A typical example of a visibility manifold is a Cartan-Hadamard manifold with sectional curvature bounded from above by a negative constant  $-a < 0$ .

To remedy the situation when  $N$  is not a visibility manifold, we defined a generalized notion of geodesic segment joining two points at infinity.

**Definition 1.1.** A geodesic segment  $\gamma$  joining a pair of points  $x$  and  $y$  in  $\overline{N}$  is the limiting set of a sequence of geodesic segments  $\{\gamma_n\}$  in  $N$  with end points  $\{x_n\}$  and  $\{y_n\}$  such that  $x_n \rightarrow x$  and  $y_n \rightarrow y$ . We will denote  $\gamma$  by  $\overline{xy}$ .

Observe that if  $\overline{xy} \cap S_\infty(N) = \{x, y\}$ , then  $\overline{xy}$  must be a geodesic line in  $N$  and hence a geodesic segment in the traditional sense. For the case of two non-antipodal points in  $S_\infty(\mathbb{R}^2)$ , the shortest arc on  $\mathbb{S}^1 = S_\infty(\mathbb{R}^2)$  joining the two points will be the geodesic segment in the sense defined above. If the two points are antipodal in  $S_\infty(\mathbb{R}^2)$ , say the northpole and the southpole, then there are infinitely many geodesic segments joining them. Each vertical line is a geodesic segment in the genuine sense. Also, both arcs on  $\mathbb{S}^1$  joining the two poles are geodesic segments joining them. Using this definition, for a pair of points in  $S_\infty(N)$ , it is possible to have more than one geodesic segments joining them. The convexity we will define will be in the sense of strictly convex.

**Definition 1.2.** A subset  $C$  of  $\overline{N}$  is a convex set if for every pair of points in  $C$ , any geodesic segment joining them is also in  $C$ .

**Definition 1.3.** For a subset  $A$  in  $\overline{N}$ , we define its convex hull  $\mathcal{C}(A)$  to be the smallest convex subset of  $\overline{N}$  containing  $A$ .

In what follows, when we say that a subset is closed, we mean that it is closed in  $\overline{N}$  unless otherwise noted. In general, we denote the closure for a subset  $A$  in  $\overline{N}$  by  $\overline{A}$ . For a given sequence of closed subsets  $\{A_n\}$  decreasing to  $A$ , it is natural to ask whether the convex hull of  $A_n$  in  $\overline{N}$  decreases to the convex hull of  $A$ . For this purpose, we introduce the following definition.

**Definition 1.4.** A Cartan-Hadamard manifold  $N$  is said to satisfy the separation property if for every closed convex subset  $A$  in  $\overline{N}$  and every point  $p$  not in  $A$ , there exists a closed convex set  $C$  properly containing  $A$  and separating  $p$  from  $A$ , i.e.,  $A \subset C$ ,  $A \cap S_\infty(N)$  is contained in the interior of  $C \cap S_\infty(N)$  and  $p$  is not in  $C$ .

For a two-dimensional visibility manifold or a Cartan-Hadamard manifold with constant negative curvature, it is easy to check that the separation property holds. In fact, for a point  $p$  not in the closed convex set  $A$ , pick up a point  $q \in A$  such that  $d(p, q) = d(p, A)$ . Then the convexity of  $A$  and the first variation formula imply that for  $z \in A$ ,  $\angle(\overline{zq}, \overline{qp}) \geq \pi/2$ . Let  $x$  be the midpoint of the geodesic segment between  $p$  and  $q$ , and

$$C = \{y \in \overline{N} : \angle(\overline{yx}, \overline{xp}) \geq \pi/2\}.$$

Then  $C$  is closed, convex as  $\partial C$  is evidently totally geodesic and  $C$  properly separates  $p$  from  $A$ .

**Lemma 1.5.** *A Cartan Hadamard manifold  $N$  satisfies the separation property if and only if for every closed subset  $A$  and monotone decreasing sequence of closed subsets  $\{A_n\}$  in  $\overline{N}$  such that  $\bigcap_{n=1}^\infty A_n = A$ , then*

$$\bigcap_{n=1}^\infty \overline{\mathcal{C}(A_n)} = \overline{\mathcal{C}(A)}.$$

*Proof.* Suppose that  $N$  satisfies the separation property. Let  $\{A_n\}$  in  $\overline{N}$  be a decreasing sequence of closed subsets with  $\bigcap_{n=1}^\infty A_n = A$ . Obviously,

$$\overline{\mathcal{C}(A)} \subset \bigcap_{n=1}^\infty \overline{\mathcal{C}(A_n)}$$

from the definition of convex hull. Assume the contrary that

$$\bigcap_{n=1}^\infty \overline{\mathcal{C}(A_n)} \neq \overline{\mathcal{C}(A)}.$$

Then there exists a point  $p \in \bigcap_{n=1}^\infty \overline{\mathcal{C}(A_n)}$  but not in  $\overline{\mathcal{C}(A)}$ . The separation property asserts that there is a closed convex subset  $C$  properly separating  $p$  from  $\overline{\mathcal{C}(A)}$ . Let

$$C_\epsilon = \{x \in N : d(x, C) \leq \epsilon\}$$

be the  $\epsilon$ -neighborhood of  $C$ . For sufficiently small  $\epsilon > 0$ ,  $\overline{C_\epsilon}$  also properly separates  $p$  from  $\overline{\mathcal{C}(A)}$ . Since  $A \cap S_\infty(N)$  is contained in the interior of  $C \cap S_\infty(N)$  and  $A_n$  is decreasing to  $A$ , we conclude that for  $n$  sufficiently large,  $A_n \subset \overline{C_\epsilon}$ . Thus,  $\overline{\mathcal{C}(A_n)} \subset \overline{C_\epsilon}$  and  $p \in \overline{C_\epsilon}$ , which is a contradiction.

Conversely, to show that  $N$  satisfies the separation property, let  $A$  be a closed convex subset and  $p$  a point not in  $A$ . We identify  $\overline{N}$  with the closed unit ball of the Euclidean space endowed the canonical metric. Let  $A_n$  be the tubular neighborhood of  $A$  of size  $1/n$ . It is then clear

that  $A_n$  is a decreasing sequence of closed subsets with  $\bigcap_{n=1}^{\infty} A_n = A$ . Hence by the assumption,

$$\bigcap_{n=1}^{\infty} \overline{\mathcal{C}(A_n)} = \overline{\mathcal{C}(A)} = A.$$

The fact that  $p \notin A$  implies that  $p \notin \overline{\mathcal{C}(A_n)}$  for sufficiently large  $n$ . By choosing  $C = \overline{\mathcal{C}(A_n)}$ , it is clear that  $C$  properly separates  $p$  from  $A$ .

q.e.d.

The above Lemma indicates that the separation property is quite natural in the study of convex sets. We should point out that it is not known if a Cartan-Hadamard manifold with strictly negative curvature (or more generally a visibility manifold) satisfies the separation property. However, using the result in [1] (also see [2]) we show that every Cartan-Hadamard manifold with sectional curvature satisfying

$$-b \leq K_N \leq -a < 0$$

must satisfy the following statement.

**Lemma 1.6.** *Let  $N$  be a Cartan-Hadamard manifold. Suppose that its sectional curvature satisfies  $-b \leq K_N \leq -a$  for some positive constants  $a$  and  $b$ . Suppose  $d_H(A, B)$  denotes the Hausdorff distance between the two sets  $A$  and  $B$  in  $\overline{N}$ . For every closed subset  $A$  and monotone decreasing sequence of closed subsets  $\{A_n\}$  in  $\overline{N}$  such that  $\bigcap_{n=1}^{\infty} A_n = A$ , then*

$$d_H(\bigcap_{n=1}^{\infty} \overline{\mathcal{C}(A_n)}, \overline{\mathcal{C}(A)}) < \infty.$$

*Proof.* For a point  $q \in N$ , let  $v \in T_q(N)$  be a tangent vector at  $q$ . We denote

$$T(v, \theta) = \{x \in N : \angle(v, \overline{qx}) < \theta\}$$

to be the cone of angle  $\theta$  around  $v$ . The truncated cone is given by

$$T(v, \theta, r) = \{x \in N : \angle(v, \overline{qx}) < \theta, d(x, q) > r\}.$$

According to [1], for any given  $0 < \alpha < \beta < \pi$ ,  $q \in N$  and nonzero tangent vector  $v$  at point  $q$ , there exists a constant  $R_0(\alpha, \beta, a, b) > 0$  independent of  $q$  and  $v$  such that one can construct a closed convex set  $C$  in  $\overline{N}$  satisfying

$$B_q(R_0) \subset C,$$

$$T(-v, \pi - \beta) \cap S_\infty(N) \subset C \cap S_\infty(N),$$

and

$$T(v, \alpha, R_0 + \eta) \cap C = \emptyset$$

for all  $\eta > 0$ . Let us fix  $0 < \alpha < \beta < \pi/2$  and the corresponding  $R_0$ . For a given closed convex set  $B$  and point  $p$  not in  $B$  with  $d(p, B) > R_0$ , we claim that there exists a closed convex set  $C$  properly separating  $p$  from  $B$ . In fact, choose  $q \in B$  such that  $d(p, q) = d(p, B)$ . Let  $\gamma$  be the geodesic segment  $\overline{qp}$  with  $\gamma(0) = q$ . Let  $v = \gamma'(0)$ . Then we have a closed convex set  $C$  such that  $B_q(R_0) \subset C$  and

$$T(v, \alpha, R_0 + \eta) \cap C = \emptyset$$

for all  $\eta > 0$ . In particular,  $p$  is not in  $C$ . We now claim that  $B \subset C$  and  $B \cap S_\infty(N)$  is contained in the interior of  $C \cap S_\infty(N)$ . Note that by the choice of  $q$  and the first variation formula, we have  $\angle(\overline{yq}, -v) \leq \pi/2$  for  $y \in B$ . Since  $\beta < \pi/2$ , one concludes that the point at infinity given by the geodesic ray emanating from  $q$  and passing through  $y$  is in  $T(-v, \pi - \beta) \cap S_\infty(N)$ . In particular, it is in the set  $C \cap S_\infty(N)$ . Since  $q \in C$  and  $C$  is convex, the whole geodesic ray must lie in  $C$ . Thus,  $y \in C$ , and  $B \subset C$ . This argument also implies that  $B \cap S_\infty(N)$  is contained in  $T(-v, \pi - \beta) \cap S_\infty(N)$ , which is evidently in the interior of  $C \cap S_\infty(N)$ . In conclusion,  $C$  properly separates  $p$  from  $B$  and the claim follows. We now apply the claim to the case  $B = \overline{\mathcal{C}(A)}$ . If there exists a point  $p \in \bigcap_{n=1}^\infty \overline{\mathcal{C}(A_n)}$  such that  $d(p, \overline{\mathcal{C}(A)}) > R_0$ , then there exists a closed convex set  $C$  properly separating  $p$  from  $\overline{\mathcal{C}(A)}$ . Since  $\bigcap_{n=1}^\infty A_n = A$ , we have  $A_n \subset C$  for sufficiently large  $n$ . In particular,  $\overline{\mathcal{C}(A_n)} \subset C$  and  $p \in C$ . This is a contradiction. So we conclude that  $d(p, \overline{\mathcal{C}(A)}) \leq R_0$  for  $p \in \bigcap_{n=1}^\infty \overline{\mathcal{C}(A_n)}$ . It is clear that  $\overline{\mathcal{C}(A)} \subset \bigcap_{n=1}^\infty \overline{\mathcal{C}(A_n)}$ . Hence, we have

$$d_H(\bigcap_{n=1}^\infty \overline{\mathcal{C}(A_n)}, \overline{\mathcal{C}(A)}) \leq R_0.$$

This completes our proof. q.e.d.

According to our definition of convex hull, it is possible that  $\overline{\mathcal{C}(K)} \cap S_\infty(N)$  is a much bigger set than  $K \cap S_\infty(N)$ . In fact, if we consider  $K$  to be the  $y$ -axis in  $\mathbb{R}^2$ , then  $K \cap S_\infty(\mathbb{R}^2)$  consists of the two poles in  $\mathbb{S}^1$ . However,  $\mathcal{C}(K) = \mathbb{R}^2$  because every line given by  $x = \text{constant}$  is a geodesic joining the two poles of  $\mathbb{S}^1$ . Hence,  $\overline{\mathcal{C}(K)} \cap S_\infty(\mathbb{R}^2) = \mathbb{S}^1$ . On the other hand, if we assume in addition that  $N$  satisfies the following separation property at infinity, then

$$\overline{\mathcal{C}(K)} \cap S_\infty(N) = K \cap S_\infty(N).$$

**Definition 1.7.** Let  $N$  be a Cartan-Hadamard manifold.  $N$  is said to satisfy the separation property at infinity if for any closed subset  $A$  of  $S_\infty(N)$  and any point  $p \in S_\infty(N) \setminus A$ , there exists a closed convex subset  $C$  in  $\overline{N}$  such that  $A$  is contained in the interior of  $C \cap S_\infty(N)$  and  $p$  not in  $C$ .

It is easy to check that a two-dimensional visibility manifold always satisfies separation property at infinity. On the other hand, upon improving a result of M. Anderson [1], A. Borbély [2] has shown that Cartan-Hadamard manifold  $N$  has separation property at infinity provided that its sectional curvature satisfies  $-Ce^{\lambda d(x)} \leq K_N(x) \leq -1$  for some constant  $C > 0$  and  $0 \leq \lambda < 1/3$ , where  $d(x)$  is the distance from point  $x$  to a fixed point  $o \in N$ . We have the following simple lemma.

**Lemma 1.8.** *Let  $N$  be a Cartan-Hadamard manifold. Then for every closed set  $K$  in  $\overline{N}$ ,*

$$\overline{\mathcal{C}(K)} \cap S_\infty(N) = K \cap S_\infty(N)$$

*if and only if  $N$  satisfies the separation property at infinity.*

*Proof.* Assume that  $N$  satisfies the separation property at infinity. For a given closed subset  $K$ , let  $A = K \cap S_\infty(N)$ . If  $A = S_\infty(N)$ , then there is nothing to prove. So we may assume this is not the case. The closeness of  $K$  implies that  $A$  is closed. Given  $p \in S_\infty(N) \setminus A$ , there is a closed convex subset  $C$  such that  $A$  is contained in the interior of  $C \cap S_\infty(N)$  and  $p$  is not in  $C$ . In particular, we conclude that

$$\sup_{x \in K} d(x, C) = R < \infty.$$

Let us consider the  $R$ -neighborhood,

$$C_R = \{x \in N : d(x, C) \leq R\},$$

of  $C$ . Then  $\overline{C_R}$  is a closed convex subset and  $K \subset \overline{C_R}$ . Moreover,

$$C \cap S_\infty(N) = \overline{C_R} \cap S_\infty(N).$$

Therefore,  $\overline{\mathcal{C}(K)} \subset \overline{C_R}$  and  $p$  is not in  $\overline{C_R}$ . In particular,  $p$  is not in  $\overline{\mathcal{C}(K)} \cap S_\infty(N)$ . This shows that  $\overline{\mathcal{C}(K)} \cap S_\infty(N) = A$ .

Conversely, to show that  $N$  satisfies the separation property at infinity, let  $A$  be a closed subset of  $S_\infty(N)$  and point  $p \in S_\infty(N) \setminus A$ . Then there exists a closed subset  $K \subset S_\infty(N)$  such that  $A$  is in the interior of

$K$  and  $p \notin K$ . Let  $C = \overline{\mathcal{C}(K)}$  and by the assumption,  $C \cap S_\infty(N) = K$ . Thus,  $p \notin C$  and  $A$  is contained in the interior of  $C \cap S_\infty(N)$ . Hence,  $N$  satisfies the separation property at infinity and the lemma is proved. q.e.d.

### 2 General harmonic maps

We are now ready to prove a general structural result concerning harmonic maps into a Cartan-Hadamard manifold.

**Theorem 2.1.** *Let  $M$  be a complete Riemannian manifold such that the dimension of the space of bounded harmonic functions  $\mathcal{H}_0(M)$  is  $k_0$ . Let  $u : M \rightarrow N$  be a harmonic map from  $M$  into a Cartan-Hadamard manifold  $N$ . Denote  $A = \overline{u(M)} \cap S_\infty(N)$ , where  $S_\infty(N)$  is the geometric boundary of  $N$ . Then there exists a set of points  $\{y_i\}_{i=1}^k \subset \overline{u(M)} \cap N$  with  $k \leq k_0$ , such that,*

$$u(M) \subset \bigcap_{\epsilon > 0} \overline{\mathcal{C}(A_\epsilon \cup \{y_i\}_{i=1}^k)},$$

where  $A_\epsilon$  is the  $\epsilon$ -neighborhood of  $A$ . If in addition we assume that either  $u$  is bounded ( $A = \emptyset$ ), or  $N$  is a two-dimensional visibility manifold, or  $N$  has strongly negative sectional curvature, then

$$u(M) \subset \overline{\mathcal{C}(A \cup \{y_i\}_{i=1}^k)}.$$

*Proof.* A domain  $\Omega$  in  $M$  is said to be massive if there exists a bounded, nonnegative, nontrivial, subharmonic function  $f$  on  $\Omega$  such that  $f = 0$  on  $\partial\Omega$ . Such a function  $f$  is called a potential function of  $\Omega$ . Note that by setting  $f = 0$  on  $M \setminus \Omega$ ,  $f$  is a bounded subharmonic on  $M$ . A result in [7] implies that  $\dim \mathcal{H}_0(M) = k_0$  if and only if  $M$  has exactly  $k_0$  disjoint massive subsets  $\Omega_1, \dots, \Omega_{k_0}$ .

Let  $\hat{M}$  be the Stone-Cěch compactification of  $M$ . Then every bounded continuous function on  $M$  can be continuously extended to  $\hat{M}$ . For each  $i \in \{1, \dots, k_0\}$ , let us define the set

$$S_i = \bigcap \{ \hat{x} \in \hat{M} \mid f(\hat{x}) = \sup f \},$$

where the intersection is taken over all the potential functions  $f$  of  $\Omega_i$ . The fact that  $f$  is subharmonic together with the maximum principle implies that  $S_i \subset \hat{M} \setminus M$ . We claim that  $S_i \neq \emptyset$ . In fact, for each

potential function  $f$  of  $\Omega_i$ , the set  $\{\hat{x} \mid f(\hat{x}) = \sup f\}$  is a closed subset of  $\hat{M} \setminus M$ . By the compactness of  $\hat{M} \setminus M$ , we need only to show that for any finitely many potential functions  $f_1, \dots, f_l$  of  $\Omega_i$ ,

$$\bigcap_{j=1}^l \{\hat{x} \mid f_j(\hat{x}) = \sup f_j\} \neq \emptyset.$$

We will argue by induction on  $l$ . It is trivially true for one potential function. Let us assume that it is true for  $l$  potential functions that

$$\bigcap_{j=1}^l \{\hat{x} \mid f_j(\hat{x}) = \sup f_j\} \neq \emptyset.$$

If we define the function  $f = f_1 + \dots + f_l$ , then we have

$$\{\hat{x} \mid f(\hat{x}) = \sup f\} = \bigcap_{j=1}^l \{\hat{x} \mid f_j(\hat{x}) = \sup f_j\}.$$

Note that both  $f$  and  $f_{l+1}$  are potential functions of  $\Omega_i$ . If

$$\{\hat{x} \mid f(\hat{x}) = \sup f\} \cap \{\hat{x} \mid f_{l+1}(\hat{x}) = \sup f_{l+1}\} = \emptyset,$$

then for sufficiently small  $\epsilon$ , the sets

$$D_1 = \{x \in M \mid f(x) > \sup f - \epsilon\}$$

and

$$D_2 = \{x \in M \mid f_{l+1}(x) > \sup f_{l+1} - \epsilon\}$$

are disjoint. Clearly both  $D_1$  and  $D_2$  are subsets of  $\Omega_i$  with the properties that  $\partial D_1 \cap \partial \Omega_i = \emptyset$  and  $\partial D_2 \cap \partial \Omega_i = \emptyset$  because  $f = f_{l+1} = 0$  on  $\partial \Omega_i$ . Also, the functions

$$g_1 = f - \sup f + \epsilon$$

and

$$g_2 = f_{l+1} - \sup f_{l+1} + \epsilon$$

are potential functions of  $D_1$  and  $D_2$ , respectively. In particular, this implies that  $M$  has  $k_0 + 1$  disjoint massive sets given by

$$\{\Omega_1, \dots, \Omega_{i-1}, D_1, D_2, \Omega_{i+1}, \dots, \Omega_{k_0}\},$$

which is a contradiction. Therefore,

$$\begin{aligned} & \bigcap_{j=1}^{l+1} \{\hat{x} \mid f_j(\hat{x}) = \sup f_j\} \\ &= \{\hat{x} \mid f(\hat{x}) = \sup f\} \cap \{\hat{x} \mid f_{l+1}(\hat{x}) = \sup f_{l+1}\} \\ &\neq \emptyset, \end{aligned}$$

and the claim that  $S_i$  is nonempty follows.

We now show that for each  $i$  there exists a potential function  $h_i$  of  $\Omega_i$  such that

$$\{\hat{x} \mid h_i(\hat{x}) = \sup h_i\} = S_i.$$

The function  $h_i$  will be called a minimal potential function of  $\Omega_i$ . For an arbitrary open set  $U$  in  $\hat{M}$  such that  $S_i \subset U$ , note that

$$\begin{aligned} \hat{M} \setminus U &\subset \hat{M} \setminus S_i \\ &= \cup \{\hat{x} \in \hat{M} \mid f(\hat{x}) < \sup f\}, \end{aligned}$$

where the union is over all potential functions  $f$  of  $\Omega_i$ . The compactness of  $\hat{M} \setminus U$  implies there exist finitely many potential functions  $f_1, \dots, f_l$  of  $\Omega_i$  such that

$$\hat{M} \setminus U \subset \cup_{j=1}^l \{\hat{x} \mid f_j(\hat{x}) < \sup f_j\}.$$

Let us define  $g = f_1 + \dots + f_l$ , which has the property that  $\{\hat{x} \mid g(\hat{x}) = \sup g\} \subset U$ . One may assume by scaling  $g$  that  $0 \leq g \leq 1$  on  $M$  and  $\sup g = 1$ . Now choose a sequence of open sets  $U_n \subset \hat{M}$ ,  $n = 1, 2, \dots$ , such that  $U_n \subset U_{n+1}$  and  $\cap_{n=1}^\infty U_n = S_i$ . For each  $U_n$ , there exists a potential function  $g_n$  of  $\Omega_i$  such that  $0 \leq g_n \leq 1$ ,  $\sup g_n = 1$  and

$$\{\hat{x} \mid g_n(\hat{x}) = \sup g_n\} \subset U_n.$$

By defining

$$h_i = \sum_{n=1}^\infty 2^{-n} g_n,$$

it is clear that  $h_i$  is a minimal potential function of  $\Omega_i$  satisfying

$$\{\hat{x} \mid h_i(\hat{x}) = \sup h_i\} = S_i.$$

From now on, we will denote  $h_i$  to be a minimal potential function of  $\Omega_i$ .

For a bounded subharmonic function  $v$  on  $M$ , consider the set

$$S = \{\hat{x} \mid v(\hat{x}) = \sup v\}.$$

We claim that  $S$  must contain some  $S_i$ . Moreover, for each  $j$ , either  $S \cap S_j = \emptyset$  or  $S_j \subset S$ . In fact, let us first argue that  $S \cap S_i \neq \emptyset$  for some  $i$ . If this is not the case, then for  $\epsilon > 0$  sufficiently small the sets

$$\Omega = \{x \in M \mid v(x) > \sup v - \epsilon\}$$

and

$$\tilde{\Omega}_i = \{x \in M \mid h_i(x) > \sup h_i - \epsilon\}$$

must satisfy  $\Omega \cap \tilde{\Omega}_i = \emptyset$ . Clearly  $\tilde{\Omega}_i \subset \Omega_i$ , and each  $\tilde{\Omega}_i$  is a massive set with potential function  $h_i - \sup h_i + \epsilon$ . Also,  $\Omega$  is a massive set with potential function  $v - \sup v + \epsilon$ . Therefore

$$\{\Omega, \tilde{\Omega}_1, \dots, \tilde{\Omega}_{k_0}\}$$

are  $k_0 + 1$  disjoint massive sets of  $M$ , which is impossible. Therefore,  $S \cap S_i \neq \emptyset$  for some  $i$ . To see that  $S_i \subset S$ , let us consider the function  $w = h_i + v$ . Note that

$$\{\hat{x} \mid w(\hat{x}) = \sup w\} = S_i \cap S \subset S_i.$$

Thus, for sufficiently small  $\epsilon > 0$ , we have the set

$$W = \{x \mid w(x) > \sup w - \epsilon\} \subset \Omega_i,$$

and  $f = w - \sup w + \epsilon$  is a potential function of this massive set  $W$ . In particular, by extending  $f$  to be zero outside  $W$ ,  $f$  is a potential function of  $\Omega_i$  with

$$\begin{aligned} \{\hat{x} \mid f(\hat{x}) = \sup f\} &= \{\hat{x} \mid w(\hat{x}) = \sup w\} \\ &= S_i \cap S. \end{aligned}$$

The minimality of  $S_i$  implies that  $S_i \subset S_i \cap S$ , hence  $S_i \subset S$ . The preceding argument also shows that for any  $j$ , either  $S \cap S_j = \emptyset$  or  $S_j \subset S$ .

We are now ready for the proof of the theorem. Let us pick a point  $y_0 \in \overline{u(M)}$ . If

$$u(M) \subset \bigcap_{\epsilon > 0} \overline{\mathcal{C}(A_\epsilon \cup \{y_0\})},$$

then we are done. Hence we may assume that there exists an  $\epsilon$ -neighborhood  $A_\epsilon$  of  $A$  in  $S_\infty(N)$  such that the set

$$u(M) \setminus \overline{\mathcal{C}(A_\epsilon \cup \{y_0\})} \neq \emptyset.$$

One can easily check that it is bounded in  $N$ . Since  $u$  is a harmonic map and the function  $d(y, \overline{\mathcal{C}(A_\epsilon \cup \{y_0\})})$  is convex, the composition function

$$f(x) = d(u(x), \overline{\mathcal{C}(A_\epsilon \cup \{y_0\})})$$

is a bounded nonconstant subharmonic function on  $M$ . Thus,  $f$  attains its maximum at every point of some  $S_i$ , say  $S_1$ . In particular, for  $\hat{x}_1 \in$

$S_1$  and a net  $\{x_\alpha\}$  in  $M$  converging to  $\hat{x}_1$  in  $\hat{M}$ , a subnet of  $\{u(x_\alpha)\}$  converges to  $y_1 \in N$ . Again, if

$$u(M) \subset \bigcap_{\epsilon > 0} \overline{\mathcal{C}(A_\epsilon \cup \{y_1\})},$$

then the theorem is true, otherwise by choosing a smaller  $\epsilon$  if necessary, the function

$$g(x) = d(u(x), \overline{\mathcal{C}(A_\epsilon \cup \{y_1\})})$$

is a bounded nonconstant subharmonic function on  $M$ . If  $g$  achieves its maximum on  $S_1$ , then  $g(\hat{x}) = \sup g$  for  $\hat{x} \in S_1$ . In particular,

$$\sup g = g(\hat{x}_1) = d(y_1, \overline{\mathcal{C}(A \cup \{y_1\})}) = 0,$$

which is impossible. Hence, we may assume  $g$  achieves its maximum on  $S_2$ .

For a net  $\{x_\alpha\}$  in  $M$  converging to a point  $\hat{x}_2$  in  $S_2$ , there exists a subnet of  $\{u(x_\alpha)\}$  that converges to  $y_2 \in N$ . Suppose that we have chosen  $l$  points  $y_1, \dots, y_l$  described in the above procedure. If

$$u(M) \subset \bigcap_{\epsilon > 0} \overline{\mathcal{C}(A_\epsilon \cup \{y_i\}_{i=1}^l)},$$

then we are done, otherwise by choosing a smaller  $\epsilon$  if necessary, we define the function

$$h(x) = d(u(x), \overline{\mathcal{C}(A_\epsilon \cup \{y_i\}_{i=1}^l)}),$$

which is a bounded nonconstant subharmonic function on  $M$ . We claim that  $h$  cannot achieve its maximum on  $\bigcup_{i=1}^l S_i$ . Indeed, if it does, then  $h$  must achieve its maximum at every point on  $S_i$  for some  $1 \leq i \leq l$ . Thus using a similar argument as before,

$$h(\hat{x}_i) = d(y_i, \overline{\mathcal{C}(A_\epsilon \cup \{y_i\}_{i=1}^l)}) = 0,$$

which is a contradiction. Hence,  $h$  achieves its maximum on some  $S_j$  with  $j > l$ . We may assume that  $j = l + 1$ .

Let us pick a point  $\hat{x}_{l+1} \in S_{l+1}$  and a net  $\{x_\alpha\}$  converging to  $\hat{x}_{l+1}$ . Suppose  $y_{l+1}$  is an accumulation point of the net  $\{u(x_\alpha)\}$ . It is clear that this process must stop after at most  $k_0$  steps since there are only  $k_0$  massive sets. In particular, there exist  $k$  points  $\{y_1, \dots, y_k\}$  with  $k \leq k_0$  such that

$$u(M) \subset \bigcap_{\epsilon > 0} \overline{\mathcal{C}(A_\epsilon \cup \{y_i\}_{i=1}^k)}.$$

Moreover,  $y_i \in \overline{u(M)}$ , and the proof of the first statement is completed.

If  $u$  is bounded, then the preceding argument readily implies that

$$u(M) \subset \overline{\mathcal{C}(\{y_i\}_{i=1}^k)}.$$

If  $N$  is a two-dimensional visibility manifold, then  $N$  has separation property and Lemma 1.5 implies that

$$\bigcap_{\epsilon > 0} \overline{\mathcal{C}(A_\epsilon \cup \{y_i\}_{i=1}^k)} = \overline{\mathcal{C}(A \cup \{y_i\}_{i=1}^k)},$$

hence  $u(M) \subset \overline{\mathcal{C}(A \cup \{y_i\}_{i=1}^k)}$ .

If  $N$  has strongly negative sectional curvature bounded by

$$-b \leq K_N \leq -a < 0,$$

then by scaling the metric we may assume  $a = 1$ . Lemma 1.6 and the fact that

$$u(M) \subset \bigcap_{\epsilon > 0} \overline{\mathcal{C}(A_\epsilon \cup \{y_i\}_{i=1}^k)}$$

imply that the function

$$w(x) = d(u(x), \overline{\mathcal{C}(A \cup \{y_i\}_{i=1}^k)})$$

is bounded. If  $w$  is identically zero, then the proof is done. If not, let us denote

$$\Omega = \overline{\mathcal{C}(A \cup \{y_i\}_{i=1}^k)}$$

and  $t = \sup w(x)$ . The convexity of  $\Omega$  asserts that  $w$  is a non-trivial, non-negative, bounded subharmonic function on  $M$ . Moreover, for any  $\delta > 0$ , there exists a point  $x_\delta \in M$  such that  $w(x_\delta) > t - \delta$ . If the sequence  $\{u(x_\delta)\}$  has an interior accumulation point  $y_{k+1} \in N$ , then obviously there exists  $S_{k+1}$  with  $w(S_{k+1}) = t$  and hence  $u(S_{k+1}) = y_{k+1}$ . This creates a new interior vertex. By repeating the above argument for the convex hull

$$\mathcal{C}(A \cup \{y_i\}_{i=1}^{k+1}),$$

either this process must stop after we pick up at most  $k_0 - k$  interior vertices or that the sequence  $\{u(x_\delta)\}$  has no interior accumulation point.

To remedy the situation when  $\{u(x_\delta)\}$  has no interior accumulation point, we will modify the function  $w$  to yield a new subharmonic function with an interior accumulation point. For convenience sake, let us denote

$$\tau(y) = d^2(y, \Omega).$$

The fact that  $K_N \leq -1$  and the comparison theorem assert that there exists a constant  $C_1 > 0$  depending only on  $t$  such that the Hessian of  $\tau$  must satisfy

$$H(\tau)(y) \geq C_1$$

on the set  $\{y \in N \mid t^2/4 < \tau(y) < 9t^2/4\}$ . Let  $\phi$  be a smooth function on  $[0, \infty)$  such that  $\phi = 1$  on  $[0, t/8]$  and  $\phi = 0$  on  $[t/4, \infty)$  with  $0 \leq \phi \leq 1$  and  $|\phi'| \leq C_2, |\phi''| \leq C_2$  for some constant  $C_2$  depending only on  $t$ . Also, for any point  $p \in N$ , the bounds  $-b \leq K_N \leq -1$  and the comparison theorem yield that the Hessian of the distance function  $d_p$  to the point  $p$  satisfies

$$|D^2 d_p(x)| \leq b \coth(bd_p(x)).$$

If we choose  $p$  to satisfy  $3t/4 < d(p, \Omega) < 5t/4$ , then it is easy to see that for sufficiently small  $\epsilon$  depending only on  $t$  and  $b$ , the Hessian of the function  $h(y) = \tau(y) + \epsilon\phi(d_p(y))$  is nonnegative everywhere. In particular, the function  $h \circ u$  is a non-negative subharmonic function on  $M$ . The fact that  $w(x) = d(u(x), \Omega)$  is bounded implies that  $h \circ u$  is bounded. Moreover,  $h \circ u = w^2$  on  $u^{-1}(N \setminus B_p(t/4))$ . For any  $\delta < \epsilon/2t$ , by choosing  $p = u(x_\delta)$ , we have

$$\begin{aligned} h \circ u(x_\delta) &= \tau(u(x_\delta)) + \epsilon \\ &> (t - \delta)^2 + \epsilon \\ &> t^2 + \delta^2. \end{aligned}$$

Hence the maximum value of  $h \circ u$  must be larger than  $t^2$ , and for those points such that  $h \circ u(x) > t^2$  we must have  $u(x) \in B_{u(x_\delta)}(t/4)$ . This creates an interior accumulation point  $y_{k+1}$  for any sequence  $\{u(x_i)\}$  providing that the sequence  $\{h \circ u(x_i)\}$  tends to the maximum value of the function  $h \circ u$ . We are now back to the situation where we have produced an interior vertex  $y_{k+1}$ . This completes the proof. q.e.d.

**Corollary 2.2.** *If  $\dim H_0(M) = 1$ , then every bounded harmonic map from  $M$  into a Cartan-Hadamard manifold must be constant.*

Recall that an open set  $\Omega$  in  $M$  is D-massive if there exists a non-negative, bounded, nonconstant subharmonic function  $u$  on  $\Omega$  such that  $u = 0$  on  $\partial\Omega$  and  $\int_\Omega |\nabla u|^2(x) dx < \infty$ . It has been shown by Grigor'yan [7] that the space  $\mathcal{H}_D(M)$  of bounded harmonic functions with finite Dirichlet integral on  $M$  has dimension  $k_D$  if and only if  $M$  has exactly  $k_D$  disjoint D-massive sets. The next theorem follows from the same argument as in Theorem 2.1.

**Theorem 2.3.** *Let  $u : M \rightarrow N$  be a harmonic map into a Cartan-Hadamard manifold with finite total energy. Suppose that  $\dim \mathcal{H}_D(M) = k_D$ . Then there exists a set of  $k$  points  $\{y_i\}_{i=1}^k \subset \overline{u(M)} \cap N$  with  $k \leq k_D$ , such that,*

$$u(M) \subset \bigcap_{\epsilon > 0} \overline{\mathcal{C}(A_\epsilon \cup \{y_i\}_{i=1}^k)},$$

where  $A = \overline{u(M)} \cap S_\infty(N)$ , and  $A_\epsilon$  is the  $\epsilon$ -neighborhood of  $A$ . In particular,

$$u(M) \subset \overline{\mathcal{C}(A \cup \{y_i\}_{i=1}^k)}$$

if either  $u$  is bounded, or  $N$  is a two-dimensional visibility manifold, or  $N$  has strongly negative sectional curvature.

We conclude this section with the following result which may be viewed as a generalized version of Liouville property for harmonic maps. Before we state the theorem, let us denote  $\mathcal{H}_+(M)$  to be the linear space spanned by the set of positive harmonic functions on  $M$ .

**Theorem 2.4.** *Suppose  $M$  is a complete manifold satisfying*

$$\dim \mathcal{H}_0(M) = \dim \mathcal{H}_+(M) = k_0 < \infty.$$

*Assume that  $u : M \rightarrow N$  is a harmonic map from  $M$  into a Cartan-Hadamard manifold  $N$  which either is a two-dimensional visibility manifold or has strongly negative sectional curvature, and that*

$$A = \overline{u(M)} \cap S_\infty(N)$$

*consists of at most one point. Then the set  $A$  is necessarily empty, and there exists a set of  $k$  points  $\{y_i\}_{i=1}^k \subset \overline{u(M)} \cap N$  with  $k \leq k_0$  such that*

$$u(M) \subset \overline{\mathcal{C}(\{y_i\}_{i=1}^k)}.$$

*In particular, if  $M$  has no nonconstant positive harmonic functions, then every such harmonic map must be a constant map.*

*Proof.* Theorem 2.1 implies that

$$u(M) \subset \overline{\mathcal{C}(A \cup \{y_i\}_{i=1}^k)}$$

for some set of  $k$  points  $\{y_i\}_{i=1}^k$  in  $N$  with  $k \leq k_0$ . If  $A$  contains exactly one point  $a$ , let  $\gamma$  be a geodesic line on  $(-\infty, +\infty)$  such that its restriction to  $(0, +\infty)$  represents  $a$ . For each  $y_i$ , there exists a unique point  $\gamma(t_i)$  such that  $d(y_i, \gamma) = d(y_i, \gamma(t_i))$ . Choose a point  $p = \gamma(t_0)$

with  $t_0 < t_i$  for  $i = 1, 2, \dots, k$ . Let  $\delta$  be the geodesic ray given by the restriction of  $\gamma$  onto  $(t_0, +\infty)$ , and denote the Busemann function associated to  $\delta$  by  $\beta$ . Recall that if  $\delta$  is parametrized by arclength, then

$$\beta(y) = \lim_{t \rightarrow \infty} (t - d(y, \delta(t))).$$

We claim that there exists a constant  $c$  such that

$$d(y, p) \leq \beta(y) + c$$

for  $y \in \overline{\mathcal{C}(\{a\} \cup \{y_i\}_{i=1}^k)}$ . In fact, by the convexity of the function  $d(y, \gamma)$  and the choice of  $p$ , one easily checks that

$$d(y, \delta) = d(y, \gamma) \leq \max_{1 \leq i \leq k} d(y_i, \gamma) = c$$

for  $y \in \overline{\mathcal{C}(\{a\} \cup \{y_i\}_{i=1}^k)}$ . Therefore, if we let  $\bar{y} \in \delta$  be the point such that  $d(y, \delta) = d(y, \bar{y})$ , then

$$\begin{aligned} d(y, p) &\leq d(y, \delta) + d(\bar{y}, p) \\ &\leq c + \beta(\bar{y}) \\ &\leq 2c + \beta(y). \end{aligned}$$

This justifies the claim that

$$d(u(x), p) \leq \beta(u(x)) + c$$

for all  $x \in M$ . Since  $u$  is a harmonic map and  $N$  is a Cartan-Hadamard manifold, the function  $d(u(x), p)$  is subharmonic and  $\beta(u(x)) + c$  superharmonic. The sub-super solution method yields a harmonic function  $f(x)$  on  $M$  such that

$$d(u(x), p) \leq f(x) \leq \beta(u(x)) + c.$$

Therefore,  $f$  is an unbounded positive harmonic function on  $M$ , contradicting to our assumption that there is no such function. In conclusion,  $A$  must be empty and

$$\overline{\mathcal{C}(u(M))} = \overline{\mathcal{C}(\{y_i\}_{i=1}^k)}.$$

This proves our first statement. The second part of the theorem follows from the first part by taking  $k_0 = 1$ .    q.e.d.

Notice that the horoball of a visibility manifold intersects the geometric boundary at exactly one point (see [3]). Thus, we obtain the following Liouville type theorem which partially generalizes the results in [14] and [16].

**Corollary 2.5.** *Suppose  $M$  satisfies  $\dim \mathcal{H}_+(M) = 1$ . Assume that  $N$  either is a two-dimensional visibility manifold or has strongly negative sectional curvature. Then every harmonic map from  $M$  into a horoball of  $N$  must be constant.*

Recall that a manifold is parabolic if it does not admit a positive Green's function. It is well-known that a parabolic manifold has no massive subsets and every positive harmonic function must be constant. Applying Theorem 2.1 to this case, we have the following corollary.

**Corollary 2.6.** *Let  $u$  be a harmonic map from a parabolic manifold  $M$  into a Cartan-Hadamard manifold  $N$ . If  $N$  either is a two-dimensional visibility manifold or has strongly negative sectional curvature, then  $u(M) \subset \overline{\mathcal{C}(A)}$ , where  $A = \overline{u(M)} \cap S_\infty(N)$ .*

*Proof.* In this case, we have  $\dim \mathcal{H}_0(M) = 1$ , hence Theorem 2.1 implies that  $u(M) \subset \overline{\mathcal{C}(A \cup \{y\})}$  for some  $y \in \overline{u(M)} \cap N$ . Let us assume the contrary that  $u(M)$  is not contained in  $\mathcal{C}(A)$ . In particular, the parabolicity of  $M$  implies that the function  $d(u(x), \mathcal{C}(A))$  is unbounded. Lemma 1.6 then asserts that  $u(M) \setminus \mathcal{C}(W)$  is non-empty for some open set  $W \subset S_\infty(N)$  which properly contains  $A$ . Let us consider the function

$$f(x) = d(u(x), \overline{\mathcal{C}(W)}),$$

which is a non-constant, non-negative, bounded subharmonic function on  $M$ . However, the parabolicity assumption on  $M$  implies that such function does not exist. This completes our proof.  $\square$

### 3 Polynomial growth harmonic maps

In this section, we will study the class of polynomial growth harmonic maps into a Cartan-Hadamard manifold with separation property at infinity. When the domain manifold satisfies a type of mean value inequality for positive subharmonic functions and a volume growth property, we will show that the image of a polynomial growth harmonic map of fixed degree must be contained in a convex hull over finitely many vertices. Moreover, the number of vertices can be estimated in terms of

the degree. This result may be viewed as a higher dimensional generalization of the results in [9] which deals with harmonic diffeomorphisms from  $\mathbb{R}^2$  into a Hadamard surface. Throughout this section, we will assume  $N$  is a Cartan-Hadamard manifold.

Let us first introduce a more general notion of massive sets.

**Definition 3.1.** An open subset  $\Omega$  of  $M$  is said to be  $\ell$ -massive if there exists a nonconstant nonnegative subharmonic function  $f$  on  $\Omega$  satisfying  $f = 0$  on  $\partial\Omega$  and  $f(x) = O(\rho^\ell(x))$  as  $x \rightarrow \infty$ , where  $\rho(x)$  is the distance to a fixed point  $p \in M$ . Such a function  $f$  is called an  $\ell$ -potential function of  $\Omega$ .

Note that a massive set is 0-massive and, in general, an  $\ell$ -massive set is  $\ell'$ -massive if  $\ell \leq \ell'$ .

**Lemma 3.2.** *Let  $M$  be a complete manifold such that the maximum number of disjoint  $\ell$ -massive sets of  $M$  is  $k_\ell$ . Suppose  $u : M \rightarrow N$  is a harmonic map from  $M$  into  $N$ , and  $N$  satisfies the separation property at infinity. Assume that there exists a point  $o \in N$  such that  $d(u(x), o) = O(\rho^\ell(x))$  as  $x \rightarrow \infty$ . Then*

$$A = \overline{u(M)} \cap S_\infty(N) = \{a_i\}_{i=1}^{k'}$$

with  $k' \leq k_\ell - k_0$ , where  $k_0$  is the maximum number of disjoint massive sets of  $M$ . If, in addition,  $N$  either is a two-dimensional visibility manifold or has strongly negative sectional curvature, then there exist  $k$  points  $\{y_j\}_{j=1}^k \subset \overline{u(M)} \cap N$  with  $k' + k \leq k_\ell$  such that

$$u(M) \subset \overline{\mathcal{C}(\{a_i\}_{i=1}^{k'} \cup \{y_j\}_{j=1}^k)}.$$

*Proof.* Let  $k_0$  be the maximum number of disjoint massive sets in  $M$ . Since a massive set is always  $\ell$ -massive, we have  $k_0 \leq k_\ell$ . Theorem 2.1 implies that there exist  $k$  points  $\{y_j\}_{j=1}^k \subset \overline{u(M)} \cap N$  with  $k \leq k_0$ , such that

$$u(M) \subset \bigcap_{\epsilon > 0} \overline{\mathcal{C}(A_\epsilon \cup \{y_j\}_{j=1}^k)}.$$

If  $A$  contains at least  $k'$  points, then there exist disjoint open sets  $\{U_i\}_{i=1}^{k'}$  in  $\overline{N}$  such that  $U_i \cap A \neq \emptyset$  for  $i = 1, 2, \dots, k'$ . Since  $N$  is assumed to satisfy the separation property at infinity, Lemma 1.8 yields that  $u(M)$  is not a subset of  $\overline{\mathcal{C}((\overline{N} \setminus U_i) \cup \{y_j\}_{j=1}^k)}$ . In particular, the function

$$f_i(x) = d(u(x), \overline{(\overline{N} \setminus U_i) \cup \{y_j\}_{j=1}^k})$$

is not identically zero on  $u^{-1}(U_i)$  and  $\sup f_i = \infty$ . Clearly,  $f_i = 0$  on the boundary of  $u^{-1}(U_i)$  and

$$f_i(x) = O(\rho^\ell(x)).$$

This implies that each set  $u^{-1}(U_i)$  is  $\ell$ -massive but not massive. In particular, since they are disjoint,  $k' \leq k_\ell - k_0$ . Thus  $A$  has at most  $k_\ell - k_0$  points, and the first conclusion follows. If, in addition,  $N$  either is a two-dimensional visibility manifold or has strongly negative sectional curvature, then Theorem 2.1 yields that

$$u(M) \subset \overline{\mathcal{C}(\{a_i\}_{i=1}^{k'} \cup \{y_j\}_{j=1}^k)},$$

and the estimate  $k' + k \leq k_\ell$  follows from the argument. This completes our proof.  $\square$

In the following, we shall adopt the argument utilized in [11] to show that for a manifold satisfying a mean value inequality and a weak volume growth property, then the number of disjoint  $\ell$ -massive sets can be explicitly bounded. Let us begin by recalling some definitions from [11].

A complete manifold  $M$  is said to satisfy the following properties if the corresponding statement holds:

- (V) Weak volume growth condition: if there exist constants  $C_0 > 0$  and  $\nu > 0$  such that

$$V_x(r') \leq C_0 \left(\frac{r'}{r}\right)^\nu V_x(r)$$

for all  $x \in M$  and  $0 < r \leq r' < \infty$ , where  $V_x(r)$  denotes the volume of the geodesic ball centered at  $x \in M$  of radius  $r$ .

- (M) Mean value inequality: if there exists a constant  $\lambda > 0$ , such that, for  $x \in M$  and  $r > 0$ , any non-negative subharmonic function  $f$  defined on  $B_x(r)$  must satisfy

$$V_x(r) f(x) \leq \lambda \int_{B_x(r)} f.$$

- (P) Weak Poincaré inequality: if there exists a constant  $C' > 0$ , such that for all  $x \in M$  and  $r > 0$ ,

$$\int_{B_x(r)} u^2 \leq C' r^2 \int_{B_x(2r)} |\nabla u|^2$$

for any  $u \in H_{1,2}(B_x(2r))$  satisfying  $\int_{B_x(r)} u = 0$ .

It is known that these conditions are valid on manifolds with non-negative Ricci curvature with  $\nu$  being the dimension of the manifold. We refer to [11] for more comments regarding those properties and their relations with each other. Our purpose here is to show that for a manifold with property  $(\mathcal{M})$  and having polynomial volume growth, the number of disjoint  $\ell$ -massive sets must be finite and can be bounded. To do this, we first recall a result of the first author in [11].

**Lemma 3.3**[11]. *Let  $K$  be a  $k$ -dimensional linear space of functions defined on  $M$ . Suppose each function  $u \in K$  is polynomial growth of at most degree  $\ell$ . Suppose that the volume growth of  $M$  satisfies  $V_p(r) = O(r^\nu)$  for some point  $p \in M$ . Then for any  $\beta > 1$ ,  $\delta > 0$ , and  $r_0 > 0$ , there exists  $r > r_0$  such that if  $\{u_i\}_{i=1}^k$  is an orthonormal basis of  $K$  with respect to the inner product  $A_{\beta r}(u, v) = \int_{B_p(\beta r)} uv$ , we have*

$$\sum_{i=1}^k \int_{B_p(r)} u_i^2 \geq k \beta^{-(2\ell+\nu+\delta)}.$$

**Theorem 3.4.** *Let  $M$  be a complete manifold satisfying condition  $(\mathcal{M})$ . Suppose that the volume growth of  $M$  satisfies  $V_p(r) = O(r^\nu)$  for some point  $p \in M$ . Then  $M$  has only finitely many disjoint  $\ell$ -massive sets and the number of disjoint  $\ell$ -massive sets is bounded from above by  $\lambda 3^{(2\ell+\nu)}$ . If  $M$  is further assumed to have property  $(\mathcal{V})$ , then there exists a constant  $C > 0$  depending only on  $C_0$  and  $\nu$  such that the number of disjoint  $\ell$ -massive sets is bounded from above by  $C \lambda \ell^{\nu-1}$ .*

*Proof.* Let  $\Omega_1, \dots, \Omega_{k_\ell}$  be  $k_\ell$  disjoint  $\ell$ -massive sets in  $M$ , and  $u_1, \dots, u_{k_\ell}$  be the corresponding potential functions. Extend each  $u_i$  to be 0 on  $M \setminus \Omega_i$ . Then  $u_i$  is a nonnegative subharmonic function on  $M$  and each  $u_i$  is of polynomial growth of degree at most  $\ell$ . Since  $\Omega_i$  are disjoint, the support of the functions  $u_i$  are disjoint also. In particular, by taking  $r_0$  to be sufficiently large such that  $\int_{B_p(r_0)} u_i^2 > 0$  for each  $i$ , the set  $\{u_i\}_{i=1}^{k_\ell}$  forms an orthogonal basis with respect to  $A_r$  for all  $r \geq r_0$ . Applying Lemma 3.3 to

$$\left\{ \frac{u_i}{A_{\beta r}(u_i, u_i)} \right\}_{i=1}^{k_\ell},$$

we conclude that there exists  $r > r_0$  with

$$(3.1) \quad \sum_{i=1}^{k_\ell} \frac{\int_{B_p(r)} u_i^2}{\int_{B_p(\beta r)} u_i^2} \geq k_\ell \beta^{-(2\ell+\nu+\delta)}.$$

On the other hand, the fact that the set of functions  $\{u_i\}$  have disjoint support implies that for each  $x \in M$  there exists some  $j \in \{1, \dots, k_\ell\}$  such that

$$\sum_{i=1}^{k_\ell} \frac{u_i^2(x)}{\int_{B_p(\beta r)} u_i^2} = \frac{u_j^2(x)}{\int_{B_p(\beta r)} u_j^2}.$$

Since the function  $\sum_{i=1}^{k_\ell} \frac{u_i^2(x)}{\int_{B_p(\beta r)} u_i^2}$  is subharmonic, the maximum principle implies that there exists a point  $q \in \partial B_p(r)$  such that

$$\sum_{i=1}^{k_\ell} \frac{u_i^2(x)}{\int_{B_p(\beta r)} u_i^2} \leq \sum_{i=1}^{k_\ell} \frac{u_i^2(q)}{\int_{B_p(\beta r)} u_i^2} = \frac{u_j^2(q)}{\int_{B_p(\beta r)} u_j^2}$$

for all  $x \in B_p(r)$ . Applying the mean value inequality and noting that  $B_p(r) \subset B_q(2r) \subset B_p(3r)$ , we get

$$V_p(r) u_j^2(q) \leq V_q(2r) u_j^2(q) \leq \lambda \int_{B_q(2r)} u_j^2 \leq \lambda \int_{B_p(3r)} u_j^2.$$

Thus,

$$\sum_{i=1}^{k_\ell} \frac{\int_{B_p(r)} u_i^2}{\int_{B_p(\beta r)} u_i^2} \leq V_p(r) \frac{u_j^2(q)}{\int_{B_p(\beta r)} u_j^2} \leq \lambda \frac{\int_{B_p(3r)} u_j^2}{\int_{B_p(\beta r)} u_j^2}.$$

Choose  $\beta = 3$ . Then from (3.1) we conclude that

$$k_\ell 3^{-(2\ell+\nu+\delta)} \leq \lambda.$$

Hence,  $k_\ell \leq \lambda 3^{(2\ell+\nu)}$  as  $\delta$  is arbitrary. This completes the proof for the case that  $M$  has polynomial volume growth.

If  $M$  has property  $(\mathcal{V})$ , we can use the argument in [11] to improve the estimate. Using the same notation as before, for  $0 < \epsilon < 1/2$ , if we denote the distance from  $p$  to  $x$  by  $\rho(x)$ , then the mean value inequality

( $\mathcal{M}$ ) implies that

$$(3.2) \quad \begin{aligned} \sum_{i=1}^{k_\ell} \frac{u_i^2(x)}{\int_{B_p(\beta r)} u_i^2} &\leq \lambda V_x^{-1}((1+\epsilon)r - \rho(x)) \frac{\int_{B_x((1+\epsilon)r - \rho(x))} u_j^2}{\int_{B_p(\beta r)} u_j^2} \\ &\leq \lambda V_x^{-1}((1+\epsilon)r - \rho(x)) \sup_{i=1}^{k_\ell} \frac{\int_{B_p((1+\epsilon)r)} u_i^2}{\int_{B_p(\beta r)} u_i^2}. \end{aligned}$$

However, condition ( $\mathcal{V}$ ) and the fact that  $\rho(x) \leq r$  yield that

$$\begin{aligned} V_x((1+\epsilon)r - \rho(x)) &\geq C_0^{-1} \left( \frac{(1+\epsilon)r - \rho(x)}{(1+\epsilon)r + \rho(x)} \right)^\nu V_x((1+\epsilon)r + \rho(x)) \\ &\geq C_0^{-1} \left( \frac{(1+\epsilon)r - \rho(x)}{2(1+\epsilon)r} \right)^\nu V_p(r). \end{aligned}$$

Hence, substituting into (3.2) and integrating over  $B_p(r)$  leads to

$$(3.3) \quad \begin{aligned} \sum_{i=1}^{k_\ell} \frac{\int_{B_p(r)} u_i^2}{\int_{B_p(\beta r)} u_i^2} &\leq \frac{C \lambda}{V_p(r)} \sup_{i=1}^{k_\ell} \frac{\int_{B_p((1+\epsilon)r)} u_i^2}{\int_{B_p(\beta r)} u_i^2} \\ &\quad \times \int_{B_p(r)} ((1+\epsilon) - r^{-1} \rho(x))^{-\nu} dx. \end{aligned}$$

On the other hand, we have (see [11])

$$(3.4) \quad \int_{B_p(r)} ((1+\epsilon) - r^{-1} \rho(x))^{-\nu} dx \leq C V_p(r) \epsilon^{\nu-1}.$$

Combining this with (3.3), we conclude that

$$(3.5) \quad \sum_{i=1}^{k_\ell} \frac{\int_{B_p(r)} u_i^2}{\int_{B_p(\beta r)} u_i^2} \leq C \lambda \epsilon^{-(\nu-1)} \sup_{i=1}^{k_\ell} \frac{\int_{B_p((1+\epsilon)r)} u_i^2}{\int_{B_p(\beta r)} u_i^2}.$$

Setting  $\beta = 1 + \epsilon$ , we obtain from (3.1) and (3.5) that

$$k_\ell (1 + \epsilon)^{-(2\ell + \nu + \delta)} \leq C \lambda \epsilon^{-(\nu-1)}.$$

Now the estimate on  $k_\ell$  follows by choosing  $\epsilon = (2\ell)^{-1}$  and observing that the quantity  $(1 + (2\ell)^{-1})^{-(2\ell + \nu + \delta)}$  is bounded from below.  $\text{q.e.d.}$

By combining Lemma 3.2 with Theorem 3.4, we hence deduce the main structural result on polynomial growth harmonic maps.

**Theorem 3.5.** *Let  $M$  be a complete manifold satisfying condition  $(\mathcal{M})$  and its volume growth  $V_p(r) = O(r^\nu)$  for some point  $p \in M$ . Suppose  $N$  is a Cartan-Hadamard manifold satisfying either one of the following conditions:*

- (1) *it has strongly negative sectional curvature;*
- (2) *it is a two-dimensional visibility manifold.*

*Let  $u : M \rightarrow N$  be a harmonic map and suppose that there exists a point  $o \in N$  such that  $d(u(x), o) = O(\rho^\ell(x))$  as  $x \rightarrow \infty$ . Then there exist sets of  $k'$  points  $\{a_i\}_{i=1}^{k'} = \overline{u(M) \cap S_\infty(N)}$  and  $k$  points  $\{y_j\}_{j=1}^k \subset \overline{u(M) \cap N}$  with  $k' + k \leq \lambda 3^{(2\ell+\nu)}$  such that*

$$u(M) \subset \overline{\mathcal{C}(\{a_i\}_{i=1}^{k'} \cup \{y_j\}_{j=1}^k)}.$$

*If  $M$  is further assumed to have property  $(\mathcal{V})$ , then we have  $k' + k \leq C\ell^{\nu-1}$ .*

Let us point out that manifolds that are quasi-isometric to a manifold with non-negative Ricci curvature satisfy conditions  $(\mathcal{V})$  and  $(\mathcal{M})$ . A manifold with Ricci curvature bounded from below and is roughly isometric to a manifold with non-negative Ricci curvature also satisfies conditions  $(\mathcal{V})$  and  $(\mathcal{M})$ . A minimal submanifold in Euclidean space with Euclidean volume growth satisfies conditions  $(\mathcal{V})$  and  $(\mathcal{M})$ . We refer the reader to [11] and [12] for more detailed discussions.

Under more restricted assumptions on the domain manifold, it is possible to show that the image of a polynomial growth harmonic map is contained in the convex hull of its points at infinity. It is still an open question whether the same conclusion is valid without restricting the map to be of polynomial growth. Let us first prove the following lemma concerning  $\ell$ -massive sets.

**Lemma 3.6.** *Let  $M$  be a complete manifold satisfying conditions  $(\mathcal{V})$  and  $(\mathcal{P})$ . Suppose  $\Omega$  is a massive set of  $M$ . Then  $M \setminus \Omega$  does not contain any  $\ell$ -massive sets.*

*Proof.* From the definition of massive set, there exists a non-negative bounded subharmonic function  $f$  whose support is in  $\Omega$ . Since  $M$  satisfies conditions  $(\mathcal{V})$  and  $(\mathcal{P})$ , a lemma in [16] asserts that

$$\lim_{r \rightarrow \infty} V_p^{-1}(r) \int_{B_p(r)} f = \sup_M f.$$

In particular, for any  $\epsilon > 0$ , there exists  $r_0$  such that for all  $r \geq r_0$ , we have

$$\begin{aligned} (1 - \epsilon) \sup_M f &\leq V_p^{-1}(r) \int_{B_p(r)} f \\ &\leq V_p^{-1}(r) V(\Omega \cap B_p(r)) \sup_M f. \end{aligned}$$

This implies that

$$(3.6) \quad \epsilon V_p(r) \geq V(B_p(r) \setminus \Omega)$$

for  $r \geq r_0$ .

Suppose  $g$  is a non-negative subharmonic function supported on  $M \setminus \Omega$ . If we define

$$s(2r) = \sup_{B_p(2r)} g,$$

then

$$\int_{B_p(2r)} g \leq s(2r) V(B_p(2r) \setminus \Omega).$$

On the other hand, the mean value inequality implies that there exists a constant  $C > 0$ , such that

$$C \int_{B_p(2r)} g \geq s(r) V_p(2r).$$

Therefore, we conclude that

$$C s(2r) V(B_p(2r) \setminus \Omega) \geq s(r) V_p(2r).$$

Combining with (3.6), we have

$$C \epsilon s(2r) \geq s(r)$$

for all  $r \geq r_0$ . Setting  $r = r_0$  and iterating this inequality  $k$  times, we arrive at the inequality

$$(3.7) \quad (C \epsilon)^k s(2^k r_0) \geq s(r_0).$$

If  $g$  is polynomial growth of at most degree  $\ell$ , then it follows that

$$s(r) \leq C_1 r^\ell.$$

Hence

$$s(2^k r_0) \leq C_1 2^{k\ell} r_0^\ell.$$

This contradicts (3.7) if we choose  $2^\ell C \epsilon < 1$ , and the lemma is proved. q.e.d.

**Theorem 3.7.** *Let  $M$  be a complete manifold satisfying conditions  $(\mathcal{V})$  and  $(\mathcal{P})$ . Then any nonconstant, polynomial growth, harmonic map  $u : M \rightarrow N$  into a Cartan-Hadamard manifold  $N$  which either is a two-dimensional visibility manifold or has strongly negative sectional curvature must satisfy  $u(M) \subset \mathcal{C}(\{a_i\}_{i=1}^{k'})$ , with  $\{a_i\}_{i=1}^{k'} = \overline{u(M)} \cap S_\infty(N)$ .*

*Proof.* Due to the fact that condition  $(\mathcal{M})$  is a consequence of conditions  $(\mathcal{P})$  and  $(\mathcal{V})$ , Theorem 3.5 applies to this case and  $A = \{a_i\}_{i=1}^{k'}$ . Let us assume the contrary that  $u$  is nonconstant and  $u(\overline{M})$  is not a subset of  $\mathcal{C}(A)$ . Then Lemma 1.6 implies that either  $d(u(x), \mathcal{C}(A))$  is bounded or there exists a tubular neighborhood  $A_\epsilon$  of  $A$  in  $\overline{N}$  with  $\epsilon > 0$  and  $u(M)$  is not contained in  $\mathcal{C}(A_\epsilon)$ . Since  $M$  satisfies  $(\mathcal{V})$  and  $(\mathcal{P})$ , the parabolic Harnack inequality holds on  $M$  by [8] and [15]. In particular,

$$(3.6) \quad \dim \mathcal{H}_0(M) = \dim \mathcal{H}_+(M) = 1.$$

Therefore, Theorem 2.1 yields that  $u(M) \subset \overline{\mathcal{C}(A \cup \{y\})}$  for some  $y \in u(M)$ . It is then easy to see that the function  $d(u(x), \mathcal{C}(A_\epsilon))$  is bounded. In either case, we conclude that there exists a closed subset  $W$  in  $\overline{N}$  such that the function

$$f(x) = d(u(x), \overline{\mathcal{C}(W)})$$

is a bounded, nonnegative, non-constant, subharmonic function on  $M$ . Moreover, the set  $C = u(M) \setminus \mathcal{C}(W)$  is a non-empty bounded set in  $N$ . Its convex hull  $\mathcal{C}(C)$  is also bounded, and  $u(M) \setminus \mathcal{C}(C)$  is non-empty because  $u$  is non-constant. The distance function

$$g(x) = d(u(x), \overline{\mathcal{C}(C)})$$

is a non-negative, non-constant, subharmonic function of polynomial growth. Also the support of  $f$  is in  $u^{-1}(C)$  and the support of  $g$  is on  $M \setminus u^{-1}(\overline{\mathcal{C}(C)})$ . This is impossible because of Lemma 3.6, and the theorem is proved. q.e.d.

**Corollary 3.8.** *Let  $u : M \rightarrow N$  be a non-constant harmonic map of polynomial growth with at most degree  $\ell$ . Suppose  $N$  is either a two-dimensional visibility manifold or a Cartan-Hadamard manifold*

with strongly negative sectional curvature. Assume that  $M^n$  is an  $n$ -dimensional manifold quasi-isometric to a manifold with non-negative Ricci curvature. Then there exists a set of  $k'$  points

$$\{a_i\}_{i=1}^{k'} = \overline{u(M)} \cap S_\infty(N)$$

with  $k' \leq C\ell^{(n-1)}$  such that  $u(M) \subset \overline{\mathcal{C}(\{a_i\}_{i=1}^{k'})}$ , where the constant  $C > 0$  depends only on the dimension of  $M$  and the quasi-isometric constant.

In [5], the authors showed that if a manifold  $M$  has Ricci curvature bounded from below and it is roughly isometric to a manifold with non-negative Ricci curvature, then  $M$  must also satisfy condition  $(\mathcal{V})$  and  $(\mathcal{P})$ . In particular, Theorem 3.7 can be applied to this case. Let us first recall the definition of rough isometry.

**Definition 3.9.** A map  $f : X \rightarrow Y$  between two metric spaces  $X$  and  $Y$  is a rough isometry if there exist constants  $a \geq 1, b > 0$ , and  $c > 0$ , such that, for all  $y \in Y$  there exists  $x \in X$  with the property that

$$d_Y(y, f(x)) \leq c,$$

and for any  $x_1, x_2 \in X$

$$a^{-1} d_X(x_1, x_2) - b \leq d_Y(f(x_1), f(x_2)) \leq a d_X(x_1, x_2) + b.$$

**Definition 3.10.** A map  $f : M \rightarrow M'$  between two manifolds  $M$  and  $M'$  is an isometry at infinity if it is a rough isometry and there exists a constant  $C > 0$  such that

$$C^{-1} V_x(1) \leq V_{f(x)}(1) \leq C V_x(1)$$

for all  $x \in M$ . In this case, the manifold  $M$  is said to be isometric at infinity to  $M'$ .

**Corollary 3.11.** Let  $u : M \rightarrow N$  be a non-constant harmonic map of polynomial growth with at most degree  $\ell$ . Suppose  $N$  is either a two-dimensional visibility manifold or a Cartan-Hadamard manifold with strongly negative sectional curvature. Assume that  $M^n$  is an  $n$ -dimensional manifold which has Ricci curvature bounded from below and it is isometric at infinity to a manifold with non-negative Ricci curvature. Then there exists a set of  $k'$  points

$$\{a_i\}_{i=1}^{k'} = \overline{u(M)} \cap S_\infty(N)$$

with  $k' \leq C\ell^{(n-1)}$  such that  $u(M) \subset \overline{\mathcal{C}(\{a_i\}_{i=1}^{k'})}$ , where the constant  $C > 0$  depends only on the dimension of  $M$  and the rough-isometric constants.

#### 4 Equivariant harmonic maps

In this section, we will apply the convex hull property to study equivariant harmonic maps which are of polynomial growth. An example of this situation comes from lifting a harmonic map between two compact manifolds to their universal coverings. In this case, if the universal covering of the domain manifold  $M$  has polynomial volume growth, and the target manifold  $N$  has negative curvature, then we can conclude that the harmonic map is either constant or its image lies on a geodesic. This allows us to conclude that any homomorphism from  $\pi_1(M)$  to  $\pi_1(N)$  must either be trivial or its image is an infinite cyclic group.

**Theorem 4.1.** *Let  $M$  be a complete manifold satisfying condition  $(\mathcal{M})$  and the volume growth  $V_p(r) = O(r^\nu)$  for some point  $p \in M$ , and  $N$  is either a two-dimensional visibility manifold or a Cartan-Hadamard manifold with strongly negative sectional curvature. Let  $u : M \rightarrow N$  be a non-constant harmonic map of polynomial growth of degree at most  $\ell$ . Suppose  $G$  and  $H$  are groups of isometries of  $M$  and  $N$ , respectively, such that  $u$  is equivariant with respect to  $G$  and  $H$ . In particular, for each  $g \in G$ , there exists  $h_g \in H$ , such that*

$$u(g(x)) = h_g(u(x)).$$

*We assume that for all  $h \in H$ ,  $h = h_g$  for some  $g \in G$ . Then any isometry in  $H$  must be either elliptic or hyperbolic. If there exists a hyperbolic isometry in  $H$ , then  $u(M)$  must be a geodesic line  $\gamma \subset N$ . Any non-trivial isometry of  $H$  must act on  $\gamma$  as translation, and  $H$  is an infinite cyclic group. If all isometries in  $H$  are elliptic, then any  $h \in H$  must act on the set of vertices  $\{a_i\}_{i=1}^{k'}$  at infinity and the set of interior vertices  $\{y_j\}_{j=1}^k$  as permutations. In particular,  $H$  has at most  $k'!k!$  elements, where  $k + k' \leq \lambda 3^{(2\ell+\nu)}$ . If  $M$  is further assumed to have property  $(\mathcal{V})$ , then  $k + k' \leq C\ell^{\nu-1}$  as given by Theorem 3.5.*

*Proof.* According to Theorem 3.5, we have

$$u(M) \subset \overline{\mathcal{C}(\{a_i\}_{i=1}^{k'} \cup \{y_j\}_{j=1}^k)},$$

where  $\overline{\{a_i\}_{i=1}^{k'}} \subset \overline{u(M)} \cap S_\infty(N)$  is a set of vertices at infinity, and  $\{y_j\}_{j=1}^k \subset \overline{u(M)} \cap N$  is a set of interior vertices with  $k + k' \leq \lambda 3^{(2\ell+\nu)}$

in general, and  $k' + k \leq C\ell^{\nu-1}$  if  $M$  has property  $(\mathcal{V})$ . For simplicity, let us denote

$$\mathcal{B} = \overline{\mathcal{C}(\{a_i\}_{i=1}^{k'} \cup \{y_j\}_{j=1}^k)}.$$

The equivariant assumption of  $u$  with respect to  $G$  and  $H$  implies that  $H$  acts invariantly on  $u(M)$ . It follows that  $H$  acts invariantly on  $\mathcal{B}$ . In particular, one deduces that the action of  $H$  also leaves the sets  $\{a_i\}_{i=1}^{k'}$  and  $\{y_j\}_{j=1}^k$  invariant.

We will first show that there are no parabolic isometries in  $H$ . To see this, assume  $h \in H$  is a parabolic isometry. In this case,  $h$  has precisely one fixed point  $y \in S_\infty(N)$ . Since there are only finitely many interior vertices and they form an invariant set under  $H$ , unless it is an empty set, there must exist an integer  $t$  such that  $h^t$  fixes at least one interior vertex. On the other hand,  $h^t$  must also be parabolic and this is impossible. Hence  $\mathcal{B}$  must not have any interior vertex, in which case,  $\mathcal{B}$  must have at least two vertices at infinity. Using the same argument, we see that for some integer  $s$ , the isometry  $h^s$  will have at least 2 fixed exterior vertices. Again, this is a contradiction.

If  $h$  is a hyperbolic isometry, then  $h$  must have precisely two fixed points in  $S_\infty(N)$ . Using a similar argument as above, we conclude that  $\mathcal{B} = \mathcal{C}(\{a_1, a_2\})$ , and that  $\{a_1, a_2\}$  is the fixed point set of  $h$ . If we denote the geodesic line joining  $a_1$  and  $a_2$  by  $\gamma$ , then the action of  $h$  on  $\gamma$  must be a translation. In particular,  $h$  is of infinite order and the connectedness of  $u(M)$  implies that  $u(M) = \gamma$ . In particular,  $\gamma$  is invariant under all isometries in  $H$ . Hence all elements in  $H$  are hyperbolic which act on  $\gamma$  as translations. It is now clear that  $H$  must be an infinite cyclic group.

Finally, we may assume that all isometries in  $H$  are elliptic. In this case, there must be an interior fixed point for any isometry in  $H$ . We claim that if the sets  $\{a_i\}_{i=1}^{k'}$  and  $\{y_j\}_{j=1}^k$  are contained in the fixed point set of  $h \in H$ , then  $\mathcal{B}$  must be in the fixed point set of  $h$ , and hence the action of  $h$  is trivial on  $u(M)$ . To see this, we define  $C_1(\{a_i\}_{i=1}^{k'} \cup \{y_j\}_{j=1}^k)$  to be the union of all geodesic segments joining any two points in  $\{a_i\}_{i=1}^{k'} \cup \{y_j\}_{j=1}^k$ . The uniqueness of geodesic implies that  $C_1(\{a_i\}_{i=1}^{k'} \cup \{y_j\}_{j=1}^k)$  is in the fixed point set of  $h$ . Inductively, we define

$$C_d(\{a_i\}_{i=1}^{k'} \cup \{y_j\}_{j=1}^k) = C_1(C_{d-1}(\{a_i\}_{i=1}^{k'} \cup \{y_j\}_{j=1}^k))$$

to be the union of all geodesic segments joining any two points in  $C_{d-1}(\{a_i\}_{i=1}^{k'} \cup \{y_j\}_{j=1}^k)$ . Similarly, we conclude that  $C_d(\{a_i\}_{i=1}^{k'} \cup \{y_j\}_{j=1}^k)$

must be in the fixed point set of  $h$ . The claim now follows by observing that

$$\mathcal{B} = \overline{\cup_{d=1}^{\infty} C_d(\{a_i\}_{i=1}^{k'} \cup \{y_j\}_{j=1}^k)}.$$

Since  $H$  acts as permutations on the sets  $\{a_i\}_{i=1}^{k'}$  and  $\{y_j\}_{j=1}^k$ , the order of  $H$  must be bounded by the product of the orders of the permutation groups of  $k'$  and  $k$  elements, and the theorem is proved. q.e.d.

**Corollary 4.2.** *Let  $M$  be a compact manifold whose universal covering  $\tilde{M}$  has polynomial volume growth. Suppose  $N$  is a compact manifold with negative sectional curvature. If  $u : M \rightarrow N$  is a harmonic map from  $M$  into  $N$ , then either  $u$  is a constant map or  $u(M)$  is contained in a geodesic of  $N$ .*

*Proof.* Let  $\tilde{M}$  and  $\tilde{N}$  be the universal coverings of  $M$  and  $N$ , respectively. By lifting  $u$  to a harmonic map  $\tilde{u} : \tilde{M} \rightarrow \tilde{N}$ , the compactness of  $M$  asserts that  $\tilde{u}$  must have bounded energy density, and hence is of at most linear growth.

The volume growth assumption on  $\tilde{M}$  together with Milnor's argument [13] implies that  $\pi_1(M)$  is at most of polynomial growth. An argument of Varopoulos [17] then asserts that the Sobolev inequality, hence the mean value inequality ( $\mathcal{M}$ ), is valid on  $\tilde{M}$ . Applying Theorem 4.1, with  $G = \pi_1(M)$  and  $H = u_*(\pi_1(M))$ , we conclude that  $H$  is either finite or infinite cyclic with  $\tilde{u}(\tilde{M}) = \gamma$ , a geodesic line in  $\tilde{N}$ . The latter yields that  $u(M)$  is a geodesic and we only need to show that when  $H$  is finite then  $u$  must be constant.

To see this, observe that the finiteness of  $u_*(\pi_1(M))$  gives that there exists a finite cover  $M'$  of  $M$  such that  $u$  can be lifted to a harmonic map  $u' : M' \rightarrow \tilde{N}$ . The compactness of  $M$  implies that  $M'$  is also compact. Since  $\tilde{N}$  is a Cartan-Hadamard manifold,  $u'$  must be constant. Hence  $u$  must be constant to begin with, and the corollary is proved. q.e.d.

The next two corollaries are obvious consequences.

**Corollary 4.3.** *Let  $M$  be a compact Kähler manifold whose universal covering  $\tilde{M}$  has polynomial volume growth. Suppose  $N$  is a compact Kähler manifold with negative sectional curvature. If  $u : M \rightarrow N$  is a holomorphic map from  $M$  into  $N$ , then  $u$  must be a constant map.*

**Corollary 4.4.** *Let  $M$  be a compact manifold with its universal covering  $\tilde{M}$  having polynomial volume growth. Suppose  $N$  is a compact manifold with negative sectional curvature. Let  $\alpha : \pi_1(M) \rightarrow \pi_1(N)$*

be a homomorphism from  $\pi_1(M)$  to  $\pi_1(N)$ . Then  $\alpha$  is either the trivial homomorphism or its image  $\alpha(\pi_1(M))$  is infinite cyclic.

*Proof.* Since  $N$  has negative sectional curvature, the existence theorem of Eells-Sampson asserts that there is a harmonic map  $u : M \rightarrow N$  such that  $u_* = \alpha$ . Corollary 4.2 now applies to  $u$  and we conclude that either  $\alpha$  is trivial or  $\alpha(\pi_1(M))$  is infinite cyclic. q.e.d.

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