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VARIATIONS OF THE BOUNDARY GEOMETRY OF 3–DIMENSIONAL HYPERBOLIC CONVEX CORES

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Dedicated to D.B.A. Epstein, on his 60th birthday.

Let M be a (connected) hyperbolic 3–manifold, namely a complete 3–dimensional Riemannian manifold of constant curvature -1, such that the fundamental group $\pi_1(M)$ is finitely generated. We exclude the somewhat degenerate case where $\pi_1(M)$ has an abelian subgroup of finite index. Then, a fundamental subset of M is its *convex core* C_M, defined as the smallest non-empty closed convex subset of M. The boundary ∂C_M of this convex core is a surface of finite topological type, and its geometry was described by W. P. Thurston [17] (see also [8]): The surface ∂C_M is almost everywhere totally geodesic, and is bent along a family of disjoint geodesics called its *pleating locus* . The path metric induced by the metric of M is hyperbolic, and the way ∂C_M is bent is completely determined by a certain measured geodesic lamination.

We want to investigate how the geometry of ∂C_M varies as we deform the metric of M. For technical reasons, in particular because we do not want the topology of ∂C_M to change, we choose to restrict attention to *quasi-isometric deformations* of M, namely hyperbolic manifolds M' for which there exists a diffeomorphism $M \to M'$ whose differential is uniformly bounded. In the language of Kleinian groups, a

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quasi-isometric deformation of M is also equivalent to a quasi-conformal deformation of its holonomy; see [17, §10]. This is not a very strong restriction. For instance, in the conjecturally generic case where M is geometrically finite without cusps, every small deformation of the metric is quasi-isometric. When M is geometrically finite, quasi-isometric deformations of the metric coincide with deformations of the holonomy $\pi_1(M) \to \mathrm{Isom}(\mathbb{H}^3)$ that respect parabolicity [12]. Also, every holomorphic family of hyperbolic manifolds homeomorphic to M consists of quasi-isometric deformations [16].

Let $\mathcal{QD}(M)$ be the space of quasi-isometric deformations of the metric of M, where we identify two deformations $M \to M'$ and $M \to M''$ when the corresponding pull back metrics on M are isotopic. This space can be parametrized by the space of conformal structures on the domain of discontinuity of M [1], [15], and in particular is a differentiable manifold of dimension $3|\chi(\partial C_M)| - c$, where $\chi(\ )$ denotes the Euler characteristic, and c is the number of cusps of ∂C_M. Given a quasi-isometric deformation M', there is a homeomorphism between ∂C_M and $\partial C_{M'}$, well defined up to isotopy. Consequently, if we consider the geometry of $\partial C_{M'}$, its hyperbolic metric defines an element $\mu(M')$ of the Teichmüller space $\mathcal{T}(\partial C_M)$, and its bending measured geodesic lamination defines an element $\beta(M')$ of the space $\mathcal{ML}(\partial C_M)$ of compact measured geodesic laminations on ∂C_M; see [17], [6], [8] for a definition of these notions.

Before going any further, we must mention that the definitions have to be adapted in the special case where the convex core C_M is a totally geodesic surface, namely when M is Fuchsian or twisted Fuchsian. To keep the correspondence between ∂C_M and the domain of discontinuity of M, we define in this case ∂C_M as the unit normal bundle of C_M in M, namely as the 'two sides' of C_M in M, whereas the topological boundary of C_M is equal to C_M. With this convention, we have as above a prefered (up to isotopy) identification between ∂C_M and $\partial C_{M'}$ for every quasi-isometric deformation $M \to M'$, and such a deformation again defines a hyperbolic metric $\mu(M') \in \mathcal{T}(\partial C_M)$ and a bending measured lamination $\beta(M') \in \mathcal{ML}(\partial C_M)$.

Theorem 1. *For every hyperbolic 3–manifold M, the map $\mu : \mathcal{QD}(M) \to \mathcal{T}(\partial C_M)$, defined by considering the hyperbolic metrics of convex core boundaries, is continuously differentiable.*

A simple example in §6 shows that the map μ is not necessarily twice differentiable.

To prove a similar differentiability property for the map

$$\beta : \mathcal{QD}(M) \to \mathcal{ML}(\partial C_M),$$

we encounter a conceptual difficulty. Indeed, the space $\mathcal{ML}(\partial C_M)$ does not have a natural differentiable structure. On the other hand, it has a natural structure of piecewise linear manifold of dimension

$$3\,|\chi(\partial C_M)| - c;$$

see for instance [17], [14]. In this context, we can use a weak notion of differentiability, namely the existence of a tangent map (see §1 for a definition).

Theorem 2. *The map $\beta : \mathcal{QD}(M) \to \mathcal{ML}(\partial C_M)$, defined by considering the bending measured laminations of convex core boundaries, is tangentiable in the sense that it admits a tangent map everywhere.*

The tangent map of β plays an important rôle in the variation of the volume of the convex core C_M, as one varies the hyperbolic metric; see [5]. A continuity property, in a weak sense, for the maps μ and β was earlier obtained by L. Keen and C. Series [11].

The proof of Theorems 1 and 2 is probably of as much interest as the results themselves. Indeed, these two statements are proved simultaneously, mixing together the differentiable and piecewise linear contexts. In particular, the 'corners' of the piecewise linear structure of $\mathcal{ML}(\partial C_M)$ account for the fact that the map μ is not C^2.

The proof goes as follows. First of all, we can restrict attention to the case where M is orientable. Indeed, if $\widehat{M}$ is its orientation covering, the spaces $\mathcal{QD}(M)$, $\mathcal{T}(\partial C_M)$ and $\mathcal{ML}(\partial C_M)$ are submanifolds (in the appropriate category) of $\mathcal{QD}(\widehat{M})$, $\mathcal{T}(\partial C_{\widehat{M}})$ and $\mathcal{ML}(\partial C_{\widehat{M}})$, respectively, and the maps μ, β for M are just the restrictions of the corresponding maps for $\widehat{M}$. Consequently, we will henceforth assume that M is orientable.

Let $S_1, \dots, S_n$ be the components of ∂C_M. For each i, let $\mathcal{R}(S_i)$ denote the space of representations $\pi_1(S_i) \to \mathrm{Isom}^+(\mathbb{H}^3)$ sending the fundamental group of each end of S_i to a parabolic subgroup of $\mathrm{Isom}^+(\mathbb{H}^3)$, where $\mathrm{Isom}^+(\mathbb{H}^3)$ denotes the group of orientation-preserving isometries of the hyperbolic 3–space $\mathbb{H}^3$, and these representations are considered modulo conjugation by elements of $\mathrm{Isom}^+(\mathbb{H}^3)$. Let $\mathcal{R}(\partial C_M)$ denote the product $\prod_{i=1}^n \mathcal{R}(S_i)$. Restricting the holonomy of a quasi-isometric deformation to the components of ∂C_M, we get a map R :

$\mathcal{QD}(M) \to \mathcal{R}(\partial C_M)$. The image of R is in the non-singular part of $\mathcal{R}(\partial C_M)$, and R is differentiable. See for instance [7], and [12], [1].

If we are given a finite area hyperbolic metric and a compactly supported measured geodesic lamination on the surface S_i, we can always realize these in a unique way as the pull back metric and the bending measured lamination of a pleated surface $f = (\widetilde{f}, \rho)$, where $\rho \in \mathcal{R}(S_i)$ is not necessarily discrete, and $\widetilde{f} : \widetilde{S}_i \to \mathbb{H}^3$ is a ρ–equivariant pleated surface from the universal covering of S_i into $\mathbb{H}^3$; see [8], [10], [4]. By considering the corresponding representations, this defines a map $\varphi : \mathcal{T}(\partial C_M) \times \mathcal{ML}(\partial C_M) \to \mathcal{R}(\partial C_M)$. Thurston showed that φ is a local homeomorphism, by establishing a correspondence between $\mathcal{T}(\partial C_M) \times \mathcal{ML}(\partial C_M)$ and the space of complex projective structures on ∂C_M; see [10], and see [9] for a description of the image of φ. In particular, there is a local inverse φ^{-1} defined near the point of $\mathcal{R}(\partial C_M)$ corresponding to the original metric of M. Then, near that metric, the product $\mu \times \beta : \mathcal{QD}(M) \to \mathcal{T}(\partial C_M) \times \mathcal{ML}(\partial C_M)$ coincides with the composition $\varphi^{-1} \circ R$.

The main technical step in the proof of Theorems 1 and 2 is to show that the map φ is tangentiable, and that its tangent map is everywhere injective. This is done in §§2–3, by locally comparing φ to the parametrization of $\mathcal{R}(\partial C_M)$ by shear-bend coordinates developed in [4]. The crucial technical step here is the growth estimate provided by Lemma 7. Then, an easy inverse function theorem (Lemma 4 in §1) shows that the local inverse φ^{-1} is tangentiable. Since $\mu \times \beta = \varphi^{-1} \circ R$ and R is differentiable, it follows that μ and β are tangentiable. In addition, the proof gives that the tangent map of μ is linear, so that μ is differentiable in the usual sense. Continuity properties for the differential of μ follow from the computation of this differential, and are proved in §5.

As a by-product of the proof, we obtain the following result for the space of complex projective structures on a connected surface S of finite type (without boundary). A *complex projective structure* on S is an atlas modelling S over open subsets of the complex projective line $\mathbb{CP}^1$, where all changes of charts extend to elements of the projective group $\mathrm{PSL}_2(\mathbb{C})$, and the atlas is maximal for this property. Let $\mathcal{P}(S)$ be the space of isotopy classes of complex projective structures on S which are of cusp type near the ends of S. When $\chi(S) < 0$, Thurston defined a homeomorphism $\psi : \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathcal{P}(S)$, by associating to each complex projective structure a locally convex pleated surface; see [10] for an exposition. Because geometric structures are locally parametrized by deformations of their monodromy, the monodromy map $\mathcal{P}(S) \to \mathcal{R}(S)$ is a local diffeomorphism. Our proof that

$$\varphi : \mathcal{T}(\partial C_M) \times \mathcal{ML}(\partial C_M) \to \mathcal{R}(\partial C_M)$$

and its local inverses are tangentiable immediately gives:

Theorem 3. *The Thurston homeomorphism*

$$\psi : \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathcal{P}(S)$$

and its inverse are tangentiable.

Again, if we compose ψ^{-1} with the projection

$$\mathcal{T}(S) \times \mathcal{ML}(S) \to \mathcal{T}(S),$$

the map $\mathcal{P}(S) \to \mathcal{T}(S)$ so obtained is C^1 but not C^2.

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1. Tangent maps

Given a map $\varphi : U \to \mathbb{R}^p$ defined on an open subset U of $\mathbb{R}^n$, its *tangent map* at $x \in U$ is, if it exists, the map $T_x\varphi : \mathbb{R}^n \to \mathbb{R}^p$ such that one of the following equivalent conditions holds:

(i) $T_x\varphi(v) = \lim_{t\to 0^+} (\varphi(x+tv) - \varphi(x))/t$, uniformly in v on compact subsets of $\mathbb{R}^n$;

(ii) for every continuous curve $\gamma : [0,\varepsilon[\to U$ with $\gamma(0) = x$ and $\gamma'(0) = v$, $T_x\varphi(v) = (\varphi\circ\gamma)'(0)$.

(iii) for every sequence of points $x_n \in \mathbb{R}^n$ and numbers $t_n > 0$ such that $\lim_{n\to\infty} t_n = 0$ and $\lim_{n\to\infty}(x_n - x)/t_n = v$, $T_x\varphi(v) = \lim_{n\to\infty}(\varphi(x_n) - \varphi(x))/t_n$ (a discrete version of (ii)).

The equivalence of these three conditions is an easy exercise. The tangent map $T_x\varphi$ is continuous and positive homogeneous of degree 1 (namely $T_x\varphi(av) = aT_x\varphi(v)$ for every $v \in \mathbb{R}^n$ and $a \geqslant 0$), but not necessarily linear. We will say that φ is *tangentiable* if it admits a tangent map at each $x \in U$.

A *tangentiable structure* on a topological manifold is a maximal atlas where all changes of charts are tangentiable. Examples of such tangentiable manifolds include differentiable manifolds, piecewise linear manifolds and products of these, as we will encounter in this paper. By the usual tricks, we can define a space T_xM of tangent vectors at each point x of a tangentiable manifold M. This tangent space T_xM is not necessarily a vector space, although it admits a law of multiplication by non-negative numbers. There is also a notion of tangentiable map between tangentiable manifolds, defined using charts, and such a tangentiable map $\varphi : M \to N$ induces a tangent map $T_x\varphi : T_xM \to T_{\varphi(x)}N$ for every $x \in M$.

Lemma 4. *Let $\varphi : M \to N$ be a homeomorphism between two tangentiable manifolds. Assume that φ admits a tangent map at $x \in M$, and that this tangent map $T_x\varphi : T_xM \to T_{\varphi(x)}N$ is injective. Then, the inverse φ^{-1} admits a tangent map at $\varphi(x)$, and $T_{\varphi(x)}\varphi^{-1} = (T_x\varphi)^{-1}$.*

Proof. Because φ is a homeomorphism, $T_x\varphi$ is surjective by a degree argument. The fact that $T_{\varphi(x)}\varphi^{-1} = (T_x\varphi)^{-1}$ easily follows by taking appropriate subsequences in Definition (iii) of tangentiability. q.e.d.

2. Proof that $\varphi : \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathcal{R}(S)$ is tangentiable

Let S be a connected oriented surface of finite topological type and negative Euler characteristic. Given a finite area hyperbolic metric m and a compactly supported measured geodesic lamination b on S, there is a unique locally convex pleated surface $f = \left(\widetilde{f}, \rho\right)$ whose pull back metric is equal to m and whose bending measured lamination is equal to b; see [8], [10], [4]. This defines a map $\varphi : \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathcal{R}(S)$. This bending map φ is also the composition of the Thurston parametrization $\psi : \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathcal{P}(S)$ with the holonomy map $\mathcal{P}(S) \to \mathcal{R}(S)$. Because ψ and the monodromy map $\mathcal{P}(S) \to \mathcal{R}(S)$ are local homeomorphisms, so is φ.

In [4], we developed another local parametrization of $\mathcal{R}(S)$ which similarly uses pleated surfaces. Fix a compact geodesic lamination λ on S. If $f = \left(\widetilde{f}, \rho\right)$ is a pleated surface with pleating locus λ, the amount by which f bends along λ is measured by a transverse finitely additive measure for λ, valued in $\mathbb{R}/2\pi\mathbb{Z}$. We call such a transverse finitely additive measure an $\mathbb{R}/2\pi\mathbb{Z}$–valued *transverse cocycle* for λ. In general, this *bending transverse cocycle* is not a transverse (countably additive) measure, unless the pleated surface is *locally convex* , namely always bends in the same direction. Let $\mathcal{H}(\lambda;\mathbb{R}/2\pi\mathbb{Z})$ denote the space of all $\mathbb{R}/2\pi\mathbb{Z}$–valued transverse cocycles for λ.

Given $m \in \mathcal{T}(S)$ and $b \in \mathcal{H}(\lambda;\mathbb{R}/2\pi\mathbb{Z})$, there is a unique pleated surface $f = \left(\widetilde{f}, \rho\right)$ with pleating locus λ, pull back metric m and bending transverse cocycle b. This defines a differentiable map $\varphi_\lambda : \mathcal{T}(S) \times \mathcal{H}(\lambda;\mathbb{R}/2\pi\mathbb{Z}) \to \mathcal{R}(S)$. If, in addition, λ is *maximal* among all compact geodesic laminations (this is equivalent to say that each component of $S - \lambda$ is, either an infinite triangle, or an annulus leading to a cusp and with exactly one spike in its boundary), then φ_λ is a local diffeomorphism; see [4].

Transverse cocycles occurred in a different context in [3]. The piecewise linear structure of $\mathcal{ML}(S)$ defines a space of tangent vectors at each of its points, as in §1. In [3], we gave an interpretation of these combinatorial tangent vectors at $a \in \mathcal{ML}(S)$ as geodesic laminations containing the support of a and endowed with transverse $\mathbb{R}$–valued cocycles. In this context, Proposition 5 below connects the infinitesimal properties of the maps $\varphi : \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathcal{R}(S)$ and $\varphi_\lambda : \mathcal{T}(S) \times \mathcal{H}(\lambda;\mathbb{R}/2\pi\mathbb{Z}) \to \mathcal{R}(S)$.

Before stating this result, it is convenient to introduce the following notation. We will often have to consider the right derivatives at $t = 0$ of various quantities a_t defined for $t \in [0, \varepsilon[$, with $\varepsilon > 0$. We will denote such a right derivative $da_t/dt^+_{|t=0}$ by $\dot{a}_0$.

Proposition 5. *Let the* 1*–parameter families* $m_t \in \mathcal{T}(S)$ *and* $b_t \in \mathcal{ML}(S)$*,* $t \in [0, \varepsilon[$*, admit tangent vectors* $\dot{m}_0$ *and* $\dot{b}_0$ *at* $t = 0$*, respectively, and let* $\rho_t = \varphi(m_t, b_t) \in \mathcal{R}(S)$*. Interpret* $\dot{b}_0$ *as a geodesic lamination with a transverse* $\mathbb{R}$*–valued cocycle, and choose a maximal geodesic lamination* λ *which contains the supports of* b_0 *and* $\dot{b}_0$*. In particular,* b_0 *and* $\dot{b}_0$ *can both be considered as elements of* $\mathcal{H}(\lambda;\mathbb{R})$*, and* $\rho_0 = \varphi_\lambda\left(m_0, \bar{b}_0\right)$ *where* $\bar{b}_0 \in \mathcal{H}(\lambda;\mathbb{R}/2\pi\mathbb{Z})$ *is the reduction of* b_0 *modulo* 2π*. Then, the family* ρ_t *admits a tangent vector* $\dot{\rho}_0$ *at* $t = 0^+$ *and* $\dot{\rho}_0 = T_{(m_0,\bar{b}_0)}\varphi_\lambda\left(\dot{m}_0, \dot{b}_0\right)$*.*

The tangent space $T_{b_0}\mathcal{ML}(S)$ admits a decomposition into linear faces. Each face is associated to a geodesic lamination λ containing the support of b_0, and the tangent vectors in this face correspond to (some) transverse cocycles in $\mathcal{H}(\lambda;\mathbb{R})$; see [3, §5]. Proposition 5 immediately implies the following corollary.

Corollary 6. *The map $\varphi : \mathcal{T}(S)\times\mathcal{ML}(S)\to\mathcal{R}(S)$ is tangentiable at each (m_0,b_0). In addition, if λ is a maximal geodesic lamination containing the support of b_0, and $\bar b_0\in\mathcal{H}(\lambda;\mathbb{R}/2\pi\mathbb{Z})$ denotes the reduction of b_0 modulo 2π, then the tangent map $T_{(m_0,b_0)}\varphi$ coincides with $T_{(m_0,\bar b_0)}\varphi_\lambda$ on the product of $T_{m_0}\mathcal{T}(S)$ and the face of $T_{b_0}\mathcal{ML}(S)$ associated to λ.*

Proof of Proposition 5. Consider the transverse cocycle $b'_t = b_0 + t\dot b_0 \in \mathcal{H}(\lambda;\mathbb{R})$ and its reduction $\bar b'_t\in\mathcal{H}(\lambda;\mathbb{R}/2\pi\mathbb{Z})$ modulo 2π. Let $\rho'_t = \varphi_\lambda(m_t,\bar b'_t)\in\mathcal{R}(S)$. Because $\dot b'_0 = \dot b_0$ and φ_λ is a differentiable map, the curve $t\mapsto\rho'_t$ admits a tangent vector $\dot\rho'_0 = T_{(m_0,\bar b_0)}\varphi_\lambda(\dot m_0,\dot b_0)$ at $t=0$. We will compare the two curves $t\mapsto\rho_t$ and $t\mapsto\rho'_t$ in $\mathcal{R}(S)$, and show that they are tangent at $t=0$.

We first make the additional assumption that, for the Hausdorff topology, the geodesic lamination λ_t underlying b_t converges to some sublamination of λ as t tends to 0^+. We will later indicate how to obtain the general case from this one.

Let $f_t = (\widetilde f_t,\rho_t)$ be the locally convex pleated surface with pull back metric m_t and bending measured lamination b_t. Similarly, let $f'_t = (\widetilde f'_t,\rho'_t)$ be the pleated surface with pull back metric m_t, pleating locus λ, and bending transverse cocycle $\overline{b}'_t$. In the universal covering $\widetilde S$, consider the preimage $\widetilde\lambda$ of λ.

So far, the metric m_t was defined only up to isotopy of S, and $\widetilde f_t$, ρ_t, $\widetilde f'_t$ and ρ'_t were only defined up to conjugacy by isometries of $\mathbb{H}^3$. We can normalize these so that the metric m_t C^∞–converges to m_0 and so that, for a choice of a base point $\widetilde x_0\in\widetilde S-\widetilde\lambda$ and of a base frame at $\widetilde x_0$, $\widetilde f_t$ and $\widetilde f'_t$ coincide with $\widetilde f_0$ at these base point and frame.

To show that the two curves $t\mapsto\rho_t$ and $t\mapsto\rho'_t$ are tangent at $t=0$ in $\mathcal{R}(S)$, it suffices to show that, for each $\xi\in\pi_1(S)$, the curves $t\mapsto\rho_t(\xi)$ and $t\mapsto\rho'_t(\xi)$ are tangent at $t=0$ in $\mathrm{Isom}^+(\mathbb{H}^3)$. For this, we first have to remind the reader of the construction of $(\widetilde f_t,\rho_t)$ and $(\widetilde f'_t,\rho'_t)$.

We begin with the totally geodesic (un-)pleated surface $(\widetilde f''_t,\rho''_t)$ with pull back metric m_t and bending measured lamination 0, normalized so that $\widetilde f''_t$ coincides with $\widetilde f_0$ at the base frame in $\widetilde S$. To fix ideas, we can arrange that $\widetilde f''_t(\widetilde S)=\mathbb{H}^2\subset\mathbb{H}^3$. Choose as base point x_0 for the fundamental group $\pi_1(S)=\pi_1(S;x_0)$ the image of the base point $\widetilde x_0\in\widetilde S$. Let $\widetilde c$ be the m_0–geodesic arc in $\widetilde S$ going from $\widetilde x_0$ to $\xi\widetilde x_0$, so that the projection of $\widetilde{c}$ to S represents $\xi \in \pi_1(S; x_0)$. Then, $\rho_t(\xi)$ and $\rho'_t(\xi)$ are defined by composition of $\rho''_t(\xi)$ with rotations around certain geodesics of $\mathbb{H}^2 \subset \mathbb{H}^3$ that are determined by ξ, λ, and b_t.

Let $U \subset S$ be a train track neighborhood carrying λ, or more precisely carrying the m_0–geodesic lamination corresponding to λ. We can choose U sufficiently small so that, if $\widetilde{U}$ is its preimage in $\widetilde{S}$, each component of $\widetilde{c} \cap \widetilde{U}$ is an arc contained in a single edge of $\widetilde{U}$. Because of our assumption that the m_0–geodesic lamination underlying b_t converges to some sublamination of λ, U will also carry this lamination for t sufficiently small. Finally, since the metric m_t converges to m_0, the m_t–geodesic representative of the geodesic lamination underlying b_t will also be carried by U for t sufficiently small.

For $r \geqslant 0$, let Γ_r be the set of all edge paths of length $2r+1$ in $\widetilde{U}$ that are centered on an edge meeting $\widetilde{c}$. We can partially order the elements of Γ_r from $\widetilde{x}_0$ to $\xi\widetilde{x}_0$ as follows. For two edge paths γ, γ' centered at different edges of $\widetilde{U}$, $\gamma \prec \gamma'$ precisely when the central edge of γ cuts $\widetilde{c}$ closer to $\widetilde{x}_0$ than the central edge of γ'. Two edge paths γ, γ' with the same central edge e follow a common edge path and diverge at 1 or 2 switches; then $\gamma \prec \gamma'$ precisely when γ diverges always on the side of γ' which contains the point of $e \cap \widetilde{c}$ that is closest to $\widetilde{x}_0$. Neither $\gamma \prec \gamma'$ nor $\gamma' \prec \gamma$ holds when γ and γ' have the same central edge and diverge on opposite sides.

To each edge path γ of $\widetilde{U}$, the transverse measure of b_t associates a number $b_t(\gamma) \geqslant 0$, namely the b_t–mass of the set of those geodesics realizing γ (whether we consider m_t– or m_0–geodesics does not matter here because the m_0–geodesic lamination and m_t–geodesic lamination underlying b_t are both carried by U). This $b_t(\gamma)$ is a piecewise linear function of $b_t \in \mathcal{ML}(S)$, and the fact that $t \mapsto b_t$ admits a tangent vector at $t = 0^+$ is equivalent to the property that $t \mapsto b_t(\gamma)$ admits a right derivative $\dot{b}_0(\gamma)$ for every edge path γ. The transverse cocycle b'_t similarly associates to γ a number $b'_t(\gamma)$ which, in our case, is equal to $b_0(\gamma) + t\dot{b}_0(\gamma)$. See [2], [3].

List all the elements of Γ_r as $\gamma_1, \gamma_2, \dots, \gamma_p$ in a way which is compatible with the partial order $\prec$, namely so that $i < j$ whenever $\gamma_i \prec \gamma_j$. For each γ_i, let g_i^t be the geodesic of $\mathbb{H}^2 \subset \mathbb{H}^3$ image under $\widetilde{f}''_t : \widetilde{S} \to \mathbb{H}^2 \subset \mathbb{H}^3$ of an m_t–geodesic of $\widetilde{S}$ that is carried by $\widetilde{U}$ and realizes γ_i. Such a geodesic may not exist for every γ_i, but it will definitely exist if at least one of $b_t(\gamma_i)$ or $b'_t(\gamma_i)$ is non-zero (for instance, a leaf of the m_t–geodesic lamination underlying b_t if $b_t(\gamma_i) \neq 0$, or a leaf of the m_t-geodesic lamination corresponding to λ if $b'_t(\gamma_i) \neq 0$), which is exactly the case in which we need it. Then,

$$\rho_t(\xi) = \lim_{r\to\infty} R_{g_1^t}^{b_t(\gamma_1)} R_{g_2^t}^{b_t(\gamma_2)} \dots R_{g_p^t}^{b_t(\gamma_p)} \rho''_t(\xi) \tag{1}$$

and

$$\rho'_t(\xi) = \lim_{r\to\infty} R_{g_1^t}^{b'_t(\gamma_1)} R_{g_2^t}^{b'_t(\gamma_2)} \dots R_{g_p^t}^{b'_t(\gamma_p)} \rho''_t(\xi) . \tag{2}$$

where $R_g^b \in \mathrm{Isom}^+(\mathbb{H}^3)$ denotes the hyperbolic rotation of angle $b \in \mathbb{R}/2\pi\mathbb{Z}$ around the oriented geodesic g, and the g_i^t are oriented to the left as seen from the base point $\widetilde{f}_0(\widetilde{x}_0)$ in $\mathbb{H}^2$. Compare [8, §3] for the case of transverse measures, and see [4, §5] for the more general case of transverse cocycles, where the convergence is much more subtle.

Identify the isometry group $\mathrm{Isom}^+(\mathbb{H}^3)$ with some matrix group, for instance $\mathrm{SO}(3,1)$, and endow the corresponding space of matrices with any of the classical norms $\| \ \|$ such that $\|AB\| \leqslant \|A\| \, \|B\|$.

We can write the difference $\rho_t(\xi) - \rho'_t(\xi)$ as

$$\rho_t(\xi) - \rho'_t(\xi) = \lim_{r\to\infty} A_r^t - B_r^t = \lim_{r\to\infty} C_r^t,$$

where

$$\begin{aligned} A_r^t &= R_{g_1^t}^{b_t(\gamma_1)} R_{g_2^t}^{b_t(\gamma_2)} \dots R_{g_p^t}^{b_t(\gamma_p)} \rho''_t(\xi) , \\ B_r^t &= R_{g_1^t}^{b'_t(\gamma_1)} R_{g_2^t}^{b'_t(\gamma_2)} \dots R_{g_p^t}^{b'_t(\gamma_p)} \rho''_t(\xi) , \end{aligned}$$

and $C_r^t = A_r^t - B_r^t$.

The following growth estimate is the technical key to the proof of Proposition 5.

Lemma 7. *There is a number $A > 0$ such that*

$$C_{r+1}^t - C_r^t = tO\left(e^{-Ar} \|\dot{b}_0\|_U\right)$$

and

$$\rho_t(\xi) - \rho'_t(\xi) = C_r^t + tO\left(e^{-Ar} \|\dot{b}_0\|_U\right),$$

where $\|\dot{b}_0\|_U$ denote the maximum of $|\dot{b}_0(e)|$ as e ranges over all edges of U, and A and the constants hidden in the symbols $O(\ )$ are independent of r and t.

Proof of Lemma 7. List the edge paths of Γ_{r+1} as $\delta_1, \dots, \delta_q$, where the indexing is chosen to be compatible with the partial order $\prec$. There is a natural map $\sigma : \Gamma_{r+1} \to \Gamma_r$, where $\sigma(\delta_i)$ is defined by chopping off the two end edges of δ_i. This map respects $\prec$ in the sense that, if $\delta \prec \delta'$, then $\sigma(\delta) \prec \sigma(\delta')$ or $\sigma(\delta) = \sigma(\delta')$. We can therefore choose the indexing so that, for every j, the set of those indices i for which $\sigma(\delta_i) = \gamma_j$ is of the form $k, k+1, \dots, k+l$. We will also denote by σ the map $\{1, \dots, q\} \to \{1, \dots, p\}$ defined by $\sigma(\delta_i) = \gamma_{\sigma(i)}$.

For each δ_i, let h_i^t be the image under $\widetilde{f}_t'' : \widetilde{S} \to \mathbb{H}^2 \subset \mathbb{H}^3$ of an m_t–geodesic of $\widetilde{S}$ that is carried by $\widetilde{U}$ and realizes δ_i, if such a geodesic exists. Then,

$$A_{r+1}^t = R_{h_1^t}^{b_t(\delta_1)} R_{h_2^t}^{b_t(\delta_2)} \dots R_{h_q^t}^{b_t(\delta_q)} \rho_t''(\xi).$$

Noting that $b_t(\gamma_j) = \sum_{\sigma(i)=j} b_t(\delta_i)$, we can rewrite A_r^t as

$$A_r^t = R_{g_{\sigma(1)}^t}^{b_t(\delta_1)} R_{g_{\sigma(2)}^t}^{b_t(\delta_2)} \dots R_{g_{\sigma(q)}^t}^{b_t(\delta_q)} \rho_t''(\xi).$$

We conclude that

$$\begin{aligned} &A_{r+1}^t - A_r^t \\ &\quad = \sum_{i=1}^{q} R_{h_1^t}^{b_t(\delta_1)} \dots R_{h_{i-1}^t}^{b_t(\delta_{i-1})} \left(R_{h_i^t}^{b_t(\delta_i)} - R_{g_{\sigma(i)}^t}^{b_t(\delta_i)} \right) R_{g_{\sigma(i+1)}^t}^{b_t(\delta_{i+1})} \dots R_{g_{\sigma(q)}^t}^{b_t(\delta_q)} \rho_t''(\xi). \end{aligned}$$

Similarly,

$$\begin{aligned} &B_{r+1}^t - B_r^t \\ &\quad = \sum_{i=1}^{q} R_{h_1^t}^{b_t'(\delta_1)} \dots R_{h_{i-1}^t}^{b_t'(\delta_{i-1})} \left(R_{h_i^t}^{b_t'(\delta_i)} - R_{g_{\sigma(i)}^t}^{b_t'(\delta_i)} \right) R_{g_{\sigma(i+1)}^t}^{b_t'(\delta_{i+1})} \dots R_{g_{\sigma(q)}^t}^{b_t'(\delta_q)} \rho_t''(\xi). \end{aligned}$$

It follows that $C_{r+1}^t - C_r^t = \left(A_{r+1}^t - A_r^t\right) - \left(B_{r+1}^t - B_r^t\right)$ can be written as a sum of q^2 terms, each of the form

$$\begin{aligned} &R_{h_1^t}^{b_t(\delta_1)} \dots R_{h_{i-1}^t}^{b_t(\delta_{i-1})} \left(R_{h_i^t}^{b_t(\delta_i)} - R_{g_{\sigma(i)}^t}^{b_t(\delta_i)} \right) R_{g_{\sigma(i+1)}^t}^{b_t(\delta_{i+1})} \dots \\ &\qquad \dots R_{g_{\sigma(j-1)}^t}^{b_t(\delta_{j-1})} \left(R_{g_{\sigma(j)}^t}^{b_t(\delta_j)} - R_{g_{\sigma(j)}^t}^{b_t'(\delta_j)} \right) R_{g_{\sigma(j+1)}^t}^{b_t'(\delta_{j+1})} \dots R_{g_{\sigma(q)}^t}^{b_t'(\delta_q)} \rho_t''(\xi), \end{aligned} \tag{3}$$

$$\begin{aligned} &R_{h_1^t}^{b_t(\delta_1)} \dots R_{h_{i-1}^t}^{b_t(\delta_{i-1})} \left(\left(R_{h_i^t}^{b_t(\delta_i)} - R_{g_{\sigma(i)}^t}^{b_t(\delta_i)} \right) \right. \\ &\qquad \left. - \left(R_{h_i^t}^{b_t'(\delta_i)} - R_{g_{\sigma(i)}^t}^{b_t'(\delta_i)} \right) \right) R_{g_{\sigma(i+1)}^t}^{b_t'(\delta_{i+1})} \dots R_{g_{\sigma(q)}^t}^{b_t'(\delta_q)} \rho_t''(\xi), \end{aligned} \tag{4}$$

or

$$(5)\quad \begin{aligned} &R^{b_t(\delta_1)}_{h^t_1}\dots R^{b_t(\delta_{j-1})}_{h^t_{j-1}}\left(R^{b_t(\delta_j)}_{h^t_j}-R^{b'_t(\delta_j)}_{h^t_j}\right)R^{b'_t(\delta_{j+1})}_{h^t_{j+1}}\dots\\ &\qquad\dots R^{b'_t(\delta_{i-1})}_{h^t_{i-1}}\left(R^{b'_t(\delta_i)}_{h^t_i}-R^{b'_t(\delta_i)}_{g^t_{\sigma(i)}}\right)R^{b'_t(\delta_{i+1})}_{g^t_{\sigma(i+1)}}\dots R^{b'_t(\delta_q)}_{g^t_{\sigma(q)}}\rho''_t(\xi)\,. \end{aligned}$$

To bound these terms, we will use the following estimate.

Lemma 8. *Let $A_1, A_2, \dots, A_n$ be square matrices, and let the number R bound the norm of all products $A_{i_1}A_{i_2}\dots A_{i_p}$ with $1 \leqslant i_1 < i_2 < \dots < i_p \leqslant n$. Then, for every matrices $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$,*

$$\|(A_1+\varepsilon_1)(A_2+\varepsilon_2)\dots(A_n+\varepsilon_n)-A_1A_2\dots A_n\| \leqslant R\left(e^{nRE}-1\right),$$

where $E = \max_i \|\varepsilon_i\|$.

Proof. If we expand $(A_1+\varepsilon_1)(A_2+\varepsilon_2)\dots(A_n+\varepsilon_n)-A_1A_2\dots A_n$, each term in the expansion is the product of k terms ε_i and $k+1$ terms $A_{i_1}A_{i_2}\dots A_{i_s}$ with $1 \leqslant i_1 < i_2 < \dots < i_s \leqslant n$, for some k between 1 and n. In addition, the number of terms with k such ε_i is equal to the binomial coefficient $\binom{n}{k}$. It follows that

$$\begin{aligned} \|(A_1+\varepsilon_1)(A_2+\varepsilon_2)\dots(A_n+\varepsilon_n)-A_1A_2\dots A_n\| &\leqslant R\left((1+RE)^n-1\right)\\ &\leqslant R\left(e^{nRE}-1\right). \end{aligned}$$

q.e.d.

Lemma 9. *In the expressions (3) to (5), the subterms of the form $R^{b_t(\delta_k)}_{h^t_k}\dots R^{b_t(\delta_l)}_{h^t_l}$, $R^{b'_t(\delta_k)}_{h^t_k}\dots R^{b'_t(\delta_l)}_{h^t_l}$, $R^{b_t(\delta_k)}_{g^t_{\sigma(k)}}\dots R^{b_t(\delta_l)}_{g^t_{\sigma(l)}}$, or $R^{b'_t(\delta_k)}_{g^t_{\sigma(k)}}\dots R^{b'_t(\delta_l)}_{g^t_{\sigma(l)}}$ are uniformly bounded (independent of r and t).*

Proof. For every i with $b_t(\delta_i) \neq 0$, there is a leaf of the m_t–geodesic lamination underlying b_t which realizes δ_i, and we can consider its image $\bar h^t_i$ under $\widetilde{f}''_t$. The main property we need is that the $\bar h^t_i$ are pairwise disjoint which, because the ordering of the δ_i is compatible with $\prec$, guarantees that $\bar h^t_i$ meets $\widetilde{f}''_t(\widetilde{c})$ closer to $\widetilde{f}''_t(\widetilde{x}_0)$ than $\bar h^t_{i'}$ if $i < i'$. For $i_1 < i_2 < \dots < i_p$ with all $b_t\left(\delta_{i_j}\right) \neq 0$, consider $R^{b_t\left(\delta_{i_1}\right)}_{\bar h^t_{i_1}}\dots R^{b_t\left(\delta_{i_p}\right)}_{\bar h^t_{i_p}}$. Because of the ordering of the intersections $\bar h^t_i \cap \widetilde{f}''_t(\widetilde{c})$, the point $R^{b_t\left(\delta_{i_1}\right)}_{\bar h^t_{i_1}}\dots R^{b_t\left(\delta_{i_p}\right)}_{\bar h^t_{i_p}}\widetilde{f}''_t(\xi\widetilde{x}_0)$ can be connected to $\widetilde{f}''_t(\widetilde{x}_0)$ by a broken arc of the same length as $\widetilde{f}''_t(\widetilde{c})$. It follows that

$R_{\bar{h}^t_{i_1}}^{b_t(\delta_{i_1})} \dots R_{\bar{h}^t_{i_p}}^{b_t(\delta_{i_p})}$ stays in a compact subset of the isometry group of $\mathbb{H}^3$; in particular, its norm is uniformly bounded by a constant $R > 0$.

Set $\varepsilon_i = R_{h^t_i}^{b_t(\delta_i)} - R_{\bar{h}^t_i}^{b_t(\delta_i)}$. Because $R_{h^t_i}^{b_t(\delta_i)}$ and $R_{\bar{h}^t_i}^{b_t(\delta_i)}$ are uniformly bounded, $\|\varepsilon_i\|$ is bounded by a constant times the distance between h^t_i and $\bar{h}^t_i$. Because h^t_i and $\bar{h}^t_i$ follow the same edge path of length $2r+1$ and the metric m_t is hyperbolic, this distance is an $O\left(e^{-Ar}\right)$ for some constant $A > 0$ depending on $\widetilde{U}$ and $\widetilde{c}$.

We are now in a position to apply Lemma 8. To prove that the product $R_{h^t_k}^{b_t(\delta_k)} \dots R_{h^t_l}^{b_t(\delta_l)}$ is uniformly bounded, Lemma 8 and the above estimate for ε_i imply that it suffices to show that $(l-k)\, e^{-Ar}$ is bounded. Although the number of edge paths $\delta \in \Gamma_{r+1}$ grows exponentially with r, the number of those for which $b_t(\delta) \neq 0$ is bounded by a polynomial function of r (this is a general fact about geodesic laminations, see for instance [3, Lemma 10]). It follows that $l - k = O(r^n)$ for some n. As a consequence, $(l-k)\, e^{-Ar}$ is bounded. By Lemma 8, we conclude that all the products $R_{h^t_k}^{b_t(\delta_k)} \dots R_{h^t_l}^{b_t(\delta_l)}$ are uniformly bounded.

The proof of Lemma 9 for the products $R_{h^t_k}^{b'_t(\delta_k)} \dots R_{h^t_l}^{b'_t(\delta_l)}$, $R_{g^t_{\sigma(k)}}^{b_t(\delta_k)} \dots R_{g^t_{\sigma(l)}}^{b_t(\delta_l)}$ and $R_{g^t_{\sigma(k)}}^{b'_t(\delta_k)} \dots R_{g^t_{\sigma(l)}}^{b'_t(\delta_l)}$ is identical. q.e.d.

Remark. One could naively think that it is possible to greatly simplify the proof of Lemma 9 by taking $h^t_i = \bar{h}^t_i$ right away. However, it is not possible to do so simultaneously for the terms involving b_t and those involving b'_t. In general, we cannot choose the h^t_i so that h^t_i is disjoint from $h^t_{i'}$ whenever $b_t(\delta_i)\, b_t(\delta_{i'}) \neq 0$ or $b'_t(\delta_i)\, b'_t(\delta_{i'}) \neq 0$.

We can now estimate $C^t_{r+1} - C^t_r$.

In a term of type (3), the quantity $R_{h^t_i}^{b_t(\delta_i)} - R_{g^t_{\sigma(i)}}^{b_t(\delta_i)}$ is bounded by a constant times the distance from h^t_i to $g^t_{\sigma(i)}$, which is an $O\left(e^{-Ar}\right)$ since these two geodesics follow the same edge path of length $2r+1$. The quantity $R_{g^t_{\sigma(j)}}^{b_t(\delta_j)} - R_{g^t_{\sigma(j)}}^{b'_t(\delta_j)}$ is bounded by a constant times $b_t(\delta_j) - b'_t(\delta_j)$. In [3, Lemma 2], we give an explicit formula expressing $b_t(\delta_j)$ in terms of the weights $b_t(e)$ it assigns to the edges e of U. Because δ_j is an edge path of length $2r+3$, from this formula it follows that

$$b_t(\delta_j) - b_0(\delta_j) = O\left(r\, \|b_t - b_0\|_U\right) = tO\big(r\big\|\dot{b}_0\big\|_U\big).$$

Similarly,

$$b_t(\delta_j) = b_0(\delta_j) + t\dot{b}_0(\delta_j) = b_0(\delta_j) + tO\big(r\big\|\dot{b}_0\big\|_U\big),$$

and we conclude that

$$b_t(\delta_j) - b'_t(\delta_j) = tO\big(r\|\dot{b}_0\|_U\big).$$

By Lemma 9, every term of type (3) therefore is of the form $tO\big(re^{-Ar}\|\dot{b}_0\|_U\big)$.

Similarly, every term of type (5) is of the form $tO\big(re^{-Ar}\|\dot{b}_0\|_U\big)$.

In a term of type (4), the quantity

$$\big(R_{h_i^t}^{b_t(\delta_i)} - R_{g^t_{\sigma(i)}}^{b_t(\delta_i)}\big) - \big(R_{h_i^t}^{b'_t(\delta_i)} - R_{g^t_{\sigma(i)}}^{b'_t(\delta_i)}\big)$$

is bounded by a constant times the product of $b_t(\delta_i) - b'_t(\delta_i)$ and the distance from h_i^t to $g^t_{\sigma(i)}$. As above, we conclude that a term of type (4) is of the form $tO\big(re^{-Ar}\|\dot{b}_0\|_U\big)$.

We saw that $C^t_{r+1} - C^t_r$ is a sum of q^2 terms of type (3), (4) or (5), and also that $q = O(r^n)$ for some n. Therefore,

$$C^t_{r+1} - C^t_r = tO\big(r^{2n+1}e^{-Ar}\|\dot{b}_0\|_U\big) = tO\big(e^{-A'r}\|\dot{b}_0\|_U\big)$$

for any $A' < A$. This proves the first statement of Lemma 7.

The second statement of Lemma 7 is obtained by summing the differences $C^t_{r+1} - C^t_r$ from r to ∞, since $\rho_t(\xi) - \rho'_t(\xi) = \lim_{r\to\infty} C^t_r$.

q.e.d.

Now, fix r and let t tend to 0^+. For $i = 1, \dots, p$,

$$\big(R_{g_i^t}^{b_t(\gamma_i)} - R_{g_i^t}^{b'_t(\gamma_i)}\big)/t = O\left(b_t(\gamma_i) - b'_t(\gamma_i)\right)/t.$$

As t tends to 0^+, each $(b_t(\gamma_i) - b'_t(\gamma_i))/t$ converges to 0 since $b'_t(\gamma_i) = b_0(\gamma_i) + t\dot{b}_0(\gamma_i)$. Therefore, for a fixed r,

$$\begin{aligned} C^t_r/t = {} & R_{g_1^t}^{b_t(\gamma_1)} R_{g_2^t}^{b_t(\gamma_2)} \dots R_{g_p^t}^{b_t(\gamma_p)} \rho''_t(\xi)/t - R_{g_1^t}^{b'_t(\gamma_1)} R_{g_2^t}^{b'_t(\gamma_2)} \\ & \dots R_{g_p^t}^{b'_t(\gamma_p)} \rho''_t(\xi)/t \\ = {} & \sum_{i=1}^{p} R_{g_1^t}^{b_t(\gamma_1)} R_{g_2^t}^{b_t(\gamma_2)} \dots R_{g_{i-1}^t}^{b_t(\gamma_{i-1})} \frac{R_{g_i^t}^{b_t(\gamma_i)} - R_{g_i^t}^{b'_t(\gamma_i)}}{t} R_{g_{i+1}^t}^{b'_t(\gamma_{i+1})} \\ & \dots R_{g_p^t}^{b'_t(\gamma_p)} \rho''_t(\xi) \end{aligned}$$

converges to 0 as t tends to 0^+.

Thus from Lemma 7 it follows that every limit point of $(\rho_t(\xi) - \rho'_t(\xi))/t$ as t tends to 0^+ is of the form $O\big(e^{-Ar}\|\dot{b}_0\|_U\big)$.

This holds for every r. If we now let r tend to ∞, we conclude that 0 is the only limit point of $(\rho_t(\xi) - \rho'_t(\xi))/t$ as t tends to 0^+, namely that the two curves $t \mapsto \rho_t(\xi)$ and $t \mapsto \rho'_t(\xi) \in \mathrm{Isom}^+(\mathbb{H}^3)$ are tangent at $t = 0$. Hence the two curves $t \mapsto \rho_t$ and $t \mapsto \rho'_t \in \mathcal{R}(S)$ are tangent at $t = 0$. As a consequence, $t \mapsto \rho_t$ has a tangent vector $\dot{\rho}_0$ at $t = 0$, which is equal to $\dot{\rho}'_0 = T_{(m_0,\bar{b}_0)}\varphi_\lambda(\dot{m}_0, \dot{b}_0)$.

This concludes the proof of Proposition 5 under the additional assumption that the geodesic laminations underlying the b_t converge to some sublamination of λ.

In the general case, let t_n, $n \in \mathbb{N}$, be a sequence converging to 0^+, such that the geodesic lamination underlying b_{t_n} converges to some lamination λ' for the Hausdorff topology. The geodesic lamination λ' must contain the supports of b_0 and $\dot{b}_0$. We can therefore consider $b'_t = b_0 + t\dot{b}_0$ as a transverse cocycle for λ' as well as for λ; the same holds for its reduction $\bar{b}'_t$ modulo 2π. Note that $\varphi_{\lambda'}(m_t, \bar{b}'_t) = \varphi_\lambda(m_t, \bar{b}'_t) = \rho'_t$. Then, the same argument as above shows that the "discrete curve" $t_n \mapsto \rho_{t_n}$ is tangent to the curve $t \mapsto \rho'_t$ at 0, in the sense that $\lim_{n\to\infty}\big(\rho_{t_n}(\xi) - \rho'_{t_n}(\xi)\big)/t_n = 0$ for every $\xi \in \pi_1(S)$. Since this property holds for any such subsequence t_n, $n \in \mathbb{N}$, this shows that the two curves $t \mapsto \rho_t$ and $t \mapsto \rho'_t$ are tangent at $t = 0$. Again, it follows that $t \mapsto \rho_t$ has a tangent vector $\dot{\rho}_0$ at $t = 0$ which is equal to $\dot{\rho}'_0 = T_{(m_0,\bar{b}_0)}\varphi_\lambda(\dot{m}_0, \dot{b}_0)$, and this completes the proof of Proposition 5.

q.e.d.

By Proposition 5, the map $\varphi : \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathcal{R}(S)$ has a tangent map $T_{(m,b)}\varphi : T_m\mathcal{T}(S) \times T_b\mathcal{ML}(S) \to T_{\varphi(m,b)}\mathcal{R}(S)$ everywhere. If, in addition, the support of b is a maximal geodesic lamination λ, then $T_b\mathcal{ML}(S) \cong \mathcal{H}(\lambda;\mathbb{R})$ and $T_{(m,b)}\varphi = T_{(m,b)}\varphi_\lambda$. Since φ_λ is a local diffeomorphism, this immediately shows that $T_{(m,b)}\varphi$ is invertible when the support of b is a maximal geodesic lamination. The general case requires more work.

3. Proof that $T_{(m,b)}\varphi : T_m\mathcal{T}(S) \times T_b\mathcal{ML}(S) \to T_{\varphi(m,b)}\mathcal{R}(S)$ is injective

Proposition 10. *The tangent map*

$$T_{(m,b)}\varphi : T_m\mathcal{T}(S) \times T_b\mathcal{ML}(S) \to T_{\varphi(m,b)}\mathcal{R}(S)$$

is injective.

Proof. Let $v' = \big(\dot m', \dot b'\big)$ and $v'' = \big(\dot m'', \dot b''\big)$ be two tangent vectors at (m_0, b_0) such that $T_{(m_0,b_0)}\varphi\,(v') = T_{(m_0,b_0)}\varphi\,(v'')$. We want to show that $v' = v''$.

By Proposition 5, $T_{(m_0,b_0)}\varphi\,(v') = T_{(m_0,b_0)}\varphi_{\lambda'}\big(\dot m', \dot b'\big)$ where λ' is any maximal geodesic lamination containing the supports of b_0 and $\dot b'$. Similarly, $T_{(m_0,b_0)}\varphi\,(v'') = T_{(m_0,b_0)}\varphi_{\lambda''}\big(\dot m'', \dot b''\big)$ where λ'' is any maximal geodesic lamination containing the supports of b_0 and $\dot b''$.

Lemma 11. *The support of $\dot b'$ does not cross the support of $\dot b''$.*

Proof. Suppose that there is a leaf g' of the support of $\dot b'$ that transversely intersects in x a leaf g'' of the support of $\dot b''$. Without loss of generality, we may assume that g' is in the boundary of $S - \lambda'$ and that g'' is in the boundary of $S - \lambda''$.

Let $\rho_t \in \mathcal{R}\,(S)$, $t \in [0, \varepsilon[$, be a family of representations with

$$\rho_0 = \varphi_{\lambda'}\,(m_0, b_0) = \varphi_{\lambda''}\,(m_0, b_0)$$

and

$$\dot\rho_0 = T_{(m_0,b_0)}\varphi_{\lambda'}\big(\dot m', \dot b'\big) = T_{(m_0,b_0)}\varphi_{\lambda''}\big(\dot m'', \dot b''\big).$$

For t small enough, ρ_t determines a pleated surface $f'_t = \big(\widetilde f'_t, \rho_t\big)$ with pleating locus λ' and a pleated surface $f''_t = \big(\widetilde f''_t, \rho_t\big)$ with pleating locus λ''. Let $m'_t \in \mathcal{T}\,(S)$ and $b'_t \in \mathcal{H}\,(\lambda'; \mathbb{R}/2\pi\mathbb{Z})$ (resp. $m''_t \in \mathcal{T}\,(S)$ and $b''_t \in \mathcal{H}\,(\lambda''; \mathbb{R}/2\pi\mathbb{Z})$) be the pull back metric and the bending cocycles of f'_t (resp. f''_t). Namely, $\rho_t = \varphi_{\lambda'}\,(m'_t, b'_t) = \varphi_{\lambda''}\,(m''_t, b''_t)$. Note that $b'_0 = b''_0 = b_0$, $\dot b'_0 = \dot b'$ and $\dot b''_0 = \dot b''$.

Lift x to a point $\widetilde x$ in the universal covering $\widetilde S$, and let $\widetilde g'$ and $\widetilde g''$ be the lifts of g' and g'' passing through $\widetilde x$, respectively. We want to compare the respective positions of the geodesics $\widetilde f'_t\,(\widetilde g'_t)$ and $\widetilde f''_t\,(\widetilde g''_t)$ of $\mathbb{H}^3$, where $\widetilde g'_t$ is the m'_t–geodesic of $\widetilde S$ corresponding to $\widetilde g'$, and $\widetilde g''_t$ is the m''_t–geodesic corresponding to $\widetilde g''$. Because $\widetilde f'_0 = \widetilde f''_0$, the geodesics $\widetilde f'_0\,(\widetilde g'_0)$ and $\widetilde f''_0\,(\widetilde g''_0)$ are coplanar and meet in one point.

If $\widehat g''_t$ denotes the m'_t–geodesic corresponding to $\widetilde g''$, $\widetilde f''_t\,(\widetilde g''_t)$ is also the geodesic of $\mathbb{H}^3$ that is asymptotic to $\widetilde f'_t\,(\widehat g''_t)$. Because g' and g'' intersect, they have to be disjoint from the support of b_0. This shows that $\dot b'\,(k'') \geqslant 0$ for every arc k'' contained in g''. Indeed, $\dot b' \in T_{b_0}\mathcal{ML}\,(S)$ is tangent to a family of measured laminations $b_t \in \mathcal{ML}\,(S)$ with $b_0\,(k) = 0$ and $b_t\,(k) \geqslant 0$; compare [3, Theorem 19]. It follows that, infinitesimally, $\widetilde f'_t\,(\widehat g''_t)$ bends everywhere in the direction of the negative side of $\widetilde f'_0\big(\widetilde S\big)$.

Intuitively, this will imply that, as t moves away from 0, $\widetilde{f}''_t(\widetilde{g}''_t)$ moves away from $\widetilde{f}'_t(\widetilde{g}'_t)$ in the direction of the negative side of $\widetilde{f}'_0(\widetilde{S})$. We need to quantify this.

By [4, Corollary 32], for every component P of $\widetilde{S} - \widetilde{\lambda}'$, the infinite triangle $\widetilde{f}'_t(P) \subset \mathbb{H}^3$ depends differentiably on the representation ρ_t. By our assumption that g' is a boundary leaf, it is seen that $\widetilde{f}'_t(\widetilde{g}'_t)$ depends differentiably on ρ_t. Since the same property holds for $\widetilde{f}''_t(\widetilde{g}''_t)$, the length l_t of the shortest geodesic arc from $\widetilde{f}'_t(\widetilde{g}'_t)$ to $\widetilde{f}''_t(\widetilde{g}''_t)$ also depends differentiably on ρ_t.

To estimate the derivative $\dot{l}_0$, normalize ρ_t and $\widetilde{f}'_t$ so that $\widetilde{f}'_t$ sends the component of $\widetilde{S} - \widetilde{\lambda}$ that is adjacent to $\widetilde{g}'$ to a fixed ideal triangle in $\mathbb{H}^2 \subset \mathbb{H}^3$. Thus, $\widetilde{f}'_t(\widehat{g}''_t)$ is obtained from the geodesic $\widetilde{f}'_0(\widetilde{g}''_0) \subset \mathbb{H}^2$ by, first moving it in $\mathbb{H}^2$ to reflect the passage from the metric m'_0 to m'_t, and then bending this geodesic by successive rotations along geodesics of $\mathbb{H}^2$, following a formula analogous to (1). Let $\widetilde{h}''$ be a half-line in $\widetilde{g}''$, which crosses the support of $\dot{b}'$, and originates in the component of $\widetilde{S} - \widetilde{\lambda}$ that is adjacent to $\widetilde{g}'$; we will denote by $\widetilde{h}''_t$, $\widehat{h}''_t$ the subsets of $\widetilde{g}''_t$, $\widehat{g}''_t$ corresponding to $\widetilde{h}''$. Let θ^+_t be the visual amount by which the end point of $\widetilde{f}'_t\big(\widehat{h}''_t\big)$ dips below $\mathbb{H}^2$, as measured from a fixed base point on $\mathbb{H}^2$.

The derivative of θ^+_t at $t = 0$ is given by the formula

$$\dot{\theta}^+_0 = \int_{\widetilde{f}'_0(\widetilde{h}''_0)} A^+(u)\; d\dot{b}'(u)\,,$$

where: $d\dot{b}'$ is the distribution induced by $\dot{b}'$ on $\widetilde{f}'_0\big(\widetilde{h}''_0\big)$, which is actually a (countably additive) measure since $\dot{b}'(k'') \geqslant 0$ for every arc k'' contained in g''; $A^+(u) > 0$ denotes the amount by which the end point of $\widetilde{f}'_0\big(\widetilde{h}''_0\big)$ dips under $\mathbb{H}^2$ when we apply to it the infinitesimal rotation around the leaf of $\widetilde{f}'_0\big(\widetilde{\lambda}'\big)$ passing through $u \in \widetilde{f}'_0\big(\widetilde{h}''_0\big)$, if it exists. This formula is easily obtained by formal computations. To justify these formal computations (and show that the integral really converges), it suffices to note that $-\log A^+(u)$ is at least a constant times the distance from u to the base point and that, for every arc k of length $\geqslant 1$ in $\widetilde{f}'_0\big(\widetilde{h}''_0\big)$, $\dot{b}(k)$ is bounded by a constant (depending on $\dot{b}$ but not k) times the length of k.

The important part here is that $\dot{\theta}^+_0 > 0$, which holds because $\widetilde{h}''$ crosses the support of $\dot{b}$. A similar formula gives that $\dot{\theta}^-_0 \geqslant 0$, where θ^-_t denotes the visual amount by which the other end point of $\widetilde{f}'_t(\widehat{g}''_t)$ dips below $\mathbb{H}^2$. Combining these two properties, it follows that $\dot{l}_0 > 0$.

This proves that, for $t > 0$, the shortest geodesic arc from $\widetilde{f}'_t(\widetilde{g}'_t)$ to $\widetilde{f}''_t(\widetilde{g}''_t)$ is non-trivial and points in the direction of the negative side of $\widetilde{f}'_0(\widetilde{S})$. But the argument is symmetric. Exchanging primes and double primes, we obtain that, for $t > 0$, the opposite shortest geodesic arc from $\widetilde{f}''_t(\widetilde{g}''_t)$ to $\widetilde{f}'_t(\widetilde{g}'_t)$ must also point in the direction of the negative side of $\widetilde{f}''_0(\widetilde{S}) = \widetilde{f}'_0(\widetilde{S})$, a contradiction. q.e.d.

By Lemma 11, the supports of $\dot{b}'$ and $\dot{b}''$ do not cross each other. Therefore, there exists a maximal geodesic lamination λ which contains the supports of b_0, $\dot{b}'$ and $\dot{b}''$. As a consequence, we can choose our geodesic laminations λ', λ'' so that $\lambda' = \lambda'' = \lambda$.

Then,

$$T_{(m_0,b_0)}\varphi_\lambda\left(v'\right) = T_{(m_0,b_0)}\varphi\left(v'\right) = T_{(m_0,b_0)}\varphi\left(v''\right) = T_{(m_0,b_0)}\varphi_\lambda\left(v''\right).$$

Since φ_λ is a diffeomorphism, its tangent map is a linear isomorphism, and it follows that $v' = v''$. q.e.d.

4. Proof of Theorems 2 and 3

Theorems 2 and 3 immediately follow from Lemma 4, Corollary 6 and Proposition 10.

Indeed, for a connected surface S of finite type and negative Euler characteristic, the map $\varphi : \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathcal{R}(S)$ is the composition of the Thurston homeomorphism $\psi : \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathcal{P}(S)$ and of the monodromy map $\theta : \mathcal{P}(S) \to \mathcal{R}(S)$. Because θ is a local diffeomorphism, φ is a local homeomorphism. By Corollary 6, φ admits a tangent map everywhere, and Proposition 10 shows that this tangent map is injective. From Lemma 4, we conclude that any local inverse φ^{-1} for φ is also tangentiable. Because θ is a local diffeomorphism, this shows that ψ and ψ^{-1} are tangentiable. This proves Theorem 3.

For a hyperbolic 3–manifold M, the map

$$\mu \times \beta : \mathcal{QD}(M) \to \mathcal{T}(\partial C_M) \times \mathcal{ML}(\partial C_M)$$

locally coincides with the composition $\varphi^{-1} \circ R$ near the metric M where, as in the introduction, $\mathcal{R}(\partial C_M)$ denotes the product $\prod_{i=1}^n \mathcal{R}(S_i)$ of the representation spaces corresponding to the components $S_1, \dots, S_n$ of ∂C_M, where $R : \mathcal{QD}(M) \to \mathcal{R}(\partial C_M)$ is defined by restriction of the holonomy map, where $\varphi : \mathcal{T}(\partial C_M) \times \mathcal{ML}(\partial C_M) \to \mathcal{R}(\partial C_M)$ is defined as the product of the bending maps $\varphi_i : \mathcal{T}(S_i) \times \mathcal{ML}(S_i) \to \mathcal{R}(S_i)$, and φ^{-1} is the local inverse defined near the representation $R(M)$ and $(\mu(M), \beta(M))$. As above, a combination of Corollary 6, Proposition 10 and Lemma 4 shows that each local inverse φ_i^{-1} is tangentiable. Therefore, the local inverse φ^{-1} is tangentiable. Since R is a differentiable map between differentiable manifolds, it follows that $\mu \times \beta$ is tangentiable. Composing with the (clearly tangentiable) projection $P : \mathcal{T}(\partial C_M) \times \mathcal{ML}(\partial C_M) \to \mathcal{ML}(\partial C_M)$, we conclude that β is tangentiable everywhere. This proves Theorem 2.

The same argument shows that μ is tangentiable everywhere. To show that μ is continuously differentiable in the usual sense, we have to show that its tangent maps are linear and vary continuously with their base point. This will be done in the next section.

5. Proof of Theorem 1

By the same arguments as in §4, Theorem 1 immediately follows from the following result.

Proposition 12. *Let S be a connected oriented surface of finite type and negative Euler characteristic. Then the composition $Q \circ \varphi^{-1}$ of any local inverse φ^{-1} for the bending map $\varphi : \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathcal{R}(S)$ and the projection $Q : \mathcal{T}(S) \times \mathcal{ML}(S) \to \mathcal{T}(S)$ is continuously differentiable.*

Proof. Let $\rho_0 \in \mathcal{R}(S)$ and $(m_0, b_0) = \varphi^{-1}(\rho_0)$. By Corollary 6, Proposition 10 and Lemma 4, φ^{-1} has a tangent map at ρ_0 and $T_{\rho_0}\varphi^{-1} = \left(T_{(m_0,b_0)}\varphi\right)^{-1}$.

By Corollary 6, the restriction of $T_{(m_0,b_0)}\varphi$ to $T_{m_0}\mathcal{T}(S) \times 0$ coincides with the restriction of $T_{(m_0,b_0)}\varphi_\lambda$ for any maximal geodesic lamination λ containing the support of b. In particular, this restriction of $T_{(m_0,b_0)}\varphi$ to $T_{m_0}\mathcal{T}(S) \times 0$ is linear. Let $P_{\rho_0} \subset T_{\rho_0}\mathcal{R}(S)$ denote the linear subspace $T_{(m_0,b_0)}\varphi\left(T_{m_0}\mathcal{T}(S) \times 0\right)$; note that P_{ρ_0} depends on ρ_0, but also on the choice of the local inverse φ^{-1}.

To consider the image of $0 \times T_{b_0}\mathcal{ML}(S)$ under $T_{(m_0,b_0)}\varphi$, we will exploit the complex structure of $\mathcal{R}(S)$ coming from the complex structure of the group $\mathrm{Isom}^+\left(\mathbb{H}^3\right) = \mathrm{PSL}_2(\mathbb{C})$. Indeed, it is showed in [4, §10] that, for every maximal geodesic lamination λ containing the support of b, the differential $T_{(m_0,b_0)}\varphi_\lambda$ sends $0 \times \mathcal{H}(\lambda; \mathbb{R})$ to the subspace iP_{ρ_0} obtained from P_{ρ_0} by multiplication by i; see also the proof of Lemma 13 below. By Corollary 6, this implies that $T_{(m_0,b_0)}\varphi$ sends $0 \times T_{b_0}\mathcal{ML}(S)$ inside iP_{ρ_0}. Because $T_{(m_0,b_0)}\varphi$ is invertible, the image of $0 \times T_{b_0}\mathcal{ML}(S)$

by $T_{(m_0,b_0)}\varphi$ is actually equal to iP_{ρ_0}. (As an aside, since $T_{(m_0,b_0)}\varphi$ identifies $0 \times T_{b_0}\mathcal{ML}(S)$ to iP_{ρ_0}, this defines on $T_{b_0}\mathcal{ML}(S)$ a linear structure which is compatible with the linear structures of the faces and depends only on m_0).

We can then compute the tangent map

$$T_{\rho_0}\left(Q \circ \varphi^{-1}\right) : T_{\rho_0}\mathcal{R}(S) \to T_{m_0}\mathcal{T}(S).$$

By Corollary 6, $T_{\rho_0}\left(Q \circ \varphi^{-1}\right)$ is just the composition $\Phi_{\rho_0}^{-1} \circ \Pi_{\rho_0}$ of the projection Π_{ρ_0} of $T_{\rho_0}\mathcal{R}(S)$ onto P_{ρ_0} parallel to iP_{ρ_0} and of the inverse of the linear isomorphism $\Phi_{\rho_0} : T_{m_0}\mathcal{T}(S) \to P_{\rho_0}$ induced by $T_{(m_0,b_0)}\varphi$. In particular, $T_{\rho_0}\left(Q \circ \varphi^{-1}\right)$ is linear, and $Q \circ \varphi^{-1}$ is differentiable in the usual sense.

It remains to show that $T_{\rho_0}\left(Q \circ \varphi^{-1}\right)$ depends continuously on ρ_0.

Lemma 13. *The linear map $\Phi_{\rho_0} : T_{m_0}\mathcal{T}(S) \to P_{\rho_0}$ depends continuously on ρ_0.*

Proof. We will again make use of the complex structure of $\mathcal{R}(S)$.

If λ is a maximal geodesic lamination containing the support of b_0, we saw that φ_λ provides a local parametrization of $\mathcal{R}(S)$ near ρ_0. This parametrization associates to each representation near ρ_0 the pull back metric $m_\rho \in \mathcal{T}(S)$ and the bending cocycle $b_\rho \in \mathcal{H}(\lambda; \mathbb{R}/2\pi\mathbb{Z})$ of the pleated surface with pleating locus λ corresponding to ρ. In [4], we also associated to m_ρ on S a *shearing cocycle* $s_\rho \in \mathcal{H}(\lambda; \mathbb{R})$, and combined s_ρ and b_ρ into a complex cocycle $s_\rho + ib_\rho \in \mathcal{H}(\lambda; \mathbb{C}/2\pi i\mathbb{Z})$ to show that this provides a biholomorphic parametrization of a neighborhood of ρ_0 by an open subset of $\mathcal{H}(\lambda; \mathbb{C}/2\pi i\mathbb{Z})$.

If U is a train track carrying λ, each transverse cocycle $a \in \mathcal{H}(\lambda; \mathbb{C}/2\pi i\mathbb{Z})$ associates to each edge e of U a weight $a(e) \in \mathbb{C}/2\pi i\mathbb{Z}$. This defines a linear isomorphism between $\mathcal{H}(\lambda; \mathbb{C}/2\pi i\mathbb{Z})$ and the space $\mathcal{W}(U; \mathbb{C}/2\pi i\mathbb{Z})$ of all such systems of edge weights that satisfy the classical *switch relations* , namely such that, at each switch of U, the sum of the weights of the edges coming on one side is equal to the sum of the weights of the edges coming on the other side; see for instance [2].

Combining these two parametrizations, we get a holomorphic map $\psi_\lambda : \mathcal{U} \to \mathcal{R}(S)$ which restricts to a homeomorphism between an open subset $\mathcal{U}$ of $\mathcal{W}(U; \mathbb{C}/2\pi i\mathbb{Z})$ and a neighborhood $\psi_\lambda(\mathcal{U})$ of ρ_0.

The main point of using edge weights instead of transverse cocycles is that we can compare these maps as we vary the geodesic lamination λ. If λ_n, $n \in \mathbb{N}$, is a sequence of geodesic lamination that converges to

λ for the Hausdorff topology as n tends to ∞, the estimates of [4, §4] show that, for n large enough, the ψ_{λ_n} are also defined on the same $\mathcal{U} \subset \mathcal{W}(U;\mathbb{C}/2\pi i\mathbb{Z})$ and uniformly converge to ψ_λ on $\mathcal{U}$. Because the ψ_{λ_n} are holomorphic, we also have uniform convergence of their tangent maps. We conclude that if, in addition, we have a sequence of edge weight systems $A_n \in \mathcal{U}$ converging to some $A \in \mathcal{U}$ and a sequence of tangent vectors $\dot{A}_n \in T_{A_n}\mathcal{U} = \mathcal{W}(U;\mathbb{C})$ converging to $\dot{A} \in T_A\mathcal{U} = \mathcal{W}(U;\mathbb{C})$ then, in $\mathcal{R}(S)$, the tangent vectors $T_{A_n}\psi_{\lambda_n}(\dot{A}_n)$ converge to $T_A\psi_\lambda(\dot{A})$ as n tends to ∞.

If we restrict attention to real cocycles (and consequently to totally geodesic pleated surfaces and Fuchsian representations), we similarly have a real analytic map $\theta_\lambda : \mathcal{V} \to \mathcal{T}(S)$ which restricts to a homeomorphism between an open subset $\mathcal{V}$ of $\mathcal{W}(U;\mathbb{R})$ and a neighborhood $\theta_\lambda(\mathcal{V})$ of $m_0 \in \mathcal{T}(S)$. Again, as λ_n converges to λ for the Hausdorff topology, θ_{λ_n} and its tangent maps uniformly converge to θ_λ and its tangent maps as n tends to ∞.

We are now ready to prove the continuity property for Φ_{ρ_0}. Let $\rho_n \in \mathcal{R}(S)$, $n \in \mathbb{N}$, be a sequence of representations converging to ρ_0. Let $(m_n, b_n) = \varphi^{-1}(\rho_n) \in \mathcal{T}(S) \times \mathcal{ML}(S)$, and let $\dot{m}_n \in T_{m_n}\mathcal{T}(S)$ be a sequence of tangent vectors converging to some $\dot{m}_0 \in T_{m_0}\mathcal{T}(S)$. We want to show that $\Phi_{\rho_n}(\dot{m}_n)$ converges to $\Phi_{\rho_0}(\dot{m}_0)$.

For each n, let λ_n be a maximal geodesic lamination containing the support of b_n. Extracting a subsequence if necessary, we can assume that λ_n converges for the Hausdorff topology to some maximal geodesic lamination λ_0 containing the support of b_0. Let U be a train track carrying λ_0. Then, by definition of all the maps involved,

$$\varphi_{\lambda_n}(m_n, b_n) = \psi_{\lambda_n}\left(\theta_{\lambda_n}^{-1}(m_n) + iB_n\right)$$

for n sufficiently large, where $B_n \in \mathcal{W}(U;\mathbb{R}/2\pi\mathbb{Z})$ is the edge weight system corresponding to $b_n \in \mathcal{H}(\lambda_n;\mathbb{R}/2\pi\mathbb{Z})$. It follows that

$$\begin{aligned}\Phi_{\rho_n}(\dot{m}_n) =& T_{(m_n,b_n)}\varphi(\dot{m}_n, 0) = T_{(m_n,b_n)}\varphi_{\lambda_n}(\dot{m}_n, 0)\\ =& T_{\left(\theta_{\lambda_n}^{-1}(m_n)+iB_n\right)}\psi_{\lambda_n}\left(T_{m_n}\theta_{\lambda_n}^{-1}(\dot{m}_n)\right).\end{aligned}$$

By uniform convergence of the tangent maps, we conclude that $\Phi_{\rho_n}(\dot{m}_n)$ converges to $\Phi_{\rho_0}(\dot{m}_0)$ as n tends to ∞.

This concludes the proof of Lemma 13. q.e.d.

By Lemma 13, Φ_{ρ_0} depends continuously on ρ_0. In particular, its image P_{ρ_0} depends continuously on ρ_0. Therefore the projection

$\Pi_{\rho_0} : T_{\rho_0}\mathcal{R}(S) \to P_{\rho_0}$ parallel to iP_{ρ_0} also depends continuously on ρ_0. This proves that the tangent map $T_{\rho_0}\left(Q \circ \varphi^{-1}\right) = \Phi_{\rho_0}^{-1} \circ \Pi_{\rho_0}$ depends continuously on ρ_0, and concludes the proof of Proposition 12 and Theorem 1. q.e.d.

6. The map μ is not necessarily twice differentiable

It is not difficult to show by explicit computations that the map μ is not necessarily twice differentiable. For instance, we can borrow such computations from [13]. Let S be a once punctured torus. On S, choose a hyperbolic metric $m_0 \in \mathcal{T}(S)$ and a pair of simple closed m_0–geodesics γ, δ on S meeting transversely in one point. If $\rho \in \mathcal{R}(S)$ is geometrically finite and M is the corresponding hyperbolic 3–manifold, then the boundary ∂C_M is the union of two copies $\partial^+ C_M$ and $\partial^- C_M$ of S, where the identification of S with $\partial^+ C_M$ (resp. $\partial^- C_M$) respects (resp. reverses) the orientation. Let $\gamma^\pm$ and $\delta^\pm$ denote the closed geodesics of $\partial^\pm C_M$ homotopic to γ and δ, respectively.

For $t \in \mathbb{R}$, let $\gamma_t \in \mathcal{H}(\lambda; \mathbb{R}/2\pi\mathbb{Z})$ be the Dirac transverse measure for γ with mass the $\operatorname{mod} 2\pi$ reduction of t, and let $\rho_t = \varphi_\gamma(m_0, \gamma_t)$. The representation ρ_0 is Fuchsian, and defines a hyperbolic 3–manifold M_0. For t close to 0, we can then consider the hyperbolic metric $M_t \in \mathcal{QD}(M_0)$ corresponding to ρ_t.

First consider the case where t is non-negative, and close to 0. Then, $\partial^+ C_{M_t}$ has induced metric m_0 and bending measured geodesic lamination γ_t. If we make the additional assumption that γ and δ meet orthogonally for the metric m_0, it is shown in [13] that $\partial^- C_M$ is bent along δ^-; this can also be seen from symmetry considerations.

For $t \leqslant 0$ close to 0, it is now $\partial^- C_{M_t}$ which has induced metric m_0 and bending measured lamination γ_{-t}, and $\partial^+ C_M$ is bent along δ^+. In addition, the central equality of [13] shows that the lengths of γ^- and δ^+ are related to t by the formula

$$\cos^2(t/2) = \cosh^2 l_t\left(\gamma^-\right) \tanh^2 l_t\left(\delta^+\right).$$

Noting that $l_t(\gamma^-) = l_0(\gamma)$, we conclude that

$$\tanh^2 l_t\left(\delta^+\right) = \cos^2(t/2) / \cosh^2 l_0(\gamma).$$

Therefore, for t small, the function $\tanh^2 l_t(\delta^+)$ is equal to

$$\tanh^2 l_0(\delta) = 1/\cosh^2 l_0(\gamma)$$

if $t \geqslant 0$, and equal to $\cos^2(t/2) / \cosh^2 l_0(\gamma)$ if $t \leqslant 0$. This function of t is not twice differentiable at 0. On the other hand, the curve $t \mapsto M_t$ is real analytic in $\mathcal{QD}(M_0)$. It follows that μ is not twice differentiable at M_0.

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