

COUNTING OPEN NEGATIVELY CURVED MANIFOLDS UP TO TANGENTIAL HOMOTOPY EQUIVALENCE

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Abstract

Under mild assumptions on a group π , we prove that the class of complete Riemannian n -manifolds of uniformly bounded negative sectional curvatures and with the fundamental groups isomorphic to π breaks into finitely many tangential homotopy types. It follows that many aspherical manifolds do not admit complete negatively curved metrics with prescribed curvature bounds.

1. Introduction

This paper is an attempt to understand the topology of complete infinite volume Riemannian manifolds of negative sectional curvature. Every smooth open manifold admits a (possibly incomplete) Riemannian metric of negative sectional curvature [23]. However, the universal cover of a complete negatively curved manifold is diffeomorphic to the Euclidean space; in particular, the homotopy type of a complete negatively curved manifold is determined by its fundamental group.

For closed (or, more generally, finite volume complete) negatively curved manifolds, the fundamental group seems to encode most of the

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topological information. By contrast, complete negatively curved manifolds of *infinite* volume with isomorphic fundamental groups may be very different topologically. For example, the total space of any vector bundle over a closed negatively curved manifold admits a complete Riemannian metric of sectional curvature pinched between two negative constants [1].

Given a group π , a positive integer n , and real numbers $a \leq b < 0$, consider the class $\mathcal{M}_{a,b,\pi,n}$ of n -manifolds with fundamental groups isomorphic to π that can be given complete Riemannian metrics of sectional curvatures within $[a, b]$.

In this paper we discuss under what assumptions on π the class $\mathcal{M}_{a,b,\pi,n}$ breaks into *finitely many* tangential homotopy types. Recall that a homotopy equivalence of smooth manifolds of the same dimension $f : N \rightarrow L$ is called tangential if the vector bundles f^*TL and TN are stably isomorphic. For example, any map that is homotopic to a diffeomorphism is a tangential homotopy equivalence. Note that, unless N is a closed manifold, the bundles f^*TL and TN are isomorphic iff they are stably isomorphic.

If the cohomological dimension of π is equal to n , every manifold from the class $\mathcal{M}_{a,b,\pi,n}$ is closed. In that case, in fact, $\mathcal{M}_{a,b,\pi,n}$ falls into finitely many diffeomorphism classes [5].

Here is a way to produce infinitely many homotopy equivalent negatively curved manifolds of the same dimension such that no two of them are tangentially homotopy equivalent. Let M be a closed negatively curved manifold with $H_{4m}(M, \mathbb{Q}) \neq 0$ for some $m > 0$. (Such examples abound. For instance, any closed complex hyperbolic or quaternion hyperbolic manifold is such. Most arithmetic closed real hyperbolic orientable manifolds have nonzero Betti numbers in all dimensions [30].) Then, by an elementary K-theoretic argument, for any $k \geq \dim(M)$ there exists an infinite sequence of rank k vector bundles over M such that no two of them have tangentially homotopy equivalent total spaces. Yet, thanks to a theorem of Anderson [1], the total space of any vector bundle over M can be given a negatively curved metric.

The following is a simplified form of our main result.

Theorem 1.1. *Let π be the fundamental group of a finite aspherical complex. Suppose that π is not virtually nilpotent and that π does not split as a nontrivial amalgamated product or an HNN-extension over a virtually nilpotent group.*

Then, for any positive integer n and any negative reals $a \leq b$, the

class $\mathcal{M}_{a,b,\pi,n}$ breaks into finitely many tangential homotopy types.

Furthermore, by an easy argument we can generalize the theorem 1.1 to certain amalgamated products and HNN-extensions. For example, suppose that π is the fundamental group of a finite graph of groups such that the edge groups are virtually nilpotent groups of cohomological dimension ≤ 2 . (Note that if $\mathcal{M}_{a,b,\pi,n} \neq \emptyset$, then any nontrivial virtually nilpotent subgroup of π of cohomological dimension ≤ 2 is isomorphic to \mathbb{Z} , $\mathbb{Z} \times \mathbb{Z}$, or the fundamental group of the Klein bottle.) Fix n and $a \leq b < 0$ and suppose that, for each vertex group π_v , the class $\mathcal{M}_{a,b,\pi_v,n}$ breaks into finitely many tangential homotopy types. Then so does the class $\mathcal{M}_{a,b,\pi,n}$.

Applying a powerful accessibility result of Delzant and Potyagailo [16], we deduce the following.

Corollary 1.2. *Let π be the fundamental group of a finite aspherical complex. Assume that any nilpotent subgroup of π has cohomological dimension ≤ 2 .*

Then, for any positive integer n and any negative reals $a \leq b$, the class $\mathcal{M}_{a,b,\pi,n}$ breaks into finitely many tangential homotopy types.

Any torsion free word-hyperbolic group is the fundamental group of a finite aspherical cell complex [15, 5.24]. Moreover, any virtually nilpotent subgroup of π is either trivial or infinite cyclic.

Corollary 1.3. *Let π be a word-hyperbolic group. Then, for any positive integer n and any negative reals $a \leq b$, the class $\mathcal{M}_{a,b,\pi,n}$ breaks into finitely many tangential homotopy types.*

It is worth mentioning that in the locally symmetric case a different (and elementary) argument yields the following.

Theorem 1.4. *Let π be a finitely presented torsion-free group and let X be a nonpositively curved symmetric space.*

Then the class of manifolds of the form $X/\rho(\pi)$, where $\rho \in \text{Hom}(\pi, \text{Isom}(X))$ is a faithful discrete representation, falls into finitely many tangential homotopy types.

Gromov posed a question (see [1]) whether an (open) thickening of a finite aspherical cell complex with word-hyperbolic fundamental group admits a complete negatively curved metric. The following corollary shows that most thickenings do *not* carry complete negatively curved metric with *prescribed* curvature bounds.

Corollary 1.5. *Let K be a finite, connected, aspherical cell complex. Let $a \leq b < 0$ and $n > \max\{4, 2 \dim(K)\}$. Suppose that the class $\mathcal{M}_{a,b,\pi_1(K),n}$ breaks into finitely many tangential homotopy types.*

*Then the set of diffeomorphism classes of open thickenings of K that belong to the class $\mathcal{M}_{a,b,\pi_1(K),n}$ is **finite**.*

The corollary follows from the fact that any two tangentially homotopy equivalent thickenings of sufficiently high dimension are diffeomorphic [29, pp.226–228]. In general, the set of diffeomorphism classes of open n -dimensional thickenings of K is infinite provided $n > \max\{4, 2 \dim(K)\}$ and $\oplus_k H_{4k}(K, \mathbb{Q}) \neq 0$. Thus, for such n and K , most open n -dimensional thickenings of K do not belong to the class $\mathcal{M}_{a,b,\pi_1(K),n}$. Combining 1.3 and 1.5, we deduce the following.

Corollary 1.6. *Let M be a smooth aspherical manifold such that $\pi_1(M)$ is word-hyperbolic. Let $a \leq b < 0$ and $n > 2 \dim(M)$.*

*Then the set of isomorphism classes of vector bundles over M whose total spaces belong to $\mathcal{M}_{a,b,\pi_1(K),n}$ is **finite**.*

Actually, we show in [6] that, in case $\dim(M) \geq 3$, the assumption $n > 2 \dim(M)$ of the corollary 1.6 is redundant.

Synopsis of the paper. In the section 2 we prove the main lemmas from the global Riemannian geometry. The 3rd section provides some K-theoretic background. In the 4th section we define some invariant of actions. The 5th section is devoted to our main results including the theorem 1.1. In the section 6 we employ some group theory to deduce the corollary 1.2. The section 7 deals with applications to thickenings. Applications to convex-cocompact groups are deduced in the section 8. The theorem 1.4 is proved in the 9th section.

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2. Two types of convergence

By an *action* of an abstract group π on a space X we mean a group homomorphism $\rho : \pi \rightarrow \text{Homeo}(X)$. An action ρ is *free* if $\rho(\gamma)(x) \neq x$ for all $x \in X$ and all $\gamma \in \pi \setminus \{\text{id}\}$. In particular, if ρ is a free action, then ρ is injective.

2.1. Equivariant pointed Lipschitz topology. Let Γ_k be a discrete subgroup of the isometry group of a complete Riemannian manifold X_k and p_k be a point of X_k . The class of all such triples $\{(X_k, p_k, \Gamma_k)\}$ can be given the so-called equivariant pointed Lipschitz topology [20]; when Γ_k is trivial this reduces to the usual pointed Lipschitz topology. For convenience of the reader we give here some definitions borrowed from [20].

For a group Γ acting on a pointed metric space (X, p, d) the set $\{\gamma \in \Gamma : d(p, \gamma(p)) < r\}$ is denoted by $\Gamma(r)$. An open ball in X of radius r with center at p is denoted by $B_r(p, X)$.

For $i = 1, 2$, let (X_i, p_i) be a pointed complete metric space with the distance function d_i and let Γ_i be a discrete group of isometries of X_i . In addition, assume that X_i is a C^∞ -manifold. Take any $\epsilon > 0$.

Then a quadruple $(f_1, f_2, \phi_1, \phi_2)$ of maps

$$f_i : B_{1/\epsilon}(p_i, X_i) \rightarrow B_{1/\epsilon}(p_{3-i}, X_{3-i})$$

together with $\phi_i : \Gamma_i(\frac{1}{3\epsilon}) \rightarrow \Gamma_{3-i}$ is called an ϵ -Lipschitz approximation between the triples (X_1, p_1, Γ_1) and (X_2, p_2, Γ_2) if the following seven condition hold:

- f_i is a diffeomorphism onto its image;
- for each $x_i \in B_{1/3\epsilon}(p_i, X_i)$ and every $\gamma_i \in \Gamma_i(\frac{1}{3\epsilon})$, $f_i(\gamma_i(x_i)) = \phi_i(\gamma_i)(f_i(x_i))$;
- for every $x_i, x'_i \in B_{1/\epsilon}(p_i, X_i)$, $e^{-\epsilon} < d_{3-i}(f_i(x_i), f_i(x'_i))/d_i(x_i, x'_i) < e^\epsilon$;
- $f_i(B_{1/\epsilon}(p_i, X_i)) \supset B_{(1/\epsilon)-\epsilon}(p_{3-i}, X_{3-i})$ and $\phi_i(\Gamma_i(\frac{1}{3\epsilon})) \supset \Gamma_{3-i}(\frac{1}{3\epsilon} - \epsilon)$;
- $f_i(B_{(1/\epsilon)-\epsilon}(p_i, X_i)) \supset B_{1/\epsilon}(p_{3-i}, X_{3-i})$ and $\phi_i(\Gamma_i(\frac{1}{3\epsilon} - \epsilon)) \supset \Gamma_{3-i}(\frac{1}{3\epsilon})$;
- $f_{3-i} \circ f_i|_{B_{(1/\epsilon)-\epsilon}(p_i, X_i)} = \text{id}$ and $\phi_{3-i} \circ \phi_i|_{\Gamma_i(\frac{1}{3\epsilon} - \epsilon)} = \text{id}$;
- $d_{3-i}(f_i(p_i), p_{3-i}) < \epsilon$.

We say a sequence of triples (X_k, p_k, Γ_k) converges to (X, p, Γ) in the equivariant pointed Lipschitz topology if for any $\epsilon > 0$ there is $k(\epsilon)$ such that for all $k > k(\epsilon)$, there exists an ϵ -Lipschitz approximation between (X_k, p_k, Γ_k) and (X, p, Γ) .

Notice that if all the groups Γ_k are trivial, then Γ is trivial; in this case we say that that (X_k, p_k) converges to (X, p) in the pointed Lipschitz topology. Moreover, if X_k is a complete Riemannian manifold for all k , then the space X is necessarily a C^∞ -manifold with a complete $C^{1,\alpha}$ -Riemannian metric [22].

Remark 2.2. For those with the Kleinian groups background we remark that the equivariant pointed Lipschitz topology is closely related to the so-called Chabauty topology [7], [14]. Since the Kleinian group theory is an important source of examples, we describe the precise relation. Let X be a complete Riemannian manifold (e.g., a hyperbolic space) with the isometry group G . The set of discrete subgroups of G has the Chabauty topology induced by a natural topology on the set of closed subsets of G , namely a sequence of closed subsets S_i converges to S if for any compact $K \subset G$, the sets $S_i \cap K$ converge to $S \cap K$ in the Hausdorff topology on K . Consider the product topology on the set of pairs (Γ, p) where Γ is a discrete subgroup of G and $p \in X$. This product topology is in fact equivalent to the equivariant pointed Lipschitz topology where (Γ, p) is thought of as a triple (X, Γ, p) .

2.3. Pointwise convergence topology. Suppose that, for some $p_k \in X_k$, the sequence (X_k, p_k) converges to (X, p) in the pointed Lipschitz topology, i.e., for any $\epsilon > 0$ there is $k(\epsilon)$ such that for all $k > k(\epsilon)$ there exists an ϵ -Lipschitz approximation (f_k, g_k) between (X_k, p_k) and (X, p) . (Note that if each X_k is a Hadamard manifold, then for any $p_k \in X_k$, the sequence (X_k, p_k) is precompact in the pointed Lipschitz topology because the injectivity radius of X_k at p_k is uniformly bounded away from zero [20].) We say that a sequence $x_k \in X_k$ converges to $x \in X$ if for some ϵ

$$d(f_k(x_k), x) \rightarrow 0 \text{ as } k \rightarrow \infty,$$

where $d(\cdot, \cdot)$ is the distance function on X , and f_k comes from the ϵ -Lipschitz approximation (f_k, g_k) between (X_k, p_k) and (X, p) . Trivial examples: if (X_k, p_k) converges to (X, p) in the pointed Lipschitz topology, then p_k converges to p ; furthermore, if $x \in X$, the sequence $g_k(x)$ converges to x .

Given a sequence of isometries $\gamma_k \in \text{Isom}(X_k)$ we say that γ_k converges, if for any sequence $x_k \in X_k$ that converges to $x \in X$, $\gamma_k(x_k)$ converges. The limiting transformation γ that takes x to the limit of $\gamma_k(x_k)$ of X is necessarily an isometry. Furthermore, if γ_k and γ'_k converge to γ and γ' respectively, then $\gamma_k \cdot \gamma'_k$ converges to $\gamma \cdot \gamma'$. In particular, γ_k^{-1}

converges to γ^{-1} since the identity maps $\text{id}_k : X_k \rightarrow X_k$ converge to $\text{id} : X \rightarrow X$.

Let $\rho_k : \pi \rightarrow \text{Isom}(X_k)$ be a sequence of isometric actions of a group π on X_k . We say that a sequence of actions (X_k, p_k, ρ_k) *converges in the pointwise convergence topology* to an action (X, p, ρ) if $\rho_k(\gamma)$ converges to $\rho(\gamma)$ for every $\gamma \in \pi$. The limiting map $\rho : \Gamma \rightarrow \text{Isom}(X)$ that takes γ to the limit of $\rho_k(\gamma)$ is necessarily a homomorphism. If π is generated by a finite set S , then in order to prove that ρ_k converges in the pointwise convergence topology it suffices to check that $\rho_k(\gamma)$ converges, for every $\gamma \in S$.

A sequence of actions (X_k, p_k, ρ_k) is called *precompact in the pointwise convergence topology* if every subsequence of (X_k, p_k, ρ_k) has a subsequence that converges in the pointwise convergence topology.

Repeating the proof in [28, 4.7], it is easy to check that a sequence of isometries $\gamma_k \in \text{Isom}(X_k)$ has a converging subsequence if, for some converging sequence $x_k \in X_k$, the sequence $d_k(x_k, \gamma_k(x_k))$ is bounded where $d_k(\cdot, \cdot)$ is the distance function on X_k .

Suppose that π is a countable group and assume that for each $\gamma \in \pi$ the sequence $d_k(p_k, \rho_k(\gamma)(p_k))$ is bounded. Then (X_k, p_k, ρ_k) is precompact in the pointwise convergence topology. (Indeed, let $\gamma_1 \dots \gamma_n \dots$ be the list of all elements of π . Take any subsequence $\rho_{k,0}$ of ρ_k . Pass to subsequence $\rho_{k,1}$ of $\rho_{k,0}$ so that $\rho_{k,1}(\gamma_1)$ converges. Then pass to subsequence $\rho_{k,2}$ of $\rho_{k,1}$ such that $\rho_{k,2}(\gamma_2)$ converges, etc. Then $\rho_{k,k}(\gamma_n)$ converges for every n .)

Note that if π is generated by a finite set S , then to prove that ρ_k is precompact it suffices to check that $d_k(p_k, \rho_k(\gamma)(p_k))$ is bounded, for all $\gamma \in S$ because it implies that $d_k(p_k, \rho_k(\gamma)(p_k))$ is bounded, for each $\gamma \in \pi$.

2.4. Motivating example. Let X be a complete Riemannian manifold. Consider the isometry group $\text{Isom}(X)$ of X and let π be a group. The space $\text{Hom}(\pi, \text{Isom}(X))$ has a natural topology (which is usually called “algebraic topology” or “pointwise convergence topology”), namely ρ_k is said to converge to ρ if, for each $\gamma \in \pi$, $\rho_k(\gamma)$ converges to $\rho(\gamma)$ in the Lie group $\text{Isom}(X)$. Note that if π is finitely generated, this topology on $\text{Hom}(\pi, \text{Isom}(X))$ coincides with the compact-open topology. Certainly, for any $p \in X$, the constant sequence (X, p) converges to itself in pointed Lipschitz topology. Then, obviously, the sequence (X, p, ρ_k) converges in the pointwise convergence topology (as defined in 2.3) if and only if $\rho_k \in \text{Hom}(\pi, \text{Isom}(X))$ converges in the

algebraic topology.

Lemma 2.5. *Let $\rho_k : \pi \rightarrow \text{Isom}(X_k)$ be a sequence of isometric actions of a discrete group π on complete Riemannian n -manifolds X_k such that $\rho_k(\pi)$ acts freely. If the sequence $(X_k, p_k, \rho_k(\pi))$ converges in the equivariant pointed Lipschitz topology to (X, Γ, p) , and (X_k, p_k, ρ_k) converges to (X, p, ρ) in the pointwise convergence topology, then:*

- (1) Γ acts freely,
- (2) $\rho(\pi) \subset \Gamma$, and
- (3) $\ker(\rho) \subset \ker(\rho_k)$, for all large k .

Proof. (1) Assume $\gamma \in \Gamma$ and $\gamma(x) = x$. Choose $\epsilon \in (0, 1/10)$ so that there is an ϵ -approximation $(f_k, g_k, \phi_k, \tau_k)$ of (X_k, p_k, ρ_k) and (X, p, Γ) and $x \in B(p, \epsilon/10)$. Then $g_k(x) = g_k(\gamma(x)) = \tau_k(\gamma)(g_k(x))$. Since $\rho_k(\pi)$ acts freely, $\tau_k(\gamma) = \text{id}$. By the same argument $\tau_k(\text{id}) = \text{id}$. Hence $\text{id} = \phi_k(\tau_k(\text{id})) = \phi_k(\text{id}) = \phi_k(\tau_k(\gamma)) = \gamma$ as desired.

(2) We need to show that $\rho(\gamma) \in \Gamma$, for any $\gamma \in \pi$. We can assume $\rho(\gamma) \neq \text{id}$. Choose $\epsilon \in (0, 1/10)$ so that the ball $B(p, \frac{1}{11\epsilon})$ contains $\rho(\gamma)(p)$, and consider an ϵ -approximation $(f_k, g_k, \phi_k, \tau_k)$ of (X_k, p_k, ρ_k) and (X, p, Γ) .

Then for all large enough k , $\rho_k(\gamma) \in B(p_k, \frac{1}{10\epsilon})$. Look at $\tau_k(\rho_k(\gamma)) \in \Gamma(\frac{1}{9\epsilon})$. Since the set $\Gamma(\frac{1}{9\epsilon})$ is finite, we can pass to subsequence to assume that $\tau_k(\rho_k(\gamma))$ is equal to $\gamma_\epsilon \in \Gamma(\frac{1}{9\epsilon})$; thus $\rho_k(\gamma) = \phi_k(\gamma_\epsilon)$.

Take an arbitrary $x \in B(p, \frac{1}{9\epsilon})$. Then $g_k(x) \rightarrow x$ and, hence, $\rho_k(\gamma)(g_k(x))$ converges to $\rho(\gamma)(x)$. Notice that

$$\rho_k(\gamma)(g_k(x)) = \phi_k(\gamma_\epsilon)(g_k(x)) \rightarrow \gamma_\epsilon(x).$$

So $\rho(\gamma)(x) = \gamma_\epsilon(x)$ for any $x \in B(p, \frac{1}{9\epsilon})$.

Thus, for any small enough ϵ , we have found $\gamma_\epsilon \in \Gamma$ that is equal to $\rho(\gamma)$ on the ball $B(p, \frac{1}{9\epsilon})$. Since Γ acts freely, $\gamma_\epsilon = \gamma_{\epsilon'}$ for all $\epsilon' \leq \epsilon$, that is, the element $\gamma_\epsilon \in \Gamma$ is independent of ϵ . Thus $\rho(\gamma) = \gamma_\epsilon$ everywhere and hence $\rho(\gamma) \in \Gamma$.

(3) Assume $\rho(\gamma) = \text{id}$. Fix any $\epsilon \in (0, 1/10)$. Take $x \in B(p, \epsilon/10)$ and consider an ϵ -approximation $(f_k, g_k, \phi_k, \tau_k)$ of (X_k, p_k, ρ_k) and (X, p, Γ) . We have $g_k(x) \rightarrow x$ and $\rho_k(\gamma)(g_k(x)) \rightarrow \rho(\gamma)(x) = x$. Remark that $d(x, \phi_k(\rho_k(\gamma))(x))$ is equal to

$$d(f_k(g_k(x)), f_k(\rho_k(\gamma)(g_k(x)))) < e^\epsilon d_k(g_k(x), \rho_k(\gamma)(g_k(x))) \xrightarrow{k \rightarrow \infty} 0.$$

Therefore, for all large k , $\phi_k(\rho_k(\gamma)) = \text{id}$, because Γ is a discrete subgroup that acts freely. Hence $\rho_k(\gamma) = \tau_k(\phi_k(\rho_k(\gamma))) = \tau_k(\text{id}) = \text{id}$ as claimed. q.e.d.

Lemma 2.6. *Let $\rho_k : \pi \rightarrow \text{Isom}(X_k)$ be a sequence of isometric actions of a discrete group π on complete Riemannian n -manifolds X_k such that $\rho_k(\pi)$ acts freely. Suppose that the sequence $(X_k, p_k, \rho_k(\pi))$ converges in the equivariant pointed Lipschitz topology to (X, Γ, p) , and that (X_k, p_k, ρ_k) converges to (X, p, ρ) in the pointwise convergence topology.*

Then, for any $\epsilon > 0$ and any finite subset $S \subset \pi$, there is $k(\epsilon, S)$ with the property that for each $k > k(\epsilon, S)$ there exists an ϵ -Lipschitz approximation $(f_k, g_k, \phi_k, \tau_k)$ between $(X_k, p_k, \rho_k(\pi))$ and (X, p, Γ) such that $\phi_k(\rho_k(\gamma)) = \rho(\gamma)$ and $\rho_k(\gamma) = \tau_k(\rho(\gamma))$ for every $\gamma \in S$.

Proof. Pick $\delta < \epsilon$ so large that the set $\rho(S)(p)$ is contained in the ball $B(\frac{1}{10\delta}, p)$. Choose a δ -Lipschitz approximation $(f_k, g_k, \phi_k, \tau_k)$ between $(X_k, p_k, \rho_k(\pi))$ and (X, p, Γ) .

Note that for every $\gamma \in S$, the sequence $\rho_k(\gamma)(g_k(p))$ converges to $\rho(\gamma)(p)$. Hence the sequence $f_k(\rho_k(\gamma)(g_k(p))) = \phi_k(\rho_k(\gamma))(f_k(g_k(p))) = \phi_k(\rho_k(\gamma))(p)$ converges to $\rho(\gamma)(p)$ in X .

Since Γ is discrete and acts freely, $\phi_k(\rho_k(\gamma)) = \rho(\gamma)$ for large enough k . Moreover, this is true for any $\gamma \in S$, because S is finite. Also $\rho_k(\gamma) = (\tau_k \circ \phi_k)(\rho_k(\gamma)) = \tau_k(\rho(\gamma))$. Thus, this δ -Lipschitz approximation has all desired properties. q.e.d.

Proposition 2.7. *Let X_k be a sequence of Hadamard manifolds with sectional curvatures in $[a, b]$ for $a \leq b < 0$, and let π be a finitely generated group that is not virtually nilpotent. Let $\rho_k : \pi \rightarrow \text{Isom}(X_k)$ be an arbitrary sequence of free and isometric actions such that (X_k, p_k, ρ_k) converges in the pointwise convergence topology.*

Then the sequence (X_k, p_k, ρ_k) is precompact in the equivariant pointed Lipschitz topology.

Proof. Let (X, p, ρ) be the limit of (X_k, p_k, ρ_k) in the pointwise convergence topology. Choose r so that the open ball $B(p, r) \subset X$ contains $\{\rho(\gamma_1)(p), \dots, \rho(\gamma_m)(p)\}$ where $\{\gamma_1, \dots, \gamma_m\}$ generate π . Passing to subsequence, we assume that $B(p_k, r)$ contains $\{\rho_k(\gamma_1)(p), \dots, \rho_k(\gamma_m)(p)\}$.

Show that, for every k , there exists $q_k \in B(p_k, r)$ such that for any $\gamma \in \pi \setminus \{\text{id}\}$, we have $\rho_k(\gamma)(q_k) \notin B(q_k, \mu_n/2)$ where μ_n is the Margulis constant. Suppose not. Then for some k , the whole ball $B(p_k, r)$ projects into the thin part $\{\text{InjRad} < \mu_n/2\}$ under the projection $\pi_k : X_k \rightarrow X_k/\rho_k(\pi)$. Thus the ball $B(p_k, r)$ lies in a connected component W of the π_k -preimage of the thin part of $X_k/\rho_k(\pi)$. According to [2, p.111] the stabilizer of W in $\rho_k(\pi)$ is virtually nilpotent and, moreover, the stabilizer contains every element $\gamma \in \rho_k(\pi)$ with

$\gamma(W) \cap W \neq \emptyset$. Therefore, the whole group $\rho_k(\pi)$ stabilizes W . Hence $\rho_k(\pi)$ must be virtually nilpotent. As ρ_k is injective, π is virtually nilpotent, a contradiction.

Thus, $(X_k, q_k, \rho_k(\pi))$ is Lipschitz precompact [20] and, hence passing to subsequence, one can assume that $(X_k, q_k, \rho_k(\pi))$ converges to some (X, q, Γ) .

It is a general fact which follows easily from definitions that whenever (X_k, q_k, Γ_k) converges to (X, q, Γ) in the equivariant pointed Lipschitz topology and a sequence of points $p_k \in X_k$ converges to $p \in X$, then (X_k, p_k, Γ_k) converges to (X, p, Γ) in the equivariant pointed Lipschitz topology. q.e.d.

Corollary 2.8. *Let X_k be a sequence of Hadamard manifolds with sectional curvatures in $[a, b]$ for $a \leq b < 0$, and let π be a finitely generated group that is not virtually nilpotent. Let $\rho_k : \pi \rightarrow \text{Isom}(X_k)$ be an arbitrary sequence of free and isometric actions such that (X_k, p_k, ρ_k) converges to (X, p, ρ) in the pointwise convergence topology.*

Then ρ is a free action, in particular ρ is injective.

Proof. Pass to a subsequence so that (X_k, p_k, ρ_k) converges to (X, p, Γ) in the equivariant pointed Lipschitz topology. By 2.5(3), ρ is injective. Furthermore, $\rho(\pi)$ acts freely because it is a subgroup of Γ . q.e.d.

Corollary 2.9. *Let X be a complete Riemannian manifold with sectional curvatures in $[a, b]$ where $a \leq b < 0$. Assume π is a torsion free, finitely generated group that is not virtually nilpotent. Consider a subset of faithful discrete representations $S \subset \text{Hom}(\pi, \text{Isom}(X))$ that is precompact in the pointwise convergence topology. Then,*

(1) *for any $p \in X$, the set $\{(X, p, \rho(\pi)) : \rho \in S\}$ is precompact in the equivariant pointed Lipschitz topology,*

(2) *the closure of S in $\text{Hom}(\pi, \text{Isom}(X))$ consists of faithful discrete representations.*

Proof. Proposition 2.7 implies (1), and Corollary 2.8 implies (2).

q.e.d.

Proposition 2.10. *Assume that π is a finitely presented discrete group, that is not virtually nilpotent and does not have a nontrivial decomposition into an amalgamated product or an HNN-extension over a virtually nilpotent group.*

Let $\rho_k : \pi \rightarrow \text{Isom}(X_k)$ be an arbitrary sequence of free and isometric actions of π on Hadamard n -manifolds X_k . Assume that the sectional curvatures of X_k lie in $[a, b]$ for $a \leq b < 0$.

Then, for some $p_k \in X_k$, the sequence (X_k, p_k, ρ_k) is precompact in the pointwise convergence topology.

Proof. Let $S \subset \pi$ be a finite subset that generates π and contains $\{\text{id}\}$. For $x \in X_k$, we denote $D_k(x)$ the diameter of the set $\rho_k(S)(x)$. Set $D_k = \inf_{x \in X_k} D_k(x)$.

Suppose D_k is unbounded. Then from a work of Bestvina and Paulin [8], [32], [33] (cf. [27]) it follows that there exists an action of π on a real tree with no proper invariant subtree and virtually nilpotent arc stabilizers. For completeness we briefly review this construction. The rescaled pointed Hadamard manifold $\frac{1}{D_k} \cdot X_k$ has sectional curvature $\leq b \cdot D_k \rightarrow -\infty$ as $k \rightarrow \infty$. Choose $p_k \in X_k$ such that $D_k(p_k) \leq D_k + 1/k$. Consider the sequence of triples $(\frac{1}{D_k} \cdot X_k, p_k, \rho_k)$. Repeating an argument of Paulin [33, §4], we can pass to subsequence which converges to a triple $(X_\infty, p_\infty, \rho_\infty)$. For the definition of the convergence see [32], [33]; Paulin calls it “convergence in the Gromov topology”.

The limit space X_∞ is a length space of curvature $-\infty$, that is a real tree. Because of the way we rescaled, the limit space has a natural isometric action ρ_∞ of π with no global fixed point [32], [33]. Then it is a standard fact that there exists a unique π -invariant subtree T of X_∞ which has no proper π -invariant subtree. In fact T is the union of all the axes of all hyperbolic elements in π . Since the sectional curvatures are uniformly bounded away from zero and $-\infty$, the Margulis lemma implies that the stabilizer of any non-degenerate segment is virtually nilpotent (cf. [32]).

Note that any increasing sequence of virtually nilpotent subgroups of π is stationary. Indeed, since a virtually nilpotent group is amenable, the union U of an increasing sequence $U_1 \subset U_2 \subset U_3 \subset \dots$ of virtually nilpotent subgroups is also an amenable group. If the fundamental group of a complete manifold of pinched negative curvature is amenable, it must be finitely generated [13], [10]. In particular, U is finitely generated, hence $U_n = U$ for some n . Thus, the π -action on the tree T is stable [9, Proposition 3.2(2)].

We summarize that the π -action on T is stable, and has virtually nilpotent arc stabilizers and no proper π -invariant subtree. Therefore, the Rips machine [9, Theorem 9.5] produces a splitting of π over a virtually solvable group. Any amenable subgroup of π is virtually nilpotent [13], [10], hence π splits over a virtually nilpotent group. This is a contradiction with the assumption that D_k is unbounded.

Thus the sequence $D_k(p_k)$ is bounded, and as we observed in 2.3,

the sequence (X_k, p_k, ρ_k) is precompact in the pointwise convergence topology. q.e.d.

3. Vector bundles and \widetilde{KO} -theory

This section provides some \widetilde{KO} -theoretic background. All of the facts below are well-known to experts; however it is usually not easy to locate a reference.

In this paper we deal with rank- n real vector bundles over open smooth n -manifolds. It is well-known that any open smooth manifold N is homotopy equivalent to CW-complex of dimension $< \dim(N)$, hence two rank- n vector bundles over N are isomorphic iff they are stably isomorphic [26, 8.1.5]. The set of stable isomorphism classes of real vector bundles over a space X forms an abelian group $\widetilde{KO}(X)$. The correspondence $X \rightarrow \widetilde{KO}(X)$ is clearly a functor. This functor is isomorphic to the functor $[-, BO]$ on the category of connected CW-complexes of uniformly bounded dimension [26, 8.4.2].

We now recall the definition of the \widetilde{KO}^* -theory. By the Bott periodicity, $\Omega^8(BO \times \mathbb{Z})$ is weakly homotopy equivalent to $BO \times \mathbb{Z}$ [37, 11.60]. We can use the spectrum $\Omega^n(BO \times \mathbb{Z})$ to define a generalized (reduced) cohomology theory \widetilde{KO}^* on the category of all pointed connected finite-dimensional CW-complexes and point-preserving maps. In other words, we set $\widetilde{KO}^n(X, x) = [X, x; \Omega^n(BO \times \mathbb{Z}), *]$ (cf. [37, 8.42]).

Let F be a forgetful map from the category of pointed connected CW-complexes to the category of connected CW-complexes. We now show that the functors $\widetilde{KO}^0(-)$ and $\widetilde{KO}(F(-))$ are isomorphic on the category of pointed connected CW-complexes of uniformly bounded dimensions. Indeed, since we deal with connected CW-complexes, the functor $\widetilde{KO}^0(-) = [-; BO \times \mathbb{Z}, *]$ is isomorphic to the functor $[-; BO, *]$. As we mentioned above $\widetilde{KO}(-) \cong [-, BO]$. Therefore, it remains to show that the transformation of functors $[-; BO, *] \rightarrow [F(-); BO]$ is an isomorphism. In other words, it suffices to check that the map $[X, x; BO, *] \rightarrow [F(X, x); BO] = [X, BO]$ of based homotopy classes into free homotopy classes is bijective. Indeed, any map $X \rightarrow BO$ is homotopic to a basepoint preserving map [36, 7.3.Lemma 2] which is unique up to based homotopy because BO is a path-connected H-space. [36, 7.3.Theorem 5]. Thus, $\widetilde{KO}^0(-) \cong \widetilde{KO}(F(-))$. Most of the time we suppress the base points and treat the functors $\widetilde{KO}^0(-)$ and $\widetilde{KO}(-)$ as

isomorphic.

Rationally the group $\widetilde{KO}^i(X)$ can be easily computed in terms of cohomology of X as explained in the lemma below. Recall that the Bott periodicity implies that, after tensoring with rationals, the \widetilde{KO} -groups of the 0-sphere are given by $\widetilde{KO}^i(S^0) \otimes \mathbb{Q} \cong \mathbb{Q}$ if $i \equiv 0 \pmod{4}$ and $\widetilde{KO}^i(S^0) \otimes \mathbb{Q} = 0$ otherwise [26, 15.12.3].

Lemma 3.1. *On the category of pointed connected CW-complexes of uniformly bounded dimension the contravariant functors $\widetilde{KO}^i(-) \otimes \mathbb{Q}$ and $\bigoplus_{n>0} H^n(-) \otimes \widetilde{KO}^{i-n}(S^0) \otimes \mathbb{Q}$ are isomorphic.*

Proof. Consider the generalized cohomology theory

$$\widetilde{H}^*(-) \otimes \widetilde{KO}^*(S^0) \otimes \mathbb{Q},$$

where $\widetilde{H}^*(-)$ is the ordinary reduced cohomology; its coefficients are isomorphic to the coefficients of the theory $\widetilde{KO}^*(-) \otimes \mathbb{Q}$. Since the coefficients are rational vector spaces, [24, 3.22(ii)] implies that the theories are naturally equivalent on the category of pointed finite-dimensional CW-complexes.

For any connected finite-dimensional complex X , $\widetilde{H}^n(X) = 0$, for all $n \leq 0$. Clearly for $n > 0$, the functors $\widetilde{H}^n(-)$ and $H^n(-)$ are isomorphic. Hence we have an isomorphism of functors

$$\begin{aligned} \widetilde{KO}^i(-) \otimes \mathbb{Q} &\cong \bigoplus_{n \geq 0} \widetilde{H}^n(-) \otimes \widetilde{KO}^{i-n}(S^0) \otimes \mathbb{Q} \\ &\cong \bigoplus_{n > 0} H^n(-) \otimes \widetilde{KO}^{i-n}(S^0) \otimes \mathbb{Q}. \quad \text{q.e.d.} \end{aligned}$$

Remark 3.2. In particular, we have isomorphisms of functors

$$\widetilde{KO}^0(-) \otimes \mathbb{Q} \cong \bigoplus_{n > 0} H^{4n}(-, \mathbb{Q}) \quad \text{and} \quad \widetilde{KO}^{-1}(-) \otimes \mathbb{Q} \cong \bigoplus_{n > 0} H^{4n-1}(-, \mathbb{Q}).$$

Corollary 3.3. *Let N be an open smooth manifold such that the torsion subgroup of $\widetilde{KO}(N)$ is finite and let $n \geq \dim(N)$.*

Then the set of isomorphism classes of rank- n vector bundles over N is infinite if and only if $H^{4k}(N, \mathbb{Q}) \neq 0$ for some $k > 0$.

Proof. Since any open manifold N is homotopy equivalent to a CW-complex of dimension $< \dim(N)$, two rank- n vector bundles over N are isomorphic iff they are stably isomorphic. The result now follows from the above remark. q.e.d.

Lemma 3.4. *Let Y be a connected finite-dimensional CW-complex such that $\pi_1(Y)$ is finitely generated and $\dim_{\mathbb{Q}} H^{4k}(Y, \mathbb{Q}) < \infty$ for all $k > 0$. Then there exists a finite connected subcomplex $X \subset Y$ such that the inclusion $i : X \rightarrow Y$ induces a surjection on the fundamental groups and an injection $\widetilde{KO}(Y) \otimes \mathbb{Q} \rightarrow \widetilde{KO}(X) \otimes \mathbb{Q}$*

Proof. It is a standard fact [19, p.117] that for any homology class $\alpha \in H_i(Y, \mathbb{Q})$ there exist a finite (not necessarily connected) CW-complex X_α and a continuous map $f_\alpha : X_\alpha \rightarrow Y$ such that $\alpha \in f_{\alpha*}(H_i(X_\alpha, \mathbb{Q}))$. For every $k > 0$, choose a finite basis in the \mathbb{Q} -vector space $H_{4k}(Y, \mathbb{Q})$, and construct such a finite CW-complex X_α for every basis element α .

Let X_0 be the the disjoint union of all these complexes over all k and all the basis elements. Since Y is finite-dimensional with finite $4k$ -th Betti numbers, the CW-complex X_0 is finite. Note that X_0 comes with a continuous map $f : X_0 \rightarrow Y$ which induces a surjection on rational $4k$ -th homology, and therefore, an injection on rational $4k$ -cohomology for all $k > 0$.

Finally, set X to be an arbitrary connected finite subcomplex of Y such that $f(X_0) \subset X$. Using that $\pi_1(Y)$ is finitely generated, we add to X finitely many 1-cells of Y to make sure the inclusion $i : X \hookrightarrow Y$ induces a surjection of fundamental groups. Moreover, by construction i induces an injection on rational $4k$ -cohomology for all $k > 0$. By the remark 3.2, $\widetilde{KO}(Y) \otimes \mathbb{Q} \rightarrow \widetilde{KO}(X) \otimes \mathbb{Q}$ is injective. q.e.d.

4. An invariant of actions

Let K be a finite-dimensional connected CW-complex with a reference point q . Denote $\tilde{K} \rightarrow K$ the universal cover of K ; choose $\tilde{q} \in \tilde{K}$ that is mapped to q by the covering projection. Consider a pointed contractible manifold X and an arbitrary action $\rho : \pi_1(K, q) \rightarrow \text{Diffeo}(X)$.

Since X is contractible, the X -bundle $\tilde{K} \times_{\rho} X$ over K has a section. Any two sections are homotopic through sections.

We now define a certain invariant of actions of $\pi_1(K, q)$ on X . Given such an action ρ , consider the vertical vector bundle $\tilde{K} \times_{\rho} TX$ over $\tilde{K} \times_{\rho} X$ and set $\tau(\rho)$ to be the pullback of the vertical bundle via an arbitrary section $s : K \rightarrow \tilde{K} \times_{\rho} X$. Clearly, two actions that are conjugate in $\text{Diffeo}(X)$ have same invariants.

Any section can be lifted to a ρ -equivariant continuous map $\tilde{K} \rightarrow \tilde{K} \times X \rightarrow X$. Any two ρ -equivariant continuous maps $\tilde{g}, \tilde{f} : \tilde{K} \rightarrow X$,

are ρ -equivariantly homotopic. Indeed, \tilde{f} and \tilde{g} descend to sections $K \rightarrow \tilde{K} \times_{\rho} X$ that must be homotopic. This homotopy lifts to a ρ -equivariant homotopy of \tilde{f} and \tilde{g} .

Assume now that $\rho(\pi_1(K, q))$ acts freely and properly discontinuously on X , so that the map $\pi : X \rightarrow X/\rho(\pi_1(K))$ is a covering. Then the map \tilde{f} descends to a continuous map $f : K \rightarrow X/\rho(\pi_1(K))$. Thus, to any action $\rho : \pi_1(K, q) \rightarrow \text{Homeo}(X)$ such that $\rho(\pi_1(K, q))$ acts freely and properly discontinuously on a contractible manifold X , we associate a (unique up to homotopy) continuous map f . Observe that if ρ is injective, then f induces an isomorphism of fundamental groups.

We now observe that, in case $\rho(\pi_1(K, q))$ acts freely and properly discontinuously on X , the invariant $\tau(\rho)$ is equal to $f^*TX/\rho(\pi_1(K))$, the pullback of the tangent bundle to $X/\rho(\pi_1(K))$ via f .

If we consider only orientation-preserving actions the vector bundle $\tau(\rho)$ comes with a natural orientation.

5. Main results

Throughout this section K is a finite-dimensional, connected CW-complex with a reference point q . Let \tilde{K} be the universal cover of K , and $\tilde{q} \in \tilde{K}$ be a preimage of $q \in K$. Using the point \tilde{q} we identify $\pi_1(K, q)$ with the group of automorphisms of the covering $\tilde{K} \rightarrow K$. Recall that τ is an invariant of actions defined in §4.

Theorem 5.1. *Let the complex K is finite. Let $\rho_k : \pi = \pi_1(K, q) \rightarrow \text{Isom}(X_k)$ be a sequence of isometric actions of $\pi_1(K)$ on Hadamard n -manifolds X_k such that $\rho_k(\pi)$ is a discrete subgroup of $\text{Isom}(X_k)$ that acts freely.*

Suppose that, for some $p_k \in X_k$, $(X_k, p_k, \rho_k(\pi))$ converges in the equivariant pointed Lipschitz topology to (X, p, Γ) , and (X_k, p_k, ρ_k) converges to (X, p, ρ) in the pointwise convergence topology.

Then $\tau(\rho_k) = \tau(\rho)$ for all large k .

Proof. Let $\tilde{q} \in F \subset \tilde{K}$ be a finite subcomplex that projects onto K . Clearly, the finite set $S = \{\gamma \in \pi_1(K, q) : \gamma(F) \cap F \neq \emptyset\}$ generates $\pi_1(K, q)$.

Note that X is contractible; indeed any spheroid in X lies in the diffeomorphic image of a metric ball in X_j . Any metric ball in a Hadamard manifold is contractible. Thus $\pi_*(X) = 1$.

Using §4, we find a ρ -equivariant continuous map $\tilde{h} : \tilde{K} \rightarrow X$. Recall that by definition $\tau(\rho)$ is the pullback via $h : K \rightarrow X/\rho(\pi_1(K, q))$ of the

tangent bundle to $X/\rho(\pi_1(K, q))$. Let $\bar{h} : K \rightarrow X/\Gamma$ be the composition of h and the covering $X/\rho(\pi_1(K, q)) \rightarrow X/\Gamma$. Therefore, $\tau(\rho)$ is also the pullback via \bar{h} of the tangent bundle to X/Γ .

Choose $\epsilon > 0$ so small that $\tilde{h}(F)$ lies in the open ball $B(p, 1/10\epsilon) \subset X$. For large k , we find an ϵ -Lipschitz approximation $(\tilde{f}_k, \tilde{g}_k, \phi_k, \tau_k)$ between $(X_k, p_k, \rho_k(\pi))$ and (X, p, Γ) . By lemma 2.6 we can assume that $\tau_k(\rho(\gamma)) = \rho_k(\gamma)$ for all $\gamma \in S$. Hence, the map $\tilde{h}_k = \tilde{g}_k \circ \tilde{h} : F \rightarrow X_k$ is ρ_k -equivariant. Extend it by equivariance to a ρ_k -equivariant map $\tilde{h}_k : \tilde{K} \rightarrow X_k$. Passing to quotients we get a map $h_k : K \rightarrow X_k/\rho_k(\pi_1(K))$ such that $\tau(\rho_k)$ is the pullback via h_k of the tangent bundle to $X_k/\rho_k(\pi_1(K))$.

By construction, $h_k = g_k \circ \bar{h}$ where g_k is the drop of \tilde{g}_k . Being a diffeomorphism g_k preserves tangent bundles. Therefore, the pullback via h_k of the tangent bundle to $X_k/\rho_k(\pi_1(K))$ is equal to the pullback via \bar{h} of the tangent bundle to $X/\rho(\pi_1(K, k))$. In other words $\tau(\rho_k) = \tau(\rho)$. The proof is complete. \square

Remark 5.2. In the above theorem it is possible to keep track of orientations provided all the actions ρ_k on X_k preserve orientations (it makes sense because being a contractible manifold X_k is orientable). Indeed, fix an orientation on X (which is also contractible) and choose orientations of X_k so that diffeomorphisms g_k preserve orientations. Then, obviously, theorem 5.1 is still true where the vector bundle isomorphism $\tau(\rho_k) = \tau(\rho)$ preserves orientations.

Theorem 5.3. *Assume that $\dim_{\mathbb{Q}} \widetilde{KO}(K) \otimes \mathbb{Q} < \infty$. Let $\pi_1(K)$ be a finitely generated group, and $\rho_k : \pi = \pi_1(K) \rightarrow \text{Isom}(X_k)$ be a sequence of isometric actions of π on Hadamard n -manifolds X_k such that $\rho_k(\pi)$ is a discrete subgroup of $\text{Isom}(X_k)$ that acts freely.*

Suppose that, for some $p_k \in X_k$, $(X_k, p_k, \rho_k(\pi))$ converges in the equivariant pointed Lipschitz topology to (X, p, Γ) , and (X_k, p_k, ρ_k) converges to (X, p, ρ) in the pointwise convergence topology.

Then, for all large k , the images of $\tau(\rho_k)$ and $\tau(\rho)$ in $\widetilde{KO}(K) \otimes \mathbb{Q}$ are equal.

Proof. By 3.4 there exists a finite connected subcomplex $X \subset K$ such that the inclusion $i : X \rightarrow K$ induces a π_1 -epimorphism and a \widetilde{KO} -monomorphism.

The sequence of isometric actions $\rho_k \circ i_*$ of $\pi_1(X, x)$ on X_k satisfies the assumptions of theorem 5.1, therefore, $\tau(\rho_k \circ i_*) = \tau(\rho \circ i_*)$ for all large k . In other words, the pullback bundles $i^* \tau(\rho_k)$ are isomorphic to

the vector bundle $i^*\tau(\rho)$. Since i induces a monomorphism of rational \widetilde{KO} -groups, the images of $\tau(\rho_k)$ and $\tau(\rho)$ in $\widetilde{KO}(K) \otimes \mathbb{Q}$ are equal for all large k . q.e.d.

Corollary 5.4. *Assume that the group $\widetilde{KO}(K)$ is finitely generated. Let $\pi_1(K)$ be a finitely generated group, and let $\rho_k : \pi = \pi_1(K) \rightarrow \text{Isom}(X_k)$ be a sequence of isometric actions on Hadamard n -manifolds X_k such that $\rho_k(\pi)$ is a discrete subgroup of $\text{Isom}(X_k)$ which acts freely. Suppose that, for some $p_k \in X_k$, (X_k, p_k, ρ_k) is precompact in both pointwise convergence topology and equivariant pointed Lipschitz topology.*

Then the set $\{\tau(\rho_k)\}$ falls into finitely many stable isomorphism classes.

Proof. Argue by contradiction. Pass to subsequence to assume that all $\tau(\rho_k)$ are different and that $(X_k, p_k, \rho_k(\pi_1(K, q)))$ converges to (X, p, Γ) in the equivariant pointed Lipschitz topology, and (X_k, p_k, ρ_k) converges to (X, p, ρ) in the pointwise convergence topology.

Note that if the group $\widetilde{KO}(K)$ is finitely generated, then $\dim_{\mathbb{Q}} \widetilde{KO}(K) \otimes \mathbb{Q} < \infty$. Thus by theorem 5.3 the images of $\tau(\rho_k)$ in $\widetilde{KO}(K) \otimes \mathbb{Q}$ are all equal for large k . In other words for large k , the images of $\tau(\rho_k)$ in $\widetilde{KO}(K)$ are all equal modulo torsion. Since the abelian group $\widetilde{KO}(K)$ is finitely generated, its torsion subgroup is finite. Therefore, passing to subsequence we can assume that the images $\tau(\rho_k)$ in $\widetilde{KO}(K)$ are the same, a contradiction. q.e.d.

Corollary 5.5. *Assume that the complex K is aspherical. Suppose that the groups $\widetilde{KO}(K)$ and $\pi_1(K)$ are finitely generated. Let $\rho_k : \pi = \pi_1(K, q) \rightarrow \text{Isom}(X_k)$ be a sequence of free, isometric actions of $\pi_1(K)$ on Hadamard n -manifolds X_k . Suppose that, for some $p_k \in X_k$, $(X_k, p_k, \rho_k(\pi))$ is precompact in the equivariant pointed Lipschitz topology, and (X_k, p_k, ρ_k) is precompact in the pointwise convergence topology.*

Then the set of manifolds $\{X_k/\rho_k(\pi)\}$ falls into finitely many tangential equivalence types.

Proof. By corollary 5.4 the set $\{\tau(\rho_k)\}$ falls into finitely many stable isomorphism classes. Let $h_k : \widetilde{K} \rightarrow X_k$ be ρ_k -equivariant maps constructed in §3. Passing to quotients we get homotopy equivalences $h_k : K \rightarrow X_k/\rho_k(\pi) = N_k$ with the property that $h_k^*TN_k \cong \tau(\rho_k)$. Therefore, X_k/ρ_k and X_j/ρ_j are tangential homotopy equivalent iff $\tau(\rho_k) \cong \tau(\rho_j)$. q.e.d.

Corollary 5.6. *Suppose that the complex K is aspherical and the group $\widetilde{KO}(K)$ is finitely generated. Let $\pi_1(K)$ be a finitely presented group that is not virtually nilpotent. Assume that $\pi_1(K)$ does not split as a nontrivial amalgamated product or an HNN-extension over a virtually nilpotent group.*

Then, for any $a \leq b < 0$ and an integer $n \geq 2$, the class $\mathcal{M}_{a,b,\pi_1(K),n}$ breaks into finitely many tangential homotopy types.

Proof. Apply corollary 5.5 and propositions 2.7, 2.10. q.e.d.

Corollary 5.7. *Let π be the fundamental group of a finite aspherical CW-complex. Suppose that π is not virtually nilpotent and that π does not split as a nontrivial amalgamated product or an HNN-extension over a virtually nilpotent group.*

Then, for any $a \leq b < 0$ and an integer $n \geq 2$, the class $\mathcal{M}_{a,b,\pi_1(K),n}$ breaks into finitely many tangential homotopy types.

Proof. Let π be the the fundamental group of a finite complex K . Then π is finitely presented and the group $\widetilde{KO}(K)$ is finitely generated [24, p.52]. Hence corollary 5.6 applies. q.e.d.

6. Graphs of groups and accessibility

In this section we explain how to generalize Theorem 1.1 to certain amalgamated products and HNN–extensions, or more generally, to some graphs of groups. Then we deduce corollary 1.2.

Recall that a graph of groups is a graph whose vertices and edges are labeled with *vertex groups* π_v and *edge groups* π_e , and such that every pair (v, e) , where the edge e is incident to the vertex v , is labeled with a group monomorphism $\pi_e \rightarrow \pi_v$. We only consider finite connected graphs of groups. To each graph of groups one can associate its fundamental group which is a result of repeated amalgamated products and HNN–extensions of vertex groups over the edge groups (see [3] for more details).

Proposition 6.1. *Let π be the fundamental group of a finite graph of groups with the vertex groups π_v . Assume that the homomorphism*

$$\widetilde{KO}(K(\pi, 1)) \rightarrow \oplus_v \widetilde{KO}(K(\pi_v, 1))$$

induced by the inclusions $\pi_v \rightarrow \pi$ has finite kernel.

Fix n and $a \leq b < 0$ and suppose that, for each vertex group π_v , the class $\mathcal{M}_{a,b,\pi_v,n}$ breaks into finitely many tangential homotopy types. Then so does the class $\mathcal{M}_{a,b,\pi,n}$.

Proof. Assume, by contradiction that there exist a sequence $N_k \in \mathcal{M}_{a,b,\pi,n}$ of manifolds that are not pairwise tangentially homotopy equivalent. It defines an infinite sequence of distinct elements in $\widetilde{KO}(K(\pi, 1))$.

The homomorphism $\widetilde{KO}(K(\pi, 1)) \rightarrow \bigoplus_v \widetilde{KO}(K(\pi_v, 1))$ has finite kernel and the set of vertices is finite. Therefore, for some vertex v we get an infinite sequence of elements in $\widetilde{KO}(K(\pi_v, 1))$. Each of elements of the infinite sequence comes from the tangent bundle to a manifold $\bar{N}_k \in \mathcal{M}_{a,b,\pi_v,n}$, namely, \bar{N}_k is the covering of N_k induced by the inclusion $\pi_v \rightarrow \pi$. Thus, for this vertex $\mathcal{M}_{a,b,\pi_v,n}$ falls into infinitely many tangential homotopy types, a contradiction. q.e.d.

Corollary 6.2. *Let π be the fundamental group of a finite graph of groups such that each edge group is a virtually nilpotent group of cohomological dimension ≤ 2 .*

Fix n and $a \leq b < 0$ and suppose that, for each vertex group π_v , the class $\mathcal{M}_{a,b,\pi_v,n}$ breaks into finitely many tangential homotopy types. Then so does the class $\mathcal{M}_{a,b,\pi,n}$.

Proof. We can assume that $\mathcal{M}_{a,b,\pi,n} \neq \emptyset$. Then $\mathcal{M}_{a,b,\pi_e,n} \neq \emptyset$ for every edge group π_e . Hence all the edge groups are finitely generated [10]. Being a finitely generated virtually nilpotent group, each edge group π_e is a fundamental group of a closed aspherical manifold [17] which has to be of dimension ≤ 2 since $\text{cd}(\pi_e) \leq 2$.

In particular, the group $\widetilde{KO}^{-1}(K(\pi_e, 1))$ is a finitely generated abelian group [24, p.52]. According to 3.2, there is an isomorphism of functors $\widetilde{KO}^{-1}(-) \otimes \mathbb{Q} \cong \bigoplus_{k>0} H^{4k-1}(-, \mathbb{Q})$ on the category of connected CW-complexes of uniformly bounded dimension. Hence, for each edge group π_e , the group $\widetilde{KO}^{-1}(K(\pi_e, 1))$ is finite.

Following [34] we can assemble the cell complexes $K(\pi_v, 1)$ and $K(\pi_e, 1) \times [-1, 1]$ into an $K(\pi, 1)$ cell complex by using edge-to-vertex monomorphisms. Consider the \widetilde{KO}^* -Mayer-Vietoris sequence applied to the $K(\pi, 1)$ complex (cf. [12, VII.9]). The group $\bigoplus_e \widetilde{KO}^{-1}(K(\pi_e, 1))$ is finite, so by exactness, $\widetilde{KO}(K(\pi, 1)) \rightarrow \bigoplus_v \widetilde{KO}(K(\pi_v, 1))$ has finite kernel. We now apply 6.1 to complete the proof. q.e.d.

6.3. An accessibility result. Delzant and Potyagailo have recently proved a powerful accessibility result which we state below for

reader's convenience. Definition 6.4 and theorem 6.5 are taken from [16].

Definition 6.4. A class \mathcal{E} of subgroups of a group π is called *elementary* provided the following four conditions hold:

- (i) \mathcal{E} is closed under conjugation in π ;
- (ii) any infinite group from \mathcal{E} is contained in a *unique* maximal subgroup from \mathcal{E} ;
- (iii) if a group from the class \mathcal{E} acts on a tree, it fixes a point, an end, or a pair of ends;
- (iv) each maximal subgroup from \mathcal{E} is equal to its normalizer in π .

Note that the condition (iii) holds if any group in \mathcal{E} is amenable [31].

Theorem 6.5 [16]. *Let π be a finitely presented group without 2-torsion, and let \mathcal{E} be an elementary class of subgroups of π . Then there exist an integer $K > 0$ and a **finite** sequence $\pi_0, \pi_1, \dots, \pi_m$ of subgroups of π such that*

- (1) $\pi_m = \pi$,
- (2) for each k with $0 \leq k < K$, the group π_k either belongs to \mathcal{E} or does not split as a nontrivial amalgamated product or an HNN-extension over a group from \mathcal{E} , and
- (3) for each k with $K \leq k \leq m$, the group π_k is the fundamental group of a finite graph of groups with edge groups from \mathcal{E} , vertex groups from $\{\pi_0, \pi_1, \dots, \pi_{k-1}\}$, and proper edge-to-vertex homomorphisms.

Proposition 6.6. *If $\mathcal{M}_{a,b,\pi,n} \neq \emptyset$, then the class of virtually nilpotent subgroup of π is elementary.*

Proof. Condition (i) is trivially satisfied. Any virtually nilpotent group is amenable, hence (iii) holds thanks to [31].

Let X be a Hadamard manifold such that $X/\pi \in \mathcal{M}_{a,b,\pi,n}$. Being the fundamental group of an aspherical manifold, the group π is torsion free. According to [11], a nontrivial torsion-free virtually nilpotent discrete isometry group Γ of X is characterized by its fixed-point-set at infinity. In fact, either the fixed-point-set is a point or it consists of two points. Conversely, any discrete torsion-free subgroup of $\text{Isom}(X)$ that fixes a point at infinity is virtually nilpotent. Thus, a virtually nilpotent subgroup N of π is maximal if and only if N contains every element of π that fixes the fixed-point-set of N . This characterization proves (ii).

Finally, we deduce (iv). Let g be an element of the normalizer of a maximal virtually nilpotent subgroup N . Then g preserves the fixed-point-set of N setwise. If the fixed-point-set is a point, we conclude $g \in N$. If the fixed-point-set consists of two points p and q and $g \notin N$, then $g(p) = q$. Hence g must fix a point on the geodesic that joins p and q , so the element g is elliptic. This is a contradiction because no discrete torsion-free group contains an elliptic element. q.e.d.

Theorem 6.7. *Let π be a finitely presented group such that any nilpotent subgroup of π has cohomological dimension ≤ 2 . Assume that the group $\widetilde{KO}(K(\pi, 1))$ is finitely generated.*

Then, for any positive integer n and any negative reals $a \leq b$, the class $\mathcal{M}_{a,b,\pi,n}$ breaks into finitely many tangential homotopy types.

Proof. Applying theorem 6.5, we get a sequence $\pi_0, \pi_1, \dots, \pi_m$ of subgroups of π . In particular, for every k with $0 \leq k < K$, the group π_k does not split as a nontrivial amalgamated product or an HNN-extension over a virtually nilpotent subgroup of π . By proposition 6.9 below, the group π_k is finitely presented and $\widetilde{KO}(K(\pi, 1))$ is finitely generated. Therefore, by theorem 5.6, the class $\mathcal{M}_{a,b,\pi_k,n}$ breaks into finitely many tangential homotopy types.

Repeatedly applying corollary 6.2, we deduce that $\mathcal{M}_{a,b,\pi,n}$ breaks into finitely many tangential homotopy types. q.e.d.

Corollary 6.8. *Let π be a word-hyperbolic group. Then for any n and $a \leq b < 0$, the class $\mathcal{M}_{a,b,\pi,n}$ breaks into finitely many tangential homotopy types.*

Proof. We can assume that $\mathcal{M}_{a,b,K(\pi,1),n} \neq \emptyset$, hence π is torsion-free. Any torsion-free word-hyperbolic group is the fundamental group of a finite CW-complex K [15, 5.24]: in particular π is finitely presented and has finitely generated $\widetilde{KO}(K)$ [24, p.52]. Any nilpotent subgroup of a torsion free word hyperbolic group is either trivial or infinite cyclic. The result now follows from the previous theorem. q.e.d.

Proposition 6.9. *Let π be the fundamental group of a finite graph of groups with virtually nilpotent edge groups. Assume $\mathcal{M}_{a,b,\pi,n} \neq \emptyset$. Then:*

- (1) π is finitely presented iff all the vertex groups are finitely presented,
- (2) $\dim_{\mathbb{Q}} \oplus_{k>0} H^{4k}(\pi, \mathbb{Q}) < \infty$ iff $\dim_{\mathbb{Q}} \oplus_{k>0} H^{4k}(\pi_v, \mathbb{Q}) < \infty$ for every vertex group π_v , and

(3) the group $\widetilde{KO}(\pi)$ is finitely generated iff for every vertex group π_v the group $\widetilde{KO}(\pi)$ is finitely generated.

Proof. All the edge groups are finitely generated [10]. Being a finitely generated virtually nilpotent group, each edge group π_e is a fundamental group of a closed aspherical manifold [17]. In particular, $\widetilde{KO}^*(\pi_e)$ as well as $H^*(\pi_e)$ is a finitely generated abelian group. Hence, the parts (2) and (3) follow from the Mayer-Vietoris sequence (cf. [12, VII.9]).

We now prove (1). It trivially follows from definitions that if all vertex groups are finitely presented, then so is π .

Assume now that π is finitely presented. Then every vertex group π_v is finitely generated [3, Lemma 13, p.158]. The fundamental group π of a graph of groups has a presentation that can be described as follows. The set of generators S for this presentation π is the union of the (finite) sets of generators of all the vertex groups; so S is finite. The set of relations R is a union of the sets of relations of all the vertex groups and (finitely many) relations coming from amalgamations and HNN-extensions over the edge groups.

Since π is finitely presented, the group π has a presentation $\langle S|R' \rangle$ on the same set of generators S where R' is a finite subset of R [3, Theorem 12, p.52]. By adding finitely many (redundant) relations, we can assume that all the relations coming from the edge groups still belong to the new set of relations R' . This defines a new graph of groups decomposition of π with the same underlying graph, same edge groups and with finitely presented vertex groups π'_v . By construction, the subgroups π'_v and π_v of π have the same set of generators, hence $\pi'_v = \pi_v$. Thus, we have found a finite presentation for every vertex group. q.e.d.

7. Finiteness for thickenings

Throughout this section K is a finite, connected CW-complex.

By an n -*thickening* of K we mean a compact smooth manifold L of dimension $\geq \dim(K) + 3$ such that L is simply homotopy equivalent to K , and the inclusion $\partial L \rightarrow L$ induces an isomorphism of fundamental groups (see [38]). To avoid low-dimensional complications, we always assume that $n > 4$.

A thickening L of K is always diffeomorphic to the regular neighborhood of a finite simplicial subcomplex $K' \subset L$ with $\dim(K') \leq \dim(K)$ [25, p.219], [35].

If $\pi_1(K)$ is isomorphic to the fundamental group of a complete manifold of sectional curvature pinched between two negative constants, any homotopy equivalence $K \approx L$ is necessarily simple because $\text{Wh}(\pi_1(K)) = 0$ [18].

By an *open n -thickening* of a finite connected CW-complex K we mean an open smooth manifold that is diffeomorphic to the interior of a thickening of K .

Example 7.1 (Totally geodesic). Let N be a complete manifold of nonpositive curvature of dimension > 4 that is homotopy equivalent to a finite cell complex K of dimension $\leq \dim(N) - 3$. Assume that N contains a (possibly noncompact) totally geodesic embedded submanifold M of dimension $\leq \dim(N) - 3$ such that the inclusion $M \rightarrow N$ is a homotopy equivalence.

Then N is an open thickening of K . (Indeed, the exponential map identifies N with the total space of the normal bundle of M in N . Then, according to [35], the manifold N has exactly one end that is π_1 -stable and the natural homomorphism of the fundamental group at infinity into $\pi_1(N)$ is an isomorphism. By [25] N is simple homotopy equivalent to a simplicial subcomplex K' of dimension $\leq \dim(K) \leq \dim(N) - 3$. Hence, N is diffeomorphic to the regular neighborhood of K' in N [35]. Thus, N is an open thickening of K .)

7.3. Existence of thickenings. For each vector bundle ξ over K of rank $n > \max\{5, \dim(K)\}$ there exist an n -thickening L of K and a simple homotopy equivalence $f : K \rightarrow L$ such that $f^*TL \cong \xi$ [38, 5.1].

7.4. Uniqueness of thickenings. Any tangential homotopy equivalence of n -thickenings is homotopic to a diffeomorphism provided $n > 5$. [38, 5.1].

7.5. Uniqueness of open thickenings Any tangential homotopy equivalence of open n -thickenings is homotopic to a diffeomorphism provided $n > 4$. [29, pp.226–228]

7.6. Theorem. *Let K be a finite, connected, aspherical CW-complex and let n be an integer with $n > \max\{4, 2 \dim(K)\}$. Suppose that the class $\mathcal{M}_{a,b,\pi_1(K),n}$ breaks into finitely many tangential homotopy types.*

*Then, for any $a \leq b < 0$, the set of diffeomorphism classes of open thickenings of K that belong to $\mathcal{M}_{a,b,\pi_1(K),n}$ is **finite**.*

Proof. Apply 7.5.

7.6. Theorem. *Let K be a finite, connected, aspherical CW-complex and let n be an integer with $n > \max\{5, 2 \dim(K)\}$. Suppose that the class $\mathcal{M}_{a,b,\pi_1(K),n}$ breaks into finitely many tangential homotopy types.*

*Then, for any $a \leq b < 0$, the set of diffeomorphism classes of thickenings of K whose interiors belong to $\mathcal{M}_{a,b,\pi_1(K),n}$ is **finite**.*

Proof. Apply 7.4.

7.7. Corollary. *Let K be a finite, connected, aspherical CW-complex and let n be an integer with $n > \max\{4, 2 \dim(K)\}$. Suppose that either*

- *any nilpotent subgroup of $\pi_1(K)$ has cohomological dimension ≤ 2 ,*
- or
- *π is not virtually nilpotent and does not split as a nontrivial amalgamated product or an HNN-extension over a virtually nilpotent group.*

*Then, for any $a \leq b < 0$, the set of diffeomorphism classes of open thickenings of K that belong to $\mathcal{M}_{a,b,\pi_1(K),n}$ is **finite**.*

Proof. Combine 5.7, 6.7, and 7.6.

7.8. Theorem. *Let K be a finite, connected, aspherical CW-complex and let n be an integer with $n > \max\{4, 2 \dim(K)\}$. Suppose ρ_k is a sequence of free isometric actions of $\pi_1(K)$ on Hadamard n -manifolds X_k such that, for some $p_k \in X_k$, (X_k, p_k, ρ_k) is precompact in both pointwise convergence topology and equivariant pointed Lipschitz topology. Assume that for each k , the manifold $X_k/\rho_k(\pi_1(K))$ is an open thickening of K .*

Then the set $\{X_k/\rho_k(\pi_1(K))\}$ breaks into finitely many diffeomorphism types.

Proof. Since K is a finite complex, the group $\widehat{KO}(K)$ is finitely generated. Hence 5.5 implies that the set of manifolds $\{X_k/\rho_k(\pi_1(K))\}$ falls into finitely many tangential homotopy types. According to 7.6, $\{X_k/\rho_k(\pi_1(K))\}$ breaks into finitely many diffeomorphism types. q.e.d.

8. Finiteness for convex-cocompact groups

First, we recall some basic facts on convex-cocompact groups that are well known in the constant negative curvature case (see [11] for more details).

Let X be a Hadamard manifold with sectional curvatures pinched between two negative constants and let Γ be a discrete subgroup of the

isometry group of X . Then Γ acts by homeomorphisms on the ideal boundary $\partial_\infty X$ of X . The set of points $\Omega(\Gamma) \subset \partial_\infty X$ where Γ acts properly discontinuously is called the *domain of discontinuity* of Γ . Its complement $\Lambda(\Gamma) = \partial_\infty X \setminus \Omega(\Gamma)$ is called the *limit set* of Γ .

Fix any $\epsilon > 0$. Let $C_\epsilon(\Gamma)$ be the closed ϵ -neighborhood of the convex hull of $\Lambda(\Gamma)$ in X .

There is a Γ -equivariant homeomorphism of $X \cup \Omega(\Gamma)$ and C_ϵ defined as follows. The fibers of orthogonal projection $p : X \rightarrow C_\epsilon(\Gamma)$ are geodesic rays orthogonal to $\partial C_\epsilon(\Gamma)$. Any point of $\Omega(\Gamma)$ is the endpoint of such a ray, and no two of these rays have the same endpoint. Thus, the map p extends to an equivariant continuous map $\bar{p} : X \cup \partial_\infty X \rightarrow C_\epsilon(\Gamma) \cup \Lambda(\Gamma)$ which is the identity on the limit set. Contracting along the rays defines an equivariant homeomorphism of $X \cup \partial_\infty X$ and $C_{2\epsilon}(\Gamma) \cup \Lambda(\Gamma)$, which descends to a homeomorphism of $(X \cup \Omega(\Gamma))/\Gamma$ and $C_{2\epsilon}(\Gamma)/\Gamma$.

We say that Γ is *convex-cocompact* if the quotient $X \cup \Omega(\Gamma)/\Gamma$ is compact.

Two convex-cocompact groups Γ_1 and Γ_2 are called *topologically equivalent* if there exists a homeomorphism $h : X_1 \cup \partial_\infty X_1 \rightarrow X_2 \cup \partial_\infty X_2$ that is equivariant with respect to a certain isomorphism of Γ_1 and Γ_2 .

Given negative reals $a \leq b$, a torsion-free group π , and an integer n , define a class of convex-cocompact groups $\mathcal{CC}_{a,b,\pi,n,\pi_1(\Omega)=1}$ as follows.

A convex-cocompact group Γ of isometries of a Hadamard manifold X is said to belong to $\mathcal{CC}_{a,b,\pi,n,\pi_1(\Omega)=1}$ provided the following three conditions hold:

- Γ is isomorphic to π ;
- $\dim(X) = n$ and the sectional curvature of X is within $[a, b]$;
- the domain of discontinuity $\Omega(\Gamma)$ of Γ is simply-connected.

Proposition. 8.1. *Let Γ be a torsion-free convex-cocompact subgroup of the isometry group of a Hadamard manifold X of dimension > 5 such that $\Omega(\Gamma)$ is simply-connected. Assume that Γ is the fundamental group of a finite aspherical CW-complex K of dimension $\leq \dim(X) - 3$.*

Then $(X \cup \Omega(\Gamma))/\Gamma$ is a thickening of K .

Proof. Since Γ is a torsion-free convex-cocompact group, the quotient $(X \cup \Omega(\Gamma))/\Gamma$ is a compact manifold with boundary that is homotopy equivalent to K .

The homotopy equivalence is necessarily simple because $\text{Wh}(\Gamma) = 0$. (Farrell and Jones [18] proved the vanishing of the Whitehead group of

the fundamental group of any complete manifold of pinched negative curvature.)

Finally, since $\Omega(\Gamma)$ is simply-connected, the inclusion

$$\Omega(\Gamma)/\Gamma \rightarrow (X \cup \Omega(\Gamma))/\Gamma$$

induces a π_1 -isomorphism. q.e.d.

Proposition. 8.2. *Let K be a finite, connected, aspherical CW-complex and let n be an integer with $n > 2 \dim(K)$ and $n > 5$. Given $i \in \{1, 2\}$, let Γ_i be a torsion-free subgroup of the isometry group of a Hadamard manifold X_i such that $\Gamma_i \in \mathcal{CC}_{a,b,\pi_1(K),n,\pi_1(\Omega)=1}$.*

If X_1/Γ_1 and X_2/Γ_2 are tangentially homotopy equivalent, then Γ_1 and Γ_2 are topologically equivalent.

Proof. Since $C_\epsilon(\Gamma_i)/\Gamma_i$ is homeomorphic to $X \cup \Omega(\Gamma_i)$, proposition 8.1 implies that $C_\epsilon(\Gamma_i)/\Gamma_i$ is a thickening of K .

Moreover, $C_\epsilon(\Gamma_i)/\Gamma_i$ is a codimension zero submanifold of X/Γ_i , hence the homotopy equivalence $C_\epsilon(\Gamma_i)/\Gamma_i \hookrightarrow X/\Gamma_i$ is tangential. Thus, $C_\epsilon(\Gamma_1)/\Gamma_1$ and $C_\epsilon(\Gamma_2)/\Gamma_2$ are tangentially homotopy equivalent thickenings, and therefore they are diffeomorphic.

The diffeomorphism of compact manifolds $C_\epsilon(\Gamma_1)/\Gamma_1 \rightarrow C_\epsilon(\Gamma_2)/\Gamma_2$ is necessarily bilipschitz. So it lifts to an equivariant bilipschitz diffeomorphism $d : C_\epsilon(\Gamma_1) \rightarrow C_\epsilon(\Gamma_2)$.

Note that $C_\epsilon(\Gamma_i)$ is a Gromov hyperbolic space with ideal boundary $\Lambda(\Gamma_i)$. Therefore, d extends to an equivariant homeomorphism $C_\epsilon(\Gamma_1) \cup \Lambda(\Gamma_1) \rightarrow C_\epsilon(\Gamma_2) \cup \Lambda(\Gamma_2)$ [15, p.35]. Finally, using equivariant homeomorphisms of $X \cup \partial_\infty X$ and $C_\epsilon(\Gamma_i) \cup \Lambda(\Gamma_i)$, we produce a topological equivalence of Γ_1 and Γ_2 . q.e.d.

8.3. Theorem. *Let K be a finite, connected, aspherical CW-complex and let n be an integer with $n > 2 \dim(K)$ and $n > 5$.*

*Then the class $\mathcal{CC}_{a,b,\pi_1(K),n,\pi_1(\Omega)=1}$ falls into **finitely many** topological equivalence classes.*

Proof. Any convex-cocompact group Γ is word-hyperbolic because it acts isometrically and cocompactly on a negatively curved space $C_\epsilon(\Gamma)$. Hence the result follows from 6.8 and 8.2. q.e.d.

9. Locally symmetric nonpositively curved manifolds up to tangential homotopy equivalence

For completeness we present a proof of the following result (cf. [4]).

Theorem 9.1. *Let π be a finitely presented torsion-free group and let X be a nonpositively curved symmetric space.*

Then the class of manifolds of the form $X/\rho(\pi)$, where

$$\rho \in \text{Hom}(\pi, \text{Isom}(X))$$

is a faithful discrete representation, falls into finitely many tangential homotopy types.

Proof. We can assume that $\text{Hom}(\pi, \text{Isom}(X))$ contains a faithful discrete representation, and let K be the corresponding quotient manifold.

First note that, for any two faithful discrete representations ρ_1 and ρ_2 that lie in the same connected component of the analytic variety $\text{Hom}(\pi_1(K), \text{Isom}(X))$, we have $\tau(\rho_1) \cong \tau(\rho_2)$. Indeed, by the covering homotopy theorem the X -bundles $\tilde{K} \times_{\rho_1} X$ and $\tilde{K} \times_{\rho_2} X$ over K are isomorphic. In particular, the pullbacks to K of the vertical bundles $\tilde{K} \times_{\rho_1} TX$ and $\tilde{K} \times_{\rho_2} TX$ are isomorphic as desired.

Thus, if the analytic variety $\text{Hom}(\pi, \text{Isom}(X))$ has finitely many connected components, the set of the manifolds of the form $X/\rho(\pi)$, where ρ is discrete and faithful, falls into finitely many tangential equivalence classes. This is the case if π is finitely presented and X is a nonpositively curved symmetric space. Indeed, represent X as a Riemannian product $Y \times \mathbb{R}^k$ where Y is a nonpositively curved symmetric space without Euclidean factors. By the de Rham's theorem this decomposition is unique, so $\text{Isom}(X) \cong \text{Isom}(Y) \times \text{Isom}(\mathbb{R}^k)$. The group $\text{Isom}(Y)$ is semisimple with trivial center, hence the analytic variety $\text{Hom}(\pi, \text{Isom}(Y))$ has finitely many connected components [21, p.567]. The same is true for $\text{Hom}(\pi, \text{Isom}(\mathbb{R}^k))$ because $\text{Isom}(\mathbb{R}^k)$ is real algebraic [21, p.567]. Therefore the analytic variety

$$\text{Hom}(\pi, \text{Isom}(X)) \cong \text{Hom}(\pi, \text{Isom}(Y)) \times \text{Hom}(\pi, \text{Isom}(\mathbb{R}^k))$$

has finitely many connected components. q.e.d.

9.2. Theorem. *Let K be a finite, connected, aspherical CW-complex and let n be an integer with $n > \max\{4, 2 \dim(K)\}$.*

*Then the set of diffeomorphism classes of open n -thickenings of K that admit complete locally symmetric metrics of nonpositive sectional curvature is **finite**.*

Proof. Note that there exist only finitely many symmetric Hadamard manifolds of a given dimension. Hence, the result follows from 7.5 and 9.1. q.e.d.

9.3. Theorem. *Let K be a finite, connected, aspherical CW-complex and let X be a symmetric negatively curved Hadamard n -manifold with $n > \max\{5, 2 \dim(K)\}$.*

Let ρ_1 and ρ_2 be injective representations of $\pi_1(K)$ into the isometry group of X that lie in the same connected component of the representation variety $\text{Hom}(\pi_1(K), \text{Isom}(X))$. Suppose that, for $i \in \{1, 2\}$, $\rho_i(\pi_1(K))$ is a convex-cocompact group with simply-connected domain of discontinuity.

Then ρ_1 and ρ_2 are conjugate by a homeomorphism of $X \cup \partial_\infty X$

Proof. Recall that there are four kinds of symmetric spaces of negative sectional curvature, namely, they are hyperbolic spaces over the reals, complex numbers, quaternions and Cayley numbers. The spaces have sectional curvatures pinched between -4 and -1 . Applying 8.1 we conclude that $X \cup \Omega(\rho_i(\pi_1(K)))/\rho_i(\pi_1(K))$ is a thickening of K for $i = 1, 2$.

It follows from the proof of 9.1 that the homotopy equivalence induced by $\rho_1 \circ (\rho_2)^{-1}$ is tangential because ρ_1 and ρ_2 lie in the same connected component. Thus, by 7.4, the homotopy equivalence is homotopic to a diffeomorphism that lifts to a smooth conjugacy of ρ_1 and ρ_2 on $X \cup \Omega(\rho_1(\pi_1(K)))$ and $X \cup \Omega(\rho_2(\pi_1(K)))$. Repeating the argument of 8.2, we deduce that the conjugacy extends to an equivariant self-homeomorphism of $X \cup \partial_\infty X$. q.e.d.

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