J. DIFFERENTIAL GEOMETRY
50 (1998) 249-267

A FIXED POINT THEOREM OF DISCRETE GROUP ACTIONS ON RIEMANNIAN MANIFOLDS

MU-TAO WANG

Abstract

We prove a fixed point theorem for a class of discrete group acting on manifolds of nonpositive curvature by isometry. These discrete groups include cocompact lattices in simply connected semisimple p-adic groups of rank at least two and large p. Hence it gives a geometric generalization of Margulis' superrigidity theorem for the Archimedean representation of these groups.

1. Introduction

Let N be a complete simply connected Riemannian manifold of nonpositive sectional curvature. N has a natural compactification $\bar{N} = N \cup \partial N$ by the sphere at infinity which is defined as the equivalent classes of geodesic rays [1]. Any group action on N by isometry extends to an action on $\bar{N}$.

Definition 1.1. A group Γ is said to have property (F) if any isometric action of Γ on any complete simply connected manifold of nonpositive sectional curvature N has a fixed point in $\bar{N}$.

In term of representations of Γ, property (F) has the following interpretation. Let H be a simple noncompact Lie group with trivial center. If Γ has property (F), then any homomorphism $\rho : \Gamma \mapsto H$ with Zariski dense image is precompact in H. This is because the symmetric space associated with H has nonpositive curvature and the image being Zariski

Received October 30, 1997, and, in revised form, July 24, 1998.
Key words and phrases. buildings, harmonic maps, superrigidity

dense implies the action has no fixed point in the boundary. A classical theorem of Cartan states that every compact group has property (F).

We consider discrete groups that can be realized as fundamental groups of simplicial complexes satisfying certain combinatorial properties. Namely, in §2, we define a notion called "admissible weight" for simplicial complexes, and "local first eigenvalue" $\lambda_{1,loc}$ for such admissible weights. These can be viewed as combinatorial analogues of "metric" and "curvature" on Riemannian manifolds. Our main theorem is the following:

Theorem 1.1. *If Γ can be realized as $\pi_1(\Sigma)$ of a finite simplicial complex Σ with admissible weight c such that $\lambda_{1,loc}(\Sigma, c) > \frac{1}{2}$, then Γ has property (F).*

A special case of our theorem can be formulated more transparently. Let Σ be a two-dimensional simplicial complex. The link $Lk(v)$ of each vertex v is a graph. Let $\delta_{ij}(v)$ denotes the incidence matrix of $Lk(v)$ and $\deg_v(i)$ the degree (or valence) of the i -th vertex in $Lk(v)$. Then we have:

Theorem 1.2. *If each simplex of Σ is contained in some 2-simplex, and the smallest nonzero eigenvalue of $Id - \frac{1}{\deg_v(i)}\delta_{ij}(v)$ is greater than $\frac{1}{2}$ for each vertex v of Σ, then $\pi_1(\Sigma)$ has property (F).*

This condition is satisfied by many graphs including all complete graphs. In [16], we prove these groups have Kazhdan's property (T). After the work in [16] was completed, we were informed by Professor Margulis that Ballmann and Swiatkowski [2] also proved property (T) for these simplicial complexes (see also [15], [20]). Our formulation is different from theirs and is well-suited for the nonlinear situation considered here.

The local first eigenvalue is a generalization of "p-adic curvature" in [8], where Garland showed the inequality in Theorem (1.1) is satisfied by p-adic buildings under the assumption that the cardinality of the residue field of the defining field is large enough. Based on this, he deduced the vanishing theorem of cohomology of cocompact lattices of p-adic groups. Our theorem is in some sense a nonlinear version of Garland's. Along this line, our theorem can also be viewed as a geometric generalization of Margulis' superrigidity theorem for the Archimedean representation of cocompact lattices in semisimple p-adic groups of rank at least two except we could not get rid of the residue field assumption.

Theorem (Margulis) [12] [19]. *Let Γ be a discrete cocompact subgroup of a simply connected semisimple p-adic group G of rank at least two. Let H be a simple Lie group with trivial center. If $\rho : \Gamma \longmapsto H$ is a representation with Zariski dense image, then $\rho(\Gamma)$ is precompact in H.*

The theorem was proved by Margulis in the early 1970's using ergodic theory and linear algebraic groups theory. Actually he proved the theorem for lattices in both semisimple real Lie groups and p-adic groups of rank at least two, and superrigidity is true for Archimedean representations as well as non-Archimedean ones. It turns out that superrigidity is also true for lattices in semisimple Lie groups of rank 1 except for $SO(n, 1)$ and $SU(n, 1)$. In the Archimedean case, this was proved by Corlette [5] , while the other case was proved by Gromov and Schoen [9] later. Their methods involve vanishing theorems for harmonic maps from manifolds with special holonomy. Then Jost-Yau [11] and independently Mok-Siu-Yeung [14], gave the first geometric proofs of Margulis superrigidity theorem for lattices in simple Lie groups. Both groups of authors used harmonic maps, and their results were deduced from a Bochner type formula together with a Matsushima vanishing argument. This kind of argument was used by Calabi [4], Weil [18], Matsushima [13], etc. in proving the vanishing theorem of cohomology groups of locally symmetric spaces. In [8], Garland found a surprising formula for Bruhat-Tits buildings, which are p-adic analogues of Riemannian symmetric spaces, and applied this to prove his vanishing theorem.

The strategy we adopt here involves "harmonic maps" on simplicial complex . The present paper is organized as follows: §2 contains a definition of the local first eigenvalue for a simplicial complex with an admissible weight. In §3, we develop harmonic map theory for simplicial complexes and prove the main theorem. In §4, we show the local first eigenvalue condition was satisfied by a class of Bruhat-Tits buildings. §5, gives new examples having property (F) which are not cocompact lattice of $p-$adic groups.

I would like to thank Professor S.-T. Yau. Without his advice and encouragement, this paper would never have been completed.

2. The admissible weight and the first eigenvalue for Simplicial Complexes

In this section, we define "admissible weight" for a simplicial complex and "local first eigenvalue" for such weights. First we fix our notation. Let Σ be a simplicial complex of dimension l, $\Sigma(i)$ the set of i simplices of Σ, $C_i(\Sigma)$ the vector space over $\mathbf{R}$ of i chains of Σ, and $C^i(\Sigma)$ the vector space of i cochains. We write $\tau < \sigma$ if τ is a face of σ. The coboundary operator d on $C^i(\Sigma)$ and boundary operator ∂ on $C_i(\Sigma)$ are defined as usual. Also for any simplex v , $St(v)$ is the subcomplex formed by simplices containing v, and $Lk(v)$ is the union of the faces of $St(v)$ that do not meet with v.

Definition 2.1. An admissible weight is a positive function c on $\cup\Sigma(i)$ such that

$$\sum_{\sigma\in\Sigma(i+1),\tau<\sigma} c(\sigma) = c(\tau) \tag{2.1}$$

for any $\tau \in \Sigma(i)$. An admissible weight defines an inner product on the vector space $C^i(\Sigma)$ by $\langle f, g\rangle = \sum_{\tau\in\Sigma(i)} c(\tau)\, f(\tau)\, g(\tau)$.

We say two cochains f, g are perpendicular if $\langle f, g\rangle = 0$.

Example 2.1. Let $m \leq n \leq l$ and $\tilde{c}(\sigma)$ be the number of n-dimensional simplices containing σ. By a simple computation, we check that $\sum_{\sigma\in\Sigma(i+1),\tau<\sigma} \tilde{c}(\sigma) = (n-i)\tilde{c}(\tau)$ for any $\tau \in \Sigma(i)$ and $i+1 \leq m$. If $\tilde{c}(\sigma) \neq 0$ for all simplices σ of the m skeleton of Σ, and let $c(\sigma) = n(n-1)\cdots(n-i)\tilde{c}(\sigma)$, then it is not hard to check that c is an admissible weight on the m skeleton of Σ .

Example 2.2. If Σ is endowed with an admissible weight c, then c induces an admissible weight $c_v(\tau) := c(v*\tau)$ on each $Lk(v)$. Here $v*\tau$ denotes the join of v and τ. Condition (2.1) is verified by the following computation:

$$\sum_{\sigma\in Lk(v)(i+1),\tau<\sigma} c_v(\sigma) = \sum c(v*\sigma) = c(v*\tau) = c_v(\tau) \tag{2.2}$$

For each vertex v , we also define a "localizing operator" on $C^1(\Sigma)$:

$$\begin{aligned} (\cdot)_v \quad &: \quad C^1(\Sigma) \mapsto C^0(Lk(v)), \\ \omega_v(u) \quad &= \quad \omega(\langle u, v\rangle) \end{aligned}$$

for any $\omega \in C^1(\Sigma)$.

An admissible weight on Σ also gives the adjoint operator to d, denoted by δ :

$$\delta\omega(v) = \frac{1}{c(v)} \sum_{\tau \in \Sigma(i)} c(\tau)\,[\tau : v]\,\omega(\tau) \tag{2.3}$$

for any $v \in \Sigma(i-1)$, where $[\tau : v]$ is the incidence number of τ and v which is 0, 1, or -1 depending on the orientation of τ and v .

In particular, if $\omega \in C^1(\Sigma)$, then

$$d\omega(v) = \frac{1}{c(v)} \sum_{u \in Lk(v)(o)} c_v(\tau)\omega_v(\tau) \tag{2.4}$$

for any v $\in \Sigma(0)$.

We also define the Laplace operator $\triangle = d\delta + \delta d$, and there are two invariants associated with the Laplace operator:

Definition 2.2. For an admissible weight c on Σ , we define the global first eigenvalue, denoted by $\lambda_1(\Sigma, c)$, to be the infimum of $\langle df, df\rangle / \langle f, f\rangle$ over all $f \in C^0(\Sigma)$ satisfying $\sum_{v\in\Sigma(0)} c(v)\, f(v) = 0$.

Definition 2.3. The local first eigenvalue of (Σ, c) denoted by $\lambda_{1,loc}(\Sigma, c)$ is defined to be the infimum of $\lambda_1(Lk(v), c_v)$ over all $v \in \Sigma(0)$.

These $\lambda_1(Lk(v), c_v)$'s are always nonnegative numbers. We remark that if Σ is finite and each $Lk(v)$ is connected, then $\lambda_{1,loc}(\Sigma, c)$ is positive.

For any inner product space V, we also consider $C^i(\Sigma, V) := C^i(\Sigma) \otimes V$ with the product metric. d and δ can be extended to operators on $C^i(\Sigma, V)$. We prove some lemma which will be used in the vanishing theorem for harmonic maps.

Lemma 2.1. *If Σ is finite, and c is an admissible weight on Σ, then for any $\omega \in C^1(\Sigma, V)$, we have*

$$\lambda_{1,loc}\left(2\,\|\omega\|^2 - \|\delta\omega\|^2\right) \leq \sum_{v\in\Sigma(0)} \|d\omega_v\|^2_{Lk(v)}\,. \tag{2.5}$$

Proof. By definition, we have

$$\langle d\omega_v, d\omega_v\rangle_{Lk(v)} = \sum_{\tau\in Lk(v)(1)} c_v(\tau)\langle d\omega_v(\tau), d\omega_v(\tau)\rangle .$$

From the definition of admissible weights it follows that

$$\sum_{u\in Lk(v)(o)} c_v(u) = \sum_{\tau\in\Sigma(1), v<\tau} c(\tau) = c(v).$$

Therefore the definition of $\delta\omega$ yields

$$\sum_{u\in Lk(v)(o)} c_v(u)(\omega_v(u) - \delta\omega(v)) = 0.$$

That is, $\omega_v - \delta\omega(v)$, as an element in $C^0(Lk(v))$, is perpendicular to constant cochains. By the definition of $\lambda_{1,loc}$ and notice that $d(\omega_v - \delta\omega(v)) = d\omega_v$, we obtain

$$\begin{aligned}&\sum_v \|d\omega_v\|^2_{Lk(v)}\\ &\qquad\geq \lambda_{1,loc}\sum_v \|\omega_v - \delta\omega(v)\|^2_{Lk(v)}\\ &\qquad= \lambda_{1,loc}(\sum_v \langle\omega_v,\omega_v\rangle_{Lk(v)} - \langle\omega_v, \delta\omega(v)\rangle_{Lk(v)}),\end{aligned}$$

where the equality follows from $\langle\omega_v - \delta\omega(v), \delta\omega(v)\rangle_{Lk(v)} = 0$. However, the first term $\sum_v \langle\omega_v,\omega_v\rangle_{Lk(v)} = 2\|\omega\|^2$ because each edge contains two vertices. While the second term,

$$\begin{aligned}&\sum_v \langle\omega_v, \delta\omega(v)\rangle_{Lk(v)}\\ &\qquad= \sum_v \delta\omega(v)\sum_u (c_v(u)\,\omega_v(u))\\ &\qquad= \sum_v \delta\omega(v)\, c(v)\,\delta\omega(v)\end{aligned}$$

by the definition of $\delta\omega$ again. The last term is $\|\delta\omega\|^2$ and the lemma is proved.

Lemma 2.2. For (Σ, c) *as above, if* $\omega \in C^1(\Sigma, V)$ *satisfies:*

$$\|\omega(\tau_1) - \omega(\tau_2)\| \leq \|\omega(\tau_3)\|, \tag{2.6}$$

whenever there exists an oriented two-simplex σ *with* $\partial\sigma = \tau_1 - \tau_2 + \tau_3$, *then*

$$\sum_{v \in \Sigma(0)} \|d\omega_v\|^2_{Lk(v)} \leq \|\omega\|^2. \tag{2.7}$$

Proof. For any $v \in \Sigma(0)$, squaring both sides of (2.6) and summing up over $Lk(v)(1)$ lead to

$$\begin{aligned} &\sum_{\langle u,w\rangle \in Lk(v)(1)} c(\langle v,u,w\rangle) \|\omega(\langle v,u\rangle) - \omega(\langle v,w\rangle)\|^2 \\ &\quad \leq \sum_{\langle u,w\rangle \in Lk(v)(1)} c(\langle v,u,w\rangle) \|\omega(\langle u,w\rangle)\|^2. \end{aligned} \tag{2.8}$$

Now summing up (2.8) over all $v \in \Sigma(0)$, we get,

$$\begin{aligned} &\sum_{v \in \Sigma(0)} \|d\omega_v\|^2_{Lk(v)} \\ &\quad \leq \sum_{v \in \Sigma(0)} \sum_{\langle u,w\rangle \in Lk(v)(1)} c(\langle v,u,w\rangle) \|\omega(\langle u,w\rangle)\|^2, \end{aligned} \tag{2.9}$$

which is

$$\begin{aligned} &\sum_{v \in \Sigma(0)} \sum_{\tau \in Lk(v)(1)} c(\langle v * \tau\rangle) \|\omega(\tau)\|^2 \\ &\qquad = \sum_{\tau \in \Sigma(1)} \Big(\sum_{v \in Lk(\tau)(0)} c(\langle v * \tau\rangle)\Big) \|\omega(\tau)\|^2 \end{aligned}$$

By the definition of admissible weights,

$$\sum_{v \in Lk(\tau)(0)} c(\langle v * \tau\rangle) = c(\tau).$$

Thus the last term is $\|\omega\|^2$, and the lemma is proved.

The relation between the global and local first eigenvalues lies in the following theorem:

Theorem 2.3. *If Σ is finite and connected, and c is an admissible weight on Σ with $\lambda_{1,loc}(\Sigma, c)$ positive, then*

$$\lambda_1(\Sigma, c) \geq 2 - \frac{1}{\lambda_{1,loc}(\Sigma, c)\,.}. \tag{2.10}$$

Proof. Let $\lambda = \lambda_1(\Sigma, c)$, and h be the corresponding eigencochain, i.e., $h \in C^0(\Sigma)$ and $\delta dh = \lambda h$. Denote dh by ω. Then $d\omega = o$. Hence (2.6) is true. Combining Lemma 2.1 and Lemma 2.2 gives

$$\lambda_{1,loc}(2\|\omega\|^2 - \|\lambda h\|^2) \leq \|\omega\|^2\,. \tag{2.11}$$

Using the facts $\delta dh = \lambda h$ and δ is adjoint to d , it is easy to check $\|\lambda h\|^2 = \lambda \|\omega\|^2$. Substituting these in (2.11), we get

$$\lambda_{1,loc}(2\|\omega\|^2 - \lambda\|\omega\|^2) \leq \|\omega\|^2\,,$$

and the result follows.

3. Harmonic maps on simplicial complexes

Let Σ be a finite dimensional finite simplicial complex equipped with an admissible weight c and $\Gamma = \pi_1(\Sigma)$. Let N be a complete simply connected Riemannian manifold of nonpositive sectional curvature, and $\rho : \Gamma \mapsto I(N)$ a homomorphism into the isometry group of N. We consider the class of maps

$$\Xi = \left\{ f : \tilde{\Sigma}(0) \mapsto N, f \text{ is equivariant with respect to } \rho \right\}.$$

By equivariance, which means that for any $v \in \tilde{\Sigma}(0)$ and $\gamma \in \Gamma$, $f(\gamma(v)) = \rho(\gamma) f(v)$, any $f \in \Xi$ is determined by the restriction to a fundamental domain of Γ on $\tilde{\Sigma}$. Therefore Ξ can be identified with m copies of N, where m is the number of vertices in Σ. We define the energy of $f \in \Xi$ to be

$$E(f) = \sum_{\langle v,u\rangle \in \Sigma(1)} c\left(\langle v,u\rangle\right) d_N^2\left(f\left(v\right), f\left(u\right)\right).$$

Definition 3.1. f is called a harmonic map if f is a critical point of the functional E.

Before proving the existence of harmonic maps, we recall the definition of simplicial distance. Given any two vertices $u, v \in \tilde{\Sigma}$, the simplicial distance $d_{\tilde{\Sigma}}(v,u)$ is defined to be the minimum of d such that there exists a sequence of vertices $v = v_0, v_1, \cdots, v_d = u$ with $\langle v_i, v_{i+1}\rangle \in \tilde{\Sigma}(1)$. Suppose the simplicial distance between v and $\gamma(v)$ is realized by a path $v = v_0, v_1 \cdots v_d = \gamma(v)$. Then

$$\begin{aligned} d_N\left(f\left(v\right), f\left(\gamma\left(v\right)\right)\right) &\leq \sum_{j=0}^{d-1} d_N\left(f\left(v_j\right), f\left(v_{j+1}\right)\right) \\ &\leq c_1 \sum_{j=0}^{d-1} \sqrt{E\left(f\right)} \\ &\leq c_1 \sqrt{E\left(f\right)} d_{\tilde{\Sigma}}\left(v, \gamma\left(v\right)\right), \end{aligned}$$

where c_1 is the maximum of $\frac{1}{c(\tau)}$ for all $\tau \in \Sigma(1)$.

Theorem 3.1. *If the action of* Γ *does not fix any point on the sphere at infinity of* N, *then the harmonic map exists in* Ξ.

Proof. We think of E as a function on N^m and we are going to prove E is proper and convex. Then E has a minimum which is a harmonic map. Convexity follows from the fact that the distance function on nonpositively curved space is convex . Suppose E is not proper. Then there exists a sequence $\{f_i\} \to \infty$ in $\Xi = N^m$ with $E(f_i) < K$, for some constant K. In particular, there exists a vertex v such that $f_i(v) \to \infty$ in N. By the above computation, we see that

$$\begin{aligned} d_N\left(f_i\left(v\right), \gamma\left(f_i\left(v\right)\right)\right) =& d_N\left(f_i\left(v\right), f_i\left(\gamma\left(v\right)\right)\right) \\ \leq& c_1\sqrt{K} d_{\tilde{\Sigma}}\left(v, \gamma\left(v\right)\right). \end{aligned} \tag{3.1}$$

The right-hand side is independent of i, therefore $f_i(v)$ and $\gamma(f_i(v))$ define the same limit in ∂N. Since this is true for all $\gamma \in \Gamma$, lim $f_i(v)$ is a fixed point in ∂N, a contradiction.

For a harmonic map, we define $f(\tau)$ to be the unique minimal geodesic segment joining v and u if $\tau = \langle v, u\rangle$

Definition 3.2. The differential of f at v_0 is defined to be :

$$Df|_{v_0} : Lk(v_0)\,(0) \mapsto T_{f(v_0)}N$$

such that $exp\,(Df|_{v_0}\,(v_1)) = f\,(v_1)$, where exp is the exponential map on $T_{f(v_0)}N$.

In particular, we have$\|Df|_{v_0}\,(v_1)\| = \|Df|_{v_1}\,(v_0)\| = d(f\,(v_0)\,, f\,(v_1))$ if $\tau = \langle v_0, v_1\rangle$.

Let us derive the harmonic map equation in our context.

Lemma 3.2. *The differential of a harmonic map f satisfies the following weighted center of mass equation:*

$$\sum_{u \in Lk(v_0)(0)} c_{v_0}(u) Df|_{v_0}\,(u) = 0 \tag{3.2}$$

for all $v_0 \in \Sigma\,(0)$.

Proof. We consider the variation at v_0 given by $X \in T_{f(v_0)}N$:

$$\frac{d}{dt}\left(\sum_{u \in Lk(v_0)(0)} c\,(\langle v_0, u\rangle)\, d^2\,(\exp\,(tX)\,, f\,(u))\right) = \sum_{u \in Lk(v_0)(0)} c\,(\langle v_0, u\rangle)\,\langle X, 2Df|_{v_0}\,(u)\rangle_{T_{f(v_0)}N}\,;$$

the equality holds since $f(\langle v_0, u\rangle)$ is a minimal geodesic,

$$\nabla d^2\,(\cdot, u)\,|_{f(v_0)} = 2Df|_{v_0}\,(u)\,.$$

Since f is a critical point, this is zero for all X. Hence the equation follows.

Now we proceed to prove our main theorem:

Theorem 1.1. *If (Σ, c) has $\lambda_{1,loc} > \frac{1}{2}$, then $\pi_1\,(\Sigma)$ has property (F).*

Proof. Given any N and isometry action of Γ on N, we may assume there is no fixed point in ∂N, otherwise we are done. Therefore the harmonic map f exists and we are going to prove that this harmonic map is actually a constant map, so that its image is the desired fixed point. Fixing a $v_0 \in \Sigma(0)$, by the nonpositive curvature assumption on N, we have

$$\|Df|_{v_0}(u) - Df|_{v_0}(w)\| \leq d(f(u), f(w))$$

for any $\langle u, w \rangle \in Lk(v_0)(1)$. Thus

$$\begin{aligned} &\sum_{\langle u,w\rangle \in Lk(v_0)(1)} c(\langle v_0, u, w\rangle) \|Df|_{v_0}(u) - Df|_{v_0}(w)\|^2 \\ &\quad \leq \sum_{\langle u,w\rangle \in Lk(v_0)(1)} c(\langle v_0, u, w\rangle)\, d^2(f(u), f(w)). \end{aligned}$$

Denoting the left hand side by $\|d(Df)_{v_0}\|^2$ and summing over all $v_0 \in \Sigma(0)$ yield

$$\begin{aligned} &\sum_{v_0} \left\|d(Df)_{v_0}\right\|^2 \\ &\quad \leq \sum_{v_0} \sum_{\langle u,w\rangle \in Lk(v_0)(1)} c(\langle v_0, u, w\rangle)\, d^2(f(u), f(w)) = E(f). \end{aligned} \tag{3.3}$$

On the other hand, since $\sum_{u \in Lk(v_0)(0)} c_{v_0}(u) Df|_{v_0}(u) = 0$, $Df|_{v_0}$ as an element in $C^0(Lk(v_0))$ is perpendicular to constant cochains, so

$$\begin{aligned} &\sum_{\langle u,w\rangle \in Lk(v_0)(1)} c(\langle v_0, u, w\rangle) \|Df|_{v_0}(u) - Df|_{v_0}(w)\|^2 \\ &\quad \geq \lambda_1(Lk(v_0)) c(\langle v_0, u\rangle) \|Df|_{v_0}(u)\|^2. \end{aligned}$$

Summing over all vertices v_0 again, we get

$$\sum_{v_0} \left\|d(Df)_{v_0}\right\|^2 \geq \lambda_{1,loc} \cdot 2E(f).$$

By (3.3) and the assumption $\lambda_{1,loc} > \frac{1}{2}$, this cannot happen unless $E(f) = 0$, i.e., f is a constant map.

4. A fixed point theorem for cocompact lattices of p-adic groups

In this section, k denotes a non-archimedean completion of either an algebraic number field or an algebraic function field in one variable over a finite field.

Let G be the group of k-rational point of a simply connected algebraic group which is defined and simple over k with k rank $l+1$, and Γ be a discrete torsion free cocompact subgroup of G. Γ acts freely on the Euclidean building $\tilde{X}$ associated to G. $\tilde{X}$ is a contractible locally finite simplicial complex of dimension l, hence Γ is the fundamental group of $X = \Gamma \backslash \tilde{X}$.

There is a natural weight function on a building. Let Σ be a l dimensional building, which may be of Euclidean or spherical type. For a simplex σ, let $\tilde{c}_\Sigma(\sigma)$ be the number of l dimensional simplices of Σ having σ as a face. Let $c_\Sigma(\sigma) = l \cdot (l-1) \cdots (l-i)\tilde{c}_\Sigma(\sigma)$ if $\sigma \in \Sigma(i+1)$. Then c_Σ is an admissible weight.

Definition 4.1. $c_\Sigma(\langle x, y\rangle)$ is called the canonical admissible weight of Σ.

Given any vertex x of Σ. On $Lk(x)$ there is a function $c_{\Sigma,x}$ defined on the set of simplices by $c_{\Sigma,x}(\tau) := c_\Sigma(x * \tau)$. $c_{\Sigma,x}$ is called the induced weight on $Lk(x)$.

Now $Lk(x)$ is isomorphic to another spherical building Σ' of dimension $l-1$. Let $\Phi : Lk(x)(0) \mapsto \Sigma'(0)$ be the isomorphism between the vertices of them. It is not hard to see

$$c_{\Sigma,x}(\sigma) = c_{\Sigma'}(\Phi(\sigma)). \tag{4.1}$$

Actually since $c_{\Sigma,x}$ and $c_{\Sigma'} \circ \Phi$ agree on any $(l-1)$-dimensional simplices and satisfy the same additive law (2.1) for admissible weights, by induction they must equal on all simplices. Therefore, we have the following:

Lemma 4.1. *The first eigenvalue of $Lk(x)$ with respect to the induced weight $c_{\Sigma,x}$ from Σ is the same as that of Σ' with respect to the canonical weight, i.e., $\lambda_1(\Sigma', c_{\Sigma'}) = \lambda_1(Lk(x), c_{\Sigma,x})$.*

Lemma 4.2. *For any spherical building Σ of dimension greater than or equal to two, we have $\lambda_1(\Sigma, c_\Sigma) > \frac{1}{2}$ if the cardinality of the residue field of k is large enough.*

Proof. By the induction on the dimension of Σ, we will show that $\lambda_1(\Sigma, c_\Sigma) > 1 - \epsilon$ if the cardinality of the residue field of k is large enough. If $l = 2$, this is a theorem of Feit-Higman [7] and Garland [8]. Suppose the theorem is true for all spherical buildings of dimensional $l - 1$. In particular, this is true for any $Lk(x)$, i.e.,

$$\lambda_1(\Sigma', c_{\Sigma'}) = \lambda_1(Lk(x), c_{\Sigma,x}) > 1 - \epsilon', \tag{4.2}$$

if the cardinality of the residue field of k is large enough. By (2.10), we have

$$\lambda_1(\Sigma, c_\Sigma) \geq 2 - \frac{1}{1-\epsilon'} = 1 - \frac{\epsilon'}{1-\epsilon'},$$

and the lemma is proved.

We are ready to prove the main theorem in this section.

Theorem 4.3. *For any integer $l > 1$, there is an integer M such that if k has residue field of cardinality at least M, G is the group of $k-$rational point of a simply connected algebraic group which is defined and simple over k with k rank $l + 1$, and Γ is a discrete cocompact subgroup of G, then Γ has property (F).*

Proof. Γ always has a finite index torsion free normal subgroup Γ' by the theorem of Garland [8]. Suppose this theorem is true for Γ'. The action of the finite group Γ/Γ' has a fixed point by Cartan's fixed point theorem. Therefore this theorem is also true for Γ. Hence we may assume Γ is torsion free. According to the vanishing theorem in the previous section, it suffices to show there is an admissible weight on Σ with $\lambda_{1,loc} > \frac{1}{2}$. By the previous lemma, the theorem is proved.

5. Examples other than p-adic buildings

We show some 2-dimensional simplicial complexes which come from " surgery" of the quotient of p-adic building. Their fundamental groups also satisfy property (F). Let Σ be a 2-dimensional finite simplicial complex. We are going to take the admissible weight in Example 2.1 when $m = n = l = 2$. We notice that the induced weight on each $Lk(v)$ is just the degree (or valency) of each vertex of $Lk(v)$ as a graph. A special case of our main theorem is the following:

Corollary 5.1. *If $c(\tau) > 0$ for all τ and the first eigenvalue of each link is greater than $\frac{1}{2}$, then $\pi_1(\Sigma)$ satisfies property (F).*

We recall our definition of first eigenvalue for a graph with admissible weight. Let the vertices be indexed by $i = 1 \cdots n$, δ_{ij} be the incidence matrix, i.e., $\delta_{ij} = 1$ if vertex i and j are joined by an edge, $\delta_{ij} = 0$ otherwise. A zero cochain is just a sequence $a = \{a_i\}$. Then

$$\lambda_1 = \min \frac{\frac{1}{2}\sum \delta_{ij}\,(a_i - a_j)^2}{\sum \deg(i) a_i^2},$$

where the minimum is taken over all $\{a_i\}$ with $\sum \deg(i) a_i = 0$. Our Laplacian can be computed in the following way:

$$\begin{aligned} \langle \Delta a, a \rangle &= \frac{1}{2}\sum \delta_{ij}\,(a_i - a_j)^2 \\ &= \sum \deg(i) a_i^2 - a_i \sum \delta_{ij} a_j \\ &= \sum \deg(i) a_i \cdot \left(a_i - \frac{1}{\deg(i)} \sum \delta_{ij} a_j \right). \end{aligned}$$

Therefore,

$$(\Delta a)_i = a_i - \frac{1}{\deg(i)} \sum \delta_{ij} a_j.$$

We are going to construct our examples from the Euclidean building $\tilde{\Sigma}$ associated with $PGL(3,\, \mathbf{Q}_p)$. For materials on building, we refer to [3]. $\tilde{\Sigma}$ is a two-dimensional simplicial complex, and the links are isomorphic to the spherical building $\mathcal{G}$ associated to $PGL(3, \mathbf{Z}/p\mathbf{Z})$. $\mathcal{G}$ is a finite homogeneous biparte graph with degree $p+1$. The vertices of $\mathcal{G}$ are parametrized by the p^2+p+1 points and p^2+p+1 lines in the projective plane over $\mathbf{Z}/p\mathbf{Z}$, the edges are determined by the incidence relations, and $PGL(3, \mathbf{Z}/p\mathbf{Z})$ acts on $\mathcal{G}$ through the projective action. Each chamber complex of $\mathcal{G}$ is a hexagon and any two vertices are contained in some chamber. A picture of $\mathcal{G}$ can be found on page 82 of [3] when $p = 2$, and in [10] when $p = 3$.

The first eigenvalues can be easily computed using Hecke operators, see [8], and the space of zero cochains which is isomorphic to $\mathbf{R}^{2(p^2+p+1)}$ has the following spectral decomposition $C^0(\mathcal{G}) = I \oplus P \oplus E_1 \oplus E_2$, where

I corresponds the one-dimension constant cochain with eigenvalue zero, P corresponds to the one-dimension parity cochain with eigenvalue two, and E_1, E_2 are each p^2+p dimension eigenspaces with eigenvalues $1+\frac{\sqrt{p}}{p+1}$ and $1-\frac{\sqrt{p}}{p+1}$, respectively. Now we consider a torsion free cocompact lattice Γ in $PGL(3,\, \mathbf{Q}_p)$. Let $\Sigma = \Gamma \setminus \tilde{\Sigma}$ be the quotient. Our new example Σ' is formed by taking away a 1-simplex τ and those 2-simplices containing τ from Σ. This has two possible effects on the links of a vertex v : either τ contains v, and $Lk(v)$ loses one vertex and $p+1$ edges containing it, or τ lies in $Lk(v)$ and $Lk(v)$ loses one edge, we called the resulting graphs type (I) and (II).

Proposition 5.2. *Both type (I) and (II) have first eigenvalue greater than $\frac{1}{2}$ when p is large.*

Proof. Type (I): The new graph is denoted by $\mathcal{G}'$ and has $2p^2+2p+1$ vertices. We assume the vertex which we take away is v_0, and the those adjacent to it are $v_1 \cdots v_{p+1}$. We may also assume v_0 corresponds to a line. $C^0(\mathcal{G}')$ has dimension $2p^2+2p+1$, and is canonically embedded in $C^0(\mathcal{G})$. Let W be the codimension $p+1$ linear subspace in $C^0(\mathcal{G})$ defined by the equations

$$x_0 = \frac{1}{p} \sum_{\delta_{ki}=1, k\neq 0} x_k, \quad \text{for } \ i = 1, \cdots, p+1. \tag{5.1}$$

Let $E_1' = \Pi\left(W \cap E_1\right)$, where Π is the projection map from $C^0(\mathcal{G})$ to $C^0(\mathcal{G}')$ by forgetting the value at v_0. We claim E_1' is an eigenspace of the Laplacian on $\mathcal{G}'$ of eigenvalue $\lambda = 1+\frac{\sqrt{p}}{p+1}$. Suppose $a \in E_1'$. Then $a_i - \frac{1}{p+1}\sum_{\delta_{ij}=1} a_j = \lambda a_i$ for all i and $a_0 = \frac{1}{p}\sum_{\delta_{ij}=1, j\neq 0} a_j$, for $i = 1, \cdots, p+1$, which implies $\frac{1}{p+1}\sum_{\delta_{ij}=1} a_j = \frac{1}{p}\sum_{\delta_{ij}=1,j\neq 0} a_j$. In particular, for $i = 1, \cdots, p+1$, $a_i - \frac{1}{p}\sum_{\delta_{ij}=1,j\neq 0} a_j = \lambda a_i$. i.e., a is an eigencochain of the Laplacian on $\mathcal{G}'$ with the same eigenvalue. Likewise $E_2' = \Pi\left(W \cap E_2\right)$ is an eigenspace of eigenvalue$1-\frac{\sqrt{p}}{p+1}$. In $C^0(\mathcal{G}')$, the constant cochain and parity cochain are still eigencochains of eigenvalues 0 and 2, respectively. Besides, we have an eigencochain which is defined to be $a_i = 0$, if i is a line, $a_i = -p$ if v_i is a adjacent to v_0 , and $a_i = 1$ otherwise. It is easy to check that this has eigenvalue 1. So far we have demonstrated linearly independent eigencochains which span a subspace of codimension at most $2p$ in $C^0(\mathcal{G}')$. We need a lemma to find the complement to this.

Lemma 5.3. *Given any a_i, $i = 1, \cdots, p+1$ such that $\sum a_i = 0$ and a_i not all zero, there exist exactly two eigencochains with the value a_i at v_i, and the corresponding eigenvalues are $1 + \frac{1}{\sqrt{p+1}}$ and $1 - \frac{1}{\sqrt{p+1}}$ respectively.*

Proof. We set for $k \neq 0$,

$$\begin{aligned} a_k &= (1-\lambda)a_i \quad \text{if } \delta_{ik} = 0 \quad \text{for some } \ i, i = 1, \cdots, p+1 \\ &= 0, \text{ otherwise.} \end{aligned}$$

Then since $a_i - \frac{1}{p} \sum_{\delta_{ki}=1, k\neq 0} a_k = a_i - (1-\lambda)a_i = \lambda a_i$, and the equation is satisfied at v_i, for $i = 1, \cdots, p+1$. For those v_k such that $\delta_{ik} = 0$ for some, $i, 1 < i < p+1$, one of its neighbor is v_i, and none of its other neighbor can be adjacent to another v_j with $1 < j < p+1$ because chamber complexes are hexagons. Therefore the equation at these v_k are

$$(1-\lambda)a_i - \frac{a_i}{p+1} = \lambda\,(1-\lambda)\,a_i.$$

Since at least one of the a_i is nonzero , we can solve for $\lambda = 1 + \frac{1}{\sqrt{p+1}}$ or $1 - \frac{1}{\sqrt{p+1}}$. For the rest vertices, their neighbors consist of v_i' , $i = 1, \cdots, p+1$ such that v_i' is adjacent to v_i. Since $\sum a_i = 0$, it is easy to see the eigencochain equation is also satisfied at these vertices.

Those eigencochains constructed in the lemma has dimension $2p$. By dimension counting, we get all the eigencochains in $C^0(\mathcal{G}')$, and their eigenvalues are all greater than $\frac{1}{2}$. q.e.d.

Proof. Type (II) : The new graph $\mathcal{G}''$ has $2p^2 + 2p + 2$ vertices and $C^0(\mathcal{G})$ and $C^0(\mathcal{G}'')$ are canonically identified. We assume the edge we take away has end vertices v_0 and v_1 . Let W now be the codimension 2 subspace defined by $x_0 = \frac{1}{p} \sum_{\delta_{1k}=1} x_k$ and $x_1 = \frac{1}{p} \sum_{\delta_{0k}=1} x_k$. We claim now $E_1' = W \cap E_1$ is an eigenspace of the Laplacian on $\mathcal{G}''$ of eigenvalue $\lambda = 1 + \frac{\sqrt{p}}{p+1}$. The verification is similar to the calculation in type I and we omit it here. We also have the subspace E_2' of eigenvalue $1 + \frac{\sqrt{p}}{p+1}$. Constant cochains and parity cochains survive too. Our task is to demonstrate a four-dimensional subspace of different eigenvalues. This is achieved by the next lemma.

Lemma 5.4. *Given a_0 and a_1 such that $a_0 - a_1 = 0$ (respectively $a_0 + a_1 = 0$), with $a_0 \neq 0$, there exist exactly two eigencochains with the prescribed values. Their eigenvalues are $1 - \left(\frac{-1+\sqrt{1+4p}}{2(p+1)}\right)$ or $1 - \left(\frac{-1-\sqrt{1+4p}}{2(p+1)}\right)$ (respectively $1 - \left(\frac{1+\sqrt{1+4p}}{2(p+1)}\right)$ or $1 - \left(\frac{1-\sqrt{1+4p}}{2(p+1)}\right)$).*

Proof. We set

$$\begin{aligned} a_k &= (1-\lambda)a_0 \text{ if } \delta_{0k} = 1 \\ &= (1-\lambda)a_1 \text{ if } \delta_{1k} = 1 \\ &= \left[(p+1)(1-\lambda)^2 - 1\right]\frac{a_0}{p} \text{ if } v_k \text{ is adjacent to some } v_l \text{ with } \delta_{0l} = 1, \\ \text{or} \quad &= \left[(p+1)(1-\lambda)^2 - 1\right]\frac{a_1}{p} \text{ if } v_k \text{ is adjacent to some } v_l \text{ with } \delta_{1l} = 1. \end{aligned}$$

The equation satisfied by λ is the following

$$\left[(p+1)(1-\lambda)^2 - 1\right]\frac{a_1}{p} - \frac{1}{p+1}\left\{(1-\lambda)a_1 + \left[(p+1)(1-\lambda)^2 - 1\right]a_0\right\} = \frac{\lambda a_1}{p}\left[(p+1)(1-\lambda)^2 - 1\right].$$

We plug in $a_0 = a_1$ to get

$$(p+1)^2(1-\lambda)^3 - p(p+1)(1-\lambda)^2 - (2p+1)(1-\lambda) + p = 0$$

Solving for λ yields $\lambda = 0, 1 - \left(\frac{-1+\sqrt{1+4p}}{2(p+1)}\right)$, $1 - \left(\frac{-1-\sqrt{1+4p}}{2(p+1)}\right)$. We throw away $\lambda = 0$ since it corresponds to constant eigencochains. As for $a_0 = -a_1$, we obtain

$$(p+1)^2(1-\lambda)^3 + p(p+1)(1-\lambda)^2 - (2p+1)(1-\lambda) - p = 0$$

In this case $\lambda = 2, 1 - \left(\frac{1+\sqrt{1+4p}}{2(p+1)}\right)$, $1 - \left(\frac{1-\sqrt{1+4p}}{2(p+1)}\right)$, and we throw away $\lambda = 2$ as it corresponds to parity eigencochains. The lemma is proved.

q.e.d.

In [17], we show this is true for any buildings, i.e., $\lambda_1 > \frac{1}{2}$ is preserved under type (I) and (II) surgeries if p is large enough.

References

[1] W. Ballmann, M. Gromov & V. Schroeder, *Manifolds of Nonpositive Curvature*, Birkhaüser, Basel, 1985.

[2] W. Ballmann & J. Swiatkowski, *On L^2-cohomology and property (T) for automorphism groups of polyhedral cell complexes*, Geom. Funct. Anal. **7** (1997) 615-645.

[3] K. Brown, *Buildings*, Springer, Berlin, 1989.

[4] E. Calabi, & E. Vesentini,*On compact, locally symmetric Kahler manifolds,* Ann. Math. **71** (1960) 472-507.

[5] K. Corlette, *Flat G-bundles with canonical metrics*, J. Differential Geom. **28** (1988) 361-382.

[6] ______,*Archimedean superrigidity and hyperbolic geometry*, Ann. of Math. **135** (1992) 165-182.

[7] W. Feit & G. Higman, *The nonexistence of certain generalized polygons*, J. Alg. **1** (1964) 114-131.

[8] H. Garland, *p-adic curvature and the cohomology of discrete subgroups of p-adic groups*, Ann. of Math. **97** (1973) 375-423.

[9] M. Gromov & R. Schoen, *Harmonic maps into singular spaces and p-adic superrigidity for lattices in groups of rank one*, Inst. Hautes Études Sci. Publ. Math. **76** (1992) 165-246.

[10] T. Hattori, *Geometry of quotient spaces of $SO(3)\backslash SL(3,R)$ by congruence subgroups*, Math. Ann. **293** (1992) 443-467.

[11] J. Jost & S-T. Yau, *Harmonic maps and superrigidity*, Proc. Sympos. Pure Math., Amer. Math. Soc., Vol. 54, Part I, 1993, 245-280.

[12] G. A. Margulis, *Discrete subgroups of semisimple Lie groups*, Springer, Berlin, 1991.

[13] Y. Matsushima, *On the first Betti number of compact quotient spaces of higher dimensional symmetric spaces*, Ann. of Math. **75** (1962) 312-330.

[14] N. Mok, Y.T. Siu & S. K. Yeung, *Geometric superrigidity*, Invent. Math. **113** (1) (1993) 57-83.

[15] P. Pansu, *Formule de Matsushima, de Garland, et propriete (T) pour des groupes agissant sur des espaces symetriques ou des immeubles*, Preprint, Orsay, 1996.

[16] M. Wang, *Fixed point theorems and property (T)*, Preprint, 1997.

[17] ______, *On a generalization of the first eigenvalue of a graph*, Preprint, 1998.

[18] A. Weil, *Discrete subgroups of Lie groups II*, Ann. of Math. **75** (1962) 578-602.

[19] R. Zimmer, *Ergodic theory and semisimple groups*, Birkhaüser, Berlin, 1984.

[20] A. Zuk, *La propriete (T) de Kazhdan pour les groupes agissant sur les polyedres*, C. R. Acad. Sci. Paris, Ser. I, **323** 1996, 453-458.

Harvard University