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INTEGRAL CURVATURE BOUNDS, DISTANCE ESTIMATES AND APPLICATIONS

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Abstract

In this paper we show how the classical diameter bound for manifolds with positive Ricci curvature generalizes to a situation where one has integral curvature bounds. In addition we also generalize the P. Levy isoperimetric constant to this situation. This gives generalizations of work of Lichnerowicz, Cheng, Croke, Gallot, Colding and more.

1. Introduction

We shall in this paper generalize the classical diameter bound for manifolds with positive curvature to a situation where one only has integral curvature bounds. The history of this problem is briefly as follows. In 1855, Bonnet in [6] showed that a surface with curvature ≥ 1 has the property that no geodesic of length $> \pi$ can be length minimizing. Given that the surface is complete, one then obtains the standard diameter bound. In 1925 Synge generalized this to arbitrary Riemannian manifolds of sectional curvature ≥ 1. However, he neglected to point out the consequence for the diameter (see [40]). He did do this later in 1935 ([41]), but only after Myers in the same year published a paper on this (see [28] and also [38]). In the meantime Hopf and Rinow had also pointed out this diameter bound in their paper on completeness of surfaces from 1931, but only for abstract surfaces ([23]). In 1941 Myers then generalized these results to the now standard situation where

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$\text{Ric} \geq (n-1)$ ([29]). There have been several subsequent generalizations of this result, among which we point out those of Ambrose, Calabi, Avez, Markvorsen, Galloway, Cheeger-Gromov-Taylor, and Itokawa, to such situations as when one assumes positivity for the integral of the Ricci curvature along all geodesics (see [1], [7], [2], [27], [19], [10], [25]). Closer to the spirit of what we are studying here, J.-Y. Wu showed in [42] that if one assumes that $\text{Ric} \geq (n-1)$ away from a small metric ε-ball on which one has $\text{Ric} \geq -(n-1)\kappa^2$, then one still gets a diameter bound of the form $\text{diam} \leq \pi + O(\varepsilon)$. This was generalized by Rosenberg and D. Yang in [37] to a situation where, instead of a metric ball, one has merely a set of small volume on which the curvature is allowed to dip down. In both of these results the second variation technique of Synge's original paper is still used, much as in the early work, but with some nice tricks thrown in. Recently the second author in [39] managed to show that if one assumes the manifold has $\text{Ric} \geq -(n-1)\kappa^2$, then it suffices to assume that the amount of Ricci curvature which lies below $(n-1)$ is small in L^1 to get a diameter bound close to π. In this paper the second variation technique is still used, but with a somewhat surprising twist in that it relies on the interesting inequality [8, Thm 2.11], which was originally introduced for completely different purposes. This inequality depends rather crucially on a pointwise lower Ricci curvature bound, and unlike many other results (see [35], [36]) has not yet been generalized to manifolds with appropriate integral curvature bounds. In this paper we use an excess estimate technique to generalize the diameter bound to such a situation. Before stating the results, we introduce some notation.

Fix a real number κ, and consider at each point of a Riemannian n-manifold M the lowest eigenvalue Ric_- for the Ricci tensor. To measure the extent to which the Ricci curvature lies below $(n-1)\kappa$, we introduce the function $\rho = \max\{((n-1)\kappa - \text{Ric}_-), 0\}$ and the quantity

$$\bar{k}(p,\kappa,R) = \sup_{x\in M} \frac{1}{\text{vol}B(x,R)} \int_{B(x,R)} \rho^p.$$

Note that $\bar{k}(p,\kappa,R) = 0$ iff $\text{Ric} \geq (n-1)\kappa$. The volume normalization is natural in our context, and makes $\left(\bar{k}(p,\kappa,R)\right)^{1/p}$ scale like curvature (see also [17], [35], [36]). As $R \to 0$ we get that the $\bar{k}(p,\kappa,R)$ converges to $\min \rho^p$, while if $R \to \infty$ one gets some (asymptotic in the non-compact case) global curvature invariant. This limit will be denoted by $\bar{k}(p,\kappa)$. In [17] it was shown by several examples that smallness of $\bar{k}(p,\kappa,R)$ for $p \leq n/2$ or general boundedness of $\bar{k}(p,\kappa,R)$, for any $p \geq 1$, does not give any interesting results.

However, if one assumes that $\bar{k}\,(p,\kappa,R)$ is small for $p > n/2$, then interesting phenomena emerge. In [17] it was shown that one gets lower bounds for certain isoperimetric constants. This gives in a standard way lower eigenvalue bounds and Sobolev constant bounds (for $L^{1,1} \subset L^{\frac{2p}{2p-1}}$), which can then, for example, be used in connection with Gallot's modification of the Bochner technique to get various interesting Betti number bounds. In [35] and [36] it was shown that smallness of $\bar{k}\,(p,\kappa,R)$ gives relative volume comparison, gradient bounds, and a generalized maximum principle. This was enough to generalize almost all results, new and old, on manifolds with lower Ricci curvature bounds to the situation in which $\bar{k}\,(p,\kappa,R)$ is small. One nagging problem, however, that remained from all of these generalizations, was that there was still no way that one could get diameter bounds as described above. The problem, in all its simplicity, was that one could not control negative mean curvature, or in other words ensure that it became negative in the necessary places.

In this paper we take care of these problems and then go on to explain that virtually all old and new results on manifolds with positive Ricci curvature carry over to the integral situation. We also improve Gallot's isoperimetric inequality. Namely, we generalize the Heintze-Karcher-Gromov generalization of P. Levy's inequality. In particular, one gets almost optimal lower eigenvalue bounds in positive curvature and a bound for the classical Sobolev constant coming from the embedding $L^{1,1} \subset L^{\frac{n}{n-1}}$(this Sobolev constant then yields bounds for all other Sobolev constants).

Now let us mention a selection of the results we prove here. First the diameter bound:

Theorem 1.1. *Let $n \geq 2$ be an integer, $p > n/2$, $R > 0$, and $\kappa > 0$. For every $\delta > 0$ there is an $\varepsilon\,(n,p,\kappa,R) > 0$ such that any complete Riemannian n-manifold M with $\bar{k}\,(p,\kappa,R) \leq \varepsilon$ has* $\operatorname{diam} M \leq$ *$(\pi/\sqrt{\kappa}) + \delta$. In particular, M is compact.*

We would like to point out a connection with general relativity here. There is an analogous phenomenon to these diameter bounds in general relativity, known as the Raychaudhuri effect (see [22] and also [3]). This is the result which predicts a big bang from present expansion (Hubble effect), together with nonnegativity of the energy-stress tensor. Later, Penrose and Hawking then refined this technique to show that black holes should exist (see, for instance, [22]). An interesting problem in all of this work is to see if the results remain true in the presence of (small) quantum fluctuations. A very reasonable interpretation of having small quantum fluctuations is to say that the amount of negativity in the energy-stress tensor should be small in some suitable integral sense (noting that this still allows for large bursts as long as they don't persist). Our results, especially those of section 3, therefore indicate that indeed these singularity theorems do remain true in the presence of quantum fluctuations, at least in the Riemannian setting.

Using the same estimates as are used for the above theorem and the proof technique from [31], we can generalize the almost-maximal diameter theorem from that paper. Thus we can find $\delta(n,\kappa,K,R)$, $\varepsilon(n,\kappa,K,R) > 0$ such that a Riemannian n-manifold M with $\bar{k}(1,\kappa,R) \leq \varepsilon$, $\sec \geq -K^2$, and $\operatorname{diam} M \geq (\pi/\sqrt{\kappa}) - \delta$ is homeomorphic to a sphere.

Let M be closed, with diameter $\leq D$. Suppose that $H \subset M$ is a hypersurface with constant mean curvature η, which divides M into two domains, and let Ω be one of these domains. In the space form S^n_κ pick the distance sphere $\bar{H} = S(\bar{x}_0, r)$ of constant positive mean curvature $|\eta|$, and let $\bar{\Omega}$ be either $B(\bar{x}_0, D) - B(\bar{x}_0, r)$ or $B(\bar{x}_0, r)$. The desired choice is that which gives the same sign for the mean curvature as when we look into Ω from H. Using a generalization of the diameter estimate from above (see Section 3) we can, after possibly decreasing κ slightly, show the following generalization of the Heintze-Karcher volume comparison inequality (see [21]).

Theorem 1.2. *For any $\alpha > 1$, there is an $\varepsilon(n,p,\alpha,\kappa,R) > 0$ such that if $\bar{k}(p,\kappa,R) \leq \varepsilon$, then*

$$\operatorname{vol}(\Omega) \leq \alpha \frac{\operatorname{area}(H)}{\operatorname{area}(\bar{H})} \operatorname{vol}(\bar{\Omega}).$$

From this estimate it follows (see [18]) that the classical Sobolev constant is bounded, and from [5] that when $\kappa = 1$, then the first eigenvalue is bounded below by $\alpha^{-2} n$ (Recall that the unit sphere has first eigenvalue n with multiplicity $n+1$. Lichnerowicz first showed in [26] that $\lambda_1 \geq n$ for manifolds with $\operatorname{Ric} \geq (n-1)$. However, his technique doesn't seem to carry through to our context.) In addition, Croke's result from [15] (see also [4]) that almost minimal first eigenvalue gives almost maximal diameter carries over as well, with the same proof. We can also generalize Cheng's upper eigenvalue bounds. These results give, in particular, that manifolds with $\bar{k}(p,1,R)$ small and almost maximal diameter have first eigenvalue close to n.

Among more recent results, we can generalize Colding's amazing work on positive Ricci curvature from [12] and [13]. Quickly stated, this work says that for manifolds with Ric $\geq (n-1)$, the following statements are equivalent:

1) vol$M$ is close to volS_1^n,

2) M is Gromov-Hausdorff close to S_1^n,

3) M has radius close to π (where rad$M$ is the smallest closed metric ball which contains all of M).

Moreover, from work in [9] we get that any of these conditions imply that the manifold is diffeomorphic to a sphere.

Our claim is that these equivalences, appropriately qualified, remain true if $\bar{k}(p, 1, R)$ is small. To this list of equivalences it is natural to add the condition that the $(n+1)$-st eigenvalue is close to n. It is shown in [33] that in fact this condition implies Gromov-Hausdorff closeness to S_1^n, without using the work of Colding. This last condition also works in the context of smallness of $\bar{k}(p, 1, R)$. Given the work in [36] this generalization (including the diffeomorphism statement) is straightforward as long as one has the above diameter and eigenvalue bounds.

From the above-mentioned work of Cheng and Croke, one sees that for manifolds with Ric $\geq (n-1)$ it is equivalent to have almost maximal diameter and almost minimal first eigenvalue. To this Cheeger and Colding in the profound paper [8] add that these conditions are almost equivalent to the manifolds being Gromov-Hausdorff close to a sine warped product over some metric space. We can also add this last statement to our list of equivalences, but with the proviso that the manifold has a lower volume bound. It is interesting to note that our failure to address the collapsed case is precisely because we can't establish the inequality [8, Thm 2.11] also mentioned above.

It is interesting to ponder what we have not been able to generalize. Besides the inequality of Cheeger and Colding, another important issue needs to be addressed. Namely, what happens to covering spaces when $\bar{k}(p, \kappa, R)$ is small. One of the main consequences of the diameter bound in positive curvature is of course that one gets finite fundamental group. This is also the main feature of the work of Rosenberg-Yang and the more general work of [39]. In the case of purely integral curvature bounds, however, we have no such result.

This is briefly organized as follows. In section 2 we establish the necessary mean curvature comparison. This seems to be somewhat harder than the corresponding earlier results of a similar nature. In Section 3 we use this to prove the diameter bounds and also the generalization of Perel'man's diameter sphere theorem. In Section 4 we then go on to prove the generalized P. Levy isoperimetric inequality, which gives the classical Sobolev constant. Finally in Section 5 we discuss the gaps that need to be filled in order to establish the pinching theorems mentioned above. In this section we also mention how to obtain bounds on the Sobolev constant in situations where $\bar{k}(p,\kappa)$ is small and $\kappa \leq 0$.

2. Mean curvature comparison

In this section we shall generalize the integral estimates for the Laplacian of distance functions that were obtained in [35] to the situation where one has positive curvature (and negative Laplacian) in the comparison space.

The notation is as follows: We have a complete Riemannian n-manifold (M,g). At each point in this manifold we denote by Ric_- the lowest eigenvalue for the Ricci tensor. If we have a distance function $f(x) = d(x,x_0)$ (or $f(x) = d(x,H)$, where H is some hypersurface with mean curvature $\leq h_0$), then the gradient of f is denoted by ∂_r, and the Laplacian by $\Delta f = \mathrm{tr}(\mathrm{Hess} f)$ or h. Of course, h is also the mean curvature of the level sets of f. In the comparison space S^n_κ of constant curvature κ, we similarly pick a distance function f_κ, with Laplacian h_κ (in the hypersurface case this will be the signed distance to the distance sphere of mean curvature h_0, or in other words, simply the distance to a point but with values shifted). More precisely, we have in the point case that

$$h_\kappa(r) = \mathrm{ct}_\kappa(r) = \frac{\mathrm{sn}'_\kappa(r)}{\mathrm{sn}_\kappa(r)},$$

where

$$\begin{aligned} \mathrm{sn}''_\kappa(r) + \kappa \mathrm{sn}_\kappa(r) &= 0, \\ \mathrm{sn}_\kappa(0) &= 0, \\ \mathrm{sn}'_\kappa(0) &= 1, \end{aligned}$$

and in the hypersurface case that

$$\begin{aligned} h_\kappa(r) &= \mathrm{ct}_\kappa(r + r_0), \\ \mathrm{ct}_\kappa(r_0) &= h_0. \end{aligned}$$

Clearly $h_\kappa(r)$ depends only on the distance, while $h(r,\theta)$ depends on the distance and polar (or hypersurface) coordinate. Thus we can compare the two quantities $h(r,\theta)$ and $h_\kappa(r)$ at the same distance r from the reference point (or hypersurface). These mean curvatures satisfy

$$\begin{aligned}
\partial_r\omega &= h\omega,\\
\partial_r h+\frac{h^2}{n-1} &\le -\mathrm{Ric}_-,\\
\omega(0,\theta) &= 0,\ (\omega(0,\theta)=1),\\
h(0,\theta) &= +\infty,\ (h(0,\theta)\le h_0),
\end{aligned}$$

$$\begin{aligned}
\partial_r\omega_\kappa &= h_\kappa\omega_\kappa,\\
\partial_r h_\kappa+\frac{h_\kappa^2}{n-1} &= -(n-1)\kappa,\\
\omega_\kappa(0,\theta) &= 0,\ (\omega_\kappa(0,\theta)=1),\\
h_\kappa(0,\theta) &= +\infty,\ (h_\kappa(0,\theta)=h_0),
\end{aligned}$$

where ω denotes the volume form, and the initial values inside the parentheses are for the hypersurface case. From these formulae it follows that if $\mathrm{Ric}_-\ge(n-1)\kappa$, then $h(r,\theta)\le h_\kappa(r)$. In order to generalize this, we define

$$\begin{aligned}
\psi(r,\theta) &= (h(r,\theta)-h_\kappa(r))_+,\\
\rho(r,\theta) &= ((n-1)\kappa-\mathrm{Ric}_-)_+,
\end{aligned}$$

where $(u)_+=\max\{0,u\}$ is the positive part of the function. These quantities, in both the point and hypersurface situations, satisfy

$$\begin{aligned}
\partial_r\omega &\le \psi\omega+h_\kappa\omega,\\
\partial_r\psi+\frac{\psi^2}{n-1}+\frac{2h_\kappa\psi}{n-1} &\le \rho,\\
\psi(0,\theta) &= 0.
\end{aligned}$$

We can now prove the desired mean curvature estimates.

Theorem 2.1. *With notation as above, we have for all $n\ge 2$, $p>n/2$, $\kappa>0$, $r+r_0<\pi/\sqrt{\kappa}$ an estimate of the form*

$$\int_0^r\psi^{2p}(t,\theta)\,\omega dt\le C(n,p,\kappa,r)\int_0^r\rho^p(t,\theta)\,\omega dt,$$

where $C(n,p,\kappa,r)$ is an explicit constant depending only on the variables indicated, and θ is fixed.

Proof. There is no difference in treating the point and hypersurface cases. Nevertheless, for simplicity we only worry about the point case and thus assume that $r_0 = 0$. We shall use the inequality

$$\psi' + \frac{1}{n-1}\psi^2 + \frac{2}{n-1}\psi \cdot h_\kappa \leq \rho. \tag{2.1}$$

If we multiply this by $\psi^{2p-2}\omega$ and integrate from 0 to r, we obtain

$$\begin{aligned} \frac{1}{2p-1}\psi^{2p-1}(r)\cdot\omega(r) + \left(\frac{1}{n-1} - \frac{1}{2p-1}\right)\int_0^r \psi^{2p}\cdot\omega \\ + \left(\frac{2}{n-1} - \frac{1}{2p-1}\right)\int_0^r \psi^{2p-1}\cdot h_\kappa \cdot \omega \\ \leq \int_0^r \rho \cdot \psi^{2p-2}\cdot\omega. \end{aligned} \tag{2.2}$$

Here, all but the third term on the left-hand side are positive. Thus we have that

$$\begin{aligned} \left(\frac{1}{n-1} - \frac{1}{2p-1}\right)\int_0^r \psi^{2p}\cdot\omega \\ \leq \int_0^r \rho\cdot\psi^{2p-2}\cdot\omega \\ -h_\kappa(r)\left(\frac{2}{n-1} - \frac{1}{2p-1}\right)\int_0^r \psi^{2p-1}\cdot\omega \\ \leq \left(\int_0^r \rho^p\cdot\omega\right)^{\frac{1}{p}}\left(\int_0^r \psi^{2p}\cdot\omega\right)^{1-\frac{1}{p}} \\ -h_\kappa(r)\left(\frac{2}{n-1} - \frac{1}{2p-1}\right)\int_0^r \psi^{2p-1}\cdot\omega \end{aligned} \tag{2.3}$$

Note that if $r \leq \pi/(2\sqrt{\kappa})$ then the last term can be ignored, since in this case it is negative. Thus one immediately gets a bound of the form (see also [35, Lemma 2.2])

$$\int_0^r \psi^{2p}\cdot\omega \leq \left(\frac{1}{n-1} - \frac{1}{2p-1}\right)^{-p}\int_0^r \rho^p\cdot\omega.$$

When $r > \pi/(2\sqrt{\kappa})$, however, considerably more work is needed.

If in the above inequality (2.1) we drop the ψ^2 term and multiply through by ψ^{2p-2}, then we obtain

$$\psi'\cdot\psi^{2p-2} + \frac{2}{n-1}\psi^{2p-1}\cdot h_\kappa \leq \rho\cdot\psi^{2p-2}.$$

Again, we can use that h_κ is decreasing on $[0, r]$ to get

$$\psi' \cdot \psi^{2p-2} + h_\kappa(r)\frac{2}{n-1}\psi^{2p-1} \le \rho \cdot \psi^{2p-2}.$$

Now we can multiply this by $(2p-1)$ and the integrating factor

$$\phi(t) = \exp\left(h_\kappa(r)\frac{2(2p-1)}{n-1}t\right),$$

and write this as

$$\begin{aligned}\left(\phi \cdot \psi^{2p-1}\right)' &\le (2p-1) \cdot \phi \cdot \rho \cdot \psi^{2p-2} \\ &\le (2p-1) \cdot \rho \cdot \psi^{2p-2}.\end{aligned}$$

If we then multiply this inequality by ω and integrate, we get

$$\left(\phi \cdot \psi^{2p-1} \cdot \omega\right)\Big|_a^r - \int_a^r h \cdot \phi \cdot \psi^{2p-1} \cdot \omega \le (2p-1)\int_a^r \rho \cdot \psi^{2p-2} \cdot \omega,$$

which can be reduced to

$$\begin{aligned}&\left(\phi \cdot \psi^{2p-1} \cdot \omega\right)\Big|_a^r \\ &\qquad \le (2p-1)\left(\int_a^r h_+ \cdot \psi^{2p-1} \cdot \omega + \int_a^r \rho \cdot \psi^{2p-2} \cdot \omega\right).\end{aligned} \tag{2.4}$$

We now let $a = \pi/2\sqrt{\kappa}$ and try to estimate the two terms

$$\left(\psi^{2p-1} \cdot \omega\right)\Big|_{\pi/(2\sqrt{\kappa})}$$

and

$$\int_a^r h_+ \cdot \psi^{2p-1} \cdot \omega.$$

The first one is handled by 2.2:

$$\begin{aligned}&\frac{1}{2p-1}\left(\psi^{2p-1} \cdot \omega\right)\left(\pi/(2\sqrt{\kappa})\right) + \left(\frac{1}{n-1} - \frac{1}{2p-1}\right)\int_0^{\pi/(2\sqrt{\kappa})} \psi^{2p} \cdot \omega \\ &\qquad + \left(\frac{2}{n-1} - \frac{1}{2p-1}\right)\int_0^{\pi/(2\sqrt{\kappa})} \psi^{2p-1} \cdot h_\kappa \cdot \omega \\ &\qquad \le \int_0^{\pi/(2\sqrt{\kappa})} \rho \cdot \psi^{2p-2} \cdot \omega.\end{aligned}$$

Now, h_κ is positive on the given interval, so the second and third terms on the left-hand side can be eliminated to get that

$$\begin{aligned}
&\frac{1}{2p-1}\left(\psi^{2p-1}\cdot\omega\right)\left(\pi/(2\sqrt{\kappa})\right)\\
&\quad\le \int_0^{\pi/(2\sqrt{\kappa})}\rho\cdot\psi^{2p-2}\cdot\omega\\
&\quad\le \left(\int_0^{\pi/(2\sqrt{\kappa})}\rho^p\cdot\omega\right)^{\frac{1}{p}}\left(\int_0^{\pi/(2\sqrt{\kappa})}\psi^{2p}\cdot\omega\right)^{1-\frac{1}{p}}\\
&\quad\le \left(\int_0^{r}\rho^p\cdot\omega\right)^{\frac{1}{p}}\left(\int_0^{r}\psi^{2p}\cdot\omega\right)^{1-\frac{1}{p}}.
\end{aligned}\tag{2.5}$$

For the other term we introduce the auxiliary functions

$$\tilde{h}_\kappa=\begin{cases} h_\kappa & t\le \pi/(2\sqrt{\kappa})\\ 0 & t\ge \pi/(2\sqrt{\kappa})\end{cases}$$

and

$$\tilde{\psi}=\left(h-\tilde{h}_\kappa\right)_+.$$

Thus $\tilde{\psi}=h_+$ whenever $t\ge\pi/(2\sqrt{\kappa})$. Moreover since h_+ satisfies

$$h_+'+\frac{1}{n-1}h_+^2\le(-\operatorname{Ric}_-)_+,$$

we have

$$\begin{aligned}
\tilde{\psi}'+\frac{1}{n-1}\tilde{\psi}^2+\frac{2}{n-1}\tilde{\psi}\cdot\tilde{h}_\kappa &\le \rho,\\
\tilde{\psi}(0) &= 0.
\end{aligned}$$

Multiplying this equation by $\tilde{\psi}^{2p-1}\omega$ and integrating, we obtain as above that

$$\int_0^r\tilde{\psi}^{2p}\omega\le\left(\frac{1}{n-1}-\frac{1}{2p-1}\right)^{-p}\int_0^r\rho^p\omega.$$

In particular,

$$\begin{aligned}
&\int_{\pi/(2\sqrt{\kappa})}^r h_+\cdot\psi^{2p-1}\cdot\omega\\
&\quad\le\left(\int_0^r\tilde{\psi}^{2p}\cdot\omega\right)^{\frac{1}{2p}}\left(\int_0^r\psi^{2p}\cdot\omega\right)^{1-\frac{1}{2p}}\\
&\quad\le\left(\frac{1}{n-1}-\frac{1}{2p-1}\right)^{-\frac{1}{2}}\left(\int_0^r\rho^p\cdot\omega\right)^{\frac{1}{2p}}\left(\int_0^r\psi^{2p}\cdot\omega\right)^{1-\frac{1}{2p}}.
\end{aligned}\tag{2.6}$$

Thus by 2.4, 2.5, and 2.6 we obtain

$$\begin{aligned}
&\left(\phi \cdot \psi^{2p-1} \cdot \omega\right)(r) \\
&\quad \leq \left(\psi^{2p-1} \cdot \omega\right)\left(\pi/(2\sqrt{\kappa})\right) \\
&\qquad + (2p-1)\left(\int_{\pi/(2\sqrt{\kappa})}^{r} h_+ \cdot \psi^{2p-1} \cdot \omega + \int_0^r \rho \cdot \psi^{2p-2} \cdot \omega\right) \\
&\quad \leq (2p-1)\left(\int_0^r \rho^p \cdot \omega\right)^{\frac{1}{p}}\left(\int_0^r \psi^{2p} \cdot \omega\right)^{1-\frac{1}{p}} \\
&\qquad + (2p-1)\left(\frac{1}{n-1} - \frac{1}{2p-1}\right)^{-\frac{1}{2}}\left(\int_0^r \rho^p \cdot \omega\right)^{\frac{1}{2p}} \\
&\qquad\quad \cdot \left(\int_0^r \psi^{2p} \cdot \omega\right)^{1-\frac{1}{2p}} \\
&\qquad + (2p-1)\left(\int_0^r \rho^p \cdot \omega\right)^{\frac{1}{p}}\left(\int_0^r \psi^{2p} \cdot \omega\right)^{1-\frac{1}{p}},
\end{aligned}$$

at least for $r > \pi/(2\sqrt{\kappa})$. However, for $r \leq \pi/(2\sqrt{\kappa})$ we already know that

$$\left(\psi^{2p-1} \cdot \omega\right)(r) \leq (2p-1)\left(\int_0^r \rho^p \cdot \omega\right)^{\frac{1}{p}}\left(\int_0^r \psi^{2p} \cdot \omega\right)^{1-\frac{1}{p}},$$

so all in all we definitely get an estimate of the form

$$\begin{aligned}
\left(\psi^{2p-1} \cdot \omega\right)(r) \quad &\leq \quad C_1(p,n,r,\kappa)\left(\left(\int_0^r \rho^p \cdot \omega\right)^{\frac{1}{p}}\left(\int_0^r \psi^{2p} \cdot \omega\right)^{1-\frac{1}{p}}\right. \\
&\qquad \left. + \left(\int_0^r \rho^p \omega\right)^{\frac{1}{2p}}\left(\int_0^r \psi^{2p} \cdot \omega\right)^{1-\frac{1}{2p}}\right).
\end{aligned}$$

To obtain the desired estimate for $\int_0^r \psi^{2p} \cdot \omega$, we use the original equation

(2.3):

$$\begin{aligned}
&\left(\frac{1}{n-1}-\frac{1}{2p-1}\right)\int_0^r \psi^{2p}\cdot\omega \\
&\quad\leq \left(\int_0^r \rho^p\cdot\omega\right)^{\frac{1}{p}}\left(\int_0^r \psi^{2p}\cdot\omega\right)^{1-\frac{1}{p}} \\
&\qquad - h_\kappa(r)\left(\frac{2}{n-1}-\frac{1}{2p-1}\right)\int_0^r \psi^{2p-1}\cdot\omega \\
&\quad\leq \left(\int_0^r \rho^p\cdot\omega\right)^{\frac{1}{p}}\left(\int_0^r \psi^{2p}\cdot\omega\right)^{1-\frac{1}{p}} \\
&\qquad - h_\kappa(r)\left(\frac{2}{n-1}-\frac{1}{2p-1}\right)\cdot r\cdot C_1(p,n,r,\kappa) \\
&\qquad\quad \cdot\left(\int_0^r \rho^p\cdot\omega\right)^{\frac{1}{p}}\left(\int_0^r \psi^{2p}\cdot\omega\right)^{1-\frac{1}{p}} \\
&\qquad - h_\kappa(r)\left(\frac{2}{n-1}-\frac{1}{2p-1}\right)\cdot r\cdot C_1(p,n,r,\kappa)\left(\int_0^r \rho^p\omega\right)^{\frac{1}{2p}} \\
&\qquad\quad \cdot\left(\int_0^r \psi^{2p}\cdot\omega\right)^{1-\frac{1}{2p}} \\
&\quad\leq C_2(p,n,r,\kappa)\left(\int_0^r \rho^p\cdot\omega\right)^{\frac{1}{p}}\left(\int_0^r \psi^{2p}\cdot\omega\right)^{1-\frac{1}{p}} \\
&\qquad + C_2(p,n,r,\kappa)\left(\int_0^r \rho^p\omega\right)^{\frac{1}{2p}}\left(\int_0^r \psi^{2p}\cdot\omega\right)^{1-\frac{1}{2p}}.
\end{aligned}$$

This is an inequality of the form

$$ax\leq cx^{1-\frac{1}{p}}+bx^{1-\frac{1}{2p}},$$

where all of the coefficients are positive. If we multiply this by $x^{-1+\frac{1}{p}}$, we get

$$ax^{\frac{1}{p}}\leq c+bx^{\frac{1}{2p}}.$$

The quadratic equation then tells us that

$$\begin{aligned}
x^{\frac{1}{2p}} &\leq \frac{b+\sqrt{b^2+4ac}}{2a} \\
&\leq \frac{b}{2a}+\frac{\max\left\{b,\sqrt{4ac}\right\}}{a}.
\end{aligned}$$

In the situation at hand, we have that

$$\begin{aligned} a &= \left(\frac{1}{n-1}-\frac{1}{2p-1}\right), \\ c &= C_2\left(p,n,r,\kappa\right)\left(\int_0^r \rho^p\cdot\omega\right)^{\frac{1}{p}}, \\ b &= C_2\left(p,n,r,\kappa\right)\left(\int_0^r \rho^p\cdot\omega\right)^{\frac{1}{2p}}, \end{aligned}$$

so that

$$\left(\int_0^r \psi^{2p}\cdot\omega\right)^{\frac{1}{2p}} \leq C_3\left(p,n,r,\kappa\right)\left(\int_0^r \rho^p\cdot\omega\right)^{\frac{1}{2p}},$$

or,

$$\int_0^r \psi^{2p}\cdot\omega \leq C_4\left(p,n,r,\kappa\right)\int_0^r \rho^p\cdot\omega,$$

as desired. q.e.d.

3. Distance bounds

First, we need to recall a result from [36, Section 2]. Given that $\bar{k}\left(p,\kappa,R_1\right)$ is small for some R_1, one can for any other R_2 show that $\bar{k}\left(p,\kappa,R_2\right) \leq C\left(n,p,R_1,R_2\right)\cdot\bar{k}\left(p,\kappa,R_1\right)$. Thus it doesn't matter on which scale we make averaged curvature assumptions. The other thing that we must use is the generalized maximum principle from [36, Section 3].

Theorem 3.1. *Let $\Omega\subset M$ be a bounded domain in an n-dimensional Riemannian manifold. Then for any function u on Ω with $\Delta u \geq -f$, where f is nonnegative on Ω satisfies*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + K\cdot\left(\frac{1}{\mathrm{vol}\Omega}\int_{\Omega} f^q\right)^{\frac{1}{q}} \tag{3.1}$$

for any $q>n/2$. Here K depends on $n, q>p\geq n/2$, and on the constant that appears in the Sobolev embedding $C^0\left(\Omega\right)\subset L^{\frac{2p}{2p-1},1}\left(\Omega\right)$ for functions which vanish on $\partial\Omega$. Moreover, in case $p>n/2$ and $\bar{k}\left(p,\kappa,R\right)$ is small for some R we have a bound for this Sobolev constant from [17].

Actually the above-mentioned bound for the Sobolev constant for Dirichlet boundary conditions is most easily gotten by assuming that one has a global Sobolev constant bound for the entire manifold and that $\mathrm{vol}\Omega < (\mathrm{vol}M)/2$. In fact Gallot gets a bound for the global Sobolev constant provided we have a diameter bound and smallness for $\overline{k}$. Thus we include a preliminary lemma which bounds the diameter of the manifold by a value close to 2π. After this is done we can then go on to establish the optimal diameter bounds mentioned in the introduction.

Lemma 3.2 *Let $n \geq 2$, $p > \frac{n}{2}$, $R > 0$, and $\kappa > 0$. Then there is $D = D(n, \kappa)$, $\varepsilon = \varepsilon(n, p, \kappa, R)$ such that any complete Riemannian manifold with $\overline{k}(p, \kappa, R) < \varepsilon$ satisfies $\mathrm{diam}M \leq D$.*

Proof. Let $y, z \in M$, x_0 be a midpoint between y and z, and consider the excess function

$$e(\cdot) = d(y, \cdot) + d(\cdot, z) - d(y, z). \tag{3.2}$$

Note that $e \geq 0$ on M and $e \leq 2r$ on $B(x_0, r)$ by the triangle inequality.

From the above mean curvature estimates, by using a suitably large comparison sphere we may choose D large enough so that if $d(y, z) > D$, then $\Delta e \leq -K + \psi_1$, on $B(x_0, r)$, where K is a large positive constant to be determined, and ψ_i denotes an error term controlled in L^{2p} by $\overline{k}$ as before.

Let Ω_i be a sequence of smooth star-shaped domains which converge to $B(x_0, r) - Cut(x_0)$, f_j a sequence of smooth functions such that $|f_j - e| < j^{-1}$, $|\nabla f_j|^2 \leq 2 + j^{-1}$, and $\Delta f_j \leq \Delta e + j^{-1}$ on $B(x_0, r)$ (see, for instance [14]).

Letting $h = d^2(x_0, \cdot) - r^2$, we have that h is smooth on Ω_i, so by Green's Theorem

$$\int_{\Omega_i} (\Delta f_j)h - \int_{\Omega_i} f_j(\Delta h) = \int_{\partial\Omega_i} h(\nu f_j) - \int_{\partial\Omega_i} f_j(\nu h), \tag{3.3}$$

where ν is the outward unit normal to Ω_i. Thus,

$$\begin{aligned} &\int_{\Omega_i} (-K + \psi_1 + j^{-1})h - 3r \int_{\Omega_i} (2n + \psi_2) \\ &\qquad \leq \int_{\partial\Omega_i} h(2 + j^{-1}) - \int_{\partial\Omega_i} f_j(\nu h). \end{aligned} \tag{3.4}$$

By the dominated convergence theorem, the two integrals on the right-hand side of the above converge as $j \to \infty$, and we get

$$\int_{\Omega_i} (-K+\psi_1)h - 3r\int_{\Omega_i}(2n+\psi_2) \le 2\int_{\partial\Omega_i} h - \int_{\partial\Omega_i} e(\nu h). \tag{3.5}$$

Now,

$$\begin{aligned}\int_{B(x_0,r)} &(-K+\psi_1)h - 3r\int_{B(x_0,r)}(2n+\psi_2)\\ &\ge \left(3\frac{r}{2}\right)^2 K\mathrm{vol}(B(x_0,r/2)) - 6nr\mathrm{vol}(B(x_0,r))\\ &\quad - \int_{B(x_0,r)} r\psi_1 + 3r\psi_2.\end{aligned} \tag{3.6}$$

By relative volume comparison ([36]) we have that for $\overline{k}(p,0,R)$ small enough, $\mathrm{vol}(B(x_0,r/2)) \ge 2^{-(n+1)}\mathrm{vol}(B(x_0,r))$. Moreover, we can choose $\overline{k}(p,\kappa,R)$ small enough so that $\int_{B(x_0,r)} r\psi_1 + 3r\psi_2 \le nr\mathrm{vol}(B(x_0,r))$.

Thus for $K > n2^{(n+5)}r^{-1}$, the above quantity is positive. Therefore $2\int_{\partial\Omega_i} h - \int_{\partial\Omega_i} e(\nu h)$ becomes positive as $i \to \infty$. However the first of these integrals goes to 0 as $i \to \infty$, while in the second we have that $\nu h \ge 0$ on $\partial\Omega_i$ for all i, as Ω_i is star-shaped. This implies that e must be become negative on $B(x_0,r)$, which is a contradiction. So $d(y,z)$ must be $\le D$. q.e.d.

There are two types of diameter bounds that we are after when $\bar{k}(p,\kappa,R)$ is small. One is a global diameter bound, while the other is for domains in general. Since the latter is more general and needed for our later results, we emphasize this here. Suppose that we have a domain $\Omega \subset M$ with smooth boundary $\partial\Omega$, and that for some $p > n/2$ and $R > 0$ we have that $\bar{k}(p,\kappa,R) \le \varepsilon$ is small. Furthermore, assume that the mean curvature with respect to the normal pointing into Ω is $\le h_0$. In the space form S^n_κ choose a ball $B(\bar{x}_0, r_0)$ whose boundary has mean curvature h_0 with respect to the inward pointing normal. From standard Ricci curvature comparison it is easily seen that $\Omega \subset B(\partial\Omega, r_0)$, provided that $\mathrm{Ric} \ge (n-1)\kappa$. Our main result asserts that this almost holds when $\bar{k}(p,\kappa,R) \le \varepsilon$ is small.

Theorem 3.3. *For every $\delta > 0$ there is an $\varepsilon(n,p,\kappa,R,r_0) > 0$ such that if $\bar{k}(p,\kappa,R) \le \varepsilon$, then $\Omega \subset B(\partial\Omega, r_0+\delta)$.*

Proof. Take a point $y \in \Omega$ such that $d(y,\partial\Omega) \ge r_0 + \delta > r_0$, and select a segment σ from $\partial\Omega$ to y parametrized by arclength. Now consider the two distance functions $f_1(x) = d(\partial\Omega, x)$ and $f_2(x) =$

$d(x,y)$, and with those the excess function $e(x) = f_1(x) + f_2(x) - d(y, \partial\Omega)$. If we consider a ball $B(x_0, r)$ where x_0 is chosen near the midpoint of σ and $r \leq r_0/4$, then this excess function is ≥ 0 on $B(x_0, r)$, and attains its minimum 0 at x_0. Thus $\Delta e(x_0) \geq 0$. Using the mean curvature estimates from the previous section, we shall now show that e cannot have an interior minimum on $B(x_0, r)$, provided that ε is sufficiently small. Note that x_0 can be chosen so that $\mathrm{vol}B(x_0, r) < \frac{1}{2}\mathrm{vol}M$.

From above we have the following Laplacian estimates for f_i on Ω:

$$\begin{aligned} \Delta f_1 &\leq \mathrm{ct}_\kappa(f_1 - r_0) + \psi_1, \\ \Delta f_2 &\leq \mathrm{ct}_\kappa(f_2) + \psi_2, \end{aligned}$$

where $\mathrm{ct}_\kappa(-r_0) = h_0$, and

$$\begin{aligned} \int_{B(\partial\Omega,R)} \psi_1^{2p} &\leq C_1(n,p,\kappa,R) \int_{B(\partial\Omega,R)} \rho^p, \\ \int_{B(y,R)} \psi_2^{2p} &\leq C_1(n,p,\kappa,R) \int_{B(y,R)} \rho^p, \end{aligned}$$

where since $\kappa > 0$, there is a restriction on how large R can be. Since we are interested in what happens on $B(x_0, r)$ and δ is assumed to be small, we can by picking $r \leq \min\{r_0/3, R\}$ ensure that ψ_i are bounded in L^{2p} on this ball in terms of the curvature. Thus we have that

$$\begin{aligned} \frac{1}{\mathrm{vol}B(x_0,r)} \int_{B(x_0,r)} \psi_i^{2p} &\leq C_2(n,p,\kappa,R,r) \cdot \frac{1}{\mathrm{vol}B(x_0,R)} \int_{B(x_0,R)} \rho^p \\ &\leq C_2(n,p,\kappa,R,r) \cdot \varepsilon. \end{aligned}$$

The excess function therefore satisfies

$$\Delta e \leq \mathrm{ct}_\kappa(f_1 - r_0) + \mathrm{ct}_\kappa(f_2) + \psi_1 + \psi_2.$$

Since $f_1 + f_2 \geq r_0 + \delta$ and ct_κ is decreasing we get

$$\mathrm{ct}_\kappa(f_1 - r_0) + \mathrm{ct}_\kappa(f_2) \leq \mathrm{ct}_\kappa(\delta - f_2) + \mathrm{ct}_\kappa(f_2)$$

As long as the range of f_2 is restricted to $[2\delta, (\pi/\sqrt{\kappa}) - \delta]$, this function has a negative maximum, which is $\leq -C_3(n,\kappa) \cdot \delta$. Therefore

$$\Delta e \leq -C_3(n,\kappa) \cdot \delta + \psi,$$

where

$$\frac{1}{\mathrm{vol}B\left(x_0,r\right)}\int_{B(x_0,r)}\psi^{2p}\leq C_2\cdot\varepsilon.$$

Here we should point out that in the case of $\mathrm{Ric}\geq(n-1)\,\kappa$, we would have that $\psi_i=0$. Hence we have already arrived at a contradiction at this point, since e would then have negative Laplacian at a minimum.

Now pick u such that

$$\begin{aligned}\Delta u &= C_3\left(n,\kappa\right)\cdot\delta,\\ u &= 0 \text{ on } \partial B\left(x_0,r\right).\end{aligned}$$

By the standard maximum principle such a u must be negative on $B\left(x_0,r\right)$. So if we consider the function $e+u$ then

$$\begin{aligned}\Delta\left(e+u\right) &\leq \psi,\\ e\left(x\right)+u\left(x\right) &= e\left(x\right)\geq 0 \text{ on } \partial B\left(x_0,r\right),\\ e\left(x_0\right)+u\left(x_0\right) &= u\left(x_0\right)<0.\end{aligned}$$

Thus it must have an interior minimum on $B\left(x_0,r\right)$, which by the above maximum principle (Theorem 3.1) satisfies

$$\begin{aligned}\inf_{B(x_0,r)}\left(e+u\right) &\geq \inf_{\partial B(x_0,r)}\left(e+u\right)-K\cdot\left(\frac{1}{\mathrm{vol}B\left(x_0,r\right)}\int_{B(x_0,r)}\psi^{2p}\right)^{\frac{1}{2p}}\\ &\geq \inf_{\partial B(x_0,r)}\left(e+u\right)-K\cdot\left(C_2\varepsilon\right)^{\frac{1}{2p}}\\ &\geq -K\cdot\left(C_2\varepsilon\right)^{\frac{1}{2p}}.\end{aligned}$$

Therefore, we obtain the following lower bound

$$u\left(x_0\right)=e\left(x_0\right)+u\left(x_0\right)\geq -K\cdot\left(C_2\varepsilon\right)^{\frac{1}{2p}}.$$

We shall now derive a negative upper bound for $u\left(x_0\right)$ which depends on δ. In S^n_κ consider a metric ball $B\left(\bar{x}_0,r\right)$. On this ball we have a rotationally symmetric function $\varphi\circ d\left(\bar{x}_0,\cdot\right)$ such that

$$\begin{aligned}\Delta\varphi\circ d\left(\bar{x}_0,\cdot\right) &= C_3\left(n,\kappa\right)\cdot\delta,\\ \varphi\circ d\left(\bar{x}_0,\cdot\right) &= 0 \text{ on } \partial B\left(\bar{x}_0,r\right).\end{aligned}$$

Thus $\varphi : [0, r] \to [-\infty, 0]$ is the function which satisfies

$$\begin{aligned} \ddot{\varphi} + (n-1)\,\mathrm{ct}_\kappa\,(t)\,\dot{\varphi} &= C_3\,(n,\kappa) \cdot \delta, \\ \varphi\,(r) &= 0. \end{aligned}$$

In particular, $\varphi\,(0) < 0$ is a number which only depends on n, κ, r, and δ.

In $B\,(x_0, r)$ consider the function

$$v\,(x) = \varphi \circ d\,(x_0, x)\,.$$

This function satisfies

$$\begin{aligned} \Delta v &= \ddot{\varphi} \circ d + \Delta d \cdot \dot{\varphi} \circ d \\ &\leq C_3\,(n,\kappa) \cdot \delta + \psi \cdot \dot{\varphi} \circ d \\ &\leq C_3\,(n,\kappa) \cdot \delta + \tilde{\psi}, \end{aligned}$$

where

$$\frac{1}{\mathrm{vol} B\,(x_0, r)} \int_{B(x_0,r)} \tilde{\psi}^{2p} \leq C_4(n, p, \kappa, R, r) \cdot \varepsilon.$$

Thus the difference $v - u$ satisfies

$$\begin{aligned} \Delta\,(v-u) &\leq \tilde{\psi}, \\ v - u &= 0 \text{ on } \partial B\,(x_0, r)\,. \end{aligned}$$

Again by the maximum principle (3.1) we therefore get

$$v\,(x_0) - u\,(x_0) \geq -K \cdot (C_4 \varepsilon)^{\frac{1}{2p}}\,.$$

In other words,

$$u\,(x_0) \leq \varphi\,(0) + K \cdot (C_4 \varepsilon)^{\frac{1}{2p}}\,,$$

which becomes negative as $\varepsilon \to 0$. This, together with the lower bound $u\,(x_0) \geq -K \cdot (C_2 \varepsilon)^{\frac{1}{2p}}$ we obtained above, will clearly give us a contradiction unless $\delta \to 0$ as $\varepsilon \to 0$. q.e.d.

From this theorem we get two fairly immediate consequences. One is the diameter bound in positive curvature. The other is a generalization of the sphere theorem in [31].

Corollary 3.4. *Let $n \geq 2$ be an integer, $p > n/2$, $R > 0$, and $\kappa > 0$. For every $\delta > 0$ there is an $\varepsilon(n, p, \kappa, R) > 0$ such that any complete Riemannian n-manifold M with $\bar{k}(p, \kappa, R) \leq \varepsilon$ has* $\mathrm{diam} M \leq (\pi/\sqrt{\kappa}) + \delta$. *In particular, M is compact.*

Proof. To see this we can simply use the above proof. Take two points $p, q \in M$ with $d(x_0, y_0) = (\pi/\sqrt{\kappa}) + \delta$ and consider the excess function $e(x) = d(x, x_0) + d(x, y_0) - d(x_0, y_0)$. Again we have the appropriate estimates on the Laplacian of e from the mean curvature estimates in the above section. Thus the proof works without further change. q.e.d.

Corollary 3.5. *Let $n \geq 2$ be an integer, $R > 0$, $\kappa > 0$, and $K \geq 0$. We can then find $\delta(n, \kappa, K, R)$, $\varepsilon(n, \kappa, K, R) > 0$ such that any Riemannian n-manifold M with $\bar{k}(1, \kappa, R) \leq \varepsilon$,* $\sec \geq -K^2$, *and* $\mathrm{diam} M \geq (\pi/\sqrt{\kappa}) - \delta$, *is a twisted sphere.*

Proof. The proof is as in the original paper. First note that given the pointwise lower bound for the sectional curvature we can as in [39] assume that $\bar{k}(p, \kappa, R)$ is small for some arbitrary but fixed $p > n/2$. The only modification is that if $d(x_0, y_0) = \mathrm{diam} M$ and we consider the distance function $f(x) = d(x, y_0)$, then the Laplacian will not necessarily go to $-\infty$ as we approach x_0. Instead we can use the above bound on the Laplacian which gives the same upper bound plus an error which is small in L^{2p}. We can then extract a contradiction in the same manner as in the above proof. q.e.d.

4. Volume comparison

We are now ready to prove the first part of the volume comparison results. We consider the volume of balls or (half) tubes:

$$\begin{aligned}
\mathrm{vol} B(x_0, r) &= \int_{S^{n-1}} \int_0^r \omega(t, \theta)\, dt d\theta, \\
\mathrm{vol} B(H, r) &= \int_H \int_0^r \omega(t, \theta)\, dt d\theta, \\
\mathrm{vol}(B(x_0, r) \subset S^n_\kappa) &= v(n, \kappa, r) \\
&= \int_{S^{n-1}} \int_0^r \omega_\kappa(t, \theta)\, dt d\theta \\
&= \int_{S^{n-1}} \int_0^r \mathrm{sn}^{n-1}_\kappa(t)\, dt d\theta,
\end{aligned}$$

$$\operatorname{vol}\left(B\left(H,r\right)\subset S_{\kappa}^{n}\right) = v\left(n,\kappa,r_{0}+r\right)-v\left(n,\kappa,r_{0}\right).$$

Consider the ratio

$$y\left(r\right)=\begin{cases}\frac{\operatorname{vol}B(x_{0},r)}{v(n,\kappa,r)} & \text{in the point case,}\\ \frac{\operatorname{vol}B(H,r)}{v(n,\kappa,r+r_{0})-v(n,\kappa,r_{0})} & \text{in the hypersurface case.}\end{cases}$$

It follows from L'Hospital's rule that

$$\lim_{r\to 0}y\left(r\right)=\begin{cases}1 & \text{in the point case,}\\ \frac{\operatorname{area}(H)}{\operatorname{area}(\partial B(x_{0},r_{0}))} & \text{in the hypersurface case.}\end{cases}$$

To estimate these ratios, we use that the ratios of the volume forms satisfy

$$\begin{aligned}\partial_{r}\left(\frac{\omega\left(r,\theta\right)}{\omega_{\kappa}\left(r\right)}\right) &= \left(h\left(r,\theta\right)-h_{\kappa}\left(r\right)\right)\frac{\omega\left(r,\theta\right)}{\omega_{\kappa}\left(r\right)}\leq\psi\left(r,\theta\right)\frac{\omega\left(r,\theta\right)}{\omega_{\kappa}\left(r\right)},\\ \lim_{r\to 0}\frac{\omega\left(r,\theta\right)}{\omega_{\kappa}\left(r\right)} &= 1.\end{aligned}$$

As in [35, Lemma 2.1], we see that the volume ratio satisfies

$$y'\left(r\right)\leq C\left(n,\kappa,r\right)\cdot\left(\int_{B(x_{0},r)}\psi^{2p}d\mathrm{vol}\right)^{\frac{1}{2p}}\cdot\left(v\left(n,\kappa,r\right)\right)^{-\frac{1}{2p}}\cdot\left(y\left(r\right)\right)^{1-\frac{1}{2p}},$$

where

$$C\left(n,\kappa,r\right)=\max_{t\in[0,r]}\frac{\int_{0}^{t}\max\left\{\omega_{\kappa}\left(s\right),\omega_{\kappa}\left(t\right)\right\}ds}{\int_{0}^{t}\omega_{\kappa}\left(s\right)ds}.$$

The only slight difference is that in [35] we used that $\omega_{\kappa}\left(r\right)$ is monotonically increasing, which may not be true in the present situation, and thus the need for the maximum inside the integral. Using that the quantity

$$C\left(n,\kappa,r\right)\cdot\left(\int_{B(x_{0}\text{ or }H,r)}\psi^{2p}d\mathrm{vol}\right)^{\frac{1}{2p}}$$

is increasing in r, we see that on $[0,R]$ the volume ratio y satisfies

$$\begin{aligned}y' &\leq K\cdot f\left(r\right)\cdot y^{1-\frac{1}{2p}},\\ K &= C\left(n,\kappa,R\right)\cdot\left(\int_{B(x_{0}\text{ or }H,R)}\psi^{2p}d\mathrm{vol}\right)^{\frac{1}{2p}}.\end{aligned}$$

Direct integration over $[a,b] \subset [0,R]$ then gives us that

$$\begin{aligned} y^{\frac{1}{2p}}(b) &\leq y^{\frac{1}{2p}}(a) + 2p \cdot K \cdot \int_a^b (v(n,\kappa,t))^{-\frac{1}{2p}} dt \\ &\leq y^{\frac{1}{2p}}(a) + 2p \cdot K \cdot \int_0^R (v(n,\kappa,t))^{-\frac{1}{2p}} dt. \end{aligned}$$

We can now collect the three terms $2p, C(n,\kappa,r), \int_0^r (v(n,\kappa,t))^{-\frac{1}{2p}} dt$ into a new constant $C(n,p,\kappa,r)$, which is increasing in r. Therefore y satisfies

$$y^{\frac{1}{2p}}(b) \leq y^{\frac{1}{2p}}(a) + C(n,p,\kappa,R) \cdot \left(\int_{B(x_0 \text{ or } H,R)} \psi^{2p} d\mathrm{vol} \right)^{\frac{1}{2p}}.$$

With this we can now generalize the classical Heintze-Karcher volume comparison result for hypersurfaces. First some notation. Suppose that $H \subset M$ is a hypersurface, but now with constant mean curvature $\eta \geq 0$, and that H divides M into two domains $\Omega_\pm$, where Ω_+ is the domain in which the mean curvature is positive. Furthermore, select $d_\pm > 0$ such that $d_+ + d_- \leq \mathrm{diam} M \leq D$ and $\Omega_\pm \subset B(H, \pm d_\pm)$. In the space form S^n_κ, pick the distance sphere $\bar{H} = S(\bar{x}_0, r_0)$ of constant positive mean curvature η, and let $\bar{\Omega}_+ = B(\bar{x}_0, D) - B(\bar{x}_0, r_0)$, $\bar{\Omega}_- = B(\bar{x}_0, r_0)$. Finally assume that $d_+ \leq D - r_0$ and $d_- \leq r_0$. In the previous section we saw that if $\bar{k}(p,\kappa,R) \leq \varepsilon$ is small, then it follows that $d_- \leq r_0 + O(\varepsilon)$. Thus by decreasing κ slightly we can assume that the condition on $d_\pm$ is satisfied, and also that $\mathrm{diam} M \leq \mathrm{diam} S^n_\kappa + O(\varepsilon)$. So by decreasing κ further we can suppose that both of $\bar{\Omega}_\pm$ are metric balls of radii $\leq d_\pm$. In this situation we can now prove our generalization of the Heintze-Karcher volume comparison (see [21]).

Lemma 4.1. *For any $\alpha > 1$, there is an $\varepsilon(n,p,\alpha,\kappa) > 0$ such that if $\bar{k}(p,\kappa) \leq \varepsilon$, then*

$$\mathrm{vol}(\Omega_\pm) \leq \alpha \frac{\mathrm{area}(H)}{\mathrm{area}(\bar{H})} \mathrm{vol}(\bar{\Omega}_\pm).$$

Proof. Let Ω denote the domain with $\mathrm{vol}\Omega \geq \mathrm{vol}M/2$. From above we get that

$$\left(\frac{\mathrm{vol}\Omega}{\mathrm{vol}\bar{\Omega}} \right)^{\frac{1}{2p}} - \left(\frac{\mathrm{vol}B(H,r)}{\mathrm{vol}B(\bar{H},r)} \right)^{\frac{1}{2p}} \leq C \cdot (k(p,\kappa))^{\frac{1}{2p}},$$

which implies

$$\begin{aligned}\left(\frac{\mathrm{vol}B\left(\bar{H},r\right)}{\mathrm{vol}\bar{\Omega}}\right)^{\frac{1}{2p}}&-\left(\frac{\mathrm{vol}B\left(H,r\right)}{\mathrm{vol}\Omega}\right)^{\frac{1}{2p}}\\&\leq C\cdot\left(\mathrm{vol}B\left(\bar{H},r\right)\right)^{\frac{1}{2p}}\cdot\left(\frac{k\left(p,\kappa\right)}{\mathrm{vol}\Omega}\right)^{\frac{1}{2p}}\\&\leq C\cdot\left(\mathrm{vol}B\left(\bar{H},r\right)\right)^{\frac{1}{2p}}\cdot\left(2\frac{k\left(p,\kappa\right)}{\mathrm{vol}M}\right)^{\frac{1}{2p}}\\&\leq C'\cdot\left(\mathrm{vol}B\left(\bar{H},r\right)\right)^{\frac{1}{2p}}\left(\bar{k}\left(p,\kappa\right)\right)^{\frac{1}{2p}}.\end{aligned}$$

Now pick ε so that

$$C'\cdot\varepsilon^{\frac{1}{2p}}\leq\left(1-\left(\frac{1}{\alpha}\right)^{\frac{1}{2p}}\right)\left(\frac{1}{\mathrm{vol}\bar{\Omega}}\right)^{\frac{1}{2p}}.$$

Then,

$$\begin{aligned}\left(\frac{\mathrm{vol}B\left(\bar{H},r\right)}{\mathrm{vol}\bar{\Omega}}\right)^{\frac{1}{2p}}&-\left(\frac{\mathrm{vol}B\left(H,r\right)}{\mathrm{vol}\Omega}\right)^{\frac{1}{2p}}\\&\leq\left(1-\left(\frac{1}{\alpha}\right)^{\frac{1}{2p}}\right)\left(\frac{\mathrm{vol}B\left(\bar{H},r\right)}{\mathrm{vol}\bar{\Omega}}\right)^{\frac{1}{2p}},\end{aligned}$$

from which it follows that

$$\frac{\mathrm{vol}B\left(H,r\right)}{\mathrm{vol}\Omega}\geq\frac{1}{\alpha}\frac{\mathrm{vol}B\left(\bar{H},r\right)}{\mathrm{vol}\bar{\Omega}},$$

or,

$$\mathrm{vol}\Omega\leq\alpha\frac{\mathrm{vol}B\left(H,r\right)}{\mathrm{vol}B\left(\bar{H},r\right)}\mathrm{vol}\bar{\Omega}.$$

By letting $r\to 0$, we then get the desired result.

For the complementary domain $M-\Omega$ with volume $\leq\mathrm{vol}M/2$, we first have from the above that

$$\frac{\mathrm{area}H}{\mathrm{area}\bar{H}}\geq\frac{1}{\alpha}\frac{\mathrm{vol}\Omega}{\mathrm{vol}\bar{\Omega}}\geq\frac{1}{2\alpha}\frac{\mathrm{vol}M}{\mathrm{vol}\left(B\left(\bar{x}_0,D\right)\right)}.$$

Moreover,

$$\left(\frac{\mathrm{vol}\,(M-\Omega)}{\mathrm{vol}\left(B\left(\bar{x}_0,D\right)-\bar{\Omega}\right)}\right)^{\frac{1}{2p}}-\left(\frac{\mathrm{vol}B\left(H,r\right)}{\mathrm{vol}B\left(\bar{H},r\right)}\right)^{\frac{1}{2p}}\leq C\cdot\left(k\left(p,\kappa\right)\right)^{\frac{1}{2p}},$$

so if we let $r\to 0$, then

$$\left(\frac{\mathrm{vol}\,(M-\Omega)}{\mathrm{vol}\left(B\left(\bar{x}_0,D\right)-\bar{\Omega}\right)}\right)^{\frac{1}{2p}}-\left(\frac{\mathrm{area}H}{\mathrm{area}\bar{H}}\right)^{\frac{1}{2p}}\leq C\cdot\left(k\left(p,\kappa\right)\right)^{\frac{1}{2p}}.$$

Therefore, if ε is chosen so that

$$C\cdot(\varepsilon)^{\frac{1}{2p}}\leq\left(\alpha^{\frac{1}{2p}}-1\right)\left(2\alpha\mathrm{vol}B\left(\bar{x}_0,D\right)\right)^{-\frac{1}{2p}},$$

then we get

$$\begin{aligned}\left(\frac{\mathrm{vol}\,(M-\Omega)}{\mathrm{vol}\left(B\left(\bar{x}_0,D\right)-\bar{\Omega}\right)}\right)^{\frac{1}{2p}}&-\left(\frac{\mathrm{area}H}{\mathrm{area}\bar{H}}\right)^{\frac{1}{2p}}\\&\leq\left(\alpha^{\frac{1}{2p}}-1\right)\left(\frac{\mathrm{vol}M}{2\alpha\mathrm{vol}B\left(\bar{x}_0,D\right)}\right)^{\frac{1}{2p}}\\&\leq\left(\alpha^{\frac{1}{2p}}-1\right)\left(\frac{\mathrm{area}H}{\mathrm{area}\bar{H}}\right)^{\frac{1}{2p}}.\end{aligned}$$

Consequently,

$$\frac{\mathrm{vol}\,(M-\Omega)}{\mathrm{vol}\left(B\left(\bar{x}_0,D\right)-\bar{\Omega}\right)}\leq\alpha\frac{\mathrm{area}H}{\mathrm{area}\bar{H}}$$

as well. q.e.d.

5. Applications

The first result that we wish to look at is a generalization of Cheng's upper eigenvalue bounds from [11]. If we take a metric ball $B\left(\bar{x}_0,R\right)$ in S^n_κ, where we assume that $R\leq\pi/2\sqrt{\kappa}$ when $\kappa>0$, then the first eigenvalue for the Dirichlet problem is denoted by $\lambda_1^D\left(n,R,\kappa\right)$. The corresponding eigenfunctions are rotationally symmetric: $f\left(x\right)=\phi\left(r\right)$, where ϕ satisfies

$$\begin{aligned}\phi''+h_\kappa\phi'+\lambda_1\left(n,R,\kappa\right)\phi&=0,\\\phi\left(0\right)\text{ is bounded and }\phi\left(R\right)&=0.\end{aligned}$$

For convenience, we take ϕ to be the solution where $\phi(0) = 1$. This means that $0 \le \phi \le 1$, since $\phi' \le 0$ on $[0, R]$.

Theorem 5.1. *For every $\delta > 0$, there is an $\varepsilon(n, p, \kappa, R) > 0$ such that any Riemannian n-manifold M with $\bar{k}(p, \kappa, R) \le \varepsilon$ has the property that*

$$\lambda_1^D(B(x_0, R)) \le (1 + \delta)\,\lambda_1^D(n, \kappa, R).$$

Moreover, when $\kappa > 0$ we have that

$$\lambda_1(M) \le (1 + \delta)\,\lambda_1^D\left(n, \kappa, \frac{\mathrm{diam} M}{2}\right),$$

and $\lambda_1^D(n, \kappa, R) \to n\kappa$ as $R \to \pi/(2\sqrt{\kappa})$.

Proof. To get the desired upper eigenvalue bound, it suffices to compute the Rayleigh quotient of $f(x) = \phi(d(x_0, x))$. First we compute the square norm of the gradient:

$$\begin{aligned}
\int_{B(x_0,R)} |\nabla f|^2 \,d\mathrm{vol} &= \int_{S^{n-1}} \int_0^R (\phi')^2\, \omega(t, \theta)\, dt d\theta \\
&= \int_{S^{n-1}} \left(\phi\phi'\omega|_0^R - \int_0^R \phi\, (\phi'\omega)'\, dt \right) d\theta \\
&= -\int_{S^{n-1}} \int_0^R \phi\,(\phi'' + h\phi')\,\omega \\
&= -\int_{S^{n-1}} \int_0^R \phi\,(\phi'' + h_\kappa\phi')\,\omega \\
&\quad -\int_{S^{n-1}} \int_0^R (h - h_\kappa)\,\phi\phi'\omega \\
&\le \lambda_1^D(n, \kappa, R) \int_{S^{n-1}} \int_0^R \phi^2\omega \\
&\quad + \int_{S^{n-1}} \int_0^R (h - h_\kappa)_+ |\phi'|\,\omega.
\end{aligned}$$

The Rayleigh quotient then satisfies

$$\begin{aligned}
\mathrm{Q} &= \frac{\int_{B(x_0,R)} |\nabla f|^2 \,d\mathrm{vol}}{\int_{B(x_0,R)} f^2 d\mathrm{vol}} \\
&\le \lambda_1^D(n, \kappa, R) + \frac{\int_{S^{n-1}} \int_0^R (h - h_\kappa)_+ |\phi'|\,\omega}{\int_{S^{n-1}} \int_0^R \phi^2\omega}.
\end{aligned}$$

Now if we take $r = r(n, \kappa, R)$ to be the first value where $\phi(r) = 1/2$, then the error term can be estimated as follows:

$$\begin{aligned}
\frac{\int_{S^{n-1}} \int_0^R (h - h_\kappa)_+ |\phi'| \omega}{\int_{S^{n-1}} \int_0^R \phi^2 \omega} &\leq \frac{\sqrt{\int_{S^{n-1}} \int_0^R (h - h_\kappa)_+^2 \omega} \sqrt{\int_{S^{n-1}} \int_0^R |\phi'|^2 \omega}}{\sqrt{\frac{1}{4} \mathrm{vol} B(x_0, r)} \sqrt{\int_{S^{n-1}} \int_0^R \phi^2 \omega}} \\
&\leq 2 \sqrt{\frac{\int_{S^{n-1}} \int_0^R (h - h_\kappa)_+^2 \omega}{\mathrm{vol} B(x_0, r)}} \sqrt{\mathrm{Q}}.
\end{aligned}$$

We now need to use the smallness of $\bar{k}(p, \kappa, R)$ to conclude that we have a relative volume comparison estimate of the form

$$\frac{1}{\mathrm{vol} B(x_0, r)} \leq 4 \frac{v(n, \kappa, R)}{v(n, \kappa, r)} \frac{1}{\mathrm{vol} B(x_0, R)}.$$

Inserting this in the above estimate for the error yields

$$\begin{aligned}
&\frac{\int_{S^{n-1}} \int_0^R (h - h_\kappa)_+ |\phi'| \omega}{\int_{S^{n-1}} \int_0^R \phi^2 \omega} \\
&\qquad \leq 4 \sqrt{\frac{v(n, \kappa, R)}{v(n, \kappa, r)}} \sqrt{\frac{\int_{B(x_0, R)} (h - h_\kappa)_+^2 d\mathrm{vol}}{\mathrm{vol} B(x_0, R)}} \sqrt{\mathrm{Q}} \\
&\qquad \leq 4 \sqrt{\frac{v(n, \kappa, R)}{v(n, \kappa, r)}} \left(\frac{\int_{B(x_0, R)} (h - h_\kappa)_+^{2p} d\mathrm{vol}}{\mathrm{vol} B(x_0, R)} \right)^{\frac{1}{2p}} \sqrt{\mathrm{Q}} \\
&\qquad \leq C(n, p, \kappa, R) \left(\bar{k}(p, \kappa, R) \right)^{\frac{1}{2p}} \sqrt{\mathrm{Q}}.
\end{aligned}$$

Thus we have an equality of the form

$$\mathrm{Q} \leq \lambda_1^D(n, \kappa, R) + C(n, p, \kappa, R) \varepsilon^{\frac{1}{2p}} \sqrt{\mathrm{Q}},$$

which is easily turned into an inequality of the form

$$\mathrm{Q} \leq (1 + \delta) \lambda_1^D(n, \kappa, R)$$

as desired. q.e.d.

Gallot already obtained lower bounds for the first eigenvalue in case $\bar{k}(p, \kappa, R)$ is small. However, the question of obtaining almost optimal lower bounds in the case $\kappa > 0$ was still left open. Given that we have already established the necessary Heintze-Karcher volume comparison result, one can now proceed as in [18] (see also [15], [5], [4]) to show

Theorem 5.2. *For any $\alpha < 1$ and $\kappa > 0$, there is an $\varepsilon(n,p,\kappa) > 0$ such that any Riemannian n-manifold with $\bar{k}(p,\kappa) \le \varepsilon$ and* $\mathrm{diam} M \le D \le \pi/\sqrt{\kappa}$ *satisfies*

$$\lambda_1 \ge \alpha n \kappa \left(\frac{\int_0^{\frac{\pi}{2\sqrt{\kappa}}} \left(\cos\left(t \cdot \sqrt{\kappa}\right)\right)^{n-1} dt}{\int_0^{\frac{D}{2}} \left(\cos\left(t \cdot \sqrt{\kappa}\right)\right)^{n-1} dt} \right)^{\frac{2}{n}}.$$

The fact that we assume $D \le \pi/\sqrt{\kappa}$ is not as bad as it looks. Namely, our diameter bounds from section 3 show that when $\bar{k}(p,\kappa) < \varepsilon$, one always has a diameter bound of the form $D \le \pi/\sqrt{\kappa} + O(\varepsilon)$. Thus we can, by decreasing κ slightly, obtain almost optimal lower eigenvalue bounds. In addition, we see that if the first eigenvalue is almost minimal, then in fact the diameter must be almost maximal. The converse follows from the above generalization of Cheng's estimates. As pointed out in the introduction, Cheeger and Colding in [8] showed that either of these two conditions implies that the manifold is Gromov-Hausdorff close to a sine warped product, provided that $\mathrm{Ric} \ge n-1$. A similar result holds in this case, but we must assume that $\mathrm{vol} M \ge v$ in addition to the smallness of $\bar{k}(p,1)$ (see [36] for more details).

In addition to the lower eigenvalue bound just mentioned, one also gets Sobolev constant bounds (see [16],[18],[24]). Specifically, the smallness of $\bar{k}(p,\kappa,R)$ gives an estimate of the form

$$\left(\frac{1}{\mathrm{vol} B(x,R)} \int_{B(x,R)} |f|^{\frac{n}{n-1}} d\mathrm{vol} \right)^{\frac{n-1}{n}} \le S(n,p,\kappa,R) \frac{1}{\mathrm{vol} B(x,R)} \int_{B(x,R)} |\nabla f| \, d\mathrm{vol},$$

for functions f which vanish on $\partial B(x,R)$, and smallness of $\bar{k}(p,\kappa)$ together with $\mathrm{diam} M \le D$ gives

$$\left(\frac{1}{\mathrm{vol} M} \int_M \left(\left| f - \bar{f} \right|^{\frac{n}{n-1}} \right) d\mathrm{vol} \right)^{\frac{n-1}{n}} \le S(n,p,\kappa,D) \frac{1}{\mathrm{vol} M} \int_M |\nabla f| \, d\mathrm{vol},$$

where $\bar{f} = \frac{1}{\mathrm{vol} M} \int_M f \cdot d\mathrm{vol}$ is the average of the function.

It is worthwhile pointing out that the Sobolev constant bounds just mentioned also hold in case $\kappa \le 0$. In fact it is possible to obtain Sobolev constant bounds without the use of Gallot's bounds for the weaker Sobolev constants. To see this requires some changes in the setup. First we observe that possibly scaling the metric it suffices to consider the case where $\kappa = 0$. We then consider a manifold M with $\bar{k}(p,0) \leq \varepsilon$ and diam $\leq D$. Next we fix a hypersurface $H \subset M$ of constant mean curvature $\eta \geq 0$ which divides M in to two parts $\Omega_{\pm}$ of diameter $d_{\pm}$, where $d_{+} + d_{-} \leq D$. We now have to estimate the volume ratios

$$\frac{\mathrm{vol}H}{\mathrm{vol}\Omega_{\pm}}.$$

There is not necessarily a clearly defined ratio in Euclidean space we can compare with as in the above situation. However, it is easy to find an explicit expression we can compare it with. Namely, consider the integrals

$$\mathrm{vol}H \int_0^{d_{\pm}} (1 \pm \eta t)_+^{n-1}\, dt$$

that correspond to the volumes of the corresponding domains in $(-d_{-}, d_{+}) \times H$ if we introduce a radially flat metric on this space such that the second fundamental form of $\{0\} \times H$ is $\eta/(n-1)$. On slight problem occurs if $d_{-} > \eta^{-1}$, corresponding to the fact that we haven't derived a diameter estimate for the region Ω_{-}. In that case we use as comparion integral for $\mathrm{vol}\Omega_{-}$ the expression

$$\mathrm{vol}H \int_0^{d_{-}} \left(1 - \frac{1}{d_{-}} t\right)_+^{n-1} dt,$$

which is actually larger than

$$\mathrm{vol}H \int_0^{d_{-}} (1 - \eta t)_+^{n-1}\, dt.$$

We now claim that for any $\alpha > 1$ we can choose $\varepsilon(n,p,D) > 0$ so small that

$$\begin{aligned} \mathrm{vol}\Omega_{+} &\leq \alpha \mathrm{vol}H \int_0^{d_{+}} (1 + \eta t)_+^{n-1}\, dt, \\ \mathrm{vol}\Omega_{-} &\leq \alpha \mathrm{vol}H \max\left\{ \int_0^{d_{-}} \left(1 - \frac{1}{d_{-}} t\right)_+^{n-1} dt, \int_0^{d_{-}} (1 - \eta t)_+^{n-1}\, dt \right\}. \end{aligned}$$

From these estimates one can then in standard fashion derive estimates for the classical Sobolev constant. To obtain these estimates one simply has to redo the mean curvature estimates from Section 2 with $\kappa = 0$.The proofs of these estimates are virtually identical.

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