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RIGIDITY FOR ASPHERICAL MANIFOLDS WITH $\pi_1 \subset GL_m(\mathbb{R})$*

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0. Introduction. Recall that a closed manifold M is *topologically rigid* if any homotopy equivalence $h : N \to M$ from another manifold N is homotopic to a homeomorphism. The Poincaré conjecture states that the n-sphere S^n is topologically rigid; however most simply connected closed manifolds are obstructed from being topologically rigid by the Pontryagin classes of the manifold. A non-simply connected manifold M is further obstructed from being topologically rigid by Wall surgery group invariants and Whitehead torsion invariants for its fundamental group. Considering all these possible obstructions to rigidity it is remarkable that the following conjecture of A. Borel is still open.

BOREL'S CONJECTURE. *Every closed aspherical manifold is topologically rigid.*

The authors have previously verified topological rigidity for all closed non-positively curved Riemannian manifolds of dim $\neq 3, 4$ in [20], [23], and for every closed aspherical manifold of dim $\neq 3$ whose fundamental group is virtually poly-$\mathbb{Z}$ in [19]. This last class is consequently the same up to homeomorphism as the class of all closed infrasolvmanifolds of dim $\neq 3$; cf. [21, Th. 2.12, 2.16 and Def. 2.10, 2.11]. (We wish here to correct a misprint in [21, p. 18, lines 4 and 5] where the word "solvable" should be replaced by "poly-$\mathbb{Z}$".) We verify in this paper topological rigidity for another class of classical aspherical manifolds; namely we show that all compact complete affine flat manifolds (of dim $\neq 3, 4$) are topologically rigid; cf. Corollary 0.2. We noted in [19] that this rigidity result would follow directly from the main result of [19], stated above, provided a conjecture of Milnor was verified; cf. [19], [21, Conjecture 2.18]. He conjectured in [47] that the fundamental group of every compact complete affine flat manifold is virtually poly-$\mathbb{Z}$. However, this conjecture is still open. (Also Margulis constructed a counterexample [46] to Milnor's conjecture in which the compactness condition was not required.) The proof of this new rigidity result for affine flat manifolds is consequently much more complicated.

We say that a compact manifold M with boundary ∂M is *topologically rigid* if any homotopy equivalence $h : (N, \partial N) \to (M, \partial M)$ where N is another compact manifold and the restriction $h \mid \partial N$ is a homeomorphism satisfies the following. There exists a homotopy $h_t : (N, \partial N) \to (M, \partial M)$ from h to a homeomorphism with $h_t \mid \partial N = h \mid \partial N$. (Where $\partial M = \phi$, this agrees with the notion of topological rigidity already defined above.)

Let E be the total space of any closed n-ball bundle over a closed manifold M^m. Then E^{m+n} is a compact manifold with boundary. We previously proved topological rigidity for all such E provided $m+n \geq 5$ and M is a non-positively curved Riemannian manifold [20], [23] or M is aspherical and $\pi_1 M$ is virtually poly-$\mathbb{Z}$ [19]. We moreover prove in this paper that E is topologically rigid when $m + n \geq 5$ and M is a complete affine flat manifold; cf. Corollary 0.2.

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Compact manifolds with boundary also occur in geometry as manifold compactifications of complete Riemannian manifolds. Recall a *manifold compactification* of a manifold M is a compact manifold $\bar{M}$ whose interior is M; i.e., $M = \bar{M} - \partial\bar{M}$. Manifold compactifications for certain classes of non-positively curved Riemannian manifolds have been constructed by Raghunathan [55], Borel-Serre [4], Gromov [34], Heintze [36], Margulis, cf. [34], among others. Let M be in particular a complete (connected) Riemannian manifold with finite volume and pinched sectional curvature, then it has a smooth manifold compactification in the following two situations:

1. M has negative sectional curvature bounded away from 0 (and $-\infty$);
2. M is a real analytic manifold of non-positive sectional curvature.

(See [3, Th. 10.5, 13.1].) Note that class (2) includes all complete (connected) non-positively curved locally symmetric spaces with finite volume.

We next formulate the main result of this paper. To do this we must recall the definition of A-regular; cf. [9, p. 334]. A Riemannian manifold is *A-regular* if for some nonnegative sequence $A = \{A_i\}$ we have

$$| \nabla^i R| \leq A_i$$

where the indices i vary over the natural numbers and $\nabla^i R$ is the i-th covariant derivative of the curvature tensor. Note the 0-th condition means the sectional curvatures are pinched; i.e., bounded away from $\pm\infty$. For example, every connected locally symmetric space is A-regular since these manifolds are defined by the condition $\nabla R = 0$. Also, Shi [62] and Abresch [1] have shown that any complete Riemannian metric g with pinched sectional curvature can be approximated by a complete A-regular metric $\bar{g}$. Furthermore, if the sectional curvatures of g are contained in the interval $[a, b]$, then, given $\epsilon > 0$, $\bar{g}$ can be found so that its sectional curvatures are contained in $[a - \epsilon, b + \epsilon]$. We are very grateful to Professor Shi for showing us how this is done. This fact will be used often in Section 7.

THEOREM 0.1. *Let M be a compact aspherical manifold with possibly non-empty boundary. Suppose that* dim $M \neq 3, 4$ *and $\pi_1 M$ is isomorphic to the fundamental group of an A-regular complete non-positively curved Riemannian manifold. Then M is topologically rigid.*

Return now to the definition of a compact manifold M with boundary being topologically rigid. And drop the conditions that $h \mid \partial N$ is a homeomorphism and $h_t \mid \partial N = h \mid \partial N$. The resulting concept is called *absolute topological rigidity.* Absolute topological rigidity is of course the same as topological rigidity when $\partial M = \phi$. And we can deduce absolute topological rigidity from Theorem 0.1 in certain other cases. For example, let M be a complete pinched negatively curved Riemannian manifold with finite volume (i.e., a manifold from class (1) above) with dim $M \geq 5$, then every manifold compactification of M is absolutely topologically rigid. This result and variants of it are proven in Section 7. We, in particular, show there that any isomorphism between fundamental groups of two manifolds from class (1) with dim ≥ 5 is induced by a homeomorphism (see Corollary 7.5). On the other hand, absolute topological rigidity can be obstructed by Pontryagin classes for many manifolds of class (2). For example, it is shown in [18, §4] that the Borel-Serre compactification of the double coset space

$$\Gamma_m \backslash SL_m(\mathbb{R})/SO(m)$$

where Γ_m is any torsion-free subgroup of finite index in $SL_m(\mathbb{Z})$ is not absolutely topologically rigid when $m > 200$. The question of the uniqueness of manifold compactifications of manifolds of classes (1) and (2) is also addressed in Section 7.

Applications to complete affine flat manifolds. Let $\mathbb{A}(m)$ denote the group of all affine motions of m-dimensional Euclidean space $\mathbb{R}^m$. Then $\mathbb{A}(m)$ can be identified with the closed subgroup of $GL_{m+1}(\mathbb{R})$ consisting of all matrices B such that

$$B_{m+1,i} = \begin{cases} 0 & \text{if } i < m+1 \\ \\ 1 & \text{if } i = m+1. \end{cases}$$

Recall that a complete affine flat manifold M is an orbit space $M = \Gamma\backslash\mathbb{R}^m$ where Γ is a subgroup of $\mathbb{A}(m)$ which acts fixed point freely and properly discontinuously on $\mathbb{R}^m$. Consequently, Γ is a discrete torsion-free subgroup of $GL_{m+1}(\mathbb{R})$ and the double coset space

$$\Gamma\backslash GL_{m+1}(\mathbb{R})/O(m+1)$$

is a complete non-positively curved Riemannian locally symmetric space whose fundamental group Γ is isomorphic to $\pi_1 M$. Hence we may apply Theorem 0.1 to obtain the following corollary mentioned in the second paragraph of this introduction.

COROLLARY 0.2. *Let M^m be any compact complete affine flat manifold (where $m = \dim M$) and let E be the total space of any closed n-ball bundle whose base space is M^m. If $m+n \geq 5$, then E is topologically rigid. Consequently, M^m is topologically rigid when $m \neq 3, 4$. In particular, if N is any other compact complete affine flat manifold, then any isomorphism $\pi_1 N \to \pi_1 M$ is induced by a homeomorphism $N \to M$ provided* $\dim M \neq 3, 4$.

It is an open problem to determine whether an isomorphism $\pi_1 N \to \pi_1 M$ between the fundamental groups of compact complete affine flat manifolds can always be induced by a diffeomorphism.

Application to double coset spaces. Recall that a *linear Lie group* is a (virtually connected) Lie group which admits a faithful representation into $GL_n(\mathbb{R})$ for some integer n. There are Lie groups which are not linear; e.g., the universal covering group of $SL_n(\mathbb{R})$ is not linear when $n \geq 2$; cf. [66, Chapter 2, exercise 15]. Ado's theorem on the other hand states that any Lie group is locally isomorphic to a linear Lie group; cf. [56, p. 3]. Let G be a linear Lie group and $\rho : G \to GL_n(\mathbb{R})$ be a faithful representation. Although image(ρ) may not be a closed subgroup of $GL_n(\mathbb{R})$, both $[G^0, G^0]$ and $\rho([G^0, G^0])$ are respectively closed subgroups of G and $GL_n(\mathbb{R})$ where G^0 is the component of G containing the identity element; cf. [66, chap. 3, exercises 41 and 45]. Let $\mathbb{E}(m)$ denote the group of all rigid motions of Euclidean space $\mathbb{R}^m$ and recall that $\mathbb{E}(m)$ can be identified with a closed subgroup of $GL_{m+1}(\mathbb{R})$. Since $G^0/[G^0, G^0]$ is abelian, one easily constructs a representation $\tau : G/[G^0, G^0] \to \mathbb{E}(m)$, for some integer m, with closed image and compact kernel. Let ψ denote the composition

$$G \to G/[G^0, G^0] \xrightarrow{\tau} \mathbb{E}(m) \subset GL_{m+1}(\mathbb{R}).$$

Then the representation ξ which is the composition

$$G \xrightarrow{\rho\times\psi} GL_n(\mathbb{R}) \times GL_{m+1}(\mathbb{R}) \subset GL_{n+m+1}(\mathbb{R})$$

is both faithful and has closed image. If Γ is a discrete torsion-free subgroup of G, then the double coset space

$$\xi(\Gamma)\backslash GL_{n+m+1}(\mathbb{R})/O(n+m+1)$$

is a complete non-positively curved locally symmetric space whose fundamental group is isomorphic to Γ. Consequently, any discrete torsion-free subgroup of a linear Lie group is isomorphic to the fundamental group of an A-regular complete non-positively curved Riemannian manifold. Hence an application of Theorem 0.1 yields the following result.

COROLLARY 0.3. *Every compact aspherical manifold M (with possibly non-empty boundary) such that $\pi_1 M$ is isomorphic to a discrete subgroup of a linear Lie group is topologically rigid provided* $\dim M \neq 3, 4$. *In particular, let G be a linear Lie group, K a maximal compact subgroup of G, and Γ a torsion-free discrete subgroup of G. Then any manifold compactification of the double coset space $\Gamma \backslash G/K$ is topologically rigid provided* $\dim(G/K) \neq 3, 4$. *Consequently if Γ is cocompact in G and* $\dim(G/K) \neq 3, 4$, *then $\Gamma \backslash G/K$ is topologically rigid.*

REMARK 0.4. Consider the situation of two double coset spaces $M_i = \Gamma_i \backslash G_i / K_i$, $i = 1, 2$, where G_i is a virtually connected Lie group, K_i is a maximal compact subgroup of G_i, and Γ_i is a discrete torsion-free cocompact subgroup of G_i. Then M_i is a smooth closed aspherical manifold with $\pi_1 M_i = \Gamma_i$. Let $\alpha : \Gamma_1 \to \Gamma_2$ be an isomorphism. Then Mostow [51], [52] showed that M_1 and M_2 are differentiably commensurable; i.e., they have a common smooth finite sheeted covering space. He also showed that α is induced by a diffeomorphism when either both G_i are linear semi-simple Lie groups [50] or when both G_i are solvable analytic groups [49]. Corollary 0.3 augments Mostow's results a bit. Namely, it shows that α is induced by a homeomorphism when G_1 is linear and $\dim M_1 \neq 3, 4$. We believe an interesting problem is to decide whether α is always induced by a diffeomorphism.

Let M be an arbitrary manifold; i.e., it can be non-compact and can have non-empty boundary. We say that M is *topologically rigid* if it has the following property. Let $h : (N, \partial N) \to (M, \partial M)$ be any proper homotopy equivalence where N is another manifold. Suppose there exists a compact subset $C \subset N$ such that the restriction of h to $\partial N \cup (N - C)$ is a homeomorphism. Then there exists a proper homotopy $h_t : (N, \partial N) \to (M, \partial M)$ from h to a homeomorphism and a perhaps larger compact subset K of N such that the restrictions of h_t and h to $\partial N \cup (N - K)$ agree for all $t \in [0, 1]$. (When M is compact, this is the same concept of topological rigidity already defined.)

We now formulate the most general version of the main result of this paper.

ADDENDUM 0.5. *Let M be an (arbitrary) aspherical manifold with* $\dim M \geq 5$. *Suppose $\pi_1 M$ is isomorphic to the fundamental group of an A-regular complete non-positively curved Riemannian manifold. (This happens for example when $\pi_1 M$ is isomorphic to a torsion free discrete subgroup of a linear Lie group.) Then M is topologically rigid. In particular, every A-regular complete non-positively curved Riemannian manifold of* $\dim \geq 5$ *is topologically rigid.*

Further applications. We discuss further applications of Theorem 0.1 and Addendum 0.5 in Section 7 of this paper. Uniqueness of manifold compactifications is investigated there as well as the absolute rigidity results mentioned earlier. We also verify in Section 7 a special case of the following well known conjecture concerning the class of fundamental groups of closed aspherical manifolds; cf. [10].

CONJECTURE 0.6. *Let Γ be a torsion-free group which contains a subgroup of finite index isomorphic to the fundamental group of a closed aspherical manifold. Then Γ is the fundamental group of a closed aspherical manifold.*

We then apply this partial verification to obtain results on the following Nielsen type problem; cf. [10]. Let $\mathrm{Top}(M)$ denote the group of all homeomorphisms of a manifold and denote the group of all outer automorphisms of $\pi_1 M$ by $\mathrm{Out}(\pi_1 M)$.

PROBLEM 0.7. *Let M be a closed aspherical manifold and F be a finite subgroup of* $\mathrm{Out}(\pi_1 M)$. *Does F split back to* $\mathrm{Top}(M)$*; i.e., does there exist a finite subgroup $\bar{F}$ of* $\mathrm{Top}(M)$ *which maps isomorphically onto F under the natural homomorphism* $\mathrm{Top}(M) \to \mathrm{Out}(\pi_1 \mathrm{M})$*?*

In the remainder of this section we give an outline of the proof of Theorem 0.1. This proof will be carried out in detail in Sections 1-6. Addendum 0.5 can be proven by a routine modification of the argument given here for Theorem 0.1; hence we omit its details.

The proof of Theorem 0.1 relies on the following two propositions. Proposition 0.10 will be proven in Section 2. Proposition 0.8 is a paraphrasing of results of Kirby-Siebenman [43], Ranicki [58], Sullivan [65], and Wall [67] (cf. Remark 0.9). We let $\mathcal{L}_*$ denote the surgery spectrum for a point in the propositions; i.e., $\mathcal{L}_* = \{\mathcal{L}_i \mid i \in \mathbb{Z}\}$ where the loop space $\Omega\mathcal{L}_{i+1}$ is homotopy equivalent to $\mathcal{L}_i$ and where the homotopy groups of $\mathcal{L}_0$ are just the surgery groups for simply connected surgery. Let $\mathbb{Z}_2$ denote the additive group of the ring of integers mod 2. For any path connected space M equipped with a homomorphism $w : \pi_1 M \to \mathbb{Z}_2$, there is an "assembly map"

$$A_* : H_*(M, \mathcal{L}_*) \to L_*(\pi_1 M, w)$$

from the generalized homology groups for M with (twisted) coefficients in $\mathcal{L}_*$ into the surgery groups for the fundamental group data $w : \pi_1 M \to \mathbb{Z}_2$. (The twisting for the $\mathcal{L}_*$ over M is also induced by w. If $w = 0$, then there is no twisting.)

PROPOSITION 0.8. *The assembly map A_* is uniquely determined by the homotopy type of M and the homomorphism w. Suppose M is a compact (connected) manifold with* $\dim M \geq 5$ *and possibly non-empty boundary, and w is the 1st Stiefel-Whitney class $\omega_1(M)$ of M. Then M is topologically rigid provided A_* is an isomorphism and* $Wh(\pi_1 M) = 0$.

REMARK 0.9. According to Kirby-Siebenmann [43] we have the following when M is a compact (connected) manifold. There is a (dual) assembly map

$$A^* : H^*(M, \partial M, G/\mathrm{TOP}_*) \to L_{m-*}(\pi_1 M, \omega_1(M))$$

from the generalized cohomology for M with (untwisted) coefficients in the spectrum G/TOP_* which has G/TOP as its zero'th level space. If A^* is an isomorphism then the closed manifold M is topologically rigid provided $\dim M \geq 5$. The assembly map A_* is gotten by composing A^* with the Lefschetz duality map

$$H_*(M, G/\mathrm{TOP}_*) \cong H^{m-*}(M, \partial M, G/\mathrm{TOP}_*)$$

and using the canonical identification $G/\mathrm{TOP}_* \cong \mathcal{L}_*$.

PROPOSITION 0.10. *Let X denote an A-regular complete (connected but not necessarily compact) non-positively curved Riemannian manifold. Then the assembly map*

$$A_* : H_*(X, \mathcal{L}_*) \to L_*(\pi_1 X, \omega_1(X))$$

is an isomorphism; moreover $Wh(\pi_1(X)) = 0$.

REMARK 0.11. The authors have proven 0.10 for compact X [23], [24]. Bizhong Hu has proven a version of 0.10 for X equal a non-positively curved finite simplicial complex [38], [39], [40]. The Novikov conjecture for X of 0.10, which was proven by Miscenko [48] and Kasparov [41], is equivalent to the statement that A_* is rationally injective. The stronger result, that A_* is integrally injective, was proven by Farrell and Hsiang [17] and by Ferry and Weinberger [32]. Our proof of 0.10 given in Section 2 does not rely on these injectivity results, but does use the ideas from the authors' proofs for rigidity in [20], [23].

Proof of Theorem 0.1. We may assume that dim $M \geq 5$ since the case where dim $M \leq 2$ is already known. Let X be an A-regular (connected) complete non-positively curved Riemannian manifold with $\pi_1 X$ isomorphic to $\pi_1 M$. We can also assume the $\omega_1(X) = \omega_1(M)$ because of Lemma 7.15 of Section 7. Proposition 0.10 states that the assembly map

$$A_* : H_*(X, \mathcal{L}_*) \to L_*(\pi_1 X, \omega_1(X))$$

is an isomorphism. Combining this result with the first sentence of Proposition 0.8 yields that the assembly map

$$A_* : H_*(M, \mathcal{L}_*) \to L_*(\pi_1 M, \omega_1(M))$$

is also an isomorphism. Hence M is topologically rigid because of the second sentence of Proposition 0.8 and the fact that $Wh(\pi_1 M) = 0$ also given in Proposition 0.10.

This completes the proof for Theorem 0.1.

In the remainder of this section we shall indicate all the other major ingredients of this paper, and describe just how they fit together to prove Proposition 0.10.

In Section 2 we prove the following two Lemmas from which Proposition 0.10 immediately follows.

LEMMA 0.12. *Let Γ be the fundamental group of an A-regular complete (connected) non-positively curved Riemannian manifold X. Then $Wh(\Gamma \times \mathbb{Z}^n) = 0$, where $\mathbb{Z}^n$ denotes the free abelian group on n generators. Consequently, $K_j(\mathbb{Z}\Gamma) = 0$ for all $j \leq -1$, $\tilde{K}_0(\mathbb{Z}\Gamma) = 0$ and $Wh(\Gamma) = 0$.*

REMARK 0.13. The authors have previously proven 0.12 for compact X in [24]. Bizhong Hu has proven a version of 0.12 for X equal a non-positively curved finite simplicial complex in [38], [39].

LEMMA 0.14. *Let X be as in 0.12. Then the assembly map*

$$A_*^\infty : H_*(X, \mathcal{L}_*) \to L_*^\infty(\pi_1 X, \omega_1(X))$$

is an isomorphism.

REMARK 0.15. The $L_*^\infty(\pi_1 X, w)$ are surgery groups in whose definition all possible Whitehead torsion difficulties are systematically ignored (cf. Ranicki [57]). If the conclusions of 0.12 hold for X then it is known that

$$L_*^\infty(\pi_1 X, w) = L_*(\pi_1 X, w)$$

and that the two assembly maps A_*^∞ and A_* for X are equal (cf. Ranicki [57]). Thus 0.12 and 0.14 together imply 0.10.

We have proven versions of Lemmas 0.12 and 0.14 in [23], [24], for the special case when X is assumed to be closed. These proofs consist of a "focal transfer" to gain topological control, and then an application of a topological control theorem. We use these same arguments, together with the new and improved topological control results formulated in Section 1, to prove 0.12 and 0.14 in Section 2.

Topological control theory. We formulate our new topological control results in Section 1. The space over which "control" is measured is a complete A-regular Riemannian manifold M equipped with a smooth one-dimensional foliation $\mathcal{F}$. We assume that the geometry of the foliation $\mathcal{F}$ is "bounded" (cf. 1.3), but we do not assume any lower bound on the radius of injectivity for M or for $\mathcal{F}$. (We have in effect assumed a lower bound on the radius of injectivity of M and $\mathcal{F}$ in all our previous work on control theory (cf. [25], [26]). (Dropping this assumption is the major technical achievement of this paper.) We prove control theorems for h-cobordisms (1.1, 1.3) and for homotopy equivalences (1.2, 1.4). These include both "absolute" control theorems (e.g. where we have ϵ-control in M) and "foliated" control theorems (e.g. where we have (α, δ)-control in $(M, \mathcal{F})$).

We have proven the control theorems of Section 1 in our paper [26] for the special case when M is closed. If M is not closed, but M and $\mathcal{F}$ do have lower bounds to their radii of injectivity, then the "absolute" control results of Section 1 have been proven in [25], and the foliated control results of Section 1 can be proven by a careful combining of the arguments in [25] with those in [26].

If there are no lower bounds on the injectivity radii for M and $\mathcal{F}$ then a new geometric concept is needed: the theory of "collapsing Riemannian manifolds" due to Cheeger-Fukaya-Gromov [9]. Here is how their theory works. For small enough $\epsilon > 0$ every subset $S \subset M$ of diameter less than ϵ is contained in a larger subset $T \subset M$ of diameter less than 2ϵ which is smoothly stratified with strata $\{T_i\}$, and each stratum T_i is equipped with a smooth fiber bundle projection $p_i : T_i \to B_i$ onto a ball of radius $3\epsilon/2$ which has for fibers closed infranilmanifolds and which multiplies distances by less than 2. This theory also takes into account how the projections $\{p_i : T_i \to B_i\}$ fit together, and how (for overlapping subsets T, T') the projections $\{p_i : T_i \to B_i\}$ and $\{p'_i : T'_i \to B'_i\}$ are related. As a first approach to proving the absolute control theorems of Section 1 we could try applying suitable fibered control results of Chapman-Ferry-Quinn [7], [8], [59] to that part of our h-cobordism (or homotopy equivalence) which lies over a suitable collection $\mathcal{C}$ of the $\{T\}$ which cover all of M, where control is now measured in the $\{B_i\}$ with respect to the projections $\{p_i\}$. (These fibered control results may be applied only when the fiber meets some special rigidity and Whitehead torsion criteria: that infranilmanifold fibers meet these criteria follows from work of Farrell and Hsiang [15], [16].) This approach would require an (downward) induction argument over the dimension of the strata, and the projections $p_i : T_i \to B_i$ would have to be replaced by projections $\bar{p}_i : \bar{T}_i \to B_i$ where $\bar{T}_i$ is a small tubular neighborhood for T_i in M and where $\bar{p}_i$ denotes the composition of p_i with the orthogonal projection $\bar{T}_i \to T_i$. Moreover the collection $\mathcal{C}$ of the $\{T_i\}$ would have to be chosen so as to satisfy the following property: for any $T \in \mathcal{C}$ there are at most m' other $T' \in \mathcal{C}$ which overlap with T, where m' depends only on $m = \dim M$. This last property will assure that we don't lose control as we apply the fibered control theorems over each $T \in \mathcal{C}$.

Unfortunately there seems to be no direct way to apply the collapsing theory of Cheeger-Fukaya-Gromov to the foliated situation. What we must do is to combine the geometric ideas of Cheeger-Fukaya-Gromov [9] with the geometric ideas of the authors' [26], to obtain a weaker but more general theory of collapsing Riemannian manifolds, which can be used (as in the preceding paragraph) to prove both the absolute control theorems and the foliated control theorems of Section 1. The proofs for the absolute control results are given in Section 4 using the absolute collapsing results stated in Section 3; and the proofs of the foliated control results are given in Section 6 using the foliated collapsing results stated in Section 5. We remark that in applying the fibered control results of Chapman-Ferry-Quinn to prove the foliated control results of Section 1 we are dealing with infrasolvmanifold fibers (instead of infranilmanifold fibers): that infrasolv fibers meet the necessary rigidity and Whitehead torsion criteria of fibered control theory follows from the work of Farrell and Hsiang [15] and from the work of Farrell and Jones [19].

Collapsing theory for Riemannian manifolds. Our main results for this theory are formulated in Section 3 and Section 5. We formulate both "absolute" collapsing results (cf. 3.3, 3.4, 3.5) and "foliated" collapsing results (cf. 5.3, 5.4, 5.5). These results are proven in separate papers [29], [30]. Our "absolute" collapsing results can be deduced from the much deeper results of Cheeger-Fukaya-Gromov [9]; we then deduce our foliated collapsing results from these absolute collapsing results by using arguments similar to those in the authors' paper [26].

1. Statement of control results.

We let M denote a Riemannian manifold and we let $\mathcal{F}$ denote a smooth one-dimensional foliation of M. In this section we formulate topological control results for h-cobordisms over M and homotopy equivalences over M: two such theorems (1.1, 1.2) when the objects are ϵ-controlled over M (the absolute control results); and two such theorems (1.3, 1.4) when the objects are (α, ϵ)-controlled over $(M, \mathcal{F})$ (the foliated control results).

The absolute control results. We let W denote a cobordism of M, i.e., $\partial_- W = M$. A *product structure* for W over a subset $Y \subset M$ consists of a topological embedding $h : Y \times [0,1] \to W$ such that $h(y,0) = y$ for all $y \in Y$ and $h(Y \times 1) \subset \partial_+ W$. We will say that a pair (W, h) is *ϵ-controlled over M* if there are deformation retracts $r_\pm : W \times [0,1] \to W$ of W onto $\partial_\pm W$ such that all paths

$$w \times [0,1] \subset W \times [0,1] \xrightarrow{r_\pm} W = W \times 1 \xrightarrow{r_-} M, \quad w \in W,$$

have diameter less than ϵ in M, and such that $r_-(h(y,s),t) = h(y, s(1-t))$ and $r_+(h(y,s),t) = h(y, s(1-t)+t)$ for all $y \in Y$, $s,t \in [0,1]$.

THEOREM 1.1. *Suppose that M is a complete A-regular Riemannian manifold with* $\dim M \geq 5$. *For any sufficiently small number $\delta > 0$ there is a number $\delta' > 0$ satisfying $\lim_{\delta \to 0} \delta' = 0$. Let W denote a cobordism over M and let $h : Y \times [0,1] \to W$ denote a product structure for W over a subset $Y \subset M$. Then if (W,h) is δ-controlled over M there is a product structure $g : M \times [0,1] \to W$ for W over all of M that satisfies the following properties.*

(a) $g \mid Z \times [0,1] = h \mid Z \times [0,1]$, where $Z = Y - (M - Y)^{\delta'}$. (For each subset $S \subset M$ and any number $t > 0$ we let S^t denote the set $\{x \in M : d_M(x,S) < t\}$.)

(b) g is δ'-controlled over M.

REMARK. Property (b) is equivalent to g being a proper map; i.e., that g is a homeomorphism onto W.

In the next theorem we let $p : E \to M$ denote a fiber bundle over M having a closed manifold for fiber, and for each $n = 1, 2, 3, \ldots$ denote by $p_n : E_n \to M$ the composite map

$$E \times T^n \xrightarrow{\text{proj.}} E \xrightarrow{p} M$$

where T^n is the n-dimensional torus. A proper map $h : N \to E$ from the topological manifold N is *split over a subcomplex* $L \subset K$ *of a triangulation* K *for* M if, for each $\Delta \in L$, h is in transverse position to $p^{-1}(\Delta, \partial\Delta)$ and the restricted map

$$h : h^{-1}(p^{-1}(\Delta, \partial\Delta)) \to p^{-1}(\Delta, \partial\Delta)$$

is a homotopy equivalence. When $h : N \to E$ is a proper homotopy equivalence, we say that it is ϵ*-controlled over* M if there is a proper map $g : E \to N$, and proper homotopies $H : N \times [0,1] \to N$ and $G : E \times [0,1] \to E$ of $g \circ h$ and $h \circ g$ to the identity maps $N \to N$ and $E \to E$, such that each of the paths

$$x \times [0,1] \subset E \times [0,1] \xrightarrow{G} E \xrightarrow{p} M, \quad x \in E,$$

$$y \times [0,1] \subset N \times [0,1] \xrightarrow{H} N \xrightarrow{h} E \xrightarrow{p} M, \quad y \in N,$$

has diameter less than ϵ in M. A homotopy $F : N \times [0,1] \to E$ is said to be ϵ*-controlled over* M if each of the paths

$$y \times [0,1] \subset N \times [0,1] \xrightarrow{F} E \xrightarrow{p} M, \quad y \in N,$$

has diameter less than ϵ in M.

THEOREM 1.2. *Suppose that M is a complete A-regular Riemannian manifold. For any sufficiently small number $\delta > 0$ there is a number $\delta' > 0$ satisfying $\lim_{\delta \to 0} \delta' = 0$. Suppose that K is a piecewise smooth triangulation of M such that each $\Delta \in K$ has diameter less than δ, and suppose that $h : N \to E$ is a proper homotopy equivalence which is split over a subcomplex $L \subset K$ and is δ-controlled over M. Then for sufficiently large integers n there is a proper homotopy $h_t : N \times T^n \to E_n$, $t \in [0,1]$, of $h \times \mathrm{id}_{T^n}$ that satisfies the following properties.*

(a) *Let $L' \subset L$ denote the maximal subcomplex of L such that $|L'| \subset |L| - (M - |L|)^{\delta'}$. Then each $h_t : N \times T^n \to E_n$ is split over L'.*

(b) *$h_1 : N \times T^n \to E_n$ is split over all of K.*

(c) *The homotopy $h_t : N \times T^n \to E_n$, $t \in [0,1]$, is δ'-controlled over M.*

The proofs of Theorems 1.1 and 1.2 will be given in Section 4.

The foliated control results. We shall say that the foliation $\mathcal{F}$ of the Riemannian manifold M is *A-regular*, for some non-negative sequence $A = \{A_i\}$, if for each locally defined unit cross section F of $T\mathcal{F}$ we have that

$$|\nabla^i F| \leq A_i$$

where the indices i vary over all natural numbers and where $\nabla^i F$ is the i-th covariant derivative of F.

Let W denote a cobordism over M and let $h : Y \times [0,1] \to W$ denote a product structure for W over $Y \subset M$. We will say that (W, h) is *(γ, δ)-controlled over* $(M, \mathcal{F})$ if there are deformation retracts $r_\pm : W \times [0,1] \to W$ of W onto $\partial_\pm W$ which satisfy the following properties. Firstly we must have that $r_-(h(y,s),t) = h(y, s(1-t))$ and $r_+(h(y,s),t) = h(y, s(1-t)+t)$ for all $y \in Y$ and $s,t \in [0,1]$. Next for each $w \in W$ let $p_w^\pm : [0,1] \to M$ denote the composite path

$$[0,1] = w \times [0,1] \subset W \times [0,1] \xrightarrow{r_\pm} W = W \times 1 \xrightarrow{r_-} M.$$

Then each such path $p_w^{\pm}$ must have foliated diameter less than (γ, δ) in $(M, \mathcal{F})$. (A subset $A \subset M$ is said to have *foliated diameter less than* (γ, δ) *in* $(M, \mathcal{F})$ if for any two points $x, y \in A$ there is a path $g_1 : [0,1] \to M$ with $g_1(0) = x$ whose image lies in a segment of a leaf of $\mathcal{F}$ having length less than γ, and there is a path $g_2 : [0,1] \to M$ having length less than δ with $g_2(0) = g_1(1)$ and $g_2(1) = y$.)

THEOREM 1.3. *Suppose that M is a complete A-regular Riemannian manifold with* $\dim M \geq 5$, *and suppose that $\mathcal{F}$ is an A-regular one-dimensional foliation for M. There is a number $\lambda > 1$ which depends only on* $\dim M$. *Given any $\gamma > 0$ there is for sufficiently small $\delta > 0$ a number $\delta' > 0$ satisfying $\lim\limits_{\delta \to 0} \delta' = 0$. Let W denote a cobordism over M and let $h : Y \times [0,1] \to W$ be a product structure for W over $Y \subset M$. If (W, h) is (γ, δ)-controlled over $(M, \mathcal{F})$ then there is a product structure $g : M \times [0,1] \to W$ for W over all of M that satisfies the following properties.*

(a) *$g \mid Z \times [0,1] = h \mid Z \times [0,1]$, where $Z = M - (M - Y)^{\lambda\gamma, \delta'}$. (For any subset $S \subset M$ and numbers $s, t > 0$ we denote by $S^{s,t}$ all points $y \in M$ for which there is a path $p : [0,1] \to M$, contained in a segment of a leaf of $\mathcal{F}$ which has length less than s, and with $p(0) \in S$ and $d_M(p(1), y) < t$).*

(b) *g is $(\lambda\gamma, \delta')$-controlled over $(M, \mathcal{F})$.*

REMARK. Property (b) is equivalent to g being a proper map; i.e., that g is a homeomorphism onto W.

In the next theorem we let $p : E \to M$ and $p_n : E_n \to M$ and $h : N \to E$ be as in 1.2. We say that the proper homotopy equivalence $h : N \to E$ is (γ, δ)-*controlled over* $(M, \mathcal{F})$ if there is a proper map $g : E \to N$, and proper homotopies $H : N \times [0,1] \to N$ and $G : E \times [0,1] \to E$ of $g \circ h$ and $h \circ g$ to the identity maps $N \to N$ and $E \to E$, such that each of the paths

$$x \times [0,1] \subset E \times [0,1] \xrightarrow{G} E \xrightarrow{p} M, \quad x \in E,$$

$$y \times [0,1] \subset N \times [0,1] \xrightarrow{H} N \xrightarrow{h} E \xrightarrow{p} M, \quad y \in N,$$

has foliated diameter less than (γ, δ) in $(M, \mathcal{F})$.

A homotopy $F : N \times [0,1] \to E$ is said to be (γ, δ)-controlled over $(M, \mathcal{F})$ if each of the paths

$$y \times [0,1] \subset N \times [0,1] \xrightarrow{F} E \xrightarrow{p} M, \quad y \in N,$$

has diameter less than (γ, δ) in $(M, \mathcal{F})$.

THEOREM 1.4. *Suppose that M is a complete A-regular Riemannian manifold, and suppose that $\mathcal{F}$ is an A-regular one-dimensional foliation for M. There is a number $\lambda > 1$ which depends only on* $\dim M$. *Given any $\gamma > 0$ there is for each sufficiently small $\delta > 0$ another number δ' satisfying $\lim\limits_{\delta \to 0} \delta' = 0$. Suppose that K is a piecewise smooth triangulation of M such that each $\Delta \in K$ has diameter less than δ, and suppose that $h : N \to E$ is a proper homotopy equivalence which is split over a subcomplex $L \subset K$ and is (γ, δ)-controlled over M. Then for sufficiently large integers n there is a proper homotopy $h_t : N \times T^n \to E_n$, $t \in [0,1]$, of $h \times \mathrm{id}_{T^n}$ that satisfies the following properties.*

(a) *Let $L' \subset L$ denote the maximal subcomplex of L such that $|L'| \subset M - (M - |L|)^{\lambda\gamma, \delta'}$. Then each $h_t : N \times T^n \to E_n$ is split over L'.*

(b) *$h_1 : N \times T^n \to E_n$ is split over all of K.*

(c) The homotopy $h_t : N \times T^n \to E_n$, $t \in [0,1]$, is $(\lambda\gamma, \delta')$-controlled over $(M, \mathcal{F})$.

The proofs for Theorems 1.3 and 1.4 will be given in Section 6.

2. Proof of Lemmas 0.12 and 0.14. In this section we prove Lemmas 0.12 and 0.14. As we have noted in the introduction these lemmas together imply the truth of Proposition 0.10.

Proof of Lemma 0.12. Our proof follows the pattern set out in [31], with the following modifications: we substitute for the "asymptotic transfer" used in [31] the "focal transfer" used in [24]; we use the foliated controlled h-cobordism Theorem 1.3 in place of the control arguments of [31].

Let X denote a complete A-regular Riemannian manifold having non-positive sectional curvature everywhere. Set $N = X \times \mathbb{R}$ equipped with the product Riemannian metric. There is a subbundle S^+N of the unit sphere bundle SN defined as follows. Every vector $v \in SN$ can be written as $v = (v_1, v_2\partial/\partial t)$, where v_1 is tangent to the first factor of $N = X \times \mathbb{R}$, v_2 is a real number, and $\partial/\partial t$ is the unit vector field tangent to the second factor of $N = X \times \mathbb{R}$ and pointing in the positive direction of $\mathbb{R}$. Now define S^+N to be the subbundle of all $v \in SN$ such that $v_2 \geq 0$. Note that the orbits of the geodesic flow $g^t : SN \to SN$, $t \in \mathbb{R}$, are the leaves of a one-dimensional smooth foliation $\mathcal{F}$ for SN; let $\mathcal{F}^+$ denote the restriction of $\mathcal{F}$ to S^+N. The Riemannian metric $\langle\ ,\ \rangle_N$ canonically induces a Riemannian metric $\langle\ ,\ \rangle_{SN}$ (see Section 3 of [31]). It is straightforward to deduce that both SN and $\mathcal{F}$ are A-regular with respect to the Riemannian metric $\langle\ ,\ \rangle_{SN}$.

Now let W denote an h-cobordism over N which represents an arbitrary element $\tau \in Wh(\pi_1 X)$: W comes equipped with a product structure $P : Y \times [0,1] \to W$ over a subset $Y \subset N$ with $N - Y$ compact; W also comes equipped with proper deformation retracts $r_t^{\pm} : W \to W$, $t \in [0,1]$, of W onto $\partial_{\pm} W$ such that $r_t^+(P(y,s,)) = P(y,(1-t)s+t)$ and $r_t^-(P(y,s)) = P(y,(1-t)s)$ for all $(y,s) \in Y \times [0,1]$ and all $t \in [0,1]$. Note that τ is the obstruction to finding a product structure $P' : N \times [0,1] \to W$ for all of W such that $P' |_{Z \times [1,0]} = P |_{Z\times[0,1]}$ where $Z \subset Y$ and $N - Z$ is compact.

Let $\hat{W}$ denote the total space of the pull back along $r_1^- : W \to N$ of the bundle $S^+N \xrightarrow{\text{proj}} N$. Note that $\hat{W}$ is an h-cobordism over S^+N, the product structure $P : Y \times [0,1] \to W$ pulls back canonically to a product structure $\hat{P} : \hat{Y} \times [0,1] \to \hat{W}$ where $\hat{Y}$ is the preimage of Y under the map $S^+N \xrightarrow{\text{proj}} N$, and τ is the obstruction to finding a product structure $\hat{P}' : S^+N \times [0,1] \to \hat{W}$ for all of $\hat{W}$ such that $\hat{P}' \mid \hat{Z} \times [0,1] = \hat{P} \mid \hat{Z} \times [0,1]$ where $\hat{Z} \subset \hat{Y}$ and $S^+N - \hat{Z}$ is compact. There are no canonical pull backs for the deformation retracts $r_t^{\pm} : W \to W$, $t \in [0,1]$, however we do have the following claim.

CLAIM 2.1. Given any $\epsilon > 0$ there are deformation retracts $\hat{r}_t^{\pm} : \hat{W} \to \hat{W}$, $t \in [0,1]$, which satisfy the following properties.

(a) $\hat{r}_t^+(\hat{P}(y,s)) = \hat{P}(y,(1-t)s+t)$ and $\hat{r}_t^-(\hat{P}(y,s)) = \hat{P}(y,(1-t)s)$ hold for all $(y,s) \in Y \times [0,1]$ and all $t \in [0,1]$.

(b) There is $\alpha > 0$ which is independent of ϵ, and $s > 0$ which depends on ϵ. The h-cobordism $\hat{W}$ is (α, ϵ)-controlled over $(S^+N, \mathcal{F}^+)$ with respect to the projection

$$S^+N \xrightarrow{g^s} S^+N$$

and with respect to the deformation retracts $\hat{r}_t^{\pm} : \hat{W} \times [0,1] \to \hat{W}$, $t \in [0,1]$.

Before verifying 2.1 we will use it to complete the proof of Lemma 0.12. We would like to apply the foliated controlled h-cobordism Theorem 1.3, with respect to the projection of 2.1(b), to obtain a product structure $\hat{P}' : S^+N \times [0,1] \to \hat{W}$ for all of $\hat{W}$ which agrees with $\hat{P}$ over a subset $\hat{Z} \subset \hat{Y}$ with $S^+N - \hat{Z}$ a compact set. (The phrase "with respect to the projection of 2.1(b)" means that we are applying 1.3 with respect to the foliation $g^{-s}(\mathcal{F}^+)$ of S^+N and with respect to the metric $(g^s)^*(\langle\ ,\ \rangle_{SN})$ on S^+N). It would follow that $\tau = 0$ for any $\tau \in Wh(\pi_1 X)$, and so $Wh(\pi_1 X) = 0$. Unfortunately 1.3 does not apply directly here because S^+N has a boundary ∂S^+N (whereas the control space in 1.3 has no boundary). To overcome this minor difficulty we proceed as follows. Let $\hat{W}_\partial$ denote total space of the bundle which is obtained by pulling back the sphere bundle $\partial S^+N \xrightarrow{\text{proj}} N$ along the map $r_1^- : W \to N$; then $(\hat{W}, \hat{W}_\partial)$ is an h-cobordism of the pair $(S^+N, \partial S^+N)$. We define another h-cobordism $(\tilde{W}, \tilde{W}_\partial)$ of $(S^+N, \partial S^+N)$ by setting $\tilde{W} = \hat{W}$ and letting $\tilde{W}_\partial$ denote a "short" product neighborhood for ∂S^+N in $\hat{W}_\partial$. We may extend $\tilde{W}$ to an h-cobordism $\bar{W}$ of SN by setting $\bar{W}$ equal to the product cobordism over $SN - S^+N$. We can construct a product structure $\bar{P} : \bar{Y} \times [0,1] \to \bar{W}$ and deformation retracts $\bar{r}_t^{\pm} : \bar{W} \to \bar{W}$, $t \in [0,1]$, from the $\hat{P}$ and $\hat{r}_t^{\pm}$ of 2.1, so that the following hold: $\bar{Y} = \hat{Y} \cup (SN - S^+N)$; $\bar{W}$ is $(2\alpha, 2\epsilon)$-controlled over $(SN, \mathcal{F})$ with respect to the projection $g^s : SN \to SN$ and with respect to the deformation retracts $\bar{r}_t^{\pm}$. Thus we may apply the foliated controlled h-cobordism Theorem 1.3, with respect to the projection $g^s : SN \to SN$, to get a product structure $\bar{P}' : SN \times [0,1] \to \bar{W}$ for all of $\bar{W}$ such that $\bar{P}' \mid \bar{Z} \times [0,1] = \bar{P} \mid \bar{Z} \times [0,1]$, where $SN - \bar{Z}$ is a very small neighborhood for $SN - \bar{Y}$ in SN. It follows that $\tau = 0$, and thus $Wh(\pi_1 X) = 0$. To verify the more general claim of 0.12, that $Wh(\pi_1 X \times \mathbb{Z}^n) = 0$, we just replace X by $X \times T^n$ in the preceding argument, where T^n denotes the standard flat n-torus and where $X \times T^n$ is equipped with the product Riemannian metric.

Thus to complete the proof of Lemma 0.12 it remains to verify Claim 2.1. Note that the deformation retracts $r_t^{\pm} : W \to W$, $t \in [0,1]$, can be reconstructed from the paths $g_x^{\pm} : [0,1] \to W$, $x \in W$, where $g_x^{\pm}$ is defined as the composite map

$$[0,1] = x \times [0,1] \subset W \times [0,1] \xrightarrow{r^{\pm}} W$$

where $r^{\pm}(w,t) = r_t^{\pm}(w)$ for all $(w,t) \in W \times [0,1]$. Likewise we can construct the deformation retracts $\hat{r}_t^{\pm} : \hat{W} \to \hat{W}$, $t \in [0,1]$, from a collection of paths $\hat{g}_{\hat{x}}^{\pm} : [0,1] \to \hat{W}$, $\hat{x} \in \hat{W}$, which we get as follows. Note that for each $\hat{x} \in \hat{W}$ there is $x \in W$ and $v \in S^+N_{|r_1^-(x)}$ such that $\hat{x} = (x,v)$. We set $\hat{g}_{\hat{x}}^{\pm} = (g_x^{\pm}, C_{x,v}^{\pm})$ where $C_{x,v}^{\pm} : [0,1] \to S^+N$ is a "lifting" of the composite path $r_1^- \circ g_x^{\pm} : [0,1] \to N$ as described in 3.3 of [24] with $C_{x,v}^{\pm}(0) = v$. We note that it is a consequence of 3.4 in [24] that the "liftings" $\{C_{x,v}^{\pm}\}$ may be chosen to be continuous in the variables (x,v); thus the $\hat{r}_t^{\pm}$ are continuous maps. It is a consequence of 3.5 in [24] that each composite path $g^s \circ C_{x,v}^{\pm} : [0,1] \to S^+N$ has foliated diameter less than (α, ϵ) in $(S^+N, \mathcal{F}^+)$, where α, ϵ, s are as in 2.1. Thus the $r_t^{\pm}$ satisfy the desired control properties of 2.1. (Note that part of the hypothesis of 3.5 in [24] is that the paths $r_1^- \circ g_x^{\pm}$ which are not constant paths must be contained in a sufficiently small neighborhood for $X \times 0$ in $N = X \times \mathbb{R}$. This is arranged by requiring that $N - Y$ be contained in a sufficiently small neighborhood for $X \times 0$ in N, where $Y \subset N$ is the subset over which W is equipped with the product structure $P : Y \times [0,1] \to W$.) For further details in verification of 2.1 the reader is referred to 3.3-3.5 of [24] and to §3 in [31].

This completes the proof for Lemma 0.12. □

Proof of Lemma 0.14. Our proof follows the pattern set out in [20], with the following modifications: we substitute for the "asymptotic transfer" used in [20] the "focal transfer" used in [23]; we use the absolute control Theorem 1.2 and the foliated control Theorem 1.4 in place of the control arguments of [20].

Let $X' \to X$ denote the line bundle over X which has $\mathbb{Z}_2$ as its structure group and whose total space X' is an orientable manifold. (Note if X is orientable then $X' \to X$ is the trivial line bundle.) Equip X' with the (local) product metric. Set $Y = X' \times \mathbb{R}^k$, where k is chosen such that $\dim(Y)$ is odd. We equip Y with the product Riemannian metric. For each integer $j \geq 1$ let $p_j : Y_j \to Y$ denote the fiber bundle which has the unit j-sphere in Euclidean space $\mathbb{R}^{j+1}$ for fiber, has a subgroup of order two in the orthogonal group for structure group, and has the same orientation data (in terms of $\pi_1 Y = \pi_1 X$) as does the tangent space TX.

We shall say that a proper homotopy equivalence $f : Z \to Y_j$ has *compact support* if $f \mid (Z - C)$ is an embedding for some compact subset $C \subset Z$. Let $f_i : Z_i \to Y_{j,i}$ denote the cartesian product of f with the identity map $\mathrm{id} : T^i \to T^i$, where T^i denotes the i-dimensional torus, and let $p_{j,i} : Y_{j,i} \to Y$ denote the composite projection $Y_{j,i} = Y_j \times T^i \xrightarrow{\text{proj.}} Y_j \xrightarrow{p_j} Y$. A proper homotopy $F : Z_i \times [0,1] \to Y_{j,i}$ of f_i is said to have *compact support* if for some compact subset $D \subset Z_i$ it is true that $F \mid (Z_i - D) \times t$ is an embedding for all $t \in [0,1]$.

In order to prove Lemma 0.14 it will suffice to show that for any integer $j \geq 1$, and any proper homotopy equivalence $f : Z \to Y_j$ which has compact support, there is an integer $i \geq 1$ and a proper homotopy $F : Z_i \times [0,1] \to Y_{j,i}$ having compact support from f_i to another map $g : Z_i \to Y_{j,i}$ which is split over a triangulation for Y with respect to the projection $p_{j,i} : Y_{j,i} \to Y$ (cf. [20, §9]). If there is such a homotopy F, we will say that $f : Z \to Y_j$ can be *stably split* over a triangulation for Y. Set $f' : Z' \to Y_j'$ equal to the product map $f \times \mathrm{id} : Z \times S^1 \to Y_j \times S^1$ where $\mathrm{id} : S^1 \to S^1$ is the identity map, and set $p_{j,i}' : Y_{j,i}' \to Y'$ equal to the product map $p_{j,i} \times \mathrm{id} : Y_{j,i} \times S^1 \to Y \times S^1$. Note that the $f : Z \to Y_j$ can be stably split over Y if and only if the map $f' : Z' \to Y_j'$ can be stably split over Y'. Thus to complete the proof of Lemma 0.14 it will suffice to show that the map $f' : Z' \to Y_j'$ can be stably split over Y'.

We must "transfer" the splitting problem of the preceding paragraph to bundles over Z' and Y_j' (as in [20; §1]), where we may make use of the geometry of Y' and the control theorems of Section 1 to carry out the splitting.

We begin by reviewing a bundle construction from [20; §4]. Let ξ denote a linear bundle over a manifold M, equipped with a continuous inner product on each of its fibers, and let ξ' denote the Whitney sum $\xi \oplus \epsilon$ where ϵ denotes the trivial line bundle and where we equip the fibers of ξ' with canonical inner product. Each vector $v \in \xi'$ can be written as $v = (v_1, v_2)$ where $v_1 \in \xi$ and $v_2 \in \mathbb{R}$. Let $D(\xi')$ denote the set of all unit vectors $v \in \xi'$ with $v_2 \geq 0$, and let $S(\xi)$ denote the unit sphere bundle in ξ. For each fiber $S_p(\xi)$, $p \in M$, of $S(\xi)$ there is a $\mathbb{Z}_2$-group action $\mathbb{Z}_2 \times (S_p(\xi) \times S_p(\xi)) \to S_p(\xi) \times S_p(\xi)$ gotten by sending (x,y) to (y,x): set $C(\xi) = \bigcup_{p \in M} (S_p(\xi) \times S_p(\xi))/\mathbb{Z}_2$. We note that $C(\xi)$ contains both the unit sphere bundle $S(\xi)$ and the real projective bundle $RP(\xi)$ as subbundles: the inclusion $S(\xi) \subset C(\xi)$ is gotten by identifying each fiber $S_p(\xi)$ with the fixed point set of the action $\mathbb{Z}_2 \times (S_p(\xi) \times S_p(\xi)) \to S_p(\xi) \times S_p(\xi)$;

and the inclusion $RP(\xi) \subset C(\xi)$ is gotten by identifying each fiber $RP_p(\xi)$ with the orbit space of the restricted action $\mathbb{Z}_2 \times A_p(\xi) \to A_p(\xi)$, where $A_p(\xi) = \{(x,-x) : x \in S_p(\xi)\}$. The total space $C(\xi)$ is a stratified space having the two strata $S(\xi)$ and $C(\xi) - S(\xi)$. There are many ways to choose small "tubular neighborhoods" $\tau_p(\xi)$ for $S_p(\xi)$ in $C_p(\xi)$ in such a way that $\tau(\xi) = \bigcup_{p \in M} \tau_p(\xi)$ is a small "tubular neighborhood" for $S(\xi)$ in $C(\xi)$. (Here the quotation marks are used because the fibers of each projection $\tau_p(\xi) \to S_p(\xi)$ are equal to a cone over a real projective space, instead of a ball as is case for tubular neighborhoods in manifolds.) Note that $S(\xi)$ is also a subbundle of $D(\xi')$, in fact for each $p \in M$ we have that $\partial D_p(\xi') = S_p(\xi)$. These constructions are functorial: i.e., for any $g : N \to M$ we set $\psi = g^*(\xi)$ and $\psi' = g^*(\xi')$; then there a maps $D(g) : D(\psi') \to D(\xi')$, $C(g) : C(\psi) \to C(\xi)$, $S(g) : S(\psi) \to S(\xi)$, $\tau(g) : \tau(\psi) \to \tau(\xi)$.

In the rest of this proof we assume that the ξ, ξ', ψ, ψ', g in the constructions of the preceding paragraph are as follows: $g : N \to M$ is equal to $f' : Z' \to Y_j'$; $\xi = p'^*_j(\gamma)$, where γ is the linear subbundle of all vectors in TY' which are tangent to the first factor of $Y' = Y \times S^1$; $\xi' = p'^*_j(TY')$; $\psi = f'^*(\xi)$ and $\psi' = f'^*(\xi')$. The Riemannian metrics on these choices for the bundles ξ, ξ' are induced from the Riemannian metric on Y', and the Riemannian metric on $Y' = Y \times S^1$ is just the product of the given Riemannian metric on Y with the Euclidean metric on S^1.

DEFINITION 2.2. We will say that $C(f') : C(\psi) \to C(\xi)$ and $D(f') : D(\psi') \to D(\xi')$ can be *stably split over* Y' if there is an integer $i \geq 1$ and homotopies $F : C(\psi) \times T^i \times [0,1] \to C(\xi) \times T^i$ and $G : D(\psi') \times T^i \times [0,1] \to D(\xi') \times T^i$ of $C(f') \times \mathrm{id}$ and $D(f') \times \mathrm{id}$ respectively (where $\mathrm{id} : T^i \to T^i$ denotes the identity map), such that the following properties hold.

(a) Both F and G have compact support.

(b) $F(S(\psi) \times T^i \times [0,1]) \subset S(\xi) \times T^i$,
$F((C(\psi) - S(\psi)) \times T^i \times [0,1]) \subset (C(\xi) - S(\xi)) \times T^i$,
$F(\tau(\psi) \times T^i \times [0,1]) \subset \tau(\xi) \times T^i$,
$G(S(\psi) \times T^i \times [0,1]) \subset S(\xi) \times T^i$.

(c) $F \mid (S(\psi) \times T^i \times [0,1]) = G \mid (S(\psi) \times T^i \times [0,1])$.

(d) The mapping

$$F : \tau(\psi) \times T^i \times [0,1] \to \tau(\xi) \times T^i$$

maps each fiber of the bundle $\tau(\psi) \times T^i \times [0,1] \to S(\psi) \times T^i \times [0,1]$ homeomorphically onto a fiber of the fiber bundle $\tau(\xi) \times T^i \to S(\xi) \times T^i$.

(e) $G \mid D(\psi') \times T^i \times 1$ is split over a triangulation for Y' with respect to the composite projection

$$D(\xi') \times T^i \xrightarrow{\text{proj.}} D(\xi') \xrightarrow{\text{proj.}} Y_j' \xrightarrow{p_j'} Y'.$$

$F \mid S(\psi) \times T^i \times 1$ is split over a triangulation for Y' with respect to the composite projection

$$S(\xi') \times T^i \xrightarrow{\text{proj.}} S(\xi') \xrightarrow{\text{proj.}} Y_j' \xrightarrow{p_j'} Y'.$$

$F \mid (C(\psi) - \tau^0(\psi)) \times T^i \times 1$ is split over a triangulation for Y' with respect to the composite projection

$$C(\xi) \times T^i \xrightarrow{\text{proj.}} C(\xi') \xrightarrow{\text{proj.}} Y_j' \xrightarrow{p_j'} Y'$$

where $\tau^0(\psi)$ denotes the interior of $\tau(\psi)$ in $C(\psi)$.

The rest of this proof is based on the following proposition, which has been verified in §4 and §9 of [20].

PROPOSITION 2.3. *If $C(f') : C(\psi) \to C(\xi)$ and $D(f') : D(\psi') \to D(\xi')$ can be stably split over Y', then $f' : Z' \to Y'_j$ can also be stably split over Y'.*

It follows from 2.3, and from the preceding remarks, that in order to complete the proof of Lemma 0.14 it will suffice to verify the following claim.

CLAIM 2.4. The maps $C(f')$ and $D(f')$ can be stably split over Y.

Before proceeding with the verification of Claim 2.4 we need to review (from [20] and [23]) some geometric details for $C(\gamma)$ and $D(\gamma')$, where $\gamma' = TY'$ and γ denotes the linear subbundle of all vectors in TY' which are tangent to the first factor of $Y' = Y \times S^1$.

2.5. **Foliations and flows for $D(\gamma')$, $RP(\gamma)$ and $S(\gamma)$.** The geodesic flow $g^t : SY' \to SY'$, $t \in \mathbb{R}$, on the unit sphere bundle SY' for Y', leaves the subbundles $D(\gamma')$, $S(\gamma)$ of SY' invariant; we denote these restricted flows by $g_1^t : D(\gamma') \to D(\gamma')$ and $g_2^t : S(\gamma) \to S(\gamma)$. The foliations of $D(\gamma')$ and of $S(\gamma)$ by the orbits of these flows are denoted by $\mathcal{F}_1$ and $\mathcal{F}_2$. Note that $\mathcal{F}_2$ is the covering foliation of a one-dimensional foliation $\mathcal{F}_3$ of the real projective bundle $RP(\gamma)$ (under the two fold covering projection $S(\gamma) \to RP(\gamma)$).

2.6. **The semi-flow on $C(\gamma) - S(\gamma)$.** Note that Y' is a non-positively curved complete Riemannian manifold and that each hypersurface $Y \times s \subset Y \times S^1 = Y'$ is a totally geodesic submanifold. A point $q \in C(\gamma) - S(\gamma)$ consists of an (unordered) two-tuple $\{u, v\}$ where u, v are two distinct unit vectors tangent to some hypersurface $Y \times s \subset Y \times S^1$ at the same point $p \in Y \times s$. For a fixed but arbitrary real number $r > 1$ we define a semi-flow

$$\phi_t : C(\gamma) - S(\gamma) \to C(\gamma) - S(\gamma)$$

by sending the point $q = \{u, v\}$ to the point $\phi_t(q)$ which is represented by the unordered two-tuple $\{v_t, w_t\}$ of vectors indicated in Figure 2.6.1.

Figure 2.6.1 takes place entirely within the hypersurface $Y \times s$, and assumes that Y is simply connected. (If Y is not simply connected, then to get the desired semi-flow we first construct a semi-flow on $C(\hat{\gamma}) - S(\hat{\gamma})$, where $\hat{\gamma}$ denotes the subbundle of $T(\hat{Y} \times S^1)$ of all unit vectors tangent to the first factor of $\hat{Y} \times S^1$ and where $\hat{Y}$ denotes the universal covering space for Y, and then project this semi-flow into $C(\gamma) - S(\gamma)$. The projection is a well defined semi-flow on $C(\gamma) - S(\gamma)$ since the semi-flow on $C(\hat{\gamma}) - S(\hat{\gamma})$ is π_1 Y-equivariant.) All the lines and rays indicated in 2.6.1 are geodesics or portions of geodesics. The points u', v' lie a distance r from p along the geodesic rays starting from p in the directions u, v. The line ℓ is the unique geodesic containing u' and v', and w' is the closest point on ℓ to p. The point p_t lies a distance t from p along the geodesic connecting p to w, and the points u'_t, v'_t lie on ℓ a distance r along the geodesics rays which start at p_t and are tangent to u_t, v_t. Note that $\phi_t(q)$ is defined only for all $t \in [0, r_p]$ where $r_p = d(p, w')$; note also that $0 \leq r_p < r$. (See §1 of [23] for more details.)

2.7. **The projections $\rho_1 : C(\gamma) - RP(\gamma) \to S(\gamma)$, $\rho_2 : C(\gamma) - RP(\gamma) \to S(\gamma)$, $\rho_3 : C(\gamma) - S(\gamma) \to RP(\gamma)$, and $\rho_4 : C(\gamma) \to [0, 1]$.** We will need to refer to figure 2.6.1 when constructing these projections, which assumes that Y is simply connected. (If Y is not simply connected, then we proceed as in 2.6 to carry out the following constructions with respect to $\hat{Y} \times S^1$ in place of $Y \times S^1$, and then project all constructions to the relevant $\pi_1 Y$ orbit spaces.)

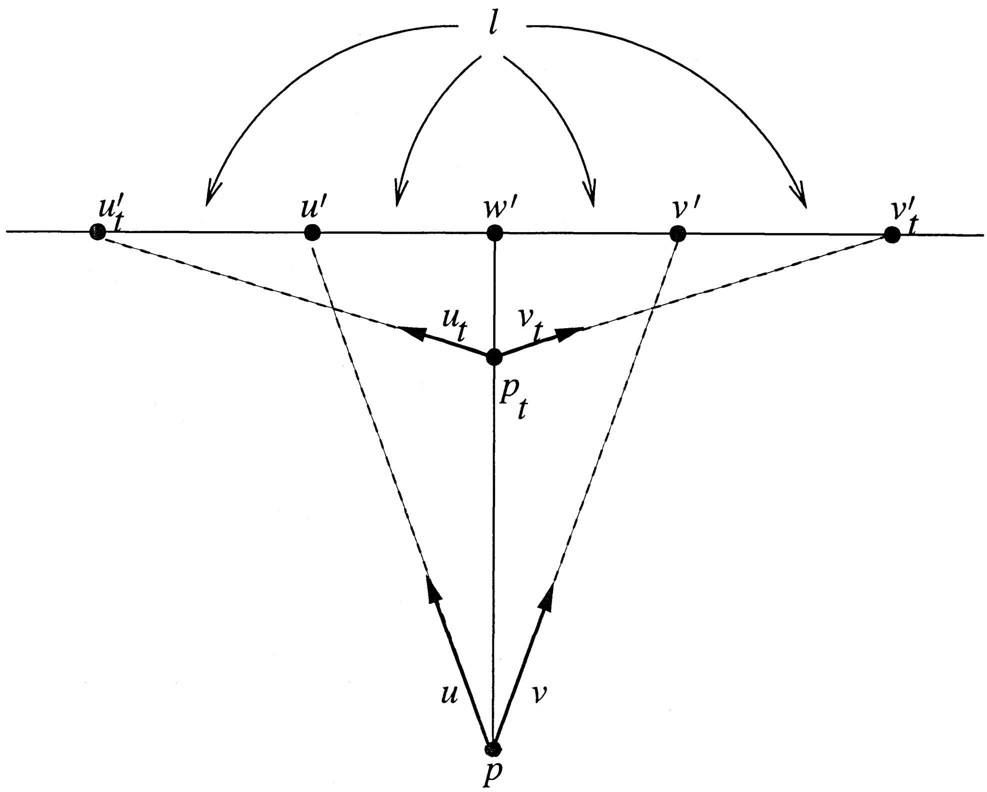

Figure 2.6.1.

Recall that $q \in C(\gamma) - (S(\gamma) \cup RP(\gamma))$ is the point represented by the unordered two-tuple $\{u, v\}$ in 2.6.1, $r > 1$ is the specified distance in 2.6.1 from p to u' or v' and from p_t to u_t' or v_t', and r_p denotes the distance in 2.6.1 from p to w'; we also let w denote the unit vector at the point p in 2.6.1 which is tangent to the geodesic ray connecting p to w'. Now we set

$$\rho_1(q) = w, \quad \rho_2(q) = g^{r_p}(w), \quad \rho_3(q) = \phi_{r_p}(q), \quad \rho_4(q) = \frac{r_p}{r}.$$

This defines ρ_1, ρ_2, ρ_3 on $C(\gamma) - (S(\gamma) \cup RP(\gamma))$, and defines ρ_4 on all of $C(\gamma)$. If $q \in S(\gamma)$ then we set

$$\rho_1(q) = q, \quad \rho_2(q) = g^r(w);$$

and if $q \in RP(\gamma)$ we set

$$\rho_3(q) = q.$$

2.8. **Focal transfers of paths.** In this subsection we discuss a specific method for "lifting" paths from Y' to $D(\gamma')$ and to $C(\gamma)$. A *lifting* of a path $\alpha : [0,1] \to Y'$ into $C(\gamma)$ consists of a path $\beta : [0,1] \to C(\gamma)$ such that the composition of β with the projection $C(\gamma) \to Y'$ is equal to the path α. Similarly we have the notion of a *lifting* of α into $D(\gamma')$. The specific liftings referred to here are constructed in detail in §1 of [23] (where they are referred to as "focal transfers"). Our purpose here is to review the main properties of these "focal transfers".

Let $\alpha : [0,1] \to Y'$ denote a continuous path in Y'. For any sufficiently large $r > 1$, and for each point $q \in C(\gamma)$ which projects to $\alpha(0)$ under the standard bundle projection $C(\gamma) \to Y'$, there is a special lifting (i.e. the "focal transfer") of α into $C(\gamma)$ denoted by $\alpha^{q,r} : [0,1] \to C(\gamma)$. Likewise for any $q \in D(\gamma')$ which projects to $\alpha(0)$ there is a lifting of α into $D(\gamma')$ denoted by $\alpha^{q,r} : [0,1] \to D(\gamma')$. These liftings satisfy the following list of topological properties (cf. §1 of [23]).

2.9. (a) $\alpha^{q,r}(0) = q$. Moreover $\alpha^{q,r}(t)$ is continuous in q, r, t.

(b) If $q \in \rho_4^{-1}([0, 1 - \frac{1}{r}])$ then $\alpha^{q,r}(t) \in \rho_4^{-1}([0, 1 - \frac{1}{r}])$ for all t. If $q \in S(\gamma)$ then $\alpha^{q,t}(0) \in S(\gamma)$ for all t.

(c) If $q \in \rho_4^{-1}([1 - \frac{1}{r}, 1])$ then the composite map $\rho_4 \circ \alpha^{q,r}$ is constant.

(d) Suppose that $q \in \rho_4^{-1}([1 - \frac{1}{r}, 1])$ and set $p = \rho_1(q)$. Then we have that $\alpha^{p,r} = \rho_1 \circ \alpha^{q,r}$. Moreover for each $t \in [0,1]$ the map

$$\rho_1^{-1}(p) \cap \rho_4^{-1}([1 - \frac{1}{r}, 1]) \xrightarrow{f_t} \rho_1^{-1}(\alpha^{p,r}(t)) \cap \rho_4^{-1}([1 - \frac{1}{r}, 1])$$

defined by $f_t(q) = \alpha^{q,r}(t)$ is a homeomorphism.

(e) If $\alpha : [0,1] \to Y'$ and $\beta : [0,1] \to Y'$ are path homotopic with end points fixed, then $\alpha^{w,r}(1) = \beta^{w,r}(1)$ holds for all $w \in C(\gamma) \cup D(\gamma')$.

The liftings $\alpha^{q,r}$ of the path α also satisfy the following important list of control properties (cf. §1 of [23]).

2.10. There is a number $\lambda > 1$. Given numbers $\sigma, \epsilon > 0$, there is a number $L > 0$. Suppose that $\alpha : [0,1] \to Y'$ is a path with $|\alpha| \leq \sigma$, where $|\alpha|$ denotes the diameter of any covering path for α in the universal covering space $\hat{Y}'$. Then the following hold provided $r > L$.

(a) If $q \in C(\gamma)$ then the path $\rho_4 \circ \alpha^{q,r}$ has diameter less than ϵ in $[0,1]$.

(b) If $q \in \rho_4^{-1}([1/2, 1])$ then the path $\rho_2 \circ \alpha^{q,r}$ has foliated diameter less than $(\lambda|\alpha|, \epsilon)$ in $(S(\gamma), \mathcal{F}_2)$. (The unit sphere bundle SY' inherits a canonical Riemannian metric from Y' as in [31, §3]. We restrict this to the submanifold $S(\gamma) \subset SY'$ to get a Riemannian metric on $S(\gamma)$, from which the distance function $d(\ ,\ )$ and diameters can be computed.)

(c) If $q \in \rho_4^{-1}([0, 1/2])$ then the path $\rho_3 \circ \alpha^{q,r}$ has foliated diameter less than $(\lambda|\alpha|, \epsilon)$ in $(RP(\gamma), \mathcal{F}_3)$. (The Riemannian metric on $S(\gamma)$ described in (b) is the pull back of a unique Riemannian metric on $RP(\gamma)$, with respect to which diameters can be computed.)

(d) If $q \in \rho_4^{-1}([1/4, 3/4])$ then the paths $\rho_2 \circ \alpha^{q,r}$ and $\rho_3 \circ \alpha^{q,r}$ have diameters less than ϵ in $S(\gamma)$ and $RP(\gamma)$ respectively.

(e) If $q \in D(\gamma')$ then the path $g^r \circ \alpha^{q,r}$ has foliated diameter less than $(\lambda|\alpha|, \epsilon)$ in $(D(\gamma'), \mathcal{F}_1)$. (Note that the submanifold $D(\gamma') \subset SY'$ inherits a Riemannian metric from SY', with respect to which distances and diameters can be computed.)

Verification for Claim 2.4. We now make use of 2.6-2.8 and the controlled splitting theorems of Section 1 to complete the verification of 2.4.

Firstly we specify the bundles $\tau(\psi) \to S(\psi)$ and $\tau(\xi) \to S(\xi)$ of 2.2. Let $\tau(\gamma) \to S(\gamma)$ denote the restricted projection $\rho_1 : \rho_4^{-1}([1 - \frac{1}{r}, 1]) \to S(\gamma)$. The projection $\tau(\gamma) \to S(\gamma)$ pulls back along the map $p'_j : Y'_j \to Y'$ to give the projection $\tau(\xi) \to S(\xi)$; and the projection $\tau(\xi) \to S(\xi)$ pulls back along the map $f' : Z' \to Y'_j$ to give the projection $\tau(\psi) \to S(\psi)$.

Next we introduce a map $g : Y'_j \to Z'$, and homotopies $F : Y'_j \times [0,1] \to Y'_j$ from $f' \circ g$ to the identity and $G : Z' \times [0,1] \to Z'$ from $g \circ f'$ to the identity, which are all compactly supported. It is important to choose these homotopies to satisfy the following property.

2.11. Let $I : Z' \times [0,1] \to Y_j'$ and $J : Z' \times [0,1] \to Y_j'$ be defined by

$$I(x,t) = f' \circ G(x,t),$$
$$J(x,t) = F(f'(x),t)$$

for all $(x,t) \in Z' \times [0,1]$. We require that the two maps I, J are homotopic thru homotopies which are constant on $Z' \times \partial[0,1]$.

We have already described liftings of the map f' to $C(f') : C(\psi) \to C(\xi)$ and to $D(f') : D(\psi') \to D(\xi')$. The focal transfers now will provide a method for "lifting" the map g to maps $C(g) : C(\xi) \to C(\psi)$ and $D(g) : D(\xi') \to D(\psi')$, for lifting the homotopy G to homotopies $C(G) : C(\psi) \times [0,1] \to C(\psi)$ and $D(G) : D(\psi') \times [0,1] \to D(\psi')$, and for lifting the homotopy F to homotopies $C(F) : C(\xi) \times [0,1] \to C(\xi)$ and $D(F) : D(\xi') \times [0,1] \to D(\xi')$.

For any $y \in C(\xi)$ we write $y \sim (x,q)$, if y projects to x under $C(\xi) \to Y_j'$ and y projects to q under $C(\xi) \to C(\gamma)$; and we let $\alpha : [0,1] \to Y'$ denote the composite path

$$[0,1] \xrightarrow{h} [0,1] = x \times [0,1] \xrightarrow{F} Y_j' \xrightarrow{p_j'} Y',$$

where $h(t) = 1 - t$. Then we set $C(F)(y, 1-t) \sim (F(x, 1-t), \alpha^{q,r}(t))$. Note that 2.9(a) assures us that $C(F)$ is a well defined homotopy with $C(F)(y,1) = y$ which has compact support. A similar approach gives $D(F)$, a homotopy with compact support which ends at the identity map.

For any $y \in C(\psi)$ we write $y \sim (x,q)$, where y maps to x under $C(\psi) \to Z'$ and y maps to q under $C(\psi) \xrightarrow{C(f')} C(\xi) \to C(\gamma)$; and we let $\alpha : [0,1] \to Y'$ denote the composite path

$$[0,1] \xrightarrow{h} [0,1] = x \times [0,1] \xrightarrow{G} Z' \xrightarrow{f'} Y_j' \xrightarrow{p_j'} Y'.$$

Now set $C(G)(y, 1-t) \sim (G(x, 1-t), \alpha^{q,r}(t))$. We have that $C(G)$ is a well defined homotopy with $C(G)(y,1) = y$ which has compact support (cf. 2.9). A similar approach gives $D(G)$, a homotopy with compact support that ends at the identity map.

Note that there is a unique map $C(g) : C(\xi) \to C(\psi)$ satisfying the following property. For each $y \in C(\xi)$ write $y \sim (x,q)$, and write $C(F)(y,0) \sim (f' \circ g(x), q')$, as in a preceding paragraph: then $C(g)(y) \sim (g(x), q')$. It is immediate that $C(g)$ has compact support and that $C(F)$ is a homotopy from $C(f') \circ C(g)$ to the identity map; an argument based on 2.11 and 2.9(e) shows that $C(G)$ is a homotopy from $C(g) \circ C(f')$ to the identity map. There is also a unique map $D(g) : D(\xi') \to D(\psi')$ satisfying analogous properties.

The following properties for the $C(f')$, $C(g)$, $C(F)$, $C(G)$ can be deduced from 2.9.

2.12. (a) $C(G)$ (or $D(G)$) is a homotopy from $C(g) \circ C(f')$ (or from $D(g) \circ D(f')$) to the identity map on $C(\psi)$ (or on $D(\psi')$). $C(F)$ (or $D(F)$) is a homotopy from $C(f') \circ C(g)$ (or from $D(f') \circ D(g)$) to the identity map on $C(\xi)$ (or on $D(\xi')$).

(b)

$$C(G) \mid S(\psi) \times [0,1] = D(G) \mid S(\psi) \times [0,1];$$
$$C(F) \mid S(\xi) \times [0,1] = D(F) \mid S(\xi) \times [0,1];$$
$$C(g) \mid S(\xi) \times [0,1] = D(g) \mid S(\xi) \times [0,1];$$
$$C(f') \mid S(\psi) \times [0,1] = D(f') \mid S(\psi) \times [0,1].$$

(c) $C(G)$ maps each fiber of $\tau(\psi) \times [0,1] \to S(\psi) \times [0,1]$ homeomorphically onto a fiber of $\tau(\psi) \to S(\psi)$; and $C(G)((C(\psi) - \tau(\psi)) \times [0,1]) \subset C(\psi) - \tau(\psi)$. $C(F)$ maps each fiber of $\tau(\xi) \times [0,1] \to S(\xi) \times [0,1]$ homeomorphically onto a fiber $\tau(\xi) \to S(\xi)$; and $C(F)((C(\xi) - \tau(\xi)) \times [0,1]) \subset C(\xi) - \tau(\xi)$.

The following control properties for $C(f')$ and $D(f')$ are computed with respect to the homotopy inverses $C(g)$ and $D(g)$ and with respect to the homotopies $C(F)$, $C(G)$, $D(F)$, $D(G)$. They can be derived directly from 2.10.

2.13. There is a number $\beta > 0$ independent of r in 2.10. There is a number $\epsilon > 0$ which satisfies $\lim_{r\to\infty} \epsilon = 0$. All the following control properties hold for $C(f')$ and $D(f')$.

(a) $C(f')$ is ϵ-controlled over $[0,1]$ with respect to the composite projection

$$C(\xi) \xrightarrow{\text{proj.}} C(\gamma) \xrightarrow{\rho_4} [0,1].$$

(For future reference we denote this composite by $\hat{\rho}_4 : C(\xi) \to [0,1]$.)

(b) The restricted map $C(f') \mid \hat{\rho}_4^{-1}([1/2,1])$ is (β,ϵ)-controlled over $(S(\gamma), \mathcal{F}_2)$ with respect to the composite projection $C(\xi) - RP(\xi) \xrightarrow{\text{proj.}} C(\gamma) - RP(\gamma) \xrightarrow{\rho_2} S(\gamma)$. (For future reference we denote this composite by $\hat{\rho}_2 : C(\xi) - RP(\xi) \to S(\gamma)$.)

(c) The restricted map $C(f') \mid \hat{\rho}_4^{-1}([0,1/2])$ is (β,ϵ)-controlled over $(RP(\gamma), \mathcal{F}_3)$ with respect to the composite projection $C(\xi) - S(\xi) \xrightarrow{\text{proj.}} C(\gamma) - S(\gamma) \xrightarrow{\rho_3} RP(\gamma)$. (For future reference we denote this composite by $\hat{\rho}_3 : C(\xi) - RP(\xi) \to S(\gamma)$.)

(d) The restricted map $C(f') \mid \hat{\rho}_4^{-1}([1/4,3/4])$ is ϵ-controlled over both $S(\gamma)$ and $RP(\gamma)$ with respect to the projections $\hat{\rho}_2$ and $\hat{\rho}_3$ respectively.

(e) The map $D(f')$ is (β,ϵ)-controlled over $(D(\gamma'), \mathcal{F}_1)$ with respect to the composite projection $D(\xi') \xrightarrow{\text{proj.}} D(\gamma') \xrightarrow{g^r} D(\gamma')$. (For future reference we will denote this composite projection by $\hat{\rho}_5 : D(\xi') \to D(\gamma')$.)

We can complete the verification of Claim 2.4 by applying the controlled splitting Theorems 1.2 and 1.4 in conjunction with the control data of 2.13.

First we apply 1.4 (as generalized in Remark 2.15 below) to $D(f')$ in conjunction with the data of 2.13(e) and with respect to the projection $\hat{\rho}_5 : D(\xi') \to D(\gamma')$ and foliation $\mathcal{F}_1$. The conclusion is that (after forming the product of $D(f')$ with the identity map $\mathrm{id} : T^i \to T^i$) there is a homotopy G of $D(f') \times \mathrm{id}$ to a map which is a split over $D(\gamma')$. Note that the restricted homotopy $G \mid S(\psi) \times T^i \times [0,1]$ can be extended to a homotopy F of all $C(f') \times \mathrm{id}$, so that F and G satisfy 2.2(a)-(d). We now replace the old maps $D(f') \times \mathrm{id}$ and $C(f') \times \mathrm{id}$ by the new maps to which the homotopies G and F have moved them. We now call these new maps $D(f')$ and $C(f')$ respectively. Note that $D(f')$ and $C(f')$ still enjoy control properties similar to 2.13(a)-(e), and are now stably split over $D(\gamma')$.

Next we apply 1.2 (as generalized in Remark 2.16 below) to $C(f')$ in conjunction with the control data of 2.13(a)(b) and with respect to the projections

$$\hat{\rho}_2 \times \hat{\rho}_4 : \hat{\rho}_4^{-1}([.59,.71]) \to S(\gamma) \times [.59,.71]$$

and

$$\hat{\rho}_3 \times \hat{\rho}_4 : \hat{\rho}_4^{-1}([.29,.41]) \to RP(\gamma) \times [.29,.41].$$

This is really two independent and simultaneous applications of 1.2. Thus the conclusion of 1.2 (as reformulated in 2.16) is that (after stabilizing the maps $C(f')$ and $D(f')$

if need be) we may vary the $C(f')$ through a suitable homotopy (as in 2.2(a)-(d)), which is the constant homotopy on $\tau(\psi) \times T^i$, to a new map $C(f')$ which is stably split over $S(\gamma) \times [.6, .7]$ with respect to $\hat{\rho}_2 \times \hat{\rho}_4$ and over $RP(\gamma) \times [.3, .4]$ with respect to $\hat{\rho}_3 \times \hat{\rho}_4$. The new $C(f')$ enjoys control properties similar to 2.13 (a)-(c) as well as the following property (d′) which is weaker than (d).

(d′) The restricted map $C(f')\big|_{\hat{\rho}_4^{-1}([\frac{1}{4},\frac{3}{4}])}$ is ϵ'-controlled over Y' with respect to $\hat{\rho}_6$. (See Remark 2.14 for the definition of $\hat{\rho}_6$.) And $\lim_{\epsilon\to 0} \epsilon' = 0$.

Now we make three simultaneous and independent applications of control Theorems 1.2 or 1.4 (as generalized in Remark 2.17 below) in conjunction with control data 2.13 (b), (c) and (d′) and with respect to the projections

$$\hat{\rho}_2 : \hat{\rho}_4^{-1}([.7, 1 - \frac{1}{r}]) \to S(\gamma)$$
$$\hat{\rho}_3 : \hat{\rho}_4^{-1}([0, .3]) \to RP(\gamma)$$
$$\hat{\rho}_6 : \hat{\rho}_4^{-1}([.4, .6]) \to Y'.$$

The conclusion is that we have (after stabilizing $C(f')$ and $D(f')$ if need be) extended the splittings for $C(f')$ to a splitting for $C(f')\big|_{\hat{\rho}_4^{-1}([0,1-\frac{1}{r}])}$ over Y' with respect to the non-standard projection $P_1 : C(\xi) \to Y'$ defined as follows. We set $P_1\big|_{\hat{\rho}_4^{-1}([\frac{1}{3},1])}$ equal to the composite projection

$$\hat{\rho}_4^{-1}([\frac{1}{3}, 1]) \xrightarrow{\hat{\rho}_2} S(\gamma) \xrightarrow{\text{proj.}} Y'$$

and we set $P_1\big|_{\hat{\rho}_4^{-1}([0,\frac{2}{3}])}$ equal to the composite projection

$$\hat{\rho}_4^{-1}([0, \frac{2}{3}]) \xrightarrow{\hat{\rho}_3} RP(\gamma) \xrightarrow{\text{proj.}} Y'.$$

It follows from Remark 2.14 that these two definitions agree on their overlapping domains. There is also a non-standard projection $P_2 : D(\xi') \to Y'$ defined by

$$P_2 = \text{Proj} \circ g^r \circ \text{proj}$$

where $\text{proj} : D(\xi') \to D(\gamma')$ and $\text{Proj} : D(\gamma') \to Y'$ denote the standard projections. At this point we have achieved that $C(f')$ and $D(f')$ are stably split with respect to these non-standard projections $P_1 : C(\xi) \to Y'$ and $P_2 : D(\xi') \to Y'$.

We leave as an exercise for the reader to show that if $C(f')$ and $D(f')$ are stably split with respect to the non-standard projections $P_1 : C(\xi) \to Y'$ and $P_2 : D(\xi') \to Y'$ (as we have just verified), then they are also stably split with respect to the standard projections $C(\xi) \to Y'$ and $D(\xi') \to Y'$. (The key to this verification is that the two pairs of projections in question are connected by a one parameter family of pairs of projections.)

This completes the verification of Claim 2.4 and also completes the proof for Lemma 0.14.

REMARK 2.14. The constructions of §1 in [23] allow us to choose the focal transfers so that the composite projections

$$\hat{\rho}_4^{-1}([1/4, 3/4]) \xrightarrow{\hat{\rho}_2} S(\gamma) \xrightarrow{\text{proj.}} Y'$$

and

$$\hat{\rho}_4^{-1}([1/4,3/4]) \xrightarrow{\hat{\rho}_3} RP(\gamma) \xrightarrow{\text{proj.}} Y'$$

are equal. We denote this composite projection by $\hat{\rho}_6$.

REMARK 2.15. There is a version of Theorem 1.4 where X has a smooth boundary ∂M and there is a collection of leaves $\partial\mathcal{F} \subset \mathcal{F}$ which foliates ∂M. This version can be deduced from 1.4 by "doubling" the given splitting problem over M, to obtain a splitting problem over the union of two copies of M glued with boundaries identified. Our hypothesis here is that the "doubling" of M is a complete and A-regular Riemannian manifold, and that the "doubling" of the foliation is also A-regular.

REMARK 2.16. There is also a version of Theorem 1.4 (and also of Theorem 1.2) for "pieces" of well controlled proper homotopy equivalences. In this version M is a product $Y \times [0,1]$, where Y is a complete and A-regular Riemannian manifold; and the leaves of $\mathcal{F}$ have the form $L \times t$ for $t \in [0,1]$ and L a leaf of a given A-regular foliation for Y. The map $h : N \to E$ of 1.4 is only required to be (γ,δ)-controlled "over a subset $S \subset X$" in the sense that the mappings $g : E \to N$, $H : N \times [0,1] \to N$, $G : E \times [0,1] \to E$ introduced in the paragraph preceding 1.4 need only exist "over S". The conclusion of this new version of Theorem 1.4 is the same as that of 1.4 provided K in 1.4 is a triangulation for a subset of S (rather than a triangulation for all of M). In addition we may assume that the homotopy $h_t : N \times T^n \to E_n$, $t \in [0,1]$, of 1.4 is constant "over" $M - S$.

Since the proof given for Theorem 1.4 in section 6 is "local" in nature, the same proof works for the "piece" version of 1.4 just described.

REMARK 2.17. There is also a version of Theorem 1.2 (and of Theorem 1.4) where the fiber of $p : E \to M$ is a compact manifold with non-empty boundary and $h : (N, \partial N) \to (E, \partial E)$ is assumed to be also split over ∂E. The conclusion of this version contains the additional assertion that $h_t\big|_{\partial N}$ is the constant homotopy. This version can be proven in the same way as 1.2 and 1.4 are proven in Sections 4 and 6, respectively.

3. Local infranil structure of the manifold M. In this section M will denote a complete A-regular Riemannian manifold. We formulate for the manifold M a weak version of the theory of "collapsing Riemannian manifolds" due to Cheeger-Fukaya-Gromov [9]. (See Lemmas 3.2-3.4.) These collapsing results will be used in Section 4 in the proofs of our absolute control Theorems 1.1 and 1.2.

3.1. **The numbers ϵ and δ_i, $i = 1, 2, \ldots, m'$.** The number m' is a positive integer depending only on $m = \dim M$. The ϵ, δ_i are fixed but arbitrarily small positive numbers such that the quotients $\delta_i/\epsilon\delta_{i+1}$ are also arbitrarily small. In Lemmas 3.2-3.4 below, for a given ϵ, the choice of δ_i will depend upon $\epsilon, \delta_{i+1}, \delta_{i+2}, \ldots, \delta_{m'}$.

The following three lemmas can be immediately deduced from [29; Theorems 0.3, 0.5].

3.2. **Topological Lemma.** *For each $p \in M$ there is an integer $k \in \{1, 2, \ldots, m'\}$, where m' comes from 3.1. There are also smooth submanifolds $C_{p,k} \subset E_{p,k} \subset M$ and smooth mappings $r_{p,k} : E_{p,k} \to B_{p,k}$ and $t_{p,k} : E_{p,k} \to [0, \delta_k]$ which satisfy the following properties.*

(a) $E_{p,k}$ is the closed tubular neighborhood of diameter δ_k for $C_{p,k}$ in M; and $B_{p,k}$ is the open ball of radius δ_k centered at the origin of some Euclidean space.

(b) The restricted map $r_{p,k} : C_{p,k} \longrightarrow B_{p,k}$ is a smooth fiber bundle projection having for fiber a closed aspherical manifold N with infranil fundamental group; $r_{p,k}$:

$E_{p,k} \to B_{p,k}$ is equal the composite map $E_{p,k} \xrightarrow{s_{p,k}} C_{p,k} \xrightarrow{r_{p,k}} B_{p,k}$, where $s_{p,k} : E_{p,k} \to C_{p,k}$ is orthogonal projection.

(c) For any $y \in E_{p,k}$ we define $t_{p,k}(y)$ to be the distance from y to $C_{p,k}$ along a geodesic in $s^{-1}_{p,k}(s_{p,k}(y))$ which meets $C_{p,k}$ perpendicularly. The product map

$$r_{p,k} \times t_{p,k} : E_{p,k} - C_{p,k} \to B_{p,k} \times (0, \delta_k]$$

is a well defined smooth fiber bundle projection having for fiber the total space of a sphere bundle over N.

3.3. **Metric Lemma.** *(a) For any $x, y \in E_{p,k}$ we have that*

$$|r_{p,k}(x) - r_{p,k}(y)| < 2d_M(x, y),$$

and

$$|t_{p,k}(x) - t_{p,k}(y)| < 2d_M(x, y),$$

where $d_M(\ ,\ )$ denotes distance in M.

(b) diameter $(E_{p,k}) < 9\delta_k$.

(c) For any $p \in M$ for which $E_{p,k}$ is well defined we have that $p \in E_{p,k}$, and

$$|r_{p,k}(p)| < \epsilon\delta_k$$
$$|t_{p,k}(p)| < \epsilon\delta_k.$$

The next lemma will be useful in comparing the bundle projections $r_{p,k}, t_{p,k}$ with $r_{p',k}, t_{p',k}$ for p' sufficiently close to p.

3.4. **Stability Lemma.** *If $E_{p,k} \cap E_{p',k} \neq \phi$ then $B_{p,k} = B_{p',k}$ and* dim $C_{p,k} =$ dim $C_{p',k}$. *Moreover for each $x \in E_{p,k} \cap E_{p',k}$ we have that*

$$|t_{p,k}(x) - t_{p',k}(x)| < \epsilon\delta_k;$$

and

$$|r_{p,k}(x) - I \circ r_{p',k}(x)| < \epsilon\delta_k$$

for some isometry $I : \mathbb{R}^n \to \mathbb{R}^n$ independent of x (where $n =$ dim *$B_{p,k}$).*

4. Proof of the absolute control theorems. In this section we prove Theorems 1.1 and 1.2. To prove these results we use the local projections of 3.2-3.4. The fact that these local projections have infranil manifolds for fibers, and that their total spaces cover M, allows us to apply fibered control results locally to prove 1.1 and 1.2. For the readers convenience we begin by reviewing the fibered control results which are used in the proofs of 1.1 and 1.2.

Fibered control results over Euclidean space. In this subsection we formulate two well known control results which will be used in the proofs for the absolute control results of Section 1.

Let $p : E \to B$ denote a fiber bundle over the manifold B (with $\partial B = \phi$) having the compact connected manifold F for fiber. Note that the restricted map $p : \partial E \to B$ is a subbundle of $p : E \to B$ with fiber ∂F.

In the next lemma we let (W, W_∂) denote a cobordism of $(E, \partial E)$, $A \subset B$ denotes an open subset of B, and $(U, \partial U) = p^{-1}(A) \cap (E, \partial E)$. We will say that (W, W_∂) is

ϵ-controlled over A if there is an open subpair $(V, V_\partial) \subset (W, W_\partial)$ with $\partial_-(V, V_\partial) = (U, \partial U)$, and there are deformation retracts $r_\pm : (V, V_\partial) \times [0,1] \to (W, W_\partial)$ of (V, V_∂) onto $\partial_\pm(V, V_\partial)$ and a retraction $\hat{r} : (W, W_\partial) \to (E, \partial E)$ such that each of the paths

$$y \times [0,1] \subset V \times [0,1] \xrightarrow{r_\pm} W = W \times 1 \xrightarrow{\hat{r}} \partial_- W = E \xrightarrow{p} B, \quad y \in V, \qquad (1)$$

has diameter less than ϵ in B. Furthermore, $\hat{r}^{-1}(U, \partial U) = (V, V_\partial)$ and $\hat{r}(x) = r_-(x, 1)$ for each $x \in V$. Let $f : (X, X \cap \partial E) \times [0,1] \to (V, V_\partial)$ denote a product structure for (W, W_∂) over the subset $X \subset U$. (See paragraph preceding Theorem 1.1.) We will say that f is *ϵ-controlled over A* if each of the paths

$$x \times [0,1] \xrightarrow{f} V = V \times 1 \xrightarrow{r_\pm} W = W \times 1 \xrightarrow{\hat{r}} \partial_- W = E \xrightarrow{p} B, \quad x \in X, \qquad (2)$$

has diameter less than ϵ in B. The following lemma is a consequence of the calculation $Wh(G \times \mathbb{Z}^n) = 0$ for any infrasolv group G derived in Farrell-Hsiang [15], and of the fibered control results of Chapman-Ferry-Quinn [7], [8], [59].

REMARK 4.1. Recall that an infrasolv group is a torsion free, finitely generated, virtually poly-$\mathbb{Z}$ group.

LEMMA 4.2. *Suppose that $\pi_1 F$ is a infrasolv group,* dim $E \geq 5$, *and that B is an open subset of Euclidean space $\mathbb{R}^n$. Then there is a number $\lambda > 1$ which depends only on n. If $X \supset \partial U$, and if W is ϵ-controlled over A, and if the product structure f is ϵ-controlled over A, then there is another product structure $g : Y \times [0,1] \to V$ for W over the subset $Y = p^{-1}(A - (\mathbb{R}^n - A)^{\lambda\epsilon})$ which satisfies the following properties:*

(a) We require that the product structure $g : Y \times [0,1] \to V$ be $\lambda\epsilon$-controlled over A.

(b) Suppose that for $C \subset A$ we have that $p^{-1}(C) \subset X$. Then we require that

$$f \mid X' \times [0,1] = g \mid X' \times [0,1]$$

where $X' = Y \cap (p^{-1}(C - (\mathbb{R}^n - C)^{\lambda\epsilon}) \cup \partial E)$.

REMARK 4.3. There is also a foliated version of 4.2. Let $\mathcal{F}$ denote the foliation of $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$ whose leaves are $\{\mathbb{R} \times p \mid p \in \mathbb{R}^{n-1}\}$. Suppose that W is (ρ, ϵ)-controlled over A, i.e. the paths in (1) have foliated diameter $< (\rho, \epsilon)$. (See "foliated control results" in Section 1.) Then there is a number $\lambda > 1$ which depends only on n, and a product structure $g : Y \times [0,1] \to V$ for W over the subset $Y = p^{-1}(A - (\mathbb{R}^n - A)^{\lambda\rho, \lambda\epsilon})$ which satisfies the following properties.

(a) g is $(\lambda\rho, \lambda\epsilon)$-controlled over A, i.e. the paths of (2) (when g, Y replace f, X) have foliated diameter $< (\lambda\rho, \lambda\epsilon)$.

(b) If the given product structure f is (ρ, ϵ)-controlled over A, then we may choose g so that $f \mid X' \times [0,1] = g \mid X' \times [0,1]$, where $X' = Y \cap (p^{-1}(C - (\mathbb{R}^n - C)^{\lambda\rho, \lambda\epsilon}) \cup \partial E)$.

REMARK 4.4. There is also a foliated version of 4.2 where B is an open subset of the half-space $[0, \infty) \times \mathbb{R}^{n-1}$ and $\mathcal{F}$ is the foliation having for leaves $\{[0, \infty) \times p \mid p \in \mathbb{R}^{n-1}\}$. In this version the mapping $p : E \to B$ is a "Seifert fibration with $\mathbb{Z}_2$-singularity along $B \cap (0 \times \mathbb{R}^{n-1})$". (See 5.2.1 below.)

In the next lemma we let $h : (D, \partial D) \to (E, \partial E)$ denote a proper map between manifolds, let $A \subset B$ denote an open subset, and set $(U, \partial U) = p^{-1}(A) \cap (E, \partial E)$. We will say that h is *ϵ-controlled over A* if the following properties hold. There are mappings $g : (U, \partial U) \to (D, \partial D)$, $H : (U, \partial U) \times [0,1] \to (E, \partial E)$, and $G : h^{-1}(U, \partial U) \times [0,1] \to (D, \partial D)$ where H is a homotopy from $h \circ g$ to id_U and where

G is a homotopy from $g \circ h$ to $\mathrm{id}_D \mid h^{-1}(U)$, and where each of the paths

$$\begin{aligned} &x \times [0,1] \subset U \times [0,1] \xrightarrow{H} E \xrightarrow{p} B, \ x \in U, \quad \text{and} \\ &x \times [0,1] \subset h^{-1}(U) \times [0,1] \xrightarrow{G} D \xrightarrow{h} E \xrightarrow{p} B, \ x \in h^{-1}(U), \end{aligned} \tag{3}$$

has diameter less than ϵ in B. For each integer $k \geq 0$ we can form the cross-product of the k-torus T^k (if $k = 0$ we let $T^k = pt.$) with the spaces U, E, D to obtain U_k, E_k, D_k, and we can form the cross-product of all these maps with id_{T^k} to obtain the maps $g_k : (U_k, \partial U_k) \to (D_k, \partial D_k)$, $h_k : (D_k, \partial D_k) \to (E_k, \partial E_k)$, $H_k : (U_k, \partial U_k) \times [0,1] \to (E_k, \partial E_k)$, and $G_k : h_k^{-1}(U_k, \partial U_k) \times [0,1] \to (D_k, \partial D_k)$. Let $p_k : E_k \to B$ denote the composite map $E_k = E \times T^k \xrightarrow{\text{proj.}} E \xrightarrow{p} B$. We shall say that a homotopy $r_k : (D_k, \partial D_k) \times [0,1] \to (E_k, \partial E_k)$ of h_k is *ϵ-controlled* over A if each of the paths

$$x \times [0,1] \subset D_k \times [0,1] \xrightarrow{r_k} E_k \xrightarrow{p_k} B, \ x \in h_k^{-1}(U_k), \tag{4}$$

has diameter less than ϵ in B. The following lemma is a consequence of work by Quinn [59]. (See Farrell and Jones [20; §6] for additional details.)

LEMMA 4.5. *Suppose that B is an open subset of Euclidean space $\mathbb{R}^n$. Then there is a number $\lambda > 1$ which depends only on n. Suppose that the proper map $h : (D, \partial D) \to (E, \partial E)$ is ϵ-controlled over A and that $h : \partial D \to \partial E$ is split over a triangulation K of a subset of A such that diameter(Δ) $< \epsilon$ for all $\Delta \in K$. Then there is an integer $k \geq 0$, which depends only on $n = \dim B$, and a homotopy*

$$r_k : (D_k, \partial D_k) \times [0,1] \to (E_k, \partial E_k)$$

of h_k which satisfies the following properties. (For each subcomplex $L \subset K$ we let L' denote the maximal subcomplex of L such that $|L'| \subset |L| - (\mathbb{R}^n - |L|)^{\lambda\epsilon}$.)

(a) We require that $r_k \mid (D_k, \partial D_k) \times 1$ is split over K'.

(b) r_k is $\lambda\epsilon$-controlled over A.

(c) $r_k \mid \partial D_k \times [0,1]$ is the constant homotopy.

(d) Suppose that $h : (D, \partial D) \to (E, \partial E)$ is already split over a subcomplex $L \subset K$. Then in addition to (a)-(c) we also require that $r_k \mid h_k^{-1}(p_k^{-1}(|L'|)) \times [0,1]$ be the constant homotopy.

REMARK 4.6. There is also a foliated version of 4.5 (compare with 4.3). Suppose that $h : (D, \partial D) \to (E, \partial E)$ is (ρ, ϵ)-controlled over A, i.e. the paths in (3) have foliated diameter $< (\rho, \epsilon)$. Then the homotopy $r_k : (D_k, \partial D_k) \times [0,1] \to (E_k, \partial E_k)$ of h_k satisfies the following properties. (For each subcomplex $L \subset K$ we let L' denote the maximal subcomplex of L satisfying $|L'| \subset |L| - (\mathbb{R}^n - |L|)^{\lambda\rho, \lambda\epsilon}$.)

(a) $r_k \mid (D_k, \partial D_k) \times 1$ is split over K'.

(b) r_k is $(\lambda\rho, \lambda\epsilon)$-controlled over A, i.e. the paths in (4) have foliated diameter $< (\lambda\rho, \lambda\epsilon)$.

(c) $r_k \mid \partial D_k \times [0,1]$ is the constant homotopy.

(d) If h is already split over a subcomplex $L \subset K$, then r_k may be chosen so that $r_k \mid h_k^{-1}(p_k^{-1}(|L'|)) \times [0,1]$ is the constant homotopy.

REMARK 4.7. There is also a foliated version of 4.5 where B is an open subset of the half-space $[0, \infty) \times \mathbb{R}^{n-1}$ and $\mathcal{F}$ is the foliation having for leaves $\{[0, \infty) \times p \mid p \in \mathbb{R}^{n-1}\}$. In this version the mapping $p : E \to B$ is a "Seifert fibration with $\mathbb{Z}_2$-singularity along $B \cap (0 \times \mathbb{R}^{n-1})$" (cf. 5.2.1 below).

We shall also need the following refinement on the conclusions of 4.5. In this next lemma we suppose that $p : E \to B$ is the composite

$$E \xrightarrow{p_1} E' \xrightarrow{p_2} B,$$

of two smooth fiber bundle projection maps $p_1 : E \to E'$ and $p_2 : E' \to B$ having compact manifolds F_1 and F_2 for fibers. For any integer $k > 0$ let $p_{1,k} : E_k \to E'$ and $p_k : E_k \to B$ denote the composite projections $E_k = E \times T^k \xrightarrow{\text{proj.}} E \xrightarrow{p_1} E'$ and $E_k = E \times T^k \xrightarrow{\text{proj.}} E \xrightarrow{p} B$. A special case of the next lemma, when F_1 is a point, follows from the authors' calculation [19] of the surgery group for any infrasolv group. A "fibered version" of the arguments used in [19] can be used to prove the following Lemma.

LEMMA 4.8. *Suppose that the fiber F_1 is a closed manifold and that the fiber F_2 is the total space of a disc bundle $\tau \to N$ over a closed aspherical manifold N with $\pi_1 N$ infrasolv. Let K denote a triangulation of a region $|K|$ in B; let K' denote a triangulation for $p_2^{-1}(|K|)$ which triangulates each subset $p_2^{-1}(\Delta)$, $p_2^{-1}(\Delta) \cap \partial E'$, $\Delta \in K$; let L denote a subcomplex of K and let L' denote the subcomplex of K' with underlying set $|L'| = p_2^{-1}(|L|)$. Suppose that the proper map $h : (D, \partial D) \to (E, \partial E)$ is split as in (a) below. Then for some integer $k > 0$ (which depends only on* $\dim(E')$*) there is a homotopy $h_{k,t} : (D_k, \partial D_k) \to (E_k, \partial E_k)$, $t \in [0,1]$, of $h_k : (D_k, \partial D_k) \to (E_k, \partial E_k)$, satisfying (b) (c).*

(a) h is split over K with respect to the projection $p : E \to B$; h is split over $L' \cup K'_\partial$ with respect to the projection $p_1 : E \to E'$, where K'_∂ is the subcomplex of K' which triangulates $|K'| \cap \partial E'$.

(b) $h_{k,t} \mid p_{1,k}^{-1}(|L'| \cup |K'_\partial|$, $t \in [0,1]$ is the constant homotopy; each $h_{k,t}$, $t \in [0,1]$, is split over K with respect to the projection $p_k : E_k \to B$.

(c) $h_{k,1}$ is split over all of K' with respect to the projection $p_{1,k} : E_k \to E'$.

We can now complete the proofs for the absolute control Theorems 1.1 and 1.2 formulated in Section 1.

Proof of Theorem 1.1. For each $k \in \{1, 2, \dots, m'\}$ we choose positive numbers $\delta_{k,i}$, $i = 1, 2, 3$, so that the following properties hold (cf. 3.1).

(a) $\delta_{k,3} = \delta_k$.
(b) All the quotients $\delta_{k,i}/\delta_{k,i+1}$are arbitrarily small. (5)
(c) $\epsilon' = \epsilon\delta_k/\delta_{k,1}$ is arbitrarily small.

For each $k \in \{1, 2, \dots, m'\}$ and any numbers $s, t \in [0, \delta_k]$, and for any $p \in M$ for which $r_{p,k}$ exists, we define a subset $B_{p,k}(s) \subset B_{p,k}$, $E_{p,k}(s,t) \subset E_{p,k}$, $E_{p,k}(s) \subset E_{p,k}$, by

$$B_{p,k}(s) = \{v \in B_{p,k} : |v| < s\},$$
$$E_{p,k}(s,t) = r_{p,k}^{-1}(B_{p,k}(s)) \bigcap t_{p,k}^{-1}([0,t)),$$
$$E_{p,k}(s) = E_{p,k}(s,s).$$

For each $k \in \{1, 2, \dots, m'\}$ and $s \in [0, \delta_k]$ we define a subset $S_k(s) \subset M$ by

$$S_k(s) = \bigcup_{p \in M} E_{p,k}(s),$$

where the union runs over all $p \in M$ such that the projection $r_{p,k} : E_{p,k} \to B_{p,k}$ exists. We define subsets $T_k \subset M$, $k \in \{0, 1, 2, \ldots, m'\}$, by

$$\begin{aligned} T_k &= \bigcup_{i \le k} [S_i(\delta_{i,2}) - \bigcup_{j>i} S_j(t_{i,j})], \quad \text{if } k > 0, \\ T_0 &= \phi, \end{aligned}$$

where $t_{i,j} = \delta_{j,1} + \frac{i}{10j}(\delta_{j,2} - \delta_{j,1})$. We note that the following properties for the $\{T_k\}$ can be deduced from Lemmas 3.2-3.4 and from 3.1 and (5).

(a) $T_0 \subset T_1 \subset T_2 \subset T_3 \subset \ldots \subset T_{m'} = M$.
(b) For any $p \in T_{k+1} - T_k$ there is $p' \in M$ such that $r_{p',k+1} : E_{p',k+1} \to B_{p',k+1}$ is defined and $p \in E_{p',k+1}(\delta_{k+1,2})$. (6)
(c) For p' as in (b) we have that $[E_{p',k+1}(\delta_{k+1}) - E_{p',k+1}(\delta_{k+1}, \delta_{k+1,2})] \subset T_k$.

Now the proof for this theorem proceeds by induction over the sequence

$$T_0 \subset T_1 \subset T_2 \subset T_3 \subset \ldots \subset T_{m'}.$$

Here is our induction hypothesis.

4.9. **(k). Induction hypothesis.** Suppose that $\delta > 0$ of 1.1 satisfies $\delta < \epsilon\delta_{1,1}$. Then there is a product structure $g_k : (Y_k \cup T_k) \times [0, 1] \to W$ for the h-cobordism W over $T_k \cup Y_k$ satisfying the following properties (see (b) for Y_k).

(a) g_k is δ_k-controlled over M.

(b) $g_k \mid Y_k \times [0, 1] = h \mid Y_k \times [0, 1]$, where $h : Y \times [0, 1] \to W$ is the product structure given in the hypothesis of Theorem 1.1 and where Y_k denotes the subset of all $x \in Y$ such that $d_M(x, M - Y) > \delta_k$.

Note that (6) and 4.9 ($k = m'$) imply the conclusions of Theorem 1.1. In fact we may identify δ' of 1.1 with $\delta_{m'}$.

Thus to complete the proof of 1.1 it will suffice to carry out the induction step 4.9 $(k) \Rightarrow$ 4.9 $(k+1)$. Roughly speaking to do this we apply the fibered control Lemma 4.2 to enough of the projections $r_{p',k+1} : E_{p',k+1}(\delta_{k+1}) \to B_{p',k+1}(\delta_{k+1})$ with $r_{p',k+1}$ as in (b) and (c) of (6). Note that (c) of (6) assures us that the part of the h-cobordism W which "lies over" a neighborhood for $E_{p',k+1}(\delta_{k+1}) - E_{p',k+1}(\delta_{k+1}, \delta_{k+1,2})$ already has a product structure (as the hypothesis of Lemma 4.2 requires). The only problem with this approach is that Lemma 4.2 may be applied to all the relevant projections *independently of one another* only if all the relevant domains $\{E_{p',k+1}(\delta_{k+1})\}$ happen to be pairwise disjoint (a rare occurrence). In order to overcome this difficulty we appeal to the inequalities of Lemmas 3.3 and 3.4, and to (b) and (c) of (6), in choosing points $\{p_i : i \in I\}$ in M for which the following properties are satisfied. (See Remark

4.11 at the end of this proof for the construction of the $\{p_i : i \in I\}$.)

$$\begin{aligned}
&\text{(a)}\quad r_{p_i,k+1} : E_{p_i,k+1} \to B_{p_i,k+1} \text{ is defined for all } i \in I; \text{ and}\\
&\qquad T_{k+1} - T_k \subset \bigcup_{i\in I} E_{p_i,k+1}(3\delta_{k+1,2}),\\
&\qquad [E_{p_i,k+1}(\delta_{k+1}) - E_{p_i,k+1}(\delta_{k+1}, \delta_{k+1,2})] \subset T_k.\\
&\text{(b)}\quad \text{For each } i \in I \text{ let } \psi(i) \text{ denote the number of sets } E_{p_j,k+1} \text{ which satisfy}\\
&\qquad E_{p_j,k+1}(9\delta_{k+1,2}) \cap E_{p_i,k+1}(9\delta_{k+1,2}) \neq \phi.
\end{aligned} \tag{7}$$

Then we have that $\psi(i) \leq \bar{m}$ where $\bar{m}$ is a positive integer depending only on $m = \dim M$. Note that (b) assures us that there are subsets $I_s \subset I$, $s = 1, 2, \ldots, \bar{m}$ which satisfy the following properties.

$$\begin{aligned}
&\text{(a)}\quad I = \bigcup_{1\leq s\leq \bar{m}} I_s; \text{ and } I_s \cap I_t = \phi \text{ if } s \neq t.\\
&\text{(b)}\quad \text{If } i, j \in I_s \text{ then } E_{p_i,k+1}(9\delta_{k+1,2}) \cap E_{p_j,k+1}(9\delta_{k+1,2}) = \phi.
\end{aligned} \tag{8}$$

And note that property (b) of (8) assures us that for each fixed $s \in \{1, 2, \ldots, \bar{m}\}$ we can apply Lemma 4.2 simultaneously to those parts of W which lie over any $E_{p_i,k+1}(9\delta_{k+1,2})$ with $i \in I_s$. If we do this first for I_1, and then for I_2, and then for I_3, etc., we become involved in a secondary induction argument the hypothesis of which is as follows.

4.10. **(k,s). Secondary induction hypothesis.** For given $s \in \{1, 2, \ldots, \bar{m}\}$ set $I^s = \bigcup_{t\leq s} I_t$ and set $T_{k,s} = T_k \bigcup \left(\bigcup_{i\in I^s} E_{p_i,k+1}(3\delta_{k+1,2})\right)$. Then there is a product structure

$$g_{k,s} : (Y_{k,s} \cup T_{k,s}) \times [0,1] \to W$$

for the h-cobordism W over $Y_{k,s} \cup T_{k,s}$ satisfying the following properties. (See (b) for $Y_{k,s}$.)

(a) Let $\epsilon' > 0$ be as in (5) and let $\lambda > 1$ be as in 4.2. For each $i \in I^s$ we have that the part of $g_{k,s}$ which lies over $E_{p_i,k+1}(9\delta_{k+1,2})$ is $(4\lambda)^s(\epsilon'\delta_{k+1,1})$-controlled over $B_{p_i,k+1}$ with respect to the projection $r_{p_i,k+1}$; and is $\epsilon'\delta_{k+1,1}$-controlled over $[\lambda_s, \delta_{k+1,2}]$ with respect to the projection $t_{p_i,k+1}$, where $\lambda_s = (\frac{1}{2} + \frac{s}{4\bar{m}})\delta_{k+1,2}$.

(b) $g_{k,s} \mid Y_{k,s} \times [0,1] = g_k \mid Y_{k,s} \times [0,1]$ where

$$Y_{k,s} = Y_k - \left(\bigcup_i E_{p_i,k+1}(9\delta_{k+1,2}, \lambda_s)\right)$$

and with the union running over all $i \in I^s$ such that $E_{p_i,k+1}(9\delta_{k+1,2}, \lambda_s)$ is not contained in Y_k.

We have already indicated that the proof of the secondary induction step 4.10 $(k, s) \Rightarrow$ 4.10 $(k, s+1)$ consists of applying the fibered control h-cobordism Lemma 4.2 simultaneously on that part of W which lies over each $E_{p_i,k+1}(9\delta_{k+1,2}, \lambda_{s+1})$ with respect to the projection $r_{p_i,k+1}$ for each $i \in I_{s+1}$: this produces a product structure for that part of W which lies over each $E_{p_i,k+1}(3\delta_{k+1,2}, \lambda_{s+1})$ which is $(4\lambda)^{s+1}(\epsilon'\delta_{k+1,1})$-controlled over each $B_{p_i,k+1}$. Note that the relative nature of Lemma 4.2 allows us to choose the new product structure to extend the old product structure g_k as indicated in 4.10 $(k, s+1)$. There is one slight difficulty here: the deformations $r_\pm : W \times [0,1] \to W$ of 1.1 do not lend themselves to this application of 4.2. So before applying 4.2 we first must equip W with a new deformation $\hat{r}_\pm : W \times [0,1] \to W$ which can be used in conjunction with 4.2. Set $U = W - \text{Image}(g_{k,s})$ and let V_1, V_2 denote open subsets of W which are "small" neighborhoods for closure (U), closure (V_1) respectively. Now set

$$\hat{r}_\pm \mid (W - V_2) \times [0,1] = \bar{r}_\pm \mid (W - V_2) \times [0,1]$$

and

$$\hat{r}_\pm \mid V_1 \times [0,1] = r_\pm \mid V_1 \times [0,1],$$

where $\bar{r}_\pm : \text{Image}(g_{k,s}) \times [0,1] \to W$ are defined by

$$\bar{r}_+(g_{k,s}(x,u),t) = g_{k,s}(x,(1-t)u+t)$$
$$\bar{r}_-(g_{k,s}(x,u),t) = g_{k,s}(x,(1-t)u).$$

We define $\hat{r}_\pm$ on $(V_2 - V_1) \times [0,1]$ to be a tapering from $r_\pm$ to $\bar{r}_\pm$. We must use both inequalities in 3.4, and the hypothesis 4.10 (k,s), in order to be assured that the tapering between $r_\pm$ and $\bar{r}_\pm$ does indeed exist, and can be chosen so that the part of $g_{k,s}$ lying over each $E_{p_i,k+1}(9\delta_{k+1,2}, \lambda_{s+1})$ is suitably controlled with respect to the projection $r_{p_i,k+1}$ and the new deformations $\hat{r}_\pm$.

This completes the proof of Theorem 1.1

REMARK 4.11. We give here some details in the construction of the $\{p_i : i \in I\}$ of (7). We let $P' \subset M$ denote the subset of all points $p' \in M$ which satisfy properties (b) and (c) of (6). Choose a maximal subset $\{p_i : i \in I\} \subset P'$ which satisfies the following: if

$$E_{p_i,k+1}(9\delta_{k+1,2}) \cap E_{p_j,k+1}(9\delta_{k+1,2}) \neq \phi,$$

then the distance in $B_{p_i,k+1}$ from $r_{p_i,k+1}(E_{p_i,k+1}(0,0))$ to $r_{p_i,k+1}(E_{p_j,k+1}(0,0))$ is greater than $\delta_{k+1,2}$. Note that it follows from (b) and (c) of (6) and from 3.4, 3.1, and (5) that the $\{p_i : i \in I\}$ satisfy property (a) of (7). Towards verifying that the $\{p_i : i \in I\}$ satisfy (b) of (7) we denote by $J_i \subset I$ the subset of all $j \in I$ such that $E_{p_i,k+1}(9\delta_{k+1,2}) \cap E_{p_j,k+1}(9\delta_{k+1,2}) \neq \phi$ for a fixed $i \in I$. It follows from 3.4 and (5) that each $r_{p_i,k+1}(E_{p_j,k+1}(9\delta_{k+1,2}))$, $j \in J_i$, is $\epsilon'\delta_{k+1,2}$-close to a ball $B(j,i)$ in $B_{p_i,k+1}$ satisfying the following properties: $B(j,i)$ has radius $9\delta_{k+1,2}$; if $j \neq j'$ then the centers of $B(j,i)$ and $B(j',i)$ are at least a distance $\frac{1}{2}\delta_{k+1,2}$ apart; $B(j,i) \cap B_{p_i,k+1}(10\delta_{k+1,2}) \neq \phi$ for all $j \in J_i$. It follows from these last properties, and the fact that $\dim(B_{p_i,k+1}) \leq \dim M$, that $|J_i| \leq \bar{m}$ where $\bar{m}$ is a positive integer which depends only on $m = \dim M$.

Proof of Theorem 1.2. This proof follows the same lines as did the proof for Theorem 1.1, except we use the fibered splitting Lemma 4.5 in place of the fibered h-cobordism Lemma 4.2 to carry out the induction steps.

The main induction hypothesis for this argument is as follows.

4.12. **(k). Induction hypothesis.** Suppose that $\delta > 0$ of 1.2 satisfies $\delta < \epsilon'\delta_{1,1}$. There is a proper homotopy $h_{k,t} : N_{n_k} \to E_{n_k}$, $t \in [0,1]$, of $h_{n_k} = h \times \mathrm{id}_{T^{n_k}}$ (where $h : N \to E$ is the proper homotopy equivalence of 1.2) which satisfies the following properties.

(a) $\{h_{k,t} : t \in [0,1]\}$ is δ_k-controlled over M.

(b) Let L, K be as in 1.2, and let L_k denote the maximal subcomplex of L such that $|L_k| \subset |L| - (M - |L|)^{\delta_k}$. Then $h_{k,t} : N_{n_k} \to E_{n_k}$, $t \in [0,1]$, is the constant homotopy over L_k.

(c) Let K_k denote the minimal subcomplex of K which contains T_k. Then $h_{k,1} : N \times T^{n_k} \to E_{n_k}$ is split over K_k.

We note that the truth of 4.12 (m') would imply the truth of Theorem 1.2. In fact we may identify δ' of 1.2 with $\delta_{m'}$.

Thus to complete the proof of 1.2 it suffices to carry out the induction step 4.12 (k) $\Rightarrow$ 4.12 $(k+1)$. To carry out this induction step we must (as in the proof for Theorem 1.1) use a secondary induction argument the hypothesis for which is as follows.

4.13. **(k,s). Secondary induction hypothesis.** There is a proper homotopy $h_{k,s,t} : N_{n_{k,s}} \to E_{n_{k,s}}$, $t \in [0,1]$, of $h_{n_{k,s}}$ which satisfies the following properties.

(a) For each $i \in I^s$ we have that the part of $\{h_{k,s,t} : t \in [0,1]\}$ which lies over $E_{p_i,k+1}(9\delta_{k+1,2})$ is $(4\lambda)^s(\epsilon'\delta_{k+1,1})$-controlled over $B_{p_i,k+1}$ with respect to the projection $r_{p_i,k+1} \circ p_{n_{k,s}}$; and is $\epsilon'\delta_{k+1,1}$-controlled over $[\lambda_s, \delta_{k+1,2}]$ with respect to the projection $t_{p_i,k+1} \circ p_{n_{k,s}}$, where $\lambda_s = (\frac{1}{2} + \frac{s}{4\bar{m}})\delta_{k+1,2}$.

(b) Let $L_{k,s}$ denote the maximal subcomplex of L_k which is contained in the complement of

$$\bigcup_i E_{p_i,k+1}(9\delta_{k+1,2}, \lambda_s),$$

where this union runs over all $i \in I^s$ such that $E_{p_i,k+1}(9\delta_{k+1,2}, \lambda_s)$ is not contained in $|L_k|$. Then each $h_{k,s,t}$ is equal to $h_{k,t}$ over $|L_{k,s}|$ for all $t \in [0,1]$.

(c) Let $K_{k,s}$ denote the minimal subcomplex of K which contains $T_{k,s}$. Then $h_{k,s,1} : N_{n_{k,s}} \to E_{n_{k,s}}$ is split over $K_{k,s}$.

We carry out the secondary induction step 4.13 $(k,s) \Rightarrow$ 4.13 $(k,s+1)$ by applying the fibered splitting Lemma 4.5 over regions of the $\{E_{p_i,k+1}(9\delta_{k+1}, \lambda_{s+1}) : i \in I_{s+1}\}$. Note that this achieves a splitting (with respect to the projections $r_{p_i,k+1} \circ p_{n_{k,s+1}}$) over triangulations of regions in the $\{B_{p_i}(4\delta_{k+1,2}) : i \in I_{s+1})$. This is not the type of splitting data needed in 4.13 $(k, s+1)$. However we can convert this splitting data into splitting data (with respect to the projection $p_{n_{k,s+1}}$) over subcomplexes of K lying in the various

$$\{E_{p_i,k+1}(4\delta_{k+1}, \lambda_{s+1}) : i \in I_{s+1}\}$$

by applying the splitting Lemma 4.8. This latter type of splitting data is the type needed in 4.13 $(k, s+1)$.

This completes the proof of Theorem 1.2.

5. Local infrasolv structure of the foliated manifold $(M, \mathcal{F})$**.** In this section M will denote a complete and A-regular Riemannian manifold and we let $\mathcal{F}$ denote an A-regular one-dimensional foliation of M. We formulate for the pair $(M, \mathcal{F})$ a foliated version of the theory of "collapsing Riemannian manifolds" due to Cheeger-Fukaya-Gromov [9]. (See Lemmas 5.2-5.4.) These foliated collapsing results will be used together with those of Section 3 to carry out the proofs of our foliated control Theorems 1.3 and 1.4 in Section 6.

5.1. **The numbers** α, β, ϵ, **and** (α_j, δ_j), $j = 1, 2, \ldots, m'$. The number m' is a positive integer depending only on $m = \dim M$. The β, ϵ, δ_j are fixed but arbitrarily small positive numbers such that each quotient $\delta_j/\epsilon\delta_{j+1}$ is also arbitrarily small. The number α may be any positive number, and the α_j may be any numbers in $(0, \alpha)$ satisfying $100\alpha_i < \epsilon\alpha_{i+1}$. In order for Lemmas 5.2-5.4 below to hold for a given choice of α, β, ϵ and of $\{\alpha_j\}$ the choice of δ_i will depend upon all of the numbers $\alpha, \beta, \epsilon, \{\alpha_j\}, \delta_{i+1}, \delta_{i+2}, \dots, \delta_{m'}$.

We shall need the following notation in the next lemma. For each point $x \in M$ we let A_x denote the "arc" of length 2α centered at x and contained in a leaf of $\mathcal{F}$. (If the leaf L containing x has length $\leq \alpha$, then $A_x = L$ is a circle.) We denote by $M(\alpha, \beta)$ the subset of all $x \in M$ such that diameter$(A_x) > \beta$. The following lemma can be deduced from [30; 0.5].

5.2. **Topological Lemma.** *For each $p \in M(\alpha, \beta)$ there is an integer $k \in \{1, 2, \dots, m'\}$, where m' comes from 5.1. There are also smooth submanifolds $C_{p,k} \subset E_{p,k} \subset M$ and smooth mappings $r_{p,k} : E_{p,k} \to B_{p,k}$ and $t_{p,k} : E_{p,k} \to [0, \delta_k]$. These manifolds and mappings satisfy all the following properties.*

(a) $E_{p,k}$ is the closed tubular neighborhood of diameter δ_k for $C_{p,k}$ in M. $B_{p,k}$ is equal to the product $B_{p,k,1} \times B_{p,k,2}$ where the factors are one of the three following types.

Type I: *$B_{p,k,1}$ is a point and $B_{p,k,2}$ is an open ball of radius δ_k centered at the origin of some Euclidean space.*

Type II: *$B_{p,k,1} = (-\alpha_k, \alpha_k)$ and $B_{p,k,2}$ is an open ball of radius δ_k centered at the origin of some Euclidean space.*

Type III: *$B_{p,k,1} = [0, \alpha_k)$ and $B_{p,k,2}$ is an open ball of radius δ_k centered at the origin of some Euclidean space.*

(b) If $r_{p,k}$ is of type I or type II then the restricted map $r_{p,k} : C_{p,k} \to B_{p,k}$ is a smooth fiber bundle projection having for fiber a closed aspherical manifold with infrasolv fundamental group. If $r_{p,k}$ is of type III then the restricted map $r_{p,k} : C_{p,k} \to B_{p,k}$ is a "Seifert fibration" with a "$\mathbb{Z}_2$-singularity" over $0 \times B_{p,k,2}$ (cf. Remark below), having for fibers closed aspherical manifolds with infrasolv fundamental groups. For all three types we have that $r_{p,k} : E_{p,k} \to B_{p,k}$ is equal the composite map $E_{p,k} \xrightarrow{s_{p,k}} C_{p,k} \xrightarrow{r_{p,k}} B_{p,k}$, where $s_{p,k}$ denotes orthogonal projection.

(c) For any $y \in E_{p,k}$ we define $t_{p,k}(y)$ to be the distance from y to $s_{p,k}(y)$ in $s^{-1}_{p,k}(s_{p,k}(y))$. The product map $r_{p,k} \times t_{p,k} : E_{p,k} - C_{p,k} \to B_{p,k} \times (0, \delta_k]$ is a well defined smooth fiber bundle projection if $r_{p,k}$ is of type I or II, and is a "Seifert fibration" with a "$\mathbb{Z}_2$-singularity" over $0 \times (B_{p,k,2} \times (0, \delta_k])$ if $r_{p,k}$ is of type III.

REMARK. A map $f : E \to [0, \delta) \times B$ is called a *Seifert fibration* with a $\mathbb{Z}_2$-*singularity* over $0 \times B$ if the following properties hold. First we must have that $f : f^{-1}((0, \delta) \times B) \to (0, \delta) \times B$ is a smooth fiber bundle projection. Next, for each point $p \in 0 \times B$, there must be an open neighborhood U for p in $[0, \delta) \times B$ of the form $U = [0, \epsilon) \times V$, a two fold covering space $\rho : X \to f^{-1}(U)$, and a smooth fiber bundle projection $g : X \to (-\epsilon, \epsilon) \times B$. We further require that g commutes with the $\mathbb{Z}_2$ actions $\psi_1 : \mathbb{Z}_2 \times X \to X$ and

$$\psi_2 : \mathbb{Z}_2 \times ((-\epsilon, \epsilon) \times B) \to (-\epsilon, \epsilon) \times B,$$

where ψ_1 is the action by the covering transformations for ρ, and where $\psi_2(1, (t, b)) = (-t, b)$ for all $(t, b) \in (-\epsilon, \epsilon) \times B$. Finally we require that the restricted map $f : f^{-1}(U) \to U$ be equal to the quotient of the bundle projection $g : X \to (-\epsilon, \epsilon) \times B$ under the two actions ψ_1, ψ_2.

We will need the following notation in the next lemma. For any $x, y \in M$ we will write

$$d_{\mathcal{F}}(x,y) < (\alpha, \delta),$$

for some real numbers $\alpha, \delta > 0$, if there is a path $g_1 : [0,1] \to M$ with $g_1(0) = x$ whose image lies in a segment of a leaf of $\mathcal{F}$ having length less than α, and if there is a second path $g_2 : [0,1] \to M$ having length less than δ such that $g_2(0) = g_1(1)$ and $g_2(1) = y$. For any subset $X \subset M$ we will write

$$D_{\mathcal{F}}(X) < (\alpha, \delta),$$

for some real numbers $\alpha, \delta > 0$, if for any $x, y \in X$ we have that $d_{\mathcal{F}}(x,y) < (\alpha, \delta)$. Note that $D_{\mathcal{F}}(\ )$ is just the "foliated diameter" of X discussed in "the foliated control results" of Section 1.

5.3. **Metric Lemma.** *The maps $r_{p,k}$ and $t_{p,k}$ satisfy all of the following metric conditions, where $\omega > 1$ is a number which depends only on $\alpha, \beta, \langle\ ,\ \rangle_M, \mathcal{F}$.*

(a) $D_{\mathcal{F}}(E_{p,k}) < (8\alpha_k, \omega\delta_k)$.

(b) Suppose that $x, y \in E_{p,k}$ satisfy

$$d_{\mathcal{F}}(x,y) < (\rho, \tau)$$

for some $(\rho, \tau) \in [0, 8\alpha_k] \times [0, \omega\delta_k]$. Then we have that

$$\begin{aligned} |r_{p,k,1}(x) - r_{p,k,1}(y)| &< \rho + \epsilon\alpha_k, \\ |r_{p,k,2}(x) - r_{p,k,2}(y)| &< \omega\tau + \omega t_{p,k}(x)\rho + \epsilon\delta_k, \\ |t_{p,k}(x) - t_{p,k}(y)| &< \tau + \omega t_{p,k}(x)\rho + \epsilon\delta_k \end{aligned}$$

also hold.

(c) For any $p \in M(\alpha, \beta)$ for which $E_{p,k}$ is well defined we have that $p \in E_{p,k}$ and

$$\begin{aligned} |r_{p,k,1}(p)| &< \epsilon\alpha_k \\ |r_{p,k,2}(p)| &< \epsilon\delta_k, \\ |t_{p,k}(p)| &< \epsilon\delta_k. \end{aligned}$$

The next lemma will be useful in comparing the bundle projections $r_{p,k}$, $t_{p,k}$ with $r_{p',k}$, $t_{p',k}$ on overlapping regions of their domains. It is an immediate consequence of [30; 0.6].

5.4. **Stability Lemma.** *If $E_{p,k} \cap E_{p',k} \neq \phi$ then the following properties hold.*

(a) $r_{p,k}$ and $r_{p',k}$ are of the same type; $\dim C_{p,k} = \dim C_{p',k}$; $B_{p,k} = B_{p',k}$.

(b) For all $x \in E_{p,k} \cap E_{p',k}$ we have that

$$|t_{p,k}(x) - t_{p',k}(x)| < \epsilon\delta_k.$$

(c) There is an affine isomorphism $\mathbf{A}_2 : \mathbb{R}^n \to \mathbb{R}^n$ (where $n = \dim B_{p,k,2}$) which satisfies

$$|\mathbf{A}_2| < \kappa, \quad |\mathbf{A}_2^{-1}| < \kappa,$$

and

$$|\mathbf{A}_2 \circ r_{p,k,2}(x) - r_{p',k,2}(x)| < \epsilon\delta_k$$

for all $x \in E_{p,k} \cap E_{p',k}$, where $\kappa > 0$ depends only on $\alpha, \beta, \langle\ ,\ \rangle_M, \mathcal{F}$. Moreover if $r_{p,k}$ is of type II or type III then there is an isometry $\mathbf{A}_1 : \mathbb{R} \to \mathbb{R}$ such that

$$|\mathbf{A}_1 \circ r_{p,k,1}(x) - r_{p',k,1}(x)| < \epsilon\alpha_k$$

for all $x \in E_{p,k} \cap E_{p',k}$; and if $r_{p,k}$ is of type III then $\mathbf{A}_1 =$ identity.

6. Proof of the foliated control theorems. In this section we prove the foliated control Theorems 1.3 and 1.4. Our proofs of these theorems are modeled on the proofs given in Section 4 of the absolute control Theorems 1.1 and 1.2. We use the local projections $r_{p,k}$ and $t_{p,k}$ described in Lemmas 5.2-5.4. The fact that these local projections have infrasolv manifolds for fibers, and that the interiors of their total spaces form an open cover for M, allows us to apply the fibered control results of Section 4 over each of a suitable collection of the $\{r_{p,k} : E_{p,k} \to B_{p,k}\}$ to prove 1.3 and 1.4.

Proof of Theorem 1.3. For each $k \in \{1, 2, \ldots, m'\}$ we choose positive numbers $\delta_{k,i}$, $i = 1, 2, 3$ which satisfy the following properties (cf. 5.1).

6.1. (a) $\delta_{k,3} = \delta_k$.

(b) All the quotients $\delta_{k,i}/\delta_{k,i+1}$ are arbitrarily small.

(c) $\epsilon' = \epsilon\delta_k/\delta_{k,1}$ is arbitrarily small.

In addition we suppose that the numbers $\{\alpha_i\}$ of 5.1 satisfy 6.2(a) below; thus we may choose for each $k \in \{1, 2, \ldots, m'\}$ numbers $\alpha_{k,i}$, $i = 1, 2, 3$, which satisfy 6.2(b)-(d) below.

6.2. (a) $\lambda^{\bar{m}}\alpha_i << \alpha_{i+1}$ for all $i \in \{1, 2, \ldots, m' - 1\}$, where $\lambda > 1$ comes from 4.3, and where $\bar{m}$ is the positive integer depending only on $m = \dim M$ which is given in 6.7 below.

(b) $\alpha_{k,3} = \alpha_k$.

(c) $\lambda^{\bar{m}}\alpha_{k,i} << \alpha_{k,i+1}$.

(d) $\lambda^{\bar{m}}\alpha_{k-1} << \alpha_{k,1}$.

We also assume the following relation between $\gamma, \delta > 0$ of 1.3, the $\{\alpha_i\}$ and $\{\delta_i\}$ and ϵ of 5.1, $\lambda > 1$ of 4.3, and $\bar{m}$ of 6.7.

6.3. (a) $\lambda^{\bar{m}}\gamma << \alpha_{1,1}$.

(b) $\delta < \epsilon\delta_{1,1}$.

Finally we assume that $\delta_{m'} < \beta$. Then note that there is a number $\rho > 1$ which satisfies the following two properties.

6.4. (a) ρ depends only on $M, \mathcal{F}, \langle\ ,\ \rangle_M, \alpha$.

(b) If $E_{p,k} \cap M(\alpha, \rho^i\beta) \neq \phi$, for some $i, k \in \{1, 2, \ldots, m'\}$ and some $p \in M(\alpha, \beta)$, then $E_{p,k} \subset M(\alpha, \rho^{i-1}\beta)$.

For any $p \in M(\alpha, \beta)$ and for any $k \in \{1, 2, \ldots, m'\}$ for which $r_{p,k}$ is well defined and of type II or III, and for any numbers $s \in [0, \alpha_k)$ and $t, u \in [0, \delta_k)$, we set

$$\begin{aligned} B_{p,k}(s,t) &= \{(x,y) \in B_{p,k,1} \times B_{p,k,2} : |x| < s \text{ and } |y| < t\}, \\ E_{p,k}(s,t,u) &= r_{p,k}^{-1}(B_{p,k}(s,t)) \bigcap t_{p,k}^{-1}([0,u)), \\ E_{p,k}(s,t) &= E_{p,k}(s,t,t). \end{aligned}$$

If $r_{p,k}$ is of type I then we set

$$\begin{aligned} B_{p,k}(s,t) &= \{x \in B_{p,k} : |x| < t\}, \\ E_{p,k}(s,t,u) &= r_{p,k}^{-1}(B_{p,k}(s,t)) \bigcap t_{p,k}^{-1}([0,u)), \\ E_{p,k}(s,t) &= E_{p,k}(s,t,t). \end{aligned}$$

We also define subsets $S_k(s,t), T_k \subset M$ by

$$S_k(s,t) = \bigcup_{p \in M(\alpha,\beta)} E_{p,k}(s,t)$$

where the union runs over all $p \in M(\alpha, \beta)$ such that the projection $r_{p,k} : E_{p,k} \to B_{p,k}$ is defined; and by

$$T_k = \bigcup_{i \le k} [S_i(\alpha_{i,2}, \delta_{i,2}) - \bigcup_{j > i} S_j(s_{i,j}, t_{i,j})]$$

if $k > 0$, and

$$T_0 = \phi,$$

where

$$s_{i,j} = \alpha_{j,1} + \frac{i}{10j}(\alpha_{j,2} - \alpha_{j,1}) \text{ and } t_{i,j} = \delta_{j,1} + \frac{i}{10j}(\delta_{j,2} - \delta_{j,1}).$$

We note that the following properties for the $\{T_k\}$ can be deduced from Lemmas 5.2-5.4 and from 5.1, 6.1, 6.2, 6.4.

6.5. (a) $T_0 \subset T_1 \subset T_2 \subset \ldots \subset T_{m'}$ and $M(\alpha, \beta) \subset T_{m'}$.

(b) For any $p \in (T_{k+1} - T_k) \cap M(\alpha, \rho^{k+1}\beta)$ there is $p' \in M(\alpha, \beta)$ such that

$$r_{p',k+1} : E_{p',k+1} \to B_{p',k+1}$$

is defined, $E_{p',k+1} \subset M(\alpha, \rho^k \beta)$, and $p \in E_{p',k+1}(\alpha_{k+1,2}, \delta_{k+1,2})$.

(c) For p' as in (b) we have that

$$[E_{p',k+1}(\alpha_{k+1}, \delta_{k+1}) - E_{p',k+1}(\alpha_{k+1}, \delta_{k+1}, \delta_{k+1,2})] \subset T_k \cap M(\alpha, \rho^k \beta).$$

Now the main part of the proof for this theorem proceeds by induction over the sequence $T_0 \subset T_1 \subset T_2 \subset T_3 \subset \ldots \subset T_{m'}$. Here is our induction hypothesis.

6.6. **(k) Induction hypothesis.** There is a product structure

$$g_k : Y_k \cup (T_k \cap M(\alpha, \rho^k \beta)) \times [0,1] \to W$$

for the h-cobordism W over $Y_k \cup (T_k \cap M(\alpha, \rho^k \beta))$ satisfying the following properties. (See (b) for Y_k.)

(a) We have that g_k is (α_k, δ_k)-controlled over $M, \mathcal{F}$.

(b) $g_k \mid Y_k \times [0,1] = h | Y_k \times [0,1]$, where $h : Y \times [0,1] \to W$ is the product structure given in the hypothesis of Theorem 1.3 and where $Y_k = X - (M - Y)^{\alpha_k, \delta_k}$. (See 1.3 for notation.)

To carry out the induction step 6.6 (k) $\Rightarrow$ 6.6 $(k+1)$ we apply the fibered control Lemma 4.2 or Remarks 4.3, 4.4 to enough of the projections

$$r_{p',k+1} : E_{p',k+1}(\alpha_{k+1}, \delta_{k+1}) \to B_{p',k+1}(\alpha_{k+1}, \delta_{k+1})$$

with $r_{p',k+1}$ as in 6.5(b)(c). Note that 6.5(c) assures us that the part of the h-cobordism W which "lies over" a neighborhood for $E_{p',k+1}(\alpha_{k+1}, \delta_{k+1}) - E_{p',k+1}(\alpha_{k+1}, \delta_{k+1,2})$ already has a product structure (as the hypotheses of 4.2-4.4 require). The only problem with this approach is that 4.2-4.4 may be applied to all the relevant projections *independently of one another* only if all the relevant domains $\{E_{p',k+1}(\alpha_{k+1}, \delta_{k+1,2})\}$ happen to be pairwise disjoint (a rare occurrence). In order to overcome this difficulty we appeal to the inequalities of Lemmas 5.3 and 5.4, and to 6.5(b)(c), in choosing points $\{p_i : i \in I\}$ in M for which the following are satisfied. (See Remark 6.10 at the end of this proof for details in the construction of the $\{p_i : i \in I\}$.)

6.7. We require that $r_{p_i,k+1} : E_{p_i,k+1} \to B_{p_i,k+1}$ is defined for all $i \in I$ and satisfies (a)-(c).

(a) $(T_{k+1} - T_k) \cap M(\alpha, \rho^{k+1}\beta) \subset \bigcup_{i \in I} E_{p_i,k+1}(\alpha_{k+1,2}, \delta_{k+1,2})$.

(b) $[E_{p_i,k+1}(\alpha_{k+1}, \delta_{k+1}) - E_{p_i,k+1}(\alpha_{k+1}, \delta_{k+1}, \delta_{k+1,2})] \subset T_k \cap M(\alpha, \rho^k\beta)$.

(c) There is an integer $\bar{m} > 0$ which depends only on $m = \dim M$. I is a disjoint union $\bigcup_{s=1}^{\bar{m}} I^s = I$ of subsets $I^s \subset I$ which satisfy: if $i, j \in I^s$ for some $s \in \{1, 2, \ldots, \bar{m}\}$ then we have that

$$E_{p_j,k+1}(2\alpha_{k+1,2}, 2\delta_{k+1,2}) \cap E_{p_i,k+1}(2\alpha_{k+1,2}2\delta_{k+1,2}) = \phi.$$

Note that 6.7(c) assures us that for each fixed $s \in \{1, 2, \ldots, \bar{m}\}$ we can apply Lemma 4.2 or Remarks 4.3, 4.4 simultaneously to those parts of W which lie over any $E_{p_i,k+1}(2\alpha_{k+1,2}2\delta_{k+1,2})$ with $i \in I_s$. If we do this first for I_1, and then for I_2, and then for I_3, etc., we become involved in a secondary induction argument the hypothesis of which is as follows.

6.8. **(k,s). Secondary induction hypothesis.** For a given $s \in \{1, 2, \ldots, \bar{m}\}$ set

$$I^s = \bigcup_{t \leq s} I_t$$

and set

$$T_{k,s} = (T_k \cap M(\alpha, \rho^k\beta)) \bigcup \left(\bigcup_{i \in I^s} E_{p_i,k+1}(\alpha_{k+1,2}, \delta_{k+1,2}) \right).$$

Then there is a product structure $g_{k,s} : (Y_{k,s} \cup T_{k,s}) \times [0,1] \to W$ for the h-cobordism W over $Y_{k,s} \cup T_{k,s}$ satisfying the following properties. (See (c) for $Y_{k,s}$.)

(a) Let $\epsilon' > 0$ be as in 6.1 and let $\lambda > 1$ come from 4.2-4.4; and let $\kappa > 1$ come from 5.4. For each $i \in I^s$ we have that $g_{k,s} \mid E_{p_i,k+1}(\alpha_{k+1,2}, \delta_{k+1,2}) \times [0,1]$ is $((4\lambda)^s\alpha_{k+1,1}, (4\lambda\kappa)^s(\epsilon'\delta_{k+1,1}))$-controlled over $B_{p_i,k+1}, \mathcal{F}_{p_i,k+1}$ with respect to the projection $r_{p_i,k+1}$. Here $\mathcal{F}_{p_i,k+1}$ denotes the foliation of $B_{p_i,k+1}$ by the $\{B_{p_i,k+1,1} \times y \mid y \in B_{p_i,k+1,2}\}$, and $B_{p_i,k+1,1}B_{p_i,k+1,2}$ denote the first and second factor for $B_{p_i,k+1}$.

(b) Let $\omega > 1$ be as in 5.3. We also have, for each $i \in I^s$, that $g_{k,s} \mid E_{p_i,k+1}(\alpha_{k+1,2}, \delta_{k+1,2}) \times [0,1]$ is $(4\omega+4)b$-controlled over $[\lambda_s, b]$ with respect to the projection $t_{p_i,k+1}$, for any number $b \in [\lambda_s, \delta_{k+1,2}]$. Here λ_s, $s \in \{1, 2, \ldots, \bar{m}\}$, is a sequence of numbers which satisfy $\epsilon'\delta_{k+1,1} << \lambda_s << \delta_{k+1,2}$ and $(100\omega + 100)\lambda_s << \lambda_{s+1}$. (Here for any small positive numbers a, b the inequality $a << b$ means that a/b is very small.)

(c) $g_{k,s} \mid Y_{k,s} \times [0,1] = g_k \mid Y_{k,s} \times [0,1]$, where

$$Y_{k,s} = Y_k - \left(\bigcup_i E_{p_i,k+1}(2\alpha_{k+1,2}, 2\delta_{k+1,2}, \lambda_s) \right)$$

with the union running over all $i \in I^s$ such that $E_{p_i,k+1}(2\alpha_{k+1,2}, 2\delta_{k+1,1}, \lambda_s)$ is not contained in Y_k.

REMARK. If in property (a) of 6.8 (k, s) the projection $r_{p_i,k+1}$ is of type I then $\mathcal{F}_{p_i,k+1}$ is simply the foliation of $B_{p_i,k+1}$ by its points, and "(a, b)-controlled over $B_{p_i,k+1}, \mathcal{F}_{p_i,k+1}$" is the same as "$b$-controlled over $B_{p_i,k+1}$".

We have already indicated that the proof of the secondary induction step

$$6.8\ (k,s) \Rightarrow 6.8\ (k,s+1)$$

consists of applying the fibered control h-cobordism Lemma 4.2 or Remarks 4.3, 4.4 simultaneously to those parts of W which lies over each $E_{p_i,k+1}(2\alpha_{k+1,2}, 2\delta_{k+1,2}, \lambda_{s+1})$ with respect to the projection $r_{p_i,k+1}$ for each $i \in I_{s+1}$: this produces a product structure for that part of W which lies over each $E_{p_i,k+1}(\alpha_{k+1,2}, \delta_{k+1,2}, \lambda_{s+1})$ that is $((4\lambda)^{s+1}\alpha_{k+1,1}, (4s\kappa)^{s+1}(\epsilon'\delta_{k+1,1}))$-controlled over each $B_{p_i,k+1}, \mathcal{F}_{p_i,k+1}$. (See Remark above if $r_{p_i,k+1}$ is of type I.) Note that the relative nature of 4.2-4.4 allows us to choose the new product structure to extend the old product structure g_k as indicated in 6.8 $(k, s+1)$. There is one slight difficulty here: the deformations $r_\pm : W \times [0,1] \to W$ of 1.3 to not lend themselves to this application of 4.2-4.4. So before applying these we first must equip W with new deformations $\hat{r}_\pm : W \times [0,1] \to W$ which can be used in conjunction with 4.2-4.4. Set $U = W - \text{Image}(g_{k,s})$ and let V_1, V_2 denote open subsets of W which are "small" neighborhoods for closure (U), closure (V_1) respectively. Now set

$$\hat{r}_\pm \mid (W - V_2) \times [0,1] = \bar{r}_\pm \mid (W - V_2) \times [0,1]$$

and

$$\hat{r}_\pm \mid V_1 \times [0,1] = r_\pm \mid V_1 \times [0,1],$$

where $\bar{r}_\pm : \text{Image}(g_{k,s}) \times [0,1] \to W$ are defined by

$$\begin{aligned} \bar{r}_+(g_{k,s}(x,u),t) &= g_{k,s}(x,(1-t)u+t) \\ \bar{r}_-(g_{k,s}(x,u),t) &= g_{k,s}(x,(1-t)u). \end{aligned}$$

We define $\hat{r}_\pm$ on $(V_2 - V_1) \times [0,1]$ to be a tapering from $r_\pm$ to $\bar{r}_\pm$. We must use all the inequalities in 5.4, and the control properties of 6.8 (k, s), in order to be assured that the tapering between $r_\pm$ and $\bar{r}_\pm$ does indeed exist, and can be chosen so that the part of $g_{k,s}$ lying over each $E_{p_i,k+1}(2\alpha_{k+1,2}, 2\delta_{k+1,2}, \lambda_{s+1})$ is suitably controlled with respect to the projection $r_{p_i,k+1}$ and the new deformations $\hat{r}_\pm$.

This completes the induction step 6.8 $(k, s) \Rightarrow$ 6.8 $(k, s+1)$. Now 6.7(a) assures us that 6.8 $(k, \bar{m}) \Rightarrow$ 6.6 $(k+1)$; which completes the induction step 6.6 $(k) \Rightarrow$ 6.6 $(k+1)$. Thus in the remainder of this proof we may assume that 6.6 (m') holds.

We note that 6.6 (m') does not directly imply the conclusions of Theorem 1.3 because $T_{m'}$ does not in general equal all of M (cf. 6.5(a)). However one may use the controlled product structure $g_{m'} : Y_{m'} \cup (T_{m'} \cap M(\alpha, \rho^{m'}\beta)) \times [0,1] \to W$ of 6.6 (m') to construct another pair of deformation retracts $\hat{r}_\pm : W \times [0,1] \to W$ for W which satisfies the following properties.

6.9. (a) $\hat{r}_\pm \mid W' \times [0,1] = r_\pm \mid W' \times [0,1]$, where $W' = r_-^{-1}(M - M(\alpha, \rho^{m'}\beta))$ and where $r_\pm : W \times [0,1] \to W$ are the original deformation retracts for W (cf. 1.3).

(b) $\hat{r}_-(g_{m'}(x,s),t) = g_{m'}(x,(1-t)s)$ and $\hat{r}_+(g_{m'}(x,s),t) = g_{m'}(x,(1-t)s+t)$, for all

$$(x,s) \in r_-^{-1}(M(\alpha, \rho^{m'+1}\beta)) \times [0,1]$$

and all $t \in [0,1]$. Thus $g_{m'}$ is 0-controlled over $M(\alpha, \rho^{m'+1}\beta)$ with respect to the new deformation retracts $\hat{r}_{\pm}$.

(c) The cobordism W is β'-controlled over M with respect to the new deformation retracts $\hat{r}_{\pm}$, where $\lim_{\beta \to 0} \beta' = 0$.

The control properties in 6.9(b)(c) allow us to apply the absolute control Theorem 1.1 to extend the product structure $g_{m'} \mid M(\alpha, \rho^{m'+1}\beta) \times [0,1]$ to a product structure $g : M \times [0,1] \to W$ for all of W which satisfies the conclusions of Theorem 1.3 (provided β is chosen small enough).

This completes the proof of Theorem 1.3.

REMARK 6.10. In this remark we will construct the points $\{p_i : i \in I\} \subset M$ which satisfy properties 6.7(a)-(c).

It will be convenient to assume the existence of another family of infrasolv structures for M, denoted by

$$\{\hat{r}_{p,k+1} : \hat{E}_{p,k+1} \to \hat{B}_{p,k+1} \mid p \in M(\alpha,\beta), 0 \leq k \leq m'-1\}$$

with associated numbers $\hat{\epsilon}$ and $\{\hat{\alpha}_{k+1}, \hat{\delta}_{k+1} \mid 0 \leq k \leq m'-1\}$ which are related to the numbers ϵ and $\{\alpha_{k+1}, \delta_{k+1} \mid 0 \leq k \leq m'-1\}$ and to the family

$$\{r_{p,k+1} : E_{p,k+1} \to B_{p,k+1} \mid p \in M(\alpha,\beta), 0 \leq k \leq m'-1\}$$

as follows. (Note it follows from [30; 0.3-0.6, 2.4.1, 3.1.1] that there is no loss of generality in making this assumption.)

(a)

$$\begin{aligned}
\hat{\epsilon}\hat{\delta}_{k+1} &<< \epsilon\delta_{k+1},\\
\hat{\alpha}_k &<< \alpha_{k+1,1} \quad \text{and} \quad \alpha_{k+1} << \hat{\alpha}_{k+1},\\
\hat{\delta}_k &<< \delta_{k+1,1} \quad \text{and} \quad \delta_{k+1} << \hat{\delta}_{k+1}.
\end{aligned}$$

(b) For each $p \in M(\alpha,\beta)$ and each $k \in \{0,1,\ldots,m'-1\}$ for which $r_{p,k+1}$ is well defined, there is another point in $\hat{p} \in M(\alpha,\beta)$ with $\hat{r}_{\hat{p},k+1}$ well defined and such that

$$\begin{aligned}
p &\in \hat{E}_{p,k+1}\left(\frac{1}{2}\hat{\alpha}_{k+1}, \frac{1}{2}\hat{\delta}_{k+1}, \epsilon\delta_{k+1}\right)\\
E_{p,k+1} &= \hat{r}^{-1}_{\hat{p},k+1}(S) \bigcap \hat{t}^{-1}_{p,k+1}([0,\delta_{k+1}]),\\
B_{p,k+1} &= S - \hat{r}_{\hat{p},k+1}(p),\\
r_{p,k+1}(q) &= \hat{r}_{\hat{p},k+1}(q) - \hat{r}_{\hat{p},k+1}(p) \quad \text{for all} \quad q \in E_{p,k+1},
\end{aligned} \tag{1}$$

where

$$S = \{(x,y) \in \hat{B}_{p,k+1,1} \times \hat{B}_{p,k+1,2} : |x - \hat{r}_{\hat{p},k+1,1}(p)| < \alpha_{k+1} \quad \text{and} \quad |y - \hat{r}_{\hat{p},k+1,2}(p)| < \delta_{k+1}\},$$

and $S - a$ denotes $\{x - a \mid x \in S\}$.

(c) Given any $\hat{p} \in M(\alpha,\beta)$ for which $\hat{r}_{\hat{p},k+1}$ exists, an infrasolv structure $r_{p,k+1}$ is well defined by the equations of (b) for any $p \in \hat{E}_{p,k+1}(\frac{1}{2}\hat{\alpha}_{k+1}, \frac{1}{2}\hat{\delta}_{k+1}, \epsilon\delta_{k+1})$.

If $\hat{E}_{p,k+1} \cap \hat{E}_{q,k+1} \neq \phi$, then it follows from 5.4 that $\hat{B}_{p,k+1} = \hat{B}_{q,k+1}$, and that there is an affine isomorphism $\hat{\mathbf{A}}_{p,q} : \mathbb{R}^n \to \mathbb{R}^n$ (where $n = \dim \hat{B}_{p,k+1}$) which satisfies the following properties.

(a) There is $\kappa > 1$ which depends only on $\alpha, \langle\ ,\ \rangle_M, \mathcal{F}$. We have that

$$\frac{1}{\kappa} < |\hat{\mathbf{A}}_{p,q}| < \kappa.$$

(b) For all $x \in \hat{E}_{p,k+1} \cap \hat{E}_{q,k+1}$ we have that

$$|\hat{\mathbf{A}}_{p,q} \circ \hat{r}_{p,k+1}(x) - \hat{r}_{q,k+1}(x)| < 2\hat{\epsilon}\hat{\delta}_{k+1}. \tag{2}$$

(c) If $\hat{r}_{p,k+1}$ is of type II or III, then there is an isometry $\hat{\mathbf{A}}_{p,q,1} : \mathbb{R} \to \mathbb{R}$ and an affine isomorphism $\hat{\mathbf{A}}_{p,q,2} : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ such that

$$\hat{\mathbf{A}}_{p,q} = \hat{\mathbf{A}}_{p,q,1} \times \hat{\mathbf{A}}_{p,q,2}.$$

Moreover if $\hat{r}_{p,k+1}$ is of type III then

$$\hat{\mathbf{A}}_{p,q,1} = \text{identity}.$$

Now apply the constructions described in [31, §7] and [25:I, §3]. (We take this opportunity to correct errata in [31, p. 564]. Change "$\bigcap_{(s,j)}$" to "$\bigcup_{(s,j)}$" on line 11; "I" to "J" on line 20, and change the phrase "there is ... $(\partial_+ Y_i \times e_i)$" on lines 26, 27 to "$(\partial_+ Y_i \times e)$".) By these constructions applied in conjunction with (2) to "long and thin" triangulations for the foliations $\hat{\mathcal{F}}_{p,k+1}$ of the $\hat{B}_{p,k+1}$ by their first factor, we can construct long and thin cell structures $C_{p,k+1}$ for the $\hat{B}_{p,k+1}, \hat{\mathcal{F}}_{p,k+1}$ which satisfy the following properties.

(a) $\hat{B}_{p,k+1}\left(\frac{1}{8}\hat{\alpha}_{k+1}, \frac{1}{8}\hat{\delta}_{k+1}\right) \subset |C_{p,k+1}| \subset \hat{B}_{p,k+1}\left(\frac{1}{4}\hat{\alpha}_{k+1}, \frac{1}{4}\hat{\delta}_{k+1}\right).$ (3)

(b) $C_{p,k+1}$ is a subcomplex of a larger "long thin" cell structure $\bar{C}_{p,k+1}$ for $\hat{B}_{p,k+1}, \hat{\mathcal{F}}_{p,k+1}$. If $\hat{E}_{p,k+1}(\frac{1}{2}\hat{\alpha}_{k+1}, \frac{1}{2}\hat{\delta}_{k+1}) \cap \hat{E}_{q,k+1}(\frac{1}{2}\hat{\alpha}_{k+1}, \frac{1}{2}\hat{\delta}_{k+1}) \neq \phi$, then for any cell $e \in C_{p,k+1}$ there is a (unique) cell $e' \in \bar{C}_{q,k+1}$ and a homeomorphism $h : \mathbf{A}_{p,q}(e) \to e'$ such that

$$|h(x) - x| < \tau\hat{\epsilon}\hat{\delta}_{k+1}$$

for all $x \in e$. Here $\tau > 1$ depends on κ of (2).

NOTE: In the case that $\hat{r}_{p,k+1}$ is of type I, then $\hat{r}_{q,k+1}$ is also of type I (cf. 5.4); and there is no "long" factor for the cell structures $\bar{C}_{p,k+1}$ and $\bar{C}_{q,k+1}$.

If the cell $e' \in \bar{C}_{q,k+1}$ in property (b) of (3) is actually contained in the smaller cell complex $C_{q,k+1}$, then we shall say that $e \in C_{p,k+1}$ is *equivalent* to $e' \in C_{q,k+1}$ and we will write $e \sim e'$. Let $e_1, e_2, e_3, \ldots$ denote a list of representatives of each such equivalence class of all the cells in all the $\{C_{p,k+1} \mid p \in M(\alpha, \beta)\}$; and denote by $C_{q_i,k+1}$ that cell structure which contains e_i. Choose a "long and thin" neighborhood $N(e_i)$ in $\hat{B}_{q_i,k+1}$ for a large compact subset of $e_i - \partial e_i$ (cf. [31; §8] and [25:I; §3]). (When $\hat{r}_{q_i,k+1}$ is of type I, then there is no "long" factor of $N(e_i)$. See Note above.) We may deduce the following properties for the $N(e_i)$ from (1), (2) and from the constructions of [31; §8] and [25:I; §3].

(a) If $\dim(e_i) = \dim(e_j)$, then we have that

$$\hat{r}^{-1}_{q_i,k+1}(N(e_i)) \bigcap \hat{r}^{-1}_{q_j,k+1}(N(e_j)) = \phi.$$

(b) Set

$$N'(e_i) = \hat{B}_{q_i,k+1} - (\hat{B}_{q_i,k+1} - N(e_i))^{\alpha_{k+1},\delta_{k+1}}$$

if $\hat{r}_{q_i,k+1}$ is of type II or III (cf. 1.3 for $A^{\alpha,\delta}$ notation); and set

$$N'(e_i) = \{y \in \hat{B}_{q_i,k+1} \mid d(y, \hat{B}_{q_i,k+1} - N(e_i)) > \delta_k\}$$

if $\hat{r}_{q_i,k+1}$ is of type I. Then we have that

$$(T^{k+1} - T^k) \bigcap M(\alpha, \rho^{k+1}\beta) \subset \bigcup \hat{r}^{-1}_{p,k+1}(N'(e_i)). \tag{4}$$

(c) For each point $x \in (T^{k+1} - T^k) \bigcap M(\alpha, \rho^{k+1}\beta)$ (cf. 6.7 for notation) there is a point $p \in M(\alpha, \beta)$, an infrasolv structure $r_{p,k+1}$ and a representative cell e_i such that

$$x \in E_{p,k+1}(\alpha_{k+1,2}, \delta_{k+1,2}),$$
$$E_{p,k+1} \subset \hat{r}^{-1}_{q_i,k+1}(N(e_i)).$$

(d) Moreover $r_{p,k+1}$ is constructed from $\hat{r}_{q_i,k+1}$ as in property (b) of (1) (with $\hat{p} = q_i$).

By using an argument similar to that contained in Remark 4.8 above, in conjunction with properties (b)-(d) of (4), we can choose for each cell e_i a finite collection $\{r_{p_j,k+1} \mid j \in {}_iI\}$ of infrasolv structures referred to in (c), (d) of (4) so that the following properties hold.

(a) Each set ${}_iI$ is a disjoint union ${}_iI = \bigcup_{s=1}^{\tilde{m}} {}_iI_s$, where $\tilde{m}$ is a positive integer which depends only on $m = \dim M$.

(b) If $j, j' \in {}_iI_s$ for some s, then we have that

$$E_{p_j,k+1}(2\alpha_{k+1,2}, 2\delta_{k+1,2}) \bigcap E_{p_{j'},k+1}(2\alpha_{k+1,2}, 2\delta_{k+1,2}) = \phi. \tag{5}$$

(c) For any point $x \in \hat{r}^{-1}_{q_i,k+1}(N'(e_i)) \bigcap (T^{k+1} - T^k) \bigcap M(\alpha, \rho^{k+1}\beta)$ there is an index $j \in {}_iI$ such that

$$x \in E_{p_j,k+1}(\alpha_{k+1,2}, \delta_{k+1,2}).$$

Now we complete the verification of 6.7 as follows. Define $\bar{m}$ of 6.7 by

$$\bar{m} = (m+1)\tilde{m};$$

and define I and I^s of 6.7 (where $s = (d,t)$) by

$$I = \bigcup {}_iI \quad \text{and} \quad I^s = \bigcup {}_iI_t,$$

where the first union runs over all ${}_iI$ and the second over all ${}_iI_t$ such that $\dim(e_i) = d$. Now properties 6.7(a)-(c) follow from (4) and (5).

Proof of the foliated control Theorem 1.4. The proof of 1.4 is a slight modification of the proof of 1.3 wherein the use of 4.5-4.8 are substituted for the use of 4.2-4.4. The details are left as an exercise for the reader. (Before carrying out this exercise the reader should review how the proof of the absolute control Theorem 1.2 is a simple modification of the proof of the absolute control Theorem 1.1.)

7. Further applications. We begin this section with a discussion of the consequences of the uniqueness question for manifold compactifications. Let us define some terms. Recall a *manifold compactification* of a manifold M was defined in Section 0 to be a compact manifold $\bar{M}$ with boundary $\partial\bar{M}$ such that $\bar{M} - \partial\bar{M} = M$. It is a smooth manifold compactification if both $\bar{M}$ and M are smooth. Two (smooth) compactifications $\bar{M}$ and $\hat{M}$ of M are (smoothly) *equivalent* if there is a (diffeomorphism) homeomorphism $f : \bar{M} \to \hat{M}$ such that f is the identity function outside some collar neighborhood of $\partial\bar{M}$. And they are *strongly equivalent* if $f \mid M = \mathrm{id}_M$. It seldom happens that all manifold compactifications of a given manifold M are strongly equivalent, as can be seen by considering the following example where $M = \mathbb{R}^2$. Let $[-\infty, \infty]$ be the endpoints compactification of $\mathbb{R}$; set $\hat{M} = [-\infty, \infty] \times [-\infty, \infty]$. And let $\bar{M}$ denote the Eberlein-O'Neill visibility sphere compactification [13] of $\mathbb{R}^2$. Recall that the points in $\bar{M} - M$ are the asymptoty classes of half lines in $\mathbb{R}^2$. Therefore, a pair of parallel but non-intersecting vertical half lines in $\mathbb{R}^2$ converge to the same point in $\bar{M}$ but to distinct points in $\hat{M}$. Consequently, $\bar{M}$ and $\hat{M}$ are not strongly equivalent.

The situation is considerably different when we consider equivalence instead of strong equivalence.

PROPOSITION 7.1. *Let $\bar{M}$ be a (smooth) manifold compactification of M. If* dim $M \geq 6$ *and* $\mathrm{Wh}(\pi_1 C) = 0$ *for each component C of $\partial\bar{M}$, then any other (smooth) manifold compactification $\hat{M}$ of M is (smoothly) equivalent to $\bar{M}$.*

Proof of 7.1. This result follows immediately from the s-cobordism theorem and the following elementary fact. There are arbitrarily short collar neighborhoods

$$\partial\bar{M} \times (0, \epsilon] \subset \partial\bar{M} \times (0, \delta]$$

such that

$$\partial\hat{M} \times (0, \delta] - \partial\bar{M} \times (0, \epsilon) \tag{1}$$

is an h-cobordism.

We mentioned in Section 0 two classes of Riemannian manifolds which have smooth manifold compactifications. The first class consisted of the complete (connected) pinched negatively curved manifolds with finite volume. Such a manifold M has a smooth compactification $\bar{M}$ constructed independently by Gromov, Heintze, and Margulis; cf. [34], [36].

COROLLARY 7.2. *Let M be a complete pinched negatively curved Riemannian manifold with finite volume. Then any (smooth) manifold compactification of M is (smoothly) equivalent to the Gromov-Heintze-Margulis compactification $\bar{M}$ provided* dim $M \neq 4, 5$. *When* dim $M = 5$*, it is also equivalent to $\bar{M}$.*

Proof of 7.2. Let C be a connected component of $\partial\bar{M}$, then $\mathrm{Wh}(\pi_1 C) = 0$ by the main result of [15] since $\pi_1 C$ is a finitely generated, torsion-free, virtually nilpotent group. Hence, Corollary 7.2 now follows from Proposition 7.1 when dim $M > 5$. And in the other posited dimensions, it will follow from the proof of Proposition 7.1 if we can show that the h-cobordism (1) is trivial. It is obviously trivial when dim $M = 2$. When dim $M = 3$, the cobordism (1) is an irreducible 3-manifold because the universal cover of M is homeomorphic to $\mathbb{R}^3$; hence the 3-dimensional h-cobordism theorem of Stallings [64] shows that (1) is trivial. Finally, Freedman and Quinn [33] showed that the 5-dimensional s-cobordism theorem is topologically true for s-cobordisms with virtually nilpotent fundamental groups. The last sentence of Corollary 7.2 follows from this additional fact.

A second class of Riemannian manifolds which have smooth manifold compactifications was mentioned in Section 0. This class contains, in particular, the double coset spaces $K\backslash G_{\mathbb{R}}/\Gamma$ where G is a semi-simple algebraic subgroup of GL_n defined over $\mathbb{Q}$, K is a maximal compact subgroup of its real points $G_{\mathbb{R}}$, and Γ is a torsion-free arithmetic subgroup of its rational points $G_{\mathbb{Q}}$. We dub such manifolds *(semi-simple) arithmetic.* When M is (semi-simple) arithmetic, Raghunathan [55] constructed a smooth manifold compactification $\hat{M}$ of it. Borel and Serre [4] later constructed in the (semi-simple) arithmetic case a topological compactification $\bar{M}$ of M and proved an important homotopy fact about $\partial\bar{M}$. They showed that $\partial\mathcal{M}$ is homotopically equivalent to a countably infinite wedge of spheres of dimensions $r-1$ where $\mathcal{M}\to\bar{M}$ is the universal cover of $\bar{M}$ and $r=\mathbb{Q}$-rank of G. (When $r=0$, $\partial\mathcal{M}=\phi$; i.e. M is compact.) Of course, $\partial\hat{M}$ and $\partial\bar{M}$ are homotopically equivalent; but it is not clear from their constructions whether or not they are homeomorphic. We have the following partial result.

COROLLARY 7.3. *Let M be a (semi-simple) arithmetic manifold, then any two (smooth) manifold compactifications of M are (smoothly) equivalent provided we assume in addition that*

(1) dim $M\geq 6$*;*

(2) $\mathbb{Q}$-rank of $G\neq 2$.

In particular, under these extra assumptions $\bar{M}$ and $\hat{M}$ are equivalent manifold compactifications of M; hence, $\partial\bar{M}$ is homeomorphic to $\partial\hat{M}$.

Proof of 7.3. We first consider the case when the $\mathbb{Q}$-rank of $G>2$. Then $\partial\mathcal{M}$ is simply connected. Hence, $\partial\bar{M}$ is connected and $\pi_1(\partial\bar{M})\simeq\pi_1(M)$. Lemma 0.12 therefore yields that $\mathrm{Wh}(\pi_1\partial\bar{M})=0$ since M is a complete non-positively curved locally symmetric space. Hence, Corollary 7.3 follows from Proposition 7.1 in this case.

When the $\mathbb{Q}$-rank of $G=1$, $\pi_1 C$ maps monomorphically onto a subgroup of $\pi_1\bar{M}$ for each component C of $\partial\bar{M}$. Let $M_C\to M$ be the connected covering space corresponding to this subgroup. Note that M_C is also a complete non-positively curved locally symmetric space. Hence, Lemma 0.12 also yields that $\mathrm{Wh}(\pi_1 M_C)=0$. But $\pi_1(M_C)\simeq\pi_1(C)$; therefore Corollary 7.3 also follows from Proposition 7.1 in this alternate case.

REMARK. More generally, let M be a complete non-positively curved locally symmetric space with finite volume. We conjecture that any two (smooth) manifold compactifications of M are (smoothly) equivalent provided dim $M\neq 4,5$.

Return now to the definition of topological rigidity for an arbitrary manifold M made in Section 0. (See the paragraph preceding Addendum 0.5.) And drop the conditions that h restricted to $\partial N\cup(N-C)$ is a homeomorphism and that h_t and h agree on $\partial N\cup(N-K)$. The resulting concept is called *absolute topological rigidity.* (When M is compact, this agrees with our earlier definition made in the paragraph following the statement of Theorem 0.1.)

THEOREM 7.4. *Let M be a complete Riemannian manifold whose sectional curvatures all lie in the closed interval $[-b^2,-a^2]$ where $0<a<b<+\infty$ and assume* dim $M\geq 5$*. Then M is topologically rigid. And when M has finite volume and* dim $M\geq 6$*, both M and $\bar{M}$ are absolutely topologically rigid where $\bar{M}$ is the Gromov-Heintze-Margulis compactification of M. ($\bar{M}$ is also absolutely topologically rigid when* dim $M=5$.*)*

Proof of 7.4. Recall that the results of Shi [62] and of Abresch [1] show that there exists an A-regular complete non-positively curved Riemannian manifold M_0 with

$\pi_1(M_0) \simeq \pi_1(M)$. (See the paragraph preceding Theorem 0.1.) Hence Addendum 0.5 shows that both M and, when M has finite volume, also $\bar{M}$ are topologically rigid.

We now assume that M has finite volume and let $f : (N, \partial N) \to (\bar{M}, \partial \bar{M})$ be a homotopy equivalence where N is a compact manifold and dim $M \geq 5$. Recall that each component of $\partial \bar{M}$ is an aspherical manifold whose fundamental group is virtually nilpotent. Hence $\partial \bar{M}$ is topologically rigid by [16]. (When dim $M = 5$, [33] is also used to show this.) We may therefore assume, after homotoping f, that $f \mid \partial N$ is a homeomorphism. But since $\bar{M}$ is topologically rigid, we can now further homotope f to a homeomorphism from N to $\bar{M}$. This shows that $\bar{M}$ is absolutely topologically rigid.

Finally, let $f : \mathcal{M} \to M$ be a proper homotopy equivalence where $\mathcal{M}$ is a manifold with $\partial \mathcal{M} = \phi$, M has finite volume, and dim $M \geq 6$. Because of [15], $\tilde{K}_0(\pi_1 C) = 0$ for each component C of $\partial \bar{M}$. Consequently, Siebenmann's thesis [63] implies that $\mathcal{M}$ has a manifold compactification $\bar{\mathcal{M}}$. It is easy to construct a homotopy equivalence $\bar{f} : (\bar{\mathcal{M}}, \partial \bar{\mathcal{M}}) \to (\bar{M}, \partial \bar{M})$ such that $\bar{f} \mid (\bar{\mathcal{M}} - \partial \bar{\mathcal{M}})$ is properly homotopic to f. Using the fact that $\bar{M}$ is absolutely topologically rigid, one now easily shows that f is properly homotopic to a homeomorphism. Hence, M is also absolutely topologically rigid.

COROLLARY 7.5. *Let M and N be complete (connected) Riemannian manifolds with finite volume and whose sectional curvatures all lie in some closed interval $[-b^2, -a^2]$ where $0 < a < b < +\infty$, and assume that* dim $M \geq 5$. *If there is an isomorphism $\alpha : \pi_1 M \to \pi_1 N$, then α is induced by a homeomorphism $f : M \to N$. Furthermore, this homeomorphism f can be constructed to be a diffeomorphism off some compact subset of M, i.e., there exists a compact subset K of M such that the restriction*

$$f : (M - K) \to (N - f(K))$$

is a diffeomorphism

REMARK 7.6. When both M and N are locally symmetric spaces, α is induced by a diffeomorphism provided dim $M \geq 3$. This is a consequence of Mostow's Strong Rigidity Theorem [50] as extended to the finite volume case by Prasad [54]. On the other hand, we constructed in [27] non-compact examples of homeomorphic but non-diffeomorphic manifolds M and N satisfying the hypotheses of Corollary 7.5. Hence the homeomorphism f of Corollary 7.5 cannot in general be a diffeomorphism.

REMARK 7.7. The assertion of Corollary 7.5 is false when dim $M = 2$; e.g., let M be a Riemann surface of genus 1 with 3 cusps and let N be a Riemann surface of genus 2 with 1 cusp. Then $\pi_1 M \simeq \pi_1 N$ but M and N are not homeomorphic.

Proof of 7.5. Suppose that M is not compact and let $\bar{M}$ be the Gromov-Heintze-Margulis compactification of M. Let C be a connected component of $\partial \bar{M}$ and recall that $\pi_1 C$ is a virtually nilpotent subgroup of $\pi_1 M$. Note that

$$cd(\pi_1 M) = cd(\pi_1 C) = \text{dim } M - 1 \geq 4 \tag{2}$$

where $cd(\Gamma)$ denotes the cohomological dimension of a group Γ. Therefore $\pi_1 N$ contains a non-trivial nilpotent subgroup which is not infinite cyclic. Consequently, N is also not compact because of Preissman's theorem; cf. [3, pg. 100, exercise (ii)]. When both M and N are compact, Corollary 7.5 was announced in [69] and proven in [23]. Hence we may now assume that both M and N are non-compact and let $\bar{N}$ denote the Gromov-Heintze-Margulis manifold compactification of N. Therefore an equation analogous to (2) holds for N; consequently, dim M = dim N.

We also have that every element in $\pi_1 C$ is a cuspidal element of M; i.e., is represented by a curve in its free homotopy class of arbitrarily short length. Let $\mathcal{N}$ be a normal nilpotent subgroup in $\pi_1 C$ of finite index and pick a non-trivial element g from the center of $\mathcal{N}$. Note that $\mathcal{N}$ is not infinite cyclic since $cd(\mathcal{N}) = cd(\pi_1 C) \geq 4$. Let $\tilde{M}$ and $\tilde{N}$ be the total spaces of the universal covers of M and N. Identify $\pi_1 M$ and $\pi_1 N$ with the groups of deck transformations of $\tilde{M}$ and $\tilde{N}$, respectively. Either $\alpha(g)$ stabilizes a unique geodesic in $\tilde{N}$ or is parabolic. If the first situation were true, then all elements in $\alpha(\mathcal{N})$ would stabilize this same geodesic since $\alpha(g)$ is in the center of $\alpha(\mathcal{N})$. Therefore, $\alpha(\mathcal{N})$ would be an infinite cyclic group contradicting the fact that $\mathcal{N}$ is not infinite cyclic. We conclude that $\alpha(g)$ is parabolic and stabilizes a unique point P on the visibility sphere $\tilde{N}(\infty)$ of $\tilde{N}$. One then easily argues that each non-trivial element in $\alpha(\pi_1 C)$ is parabolic and stabilizes the same point P. There consequently exists a connected component C_α of $\partial\bar{N}$ such that $\alpha(\pi_1 C) \subset \pi_1 C_\alpha$. Reasoning as above shows that $\alpha(\pi_1 C) = \pi_1 C_\alpha$ and that the correspondence $C \mapsto C_\alpha$ is a bijection between the components of $\partial\bar{M}$ and those of $\partial\bar{N}$. Since $\bar{M}$ and $\bar{N}$ as well as each component of $\partial\bar{M}$ and $\partial\bar{N}$ are aspherical, one can use the above information to construct a homotopy equivalence of pairs of topological spaces $\psi : (\bar{M}, \partial\bar{M}) \to (\bar{N}, \partial\bar{N})$ inducing α.

For each component C of $\partial\bar{M}$, the restricted homotopy equivalence $\psi : C \to C_\alpha$ is homotopic to a diffeomorphism because of [28, Lemmas 3 and 4] which is merely a concatenation of the earlier results from [6], [11], [34], [35], [37], [44]. Lemma 4 states that C and C_α are both infranilmanifolds and Lemma 3 is a generalization of the Bieberbach-Malcev rigidity theorems to infranilmanifolds due to K.B. Lee and F. Raymond [44]. Using this fact, we can construct from ψ a proper homotopy equivalence $\rho : M \to N$ inducing α and such that ρ is a diffeomorphism off some compact subset of M. Corollary 7.5 now follows from the fact that M is topologically rigid; cf. Theorem 7.4.

Let $\mathrm{Top}(M)$ denote the group of all self homeomorphisms of M and let $\mathrm{Out}(\pi_1 M)$ be the group of all outer automorphisms of $\pi_1 M$. There is a natural homomorphism

$$\mathrm{Top}(M) \to \mathrm{Out}(\pi_1 M).$$

We proved in [23] that this homomorphism is an epimorphism when M is a compact (connected) negatively curved Riemannian manifold with $\dim M \neq 3, 4$. (This result had been announced in [69].) Setting $M = N$ in Corollary 7.5 yields the following extension of that result.

COROLLARY 7.8. *Let M be a complete (connected) Riemannian manifold with finite volume and whose sectional curvatures all lie in some closed interval* $[-b^2, -a^2]$ *where* $0 < a < b < +\infty$. *If* $\dim M \geq 5$, *then the natural group homomorphism* $\mathrm{Top}(M) \to \mathrm{Out}(\pi_1 M)$ *is an epimorphism.*

REMARK 7.9. Let $\mathrm{Iso}(M)$ and $\mathrm{Diff}(M)$ denote respectively the groups of all (self) isometries and (self) diffeomorphisms of M. Then there are natural homomorphisms

$$\mathrm{Iso}(M) \to \mathrm{Diff}(M) \to \mathrm{Top}(M) \to \mathrm{Out}(\pi_1 M). \tag{3}$$

When the manifold of Corollary 7.8 is a locally symmetric space, the Mostow-Prasad rigidity theorem yields the much stronger conclusion that the natural homomorphism $\mathrm{Iso}(M) \to \mathrm{Out}(\pi_1 M)$ is an isomorphism. But we showed in [22] that the natural homomorphism $\mathrm{Diff}(M) \to \mathrm{Out}(\pi_1 M)$ is, in general, not an epimorphism when M is a manifold satisfying the hypothesis of Corollary 7.8.

We next formulate our partial result on Conjecture 0.6. Let M be a (connected) non-positively curved Riemannian manifold and Γ be a torsion-free group which contains $\pi_1 M$ as a subgroup with finite index. Let $\tilde{M}$ denote the universal covering space of M.

THEOREM 7.10. *If M is closed and* dim $M \neq 3, 4$, *then the deck transformation action of $\pi_1 M$ on $\tilde{M}$ extends to a topological action of Γ on $\tilde{M}$. Consequently, there exists a closed aspherical manifold N with $\pi_1 N = \Gamma$; namely, $N = \tilde{M}/\Gamma$.*

REMARK 7.11. When $\tilde{M}$ is a symmetric space without 1 or 2 dimensional factors, Γ embeds in Iso$(\tilde{M})$ extending $\pi_1 M \subset$ Iso$(\tilde{M})$; this is a consequence of Mostow's Strong Rigidity Theorem [50]. When dim $M = 2$, Theorem 7.10 is a consequence of a result due to Eckmann and Muller [14]; our proof only applies to the situation where dim $M \geq 5$.

REMARK 7.12. It follows from Theorem 7.10 that Conjecture 0.6 is true whenever Γ contains a subgroup π of finite index where π is isomorphic to the fundamental group of a closed (connected) non-positively curved Riemannian manifold of dim $\neq 3, 4$.

QUESTION 7.13. Does the manifold N constructed in Theorem 7.10 support a smooth structure? If so, then does some smooth structure on N support a non-positively curved Riemannian metric?

Define a class $\mathcal{C}$ of (discrete) groups as follows. A group $\Gamma \in \mathcal{C}$ if and only if there exists an A-regular complete (connected) non-positively curved Riemannian manifold M with $\Gamma \simeq \pi_1 M$.

LEMMA 7.14. *Let π and Γ be two groups such that $\pi \in \mathcal{C}$. If Γ is a subgroup of π, then $\Gamma \in \mathcal{C}$. Also, if Γ is torsion-free and π is a finite index subgroup of Γ, then $\Gamma \in \mathcal{C}$. That is, the class $\mathcal{C}$ is closed with respect to subgroups and torsion-free finite index supergroups.*

Proof of 7.14. Let M be an A-regular complete (connected) non-positively curved Riemannian manifold with $\pi \simeq \pi_1 M$. Identify π with the group of deck transformations of the universal cover $\tilde{M}$ of M. Consider first the case where Γ is a subgroup of π and let $N = \tilde{M}/\Gamma$. Then N is also an A-regular complete non-positively curved Riemannian manifold and $\pi_1 N = \Gamma$; consequently, $\Gamma \in \mathcal{C}$.

Consider next the situation where π is a finite index subgroup of Γ. (Because we've verified the first case, we may now assume that π is a normal subgroup of Γ.) Let s denote this index. Let X denote the s-fold metric Cartesian product of $\tilde{M}$ with itself. It is easy to see that X is also an A-regular complete non-positively curved Riemannian manifold. There is a fundamental construction due to Serre [61, Theorem 1] which gives an embedding of Γ in Iso(X) extending the diagonal action of π on X. Hence, $\mathcal{N} = X/\Gamma$ is an A-regular complete non-positively curved Riemannian manifold and $\Gamma = \pi_1 \mathcal{N}$. This shows that $\Gamma \in \mathcal{C}$ also in this case.

Let $\mathbb{Z}_2$ denote the additive group of the ring of integers mod 2. Recall that the first Stiefel-Whitney class of a (connected) manifold M is a homomorphism $\omega_1(M) : \pi_1 M \to \mathbb{Z}_2$ which determines the orientation line bundle over M.

LEMMA 7.15. *Let $\Gamma \in \mathcal{C}$ and $\omega : \Gamma \to \mathbb{Z}_2$ be a homomorphism. Then there exists an A-regular complete (connected) non-positively curved Riemannian manifold N with $\pi_1 N = \Gamma$ and $\omega_1(N) = \omega$.*

Proof of 7.15. Since $\Gamma \in \mathcal{C}$, there exists an A-regular complete non-positively curved Riemannian manifold M with $\pi_1 M = \Gamma$. Consider the homomorphism $\alpha : \Gamma \to \mathbb{Z}_2$ given by $\alpha = \omega_1(M) + \omega$. Identify $\mathbb{Z}_2$ with the subgroup Iso$(\mathbb{R})$ generated by the reflection $x \mapsto -x$. Let Γ act diagonally on the Riemannian Cartesian product $\tilde{M} \times \mathbb{R}$ and set $N = \tilde{M} \times \mathbb{R}/\Gamma$.

Proof of 7.10. Recall that a (discrete) group π is an *n-dimensional Poincare duality group* if there is an Eilenberg-MacLane space $K(\pi, 1)$ which is an n-dimensional geometric Poincare complex in the sense of Wall [68]. In particular, $\pi_1 M$ is an m-dimensional Poincare duality group with $m = \dim M$. Since $\pi_1 M$ has finite index in the torsion-free group Γ, it is easily argued using [5; Thm. 7.1, pg. 205; Prop. 10.2, pg. 224] that Γ is also an m-dimensional Poincare duality group. We note that Serre's construction [61, Thm. 1] is also crucially used in proving [5; Prop. 10.2, pg. 224].

Observe next that Lemma 7.14 yields that $\Gamma \in \mathcal{C}$ since $\pi_1 M \in \mathcal{C}$. Let $\omega : \Gamma \to \mathbb{Z}_2$ be the homomorphism representing the first Stiefel-Whitney class of the Poincare duality group Γ. (Wu's formula gives ω.) And let N^n be the A-regular complete (connected) non-positively curved Riemannian manifold with $\pi_1 N = \Gamma$ and $\omega_1(N) = \omega$ gotten by applying Lemma 7.15. Proposition 0.10 now shows that the surgery assembly map

$$A_* : H_*(N; \mathcal{L}_*) \to L_*(\Gamma, \omega)$$

is an isomorphism. And, although $n = \dim N$ may be larger than m, Proposition 0.8 says that the surgery assembly map for N depends only on $\pi_1 N = \Gamma$ and on $\omega_1(N) = \omega$. We have thus verified that Γ is an m-dimensional Novikov group in the sense of Ranicki [58, Def. 13.2]. We can hence apply Ranicki's result [58, Prop. 13.5 and Remark 13.7] to conclude that there exists a closed m-dimensional (topological) manifold $\mathcal{N}$ homotopically equivalent to N. Let $\mathcal{M} \to \mathcal{N}$ be the finite sheeted covering space corresponding to the subgroup $\pi_1 M$ of $\Gamma = \pi_1 \mathcal{N}$. Note that $\mathcal{M}$ is a closed aspherical manifold and $\pi_1 \mathcal{M} \simeq \pi_1 M$. Consequently, our topological rigidity theorem [23, Thm. 0.1] yields that $\mathcal{M}$ and M are homeomorphic. Theorem 7.10 now follows directly from the preceding 3 sentences.

We now apply Theorem 7.10 to Problem 0.7 by specializing to the situation where M is a non-positively curved Riemannian manifold such that $\pi_1 M$ has trivial center. This condition on $\pi_1 M$ always occurs when M has strictly negative sectional curvatures. Eberlein [12] shows more generally that $\pi_1 M$ has trivial center provided $\tilde{M}$ has no one-dimensional metric factor. Let F be a finite subgroup of $\mathrm{Out}(\pi_1 M)$. Then there exists a unique extension Γ_F of F by $\pi_1 M$ realizing the embedding $F \subset \mathrm{Out}(\pi_1 M)$ since $\pi_1 M$ has trivial center; cf. [5, pg. 106, Cor. 6.8]. (Recall $\pi_1 M$ is a normal subgroup of Γ_F with quotient F.)

COROLLARY 7.16. *If Γ_F is torsion-free, then F splits back to* $\mathrm{Top}(M)$*; i.e., Problem 0.7 has a positive solution in this case.*

Proof of 7.16. We obtain a representation $\Gamma_F \to \mathrm{Top}(\tilde{M})$ extending the deck transformation representation $\pi_1 M \to \mathrm{Top}(\tilde{M})$ by setting $\Gamma = \Gamma_F$ in Theorem 7.10. This naturally induces a representation $F \to \mathrm{Top}(M)$ which is the posited splitting back of F to $\mathrm{Top}(M)$.

REMARK 7.17. It is interesting to investigate how far back F splits in sequence (3). When $\tilde{M}$ is a symmetric space without 1 or 2 dimensional metric factors, Mostow [50] shows that F splits all the way back to $\mathrm{Iso}(M)$. But F does not always split back to $\mathrm{Iso}(M)$ when $\tilde{M} = \mathbb{H}^2$; however, Kerckhoff [42] shows (in this case) that F always splits back to $\mathrm{Iso}(M_F)$ where $\tilde{M}_F \simeq \mathbb{H}^2$ and M_F is diffeomorphic to M. When $\tilde{M}$ is not a symmetric space, we gave examples in [22] where F does not split back to $\mathrm{Diff}(M)$. But it is still possible that F splits back to $\mathrm{Diff}(M_F)$ where M_F is homeomorphic to M; and even that M_F supports a non-positively curved Riemannian metric with $F \subset \mathrm{Iso}(M_F)$.

We next proceed to extend Theorem 7.10 to the finite volume, pinched negatively curved situation. Fix the following notation. Let M be a complete (connected) Riemannian manifold with finite volume and whose sectional curvatures all lie in some closed interval $[-b^2, -a^2]$ where $0 < a < b < +\infty$. Also assume that dim $M \geq 5$.

THEOREM 7.18. *Let Γ be a torsion-free group which contains $\pi_1 M$ as a finite index subgroup. Then the deck transformation action of $\pi_1 M$ on the universal cover $\tilde{M}$ of M extends to a topological action of Γ on $\tilde{M}$. Moreover, there exists a compact aspherical manifold N with $\pi_1 N = \Gamma$ whose covering space corresponding to the subgroup $\pi_1 M$ of Γ is homeomorphic to the Gromov-Heintze-Margulis manifold compactification $\bar{M}$ of M.*

REMARK 7.19. When M is a locally symmetric space, Theorem 7.18 is a consequence of Prasad's generalization [54] of Mostow's Strong Rigidity Theorem [50]; in this case, $\Gamma \subset \mathrm{Iso}(\tilde{M})$.

We deduce from Theorem 7.18 the following consequence in exactly the same way that Corollary 7.16 was deduced from Theorem 7.10. Let F be a finite subgroup of $\mathrm{Out}(\pi_1 M)$ and let Γ_F be the unique extension of F by $\pi_1 M$ realizing the embedding $F \subset \mathrm{Out}(\pi_1 M)$. Recall that $\pi_1 M$ has a trivial center.

COROLLARY 7.20. *If Γ_F is torsion-free, then F splits back to* $\mathrm{Top}(M)$.

Proof of 7.18. Recall that $\pi_1 M \in \mathcal{C}$ because of [62] and [1]. Consequently, Lemma 7.14 shows that $\Gamma \in \mathcal{C}$. We can hence apply Lemma 0.12 to obtain that

$$\tilde{K}_0(\mathbb{Z}\Gamma) = \mathrm{Wh}(\Gamma) = 0.$$

Using this vanishing result together with [5, Prop. 6.6 and Thm. 7.1], we obtain a finite aspherical CW complex K with $\pi_1 K = \Gamma$.

We now make the extra assumption that $\pi_1 M$ is a normal subgroup of Γ and let F denote the quotient group. At the end of our proof, we will show how this assumption is removed.

It was shown during the proof of Corollary 7.5 that $\mathrm{Out}(\pi_1 M)$ acts on $\pi_0(\partial\bar{M})$; hence, F acts on $\pi_0(\partial\bar{M})$. Let the connected components $C_1, C_2, C_3, \ldots, C_n$ be a complete set of orbit representatives for this action and let F_i denote the isotropy subgroup of F stabilizing C_i. Let Γ_i denote the normalizer of $\pi_1(C_i)$ in Γ. Since $\pi_1(C_i)$ is its own normalizer in $\pi_1 M$, cf. [3, pgs. 110-113], we have that Γ_i is an extension of F_i by $\pi_1(C_i)$. Let us now state this more precisely. If we use ψ_i to denote the restriction to Γ_i of the canonical epimorphism $\Gamma \to F$, then $\mathrm{image}(\psi_i) = F_i$ and $\ker(\psi_i) = \pi_1(C_i)$.

Recall that each C_i is a closed aspherical manifold and $\pi_1(C_i)$ is virtually nilpotent. Hence each Γ_i is a finitely generated, torsion-free, virtually nilpotent group. It is now a consequence of Malcev's work [45] that there exists a closed smooth aspherical manifold B_i with $\pi_1 B_i = \Gamma_i$. Let B denote the disjoint union $\bigcup_{i=1}^{n} B_i$ and let $f : B \to K$ be a cellular map (with respect to some triangulation of B) such that $f_\# : \pi_1 B_i \to \pi_1 K$ identifies each $\pi_1 B_i$ with the subgroup Γ_i of $\Gamma = \pi_1 K$. Let $\mathcal{M}$ denote the mapping cylinder of f and let $p : \hat{\mathcal{M}} \to \mathcal{M}$ be its covering space corresponding to $\pi_1 M$ considered as a subgroup of $\pi_1 \mathcal{M} = \Gamma$. Setting $\hat{B} = \rho^{-1}(B)$, there is a homotopy equivalence of pairs of topological spaces

$$\psi : (\bar{M}, \partial\bar{M}) \to (\hat{\mathcal{M}}, \hat{B}).$$

Using the main result of [16] which extends to the case where dim $\hat{B} = 4$ because of [33], we can assume that the restriction

$$\psi : \partial\bar{M} \to \hat{B}$$

is a homeomorphism. In particular, $(\hat{\mathcal{M}}, \hat{B})$ is an m-dimensional Poincare pair in the sense of Wall [68]; consequently, $(\mathcal{M}, B)$ is also an m-dimensional Poincare pair since $(\hat{\mathcal{M}}, \hat{B})$ is a finite sheeted cover of $(\mathcal{M}, B)$; cf. [5; Prop. 10.2, pg. 224].

Recall $\Gamma \in \mathcal{C}$. Hence Lemma 7.15 together with Propositions 0.8 and 0.10 imply that the assembly map

$$A_* : H_*(\mathcal{M}, \mathcal{L}_*) \to L_*(\pi_1\mathcal{M}, \omega)$$

is an isomorphism, where ω is determined by the orientation data for the Poincare pair $(\mathcal{M}, B)$. We can therefore apply surgery theory to produce a compact m-dimensional manifold N and a homotopy equivalence

$$\zeta : (N, \partial N) \to (\mathcal{M}, B)$$

such that the restricted map $\zeta : \partial N \to B$ is a homeomorphism. This is done using Ranicki's work [58; §§12 and 13]; in particular, the analogue of his Remark 13.7 for Poincare pairs with manifold boundary is true. This is applicable to our situation since the finite sheeted cover $(\hat{\mathcal{M}}, \hat{B})$ of $(\mathcal{M}, B)$ has a manifold structure extending the manifold structure $\hat{B}$; namely, $\psi : (\bar{M}, \partial\bar{M}) \to (\hat{\mathcal{M}}, \hat{B})$.

Identify $\pi_1 N$ with $\pi_1\mathcal{M} = \Gamma$ via $\zeta_\#$ and consider a lift $\hat{\zeta} : (\hat{N}, \partial\hat{N}) \to (\hat{\mathcal{M}}, \hat{B})$ of ζ where $\hat{N} \to N$ is the covering space of N corresponding to the subgroup $\pi_1 M$ of Γ. Let $\tau : (\hat{\mathcal{M}}, \hat{B}) \to (\bar{M}, \partial\bar{M})$ be a homotopy inverse to ψ such that $\tau \mid \hat{B}$ is the inverse of the homeomorphism $\psi \mid \partial\bar{M}$. Recall that $\pi_1\bar{M} = \pi_1 M \in \mathcal{C}$. Hence $\bar{M}$ is topologically rigid because of Theorem 0.1. Therefore the composite

$$\tau \circ \hat{\zeta} : (\hat{N}, \partial\hat{N}) \to (\bar{M}, \partial\bar{M})$$

is homotopic rel boundary to a homeomorphism. This verifies the last sentence of Theorem 7.18.

Let $\tilde{N} \to \mathrm{Int}(N)$ be the universal cover of the interior of N, and identify Γ with its group of deck transformations. Identify $\tilde{N}$ with $\tilde{M}$ by lifting the homeomorphism $\mathrm{Int}(\hat{N}) \to M$ constructed above. This gives the action of Γ on $\tilde{M}$ posited in Theorem 7.18.

We now drop our assumption that $\pi_1 M$ is a normal subgroup of Γ. Define a smaller subgroup $\pi \subset \pi_1 M$ by $\pi = \bigcap_{\gamma\in\Gamma} \gamma(\pi_1 M)\gamma^{-1}$, and let $M_\pi \to M$ be the covering space corresponding to $\pi \subset \pi_1 M$. Since $\pi_1(M_\pi) = \pi$ is a normal subgroup with finite index in Γ, the already proven case of Theorem 7.18 gives a compact aspherical manifold N with $\pi_1 N = \Gamma$ and whose covering corresponding to $\pi \subset \Gamma$ is homeomorphic to the Gromov-Heintze-Margulis manifold compactification $\bar{M}_\pi$ of M_π. Let $\hat{N} \to N$ be the covering space corresponding to $\pi_1 M \subset \Gamma$. We can construct a bijection between the components of $\partial\bar{M}$ and $\partial\hat{N}$ with corresponding components having isomorphic fundamental groups by using the action of $\pi_1 M/\pi \subset \mathrm{Out}(\pi_1(M_\pi))$ on $\pi_0(\partial\bar{M}_\pi)$ constructed in the proof of Corollary 7.5. Therefore the manifold pairs $(\bar{M}, \partial\bar{M})$ and $(\hat{N}, \partial\hat{N})$ are homotopically equivalent and hence homeomorphic since $\bar{M}$ is absolutely topologically rigid by Theorem 7.4. The posited action of Γ on $\tilde{M}$ is then constructed as in the previous paragraph.

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