ASIAN J. MATH.
Vol. 2, No. 2, pp. 345–354, June 1998

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A LINK BETWEEN TWO ELLIPTIC QUANTUM GROUPS*

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Abstract. We consider the category $\mathcal{C}_B$ of meromorphic finite-dimensional representations of the quantum elliptic algebra $\mathcal{B}$ constructed via Belavin's R-matrix, and the category $\mathcal{C}_F$ of meromorphic finite-dimensional representations of Felder's elliptic quantum group $\mathcal{E}_{\tau,\frac{\gamma}{2}}(\mathfrak{gl}_n)$. For any fixed $c \in \mathbb{C}$, we use a version of the Vertex-IRF correspondence to construct two families of (generically) fully faithful functors $\mathcal{H}_x^c : \mathcal{C}_B \to \mathcal{D}_B$ and $\mathcal{F}_x^c : \mathcal{C}_F \to \mathcal{D}_B$ where $\mathcal{D}_B$ is a certain category of infinite-dimensional representations of $\mathcal{B}$ by difference operators. We use this to construct an equivalence between the abelian subcategory of $\mathcal{C}_B$ generated by tensor products of vector representations and the abelian subcategory of $\mathcal{C}_F$ generated by tensor products of vector representations.

1. Categories of meromorphic representations. In this section, we recall the definitions of various categories of representations of quantum elliptic algebras.

Notations: Let us fix $\tau \in \mathbb{C}$, $\mathrm{Im}(\tau) > 0$, $\gamma \in \mathbb{R}\backslash\mathbb{Q}$ and $n \geq 2$. Denote by $(v_i)_{i=1}^n$ the canonical basis of $\mathbb{C}^n$ and by $(E_{ij})_{i,j=1}^n$ the canonical basis of $\mathrm{End}(\mathbb{C}^n)$, i.e $E_{ij}v_k = \delta_{jk}v_i$. Let $\mathfrak{h} = \{\sum_i \lambda_i E_{ii} \mid \sum_i \lambda_i = 0\}$ be the space of diagonal traceless matrices. We have a natural identification $\mathfrak{h}^* = \{\sum_i \lambda_i E_{ii}^* \mid \sum_i \lambda_i = 0\}$. In particular, the weight of v_i is $\omega_i = E_{ii}^* - \frac{1}{n}\sum_k E_{kk}^*$.

Classical theta functions: The theta function $\theta_{\kappa,\kappa'}(t;\tau)$ with characteristics $\kappa, \kappa' \in \mathbb{R}$ is defined by the formula

$$\theta_{\kappa,\kappa'}(t;\tau) = \sum_{m\in\mathbb{Z}} e^{i\pi(m+\kappa)((m+\kappa)\tau+2(t+\kappa'))}.$$

It is an entire function whose zeros are simple and form the (shifted) lattice $\{\frac{1}{2} - \kappa + (\frac{1}{2} - \kappa')\tau\} + \mathbb{Z} + \tau\mathbb{Z}$.

Theta functions satisfy (and are characterized up to renormalization by) the following fundamental monodromy relations

$$\theta_{\kappa,\kappa'}(t+1;\tau) = e^{2i\pi\kappa}\theta_{\kappa,\kappa'}(t;\tau), \tag{1}$$

$$\theta_{\kappa,\kappa'}(t+\tau;\tau) = e^{-i\pi\tau-2i\pi(t+\kappa')}\theta_{\kappa,\kappa'}(t;\tau). \tag{2}$$

Theta functions with different characteristics are related to each other by shifts of t:

$$\theta_{\kappa_1+\kappa_2,\kappa_1'+\kappa_2'}(t;\tau) = e^{i\pi\kappa_2^2\tau+2i\pi\kappa_2(t+\kappa_1'+\kappa_2')}\theta_{\kappa_1,\kappa_1'}(t+\kappa_2\tau+\kappa_2';\tau). \tag{3}$$

In particular, we set $\theta(t) = \theta_{\frac{1}{2},\frac{1}{2}}(t;\tau)$.

1.1. Meromorphic representations of the Belavin quantum elliptic algebra. Consider the two $n \times n$ matrices

$$A = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \xi & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \xi^{n-1} \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \vdots & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

*Received March 20, 1998; accepted for publication June 16, 1998. The authors were supported by the NSF grant DMS-9700477. The second author would like to thank Harvard University Mathematics Department for its hospitality without which this work would not have been possible.

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where $\xi = e^{2i\pi/n}$. We have $A^n = B^n = Id$, $BA = \xi AB$, i.e A, B generate the Heisenberg group. Belavin ([3]) introduced the matrix $R^B(z) \in \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(\mathbb{C}^n)$, uniquely determined by the following properties:

1. Unitarity: $R^B(z)R^B_{21}(-z) = 1$,
2. $R^B(z)$ is meromorphic, with simple poles at $z = \gamma + \mathbb{Z} + \tau\mathbb{Z}$,
3. $R^B(0) = P : x \otimes y \mapsto y \otimes x$ for $x, y \in \mathbb{C}^n$ (permutation),
4. Lattice translation properties:

$$R^B(z+1) = A_1 R^B(z) A_1^{-1} = A_2^{-1} R^B(z) A_2,$$
$$R^B(z+\tau) = e^{-2i\pi\frac{n-1}{n}\gamma} B_1 R^B(z) B_1^{-1} = e^{-2i\pi\frac{n-1}{n}\gamma} B_2^{-1} R^B(z) B_2.$$

In particular, $R^B(z)$ commutes with $A \otimes A$ and $B \otimes B$. The matrix $R^B(z)$ satisfies the quantum Yang-Baxter equation with spectral parameters:

$$R^B_{12}(z-w)R^B_{13}(z)R^B_{23}(w) = R^B_{23}(w)R^B_{13}(z)R^B_{12}(z-w).$$

The category $\mathcal{C}_B$: Following Faddeev, Reshetikhin, Takhtajan and Semenov-Tian-Shansky, one can define an algebra $\mathcal{B}$ from $R^B(z)$, using the RLL formalism-see [4], [12]. However, we will only need to consider a certain category of modules over this algebra, defined as follows.

Let $\mathcal{C}_B$ be the category whose objects are pairs $(V, L(z))$ where V is a finite dimensional vector space and $L(z) \in \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(V)$ is an invertible meromorphic function (the L-operator) such that $L(z+n) = L(z)$ and $L(z+n\tau) = L(z)$, satisfying the following relation in the space $\mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(V) \otimes \mathrm{End}(V)$:

$$R^B_{12}(z-w)L_{13}(z)L_{23}(w) = L_{23}(w)L_{13}(z)R^B_{12}(z-w) \tag{4}$$

(as meromorphic functions of z and w); morphisms $(V, L(z)) \to (V', L'(z))$ are linear maps $\varphi : V \to V'$ such that $(1 \otimes \varphi)L(z) = L'(z)(1 \otimes \varphi)$ in the space $\mathrm{Hom}(\mathbb{C}^n \otimes V, \mathbb{C}^n \otimes V')$. The quantum Yang-Baxter relation for R^B implies that $(\mathbb{C}^n, \chi(z)R^B(z-w)) \in Ob(\mathcal{C}_B)$ for all $w \in \mathbb{C}$, where we set $\chi(z) = \frac{\theta(z-(1-\frac{1}{n})\gamma)}{\theta(z)}$. This object is called the vector representation and will be denoted simply by $V_B(w)$.

The category $\mathcal{C}_B$ is naturally a tensor category with tensor product

$$(V, L(z)) \otimes (V', L'(z)) = (V \otimes V', L_{12}(z)L'_{13}(z)) \tag{5}$$

at the level of objects and with the usual tensor product at the level of morphisms.

There is a notion of a dual representation in the category $\mathcal{C}_B$: the (right) dual of $(V, L(z))$ is $(V^*, L^*(z))$ where $L^*(z) = L^{-1}(z)^{t_2}$ (first apply inversion, then apply the transposition in the second component t_2). If $V, W \in Ob(\mathcal{C}_B)$ and $\varphi \in \mathrm{Hom}_{\mathcal{C}_B}(V, W)$ then $\varphi^t \in \mathrm{Hom}_{\mathcal{C}_B}(W^*, V^*)$, and the functor $\mathcal{C}_B \to \mathcal{C}_B$, $V \mapsto V^*$ is a contravariant equivalence of categories. Moreover, for $V, W, Z \in Ob(\mathcal{C}_B)$, we have canonical isomorphisms $(V \otimes W)^* \simeq (W^* \otimes V^*)$ and $\mathrm{Hom}_{\mathcal{C}_B}(V \otimes W, Z) \simeq \mathrm{Hom}_{\mathcal{C}_B}(V, Z \otimes W^*)$.

We will also need an extended category $\mathcal{C}_B^x$ defined as follows: objects of $\mathcal{C}_B^x$ are objects of $\mathcal{C}_B$ but we set

$$\mathrm{Hom}_{\mathcal{C}_B^x}(V, V') = \mathrm{Hom}_{\mathcal{C}_B}(V, V') \otimes M_{\mathbb{C}}$$

where $M_{\mathbb{C}}$ is the field of meromorphic functions of a complex variable x. In other words, morphisms in $\mathcal{C}_B^x$ are meromorphic 1-parameter families of morphisms in $\mathcal{C}_B$.

The category $\mathcal{D}_B$: We now define a difference-operator variant of the categories $\mathcal{C}_B, \mathcal{C}_B^x$. Let us denote by $M_{\mathfrak{h}^*}$ the field of $(n\omega_i)$-periodic meromorphic functions $\mathfrak{h}^* \to \mathbb{C}$ and by $D_{\mathfrak{h}^*}$ the $\mathbb{C}$-algebra generated by $M_{\mathfrak{h}^*}$ and shift operators $T_\mu : M_{\mathfrak{h}^*} \to M_{\mathfrak{h}^*}, f(\lambda) \mapsto f(\lambda+\mu)$ for $\mu \in \mathfrak{h}^*$. If V is a finite-dimensional vector space, we set $V_{\mathfrak{h}^*} = M_{\mathfrak{h}^*} \otimes_{\mathbb{C}} V$, and $D(V) = D_{\mathfrak{h}^*} \otimes_{\mathbb{C}} \mathrm{End}(V)$. Let $\mathcal{D}_B$ be the category whose objects are pairs $(V, L(z))$ where V is a finite-dimensional $\mathbb{C}$-vector space and $L(z) \in \mathrm{End}(\mathbb{C}^n) \otimes D(V)$ is an invertible operator with meromorphic coefficients satisfying (4) in $\mathrm{End}(\mathbb{C}^n) \otimes D(V) \otimes D(V)$; morphisms $(V, L(z)) \to (V', L'(z))$ are $(n\omega_i)$-periodic meromorphic functions $\varphi : \mathfrak{h}^* \to \mathrm{Hom}(V, V')$ such that $(1 \otimes \varphi)L(z) = L(z)(1 \otimes \varphi)$ in $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n \otimes V_{\mathfrak{h}^*}, \mathbb{C}^n \otimes V'_{\mathfrak{h}^*})$ (i.e morphisms are $M_{\mathfrak{h}^*}$-linear).

The category $\mathcal{D}_B$ is a right-module category over $\mathcal{C}_B$, i.e we have a (bi)functor $\otimes : \mathcal{D}_B \times \mathcal{C}_B \to \mathcal{D}_B$ defined by (5), and for any $V, W \in \mathcal{O}b(\mathcal{D}_B)$, $Z \in \mathcal{O}b(\mathcal{C}_B)$, we have a canonical isomorphism $\mathrm{Hom}_{\mathcal{D}_B}(V \otimes Z, W) \simeq \mathrm{Hom}_{\mathcal{D}_B}(V, W \otimes Z^*)$. The category $\mathcal{D}_B^x$ is defined in an analogous way: objects are pairs $(V, L(z,x))$ as in $\mathcal{D}_B$ but the L-operator is now a meromorphic function of z and x, and morphisms $(V, L(z,x)) \to (V', L'(z,x))$ are meromorphic maps $\varphi(\lambda, x) : \mathfrak{h}^* \times \mathbb{C} \to \mathrm{Hom}_{\mathbb{C}}(V, V')$ satisfying $(1 \otimes \varphi)L(z,x) = L(z,x)(1 \otimes \varphi)$.

1.2. Meromorphic representations of the elliptic quantum group $\mathcal{E}_{\tau,\gamma/2}(\mathfrak{gl}_n)$.

Felder's dynamical R-matrix: let us consider the functions of two complex variables

$$\alpha(z,l) = \frac{\theta(l+\gamma)\theta(z)}{\theta(l)\theta(z-\gamma)}, \qquad \beta(z,l) = \frac{\theta(z-l)\theta(\gamma)}{\theta(l)\theta(z-\gamma)}.$$

As functions of z, α and β have simple poles at $z = \gamma + \mathbb{Z} + \tau\mathbb{Z}$ and satisfy

$$\alpha(z+1,l) = \alpha(z,l), \qquad \alpha(z+\tau,l) = e^{-2i\pi\gamma}\alpha(z,l),$$
$$\beta(z+1,l) = \beta(z,l), \qquad \beta(z+\tau,l) = e^{-2i\pi(\gamma-l)}\beta(z,l).$$

Felder introduced in [5] the matrix $R^F(z,\lambda) : \mathbb{C} \times \mathfrak{h}^* \to \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(\mathbb{C}^n)$:

$$R^F(z,\lambda) = \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} \alpha(z, \lambda_i - \lambda_j) E_{ii} \otimes E_{jj} + \sum_{i \neq j} \beta(z, \lambda_i - \lambda_j) E_{ji} \otimes E_{ij}$$

where $\lambda = \sum_i \lambda_i E_{ii}^* \in \mathfrak{h}^*$.

This matrix is a solution of the quantum dynamical Yang-Baxter equation with spectral parameters

$$\begin{aligned} &R_{12}^F(z-w, \lambda - \gamma h_3) R_{13}^F(z,\lambda) R_{23}^F(w, \lambda - \gamma h_1) \\ &\qquad = R_{23}^F(w,\lambda) R_{13}^F(z, \lambda - \gamma h_2) R_{12}^F(z-w, \lambda) \end{aligned}$$

where we have used the following convention: if V_i are diagonalizable $\mathfrak{h}$-modules with weight decomposition $V_i = \bigoplus_\mu V_i^\mu$ and $a(\lambda) \in \mathrm{End}(\bigotimes_i V_i)$ then

$$a(\lambda - \gamma h_l)_{|\bigotimes_i V_i^{\mu_i}} = a(\lambda - \gamma \mu_l).$$

As usual, indices indicate the components of the tensor product on which the operators act.

In addition, $R^F(z,\lambda)$ satisfies the following two conditions:

1. Unitarity: $R_{12}^F(z,\lambda) R_{21}^F(-z,\lambda) = Id$,

2. Weight zero: $\forall h \in \mathfrak{h}$, $[h_1 + h_2, R^F(z,\lambda)] = 0$.

The category $\mathcal{C}_F$: It is possible to use $R^F(z,\lambda)$ to define an algebra by the RLL-formalism (see [5]): the elliptic quantum group $\mathcal{E}_{\tau,\gamma/2}(\mathfrak{gl}_n(\mathbb{C}))$. However, we will only need the following category of its representations $\mathcal{C}_F$, introduced by Felder in [5] and studied by Felder and Varchenko in [6]: objects are pairs $(V, L(z,\lambda))$ where V is a finite-dimensional diagonalizable $\mathfrak{h}$-module and $L(z,\lambda) : \mathbb{C}\times\mathfrak{h}^* \to \mathrm{End}(\mathbb{C}^n)\otimes\mathrm{End}(V)$ is an invertible meromorphic function which is $(n\omega_i)$-periodic in λ and which satisfies the following two conditions:

$$[h_1 + h_2, L(z,\lambda)] = 0,$$

$$\begin{aligned} &R^F_{12}(z-w,\lambda-\gamma h_3)L_{13}(z,\lambda)L_{23}(w,\lambda-\gamma h_1) \\ &\qquad =L_{23}(w,\lambda)L_{13}(z,\lambda-\gamma h_2)R^F_{12}(z-w,\lambda). \end{aligned} \tag{6}$$

Morphisms $(V, L(z,\lambda)) \to (V', L'(z,\lambda))$ are $(n\omega_i)$-periodic meromorphic weight zero maps $\varphi(\lambda) : V \to V'$ such that $L'(z,\lambda)(1\otimes\varphi(\lambda-\gamma h_1)) = (1\otimes\varphi(\lambda))L(z,\lambda)$. The quantum dynamical Yang-Baxter relation for $R^F(z,\lambda)$ implies that we have $(\mathbb{C}^n, R^F(z-w,\lambda)) \in \mathcal{O}b(\mathcal{C}_F)$ for all $w\in\mathbb{C}$. This is the vector representation and it will be denoted by $V_F(w)$.

The category $\mathcal{C}_F$ is naturally equipped with a tensor structure: it is defined on objects by

$$(V, L(z,\lambda)) \otimes (V', L'(z,\lambda)) = (V\otimes V', L_{12}(z,\lambda-\gamma h_3)L'_{13}(z,\lambda)),$$

and if $\varphi \in \mathrm{Hom}_{\mathcal{C}_F}(V,W), \varphi' \in \mathrm{Hom}_{\mathcal{C}_F}(V',W')$ then

$$(\varphi\otimes\varphi')(\lambda) = \varphi(\lambda-\gamma h_2)\otimes\varphi'(\lambda) \in \mathrm{Hom}_{\mathcal{C}_F}(V\otimes V', W\otimes W').$$

There is a notion of a dual representation in the category $\mathcal{C}_F$: the (right) dual of $(V, L(z,\lambda))$ is $(V^*, L^*(z,\lambda))$ where $L^*(z,\lambda) = L^{-1}(z,\lambda+\gamma h_2)^{t_2}$ (apply inversion, shifting and then apply the transposition in the second component t_2). If $V, W \in \mathcal{O}b(\mathcal{C}_F)$ and $\varphi(\lambda) \in \mathrm{Hom}_{\mathcal{C}_F}(V,W)$ then $\varphi^*(\lambda) := \varphi(\lambda+\gamma h_1)^t \in \mathrm{Hom}_{\mathcal{C}_F}(W^*,V^*)$, and the functor $\mathcal{C}_F \to \mathcal{C}_F$, $V\mapsto V^*$ is a contravariant equivalence of categories. Moreover, for any $V, W \in \mathcal{O}b(\mathcal{C}_F)$, there is a canonical isomorphism $(V\otimes W)^* \simeq (W^*\otimes V^*)$.

The extended category $\mathcal{C}_F^x$ is defined by $\mathcal{O}b(\mathcal{C}_F^x) = \mathcal{O}b(\mathcal{C}_F)$ and

$$\mathrm{Hom}_{\mathcal{C}_F^x}(V,V') = \mathrm{Hom}_{\mathcal{C}_F}(V,V')\otimes M_{\mathbb{C}}$$

i.e morphisms in $\mathcal{C}_F^x$ are meromorphic 1-parameter families of morphisms in $\mathcal{C}_F$.

2. The functor $\mathcal{F}_x^c : \mathcal{C}_F \to \mathcal{D}_B$. In this section, we define a family of functors from meromorphic (finite-dimensional) representations of $\mathcal{E}_{\tau,\frac{\gamma}{2}}(\mathfrak{gl}_n(\mathbb{C}))$ to infinite-dimensional representations of the quantum elliptic algebra $\mathcal{B}$.

2.1. Twists by difference operators. For any finite-dimensional diagonalizable $\mathfrak{h}$-module V, let $e^{\gamma D} \in \mathrm{End}(V_{\mathfrak{h}^*})$ denote the shift operator: $e^{\gamma D}.\sum_\mu f_\mu(\lambda)v_\mu = \sum_\mu f(\lambda+\gamma\mu)v_\mu$, $v_\mu \in V_\mu$. Now let $(V, L(z,\lambda)) \in \mathcal{C}_F$, and let $S(z,\lambda), S'(z,\lambda) : \mathbb{C}\times\mathfrak{h}^* \to \mathrm{End}(\mathbb{C}^n)$ be meromorphic and nondegenerate. Define the difference-twist of $(V, L(z,\lambda))$ to be the pair $(V, L^{S,S'}(z))$ where

$$L^{S,S'}(z) = S_1(z,\lambda-\gamma h_2)L(z,\lambda)e^{-\gamma D_1}S_1'(z,\lambda)^{-1} \in \mathrm{End}(\mathbb{C}^n)\otimes D(V). \tag{7}$$

This is a difference operator acting on $\mathbb{C}^n \otimes V_{\mathfrak{h}^*}$.

LEMMA 1. *The difference operator $L^S(z,\lambda)$ satisfies the following relation in* $\mathrm{End}(\mathbb{C}^n) \otimes D(V) \otimes D(V)$:

$$T_{12}(z,w,\lambda-\gamma h_3)L_{13}^{S,S'}(z)L_{23}^{S,S'}(w) = L_{23}^{S,S'}(w)L_{13}^{S,S'}(z)T'_{12}(z,w,\lambda)$$

where

$$T(z,w,\lambda) = S_2(w,\lambda)S_1(z,\lambda-\gamma h_2)R_{12}^F(z-w,\lambda)S_2(w,\lambda-\gamma h_1)^{-1}S_1(z,\lambda)^{-1}, \quad (8)$$
$$T'(z,w,\lambda) = S'_1(z,\lambda)S'_2(w,\lambda+\gamma h_1)R_{12}^F(z-w,\lambda)S'_1(z,\lambda+\gamma h_1)^{-1}S'_2(w,\lambda)^{-1}. \quad (9)$$

Proof. The proof is straightforward, using relation (6) for $L(z,\lambda)$ and the weight zero property of $R^F(u,\lambda)$ and $L(u,\lambda)$. □

2.2. The Vertex-IRF transform. Let $\phi_l(u) = e^{2i\pi(\frac{l^2\tau}{n}+\frac{lu}{n})}\theta_{0,0}(u+l\tau;n\tau)$ for $l=1,\dots n$. Then the vector $\Phi(u) = (\phi_1(u),\dots,\phi_n(u))$ is, up to renormalization, the unique holomorphic vector in $\mathbb{C}^n$ satisfying the following monodromy relations:

$$\Phi(u+1) = A\Phi(u), \quad (10)$$
$$\Phi(u+\tau) = e^{-i\pi\frac{\tau}{n}-2i\pi\frac{u}{n}}B\Phi(u). \quad (11)$$

Now let $S(z,\lambda) : \mathbb{C}\times\mathfrak{h}^* \to \mathrm{End}(\mathbb{C}^n)$ be the matrix whose columns are $(\Phi_1(z,\lambda),\dots,\Phi_n(z,\lambda))$ where $\Phi_j(z,\lambda) = \Phi(z-n\lambda_j)$. Using (10)-(11), it is easy to see that we have $\det(S(z,\lambda)) = \mathrm{Const}(\lambda)\theta(z)$ where $\mathrm{Const}(\lambda) \neq 0$ and hence that $S(z,\lambda)$ is invertible for $z\neq 0$ and generic λ.

LEMMA 2. *We have*

$$R^B(z-w)S_1(z,\lambda)S_2(w,\lambda-\gamma h_1) = S_2(w,\lambda)S_1(z,\lambda-\gamma h_2)R^F(z-w,\lambda),$$
$$R^B(z-w)S_2(w,\lambda)S_1(z,\lambda+\gamma h_2) = S_1(z,\lambda)S_2(w,\lambda+\gamma h_1)R^F(z-w,\lambda).$$

Proof. The first relation is equivalent to the following identities for $i,j=1,\dots n$:

$$R^B(z-w)\Phi_i(z,\lambda)\otimes\Phi_i(w,\lambda-\gamma\omega_i) = \Phi_i(z,\lambda-\gamma\omega_i)\otimes\Phi_i(w,\lambda),$$
$$R^B(z-w)\Phi_i(z,\lambda)\otimes\Phi_j(w,\lambda-\gamma\omega_i) = \alpha(z-w,\lambda_i-\lambda_j)\Phi_i(z,\lambda-\gamma\omega_j)\otimes\Phi_j(w,\lambda) + \beta(z-w,\lambda_i-\lambda_j)\Phi_j(z,\lambda-\gamma\omega_i)\otimes\Phi_i(w,\lambda).$$

These identities are proved by comparing poles and transformation properties under lattice translations as functions of z and w, and using the uniqueness of Φ. The second relation of the lemma is proved in the same way. These identities are essentially the Vertex/Interaction-Round-a-Face transform of statistical mechanics (see [9],[11] and [7] for the case $n=2$). □

The Vertex-IRF transform first appeared in the work of Baxter [1] and was subsequently generalized to the Belavin R-matrix by Jimbo, Miwa and Okado in [10].

2.3. Construction of the functor $\mathcal{F}_x^c : \mathcal{C}_F \to \mathcal{D}_B$. Let us fix some $c\in\mathbb{C}$. We now define the family of functors $\mathcal{F}_x^c : \mathcal{C}_F \to \mathcal{C}_B$ indexed by $x\in\mathbb{C}$: for $(V,L(z,\lambda)) \in \mathcal{C}_F$, set $\mathcal{F}_x^c((V,L(z,\lambda))) = (V, L^{S_x,S_{x+c}}(z))$ with $S_u(z,\lambda) = S(z-u,\lambda)$ as above and let $\mathcal{F}_x^c$ be trivial at the level of morphisms.

PROPOSITION 1. *$\mathcal{F}_x^c : \mathcal{C}_F \to \mathcal{D}_B$ is a functor.*

Proof. It follows from Lemma 2 that $(V, L^{S_x, S_{x+c}}(z)) \in \mathcal{O}b(\mathcal{D}_B)$. Furthermore, if $\varphi(\lambda) \in \mathrm{Hom}_{\mathcal{C}_F}((V, L(z,\lambda)), (V', L'(z,\lambda)))$ then by definition we have $L'(z,\lambda)(1 \otimes \varphi(\lambda - \gamma h_1)) = (1 \otimes \varphi(\lambda))L(z,\lambda)$, so that

$$\begin{aligned} S_1(z-x,&\lambda - \gamma h_2)L'(z,\lambda)e^{-\gamma D_1}S_1(z-x-c,\lambda)^{-1}(1\otimes\varphi(\lambda)) \\ &= S_1(z-x,\lambda-\gamma h_2)L'(z,\lambda)(1\otimes\varphi(\lambda-\gamma h_1))e^{-\gamma D_1}S_1(z-x-c,\lambda)^{-1} \\ &= S_1(z-x,\lambda-\gamma h_2)(1\otimes\varphi(\lambda))L'(z,\lambda)e^{-\gamma D_1}S_1(z-x-c,\lambda)^{-1} \\ &= (1\otimes\varphi(\lambda))S_1(z-x,\lambda-\gamma h_2)L'(z,\lambda)e^{-\gamma D_1}S_1(z-x-c,\lambda)^{-1} \end{aligned}$$

since $\varphi(\lambda)$ is of weight zero. Thus $\mathcal{F}_x^c(\varphi(\lambda))$ is an intertwiner in the category $\mathcal{D}_B$. □

We can also think of the family of functors $\mathcal{F}_x^c$ as a single functor $\mathcal{F}^c : \mathcal{C}_F^x \to \mathcal{D}_B^x$.

REMARK. We can think of the difference-twist and the relations in Lemma 2 as a dynamical analogue of the notion of equivalence of R-matrices due to Drinfeld and Belavin-see [2].

3. The image of the trivial representation and the functor $\mathcal{H}_x^c : \mathcal{C}_B \to \mathcal{D}_B$. Applying the functor $\mathcal{F}_x^c$ to the trivial representation $(\mathbb{C}, \mathrm{Id}) \in \mathcal{O}b(\mathcal{C}_F)$ yields

$$\mathcal{F}_x^c((\mathbb{C}, \mathrm{Id})) = (\mathbb{C}, S(z-x,\lambda)e^{-\gamma D_1}S(z-x-c,\lambda)^{-1}).$$

We will denote this object by I_x^c. For instance, when $n = 2$, we obtain a representation of the Belavin quantum elliptic algebra as difference operators acting on the space of periodic meromorphic functions in one variable λ, i.e given by an L-operator

$$L(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$$

where $a(z), b(z), c(z), d(z)$ are operators of the form $f(z)T_{-\gamma} + g(z)$ where $T_{-\gamma}$ is the shift by $-\gamma$.

Such representations of $\mathcal{B}$ by difference operators already appeared in the work of Krichever, Zabrodin ([11]) (for $n = 2$) and Hasegawa ([8],[9])(for the general case), where they were also derived by some Vertex-IRF correspondence.

DEFINITION. Let $c \in \mathbb{C}$ and let $\mathcal{H}_x^c : \mathcal{C}_B \to \mathcal{D}_B$ be the functor defined by the assignment $V \to I_x^c \otimes V$ and which is trivial at the level of morphisms. The family of functors $\mathcal{H}_x^c$ gives rise to a functor $\mathcal{H}^c : \mathcal{C}_B^x \to \mathcal{D}_B^x$.

4. Full faithfulness of the functor $\mathcal{H}_x^c : \mathcal{C}_B \to \mathcal{D}_B$. In this section, we prove the following result

PROPOSITION 2. *Let $V, V' \in \mathcal{O}b(\mathcal{C}_B)$. Then for all but finitely many values of $x \bmod \mathbb{Z} + \mathbb{Z}\tau$, the map*

$$\mathcal{H}_x^c : \mathrm{Hom}_{\mathcal{C}_B}(V, V') \tilde{\to} \mathrm{Hom}_{\mathcal{D}_B}(\mathcal{H}_x^c(V), \mathcal{H}_x^c(V'))$$

is an isomorphism.

Proof. Since $\mathrm{Hom}_{\mathcal{C}_B}(V, V') \simeq \mathrm{Hom}_{\mathcal{C}_B}(\mathbb{C}, V' \otimes V^*)$, $\mathrm{Hom}_{\mathcal{D}_B}(I_x^c \otimes V, I_x^c \otimes V') \simeq \mathrm{Hom}_{\mathcal{D}_B}(I_x^c, I_x^c \otimes V' \otimes V^*)$, it is enough to show that the map $\mathcal{H}_x^c : \mathrm{Hom}_{\mathcal{C}_B}(\mathbb{C}, W) \to \mathrm{Hom}_{\mathcal{D}_B}(I_x^c, I_x^c \otimes W)$ is an isomorphism for all $W \in \mathcal{O}b(\mathcal{C}_B)$. Since $\mathcal{H}_x^c$ is trivial at the level of morphisms, this map is injective. Now let $W \in \mathcal{O}b(\mathcal{C}_B)$ and let $\varphi(\lambda) \in \mathrm{Hom}_{\mathcal{D}_B}(I_x^c, I_x^c \otimes W)$, that is, $\varphi(\lambda)$ is a $(n\omega_i)$-periodic meromorphic function $\mathfrak{h}^* \to W$ satisfying the equation

$$\begin{aligned} \varphi_2(\lambda)&S_1(z-x,\lambda)e^{-\gamma D_1}S_1(z-x-c,\lambda)^{-1} \\ &= S_1(z-x,\lambda)e^{-\gamma D_1}S_1(z-x-c,\lambda)^{-1}L_{12}(z)\varphi_2(\lambda) \end{aligned}$$

where $L(z)$ is the L-operator of W. This is equivalent to

$$L_{12}(z)\varphi_2(\lambda) = S_1(z-x-c,\lambda)\varphi_2(\lambda+\gamma h_1)S_1(z-x-c,\lambda)^{-1} \tag{12}$$

Now $L(z)$ is an elliptic function (of periods n and $n\tau$) so it is either constant or it has a pole. Restricting W to the subrepresentation $\mathrm{Span}(\varphi(\lambda), \lambda \in \mathfrak{h}^*)$, we see that the latter case is impossible for generic x as the RHS of (12) has a pole at $z = x + c$ only; hence $L(z)$ is constant. Furthermore, from (12) we see that the matrix

$$M(\lambda) = S_1(z-x-c,\lambda)^{-1}L_{12}S_1(z-x-c,\lambda)$$

is independent of z. In particular, setting $z \mapsto z+1$ and using the transformation properties (10) of $S(z,\lambda)$, we obtain $[A_1, L_{12}] = 0$. This implies that $L = \sum_i E_{ii} \otimes D_i$ for some $D_i \in \mathrm{End}(W)$.

LEMMA 3. *Let U be a finite dimensional vector space, let $T \in \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(U)$ be an invertible solution of the equation*

$$R^B_{12}(z)T_{13}T_{23} = T_{23}T_{13}R^B_{12}(z) \tag{13}$$

such that $T = \sum_i E_{ii} \otimes D_i$ for some $D_i \in \mathrm{End}(U)$. Then $[D_i, D_j] = 0$ for all i, j and there exists $X \in \mathrm{End}(U)$ such that $X^n = 1$ and $D_{i+1} = XD_i$ for all $i = 1, \ldots n$.

Proof. Let us write $R^B(z) = \sum_{p,q,r,s} R_{p,q,r,s}(z)E_{pq} \otimes E_{rs}$. Then equation (13) is equivalent to $R_{p,q,r,s}(z)D_p \times D_q = R_{p,q,r,s}(z)D_sD_r$ for all p, q, r, s. But it follows from the general formula for $R^B(z)$ that $R_{p,q,r,s}(z) \neq 0$ if and only if $p+q \equiv r+s \pmod n$. Thus we have $[D_i, D_j] = 0$ for all i, j and $X := D_iD_{i+1}^{-1}$ is independent of i, and satisfies $X^n = 1$. □

By the above lemma, there exists $X \in \mathrm{End}(W)$ such that $X^n = 1$ and $D_{i+1} = XD_i$. Suppose that $X \neq 1$ and choose $e \in W$ such that $X(e) = \xi^k e$ with $\xi^k \neq 1$. Now we apply the transformation $z \mapsto z + \tau$ to the matrix $M(\lambda)$. Noting that, by (11), $S(z-x-c+\tau,\lambda) = e^{-i\pi\tau/2-2i\pi(z-x-c)/n}BS(z-x-c,\lambda)F(\lambda)$ where $F(\lambda) = \mathrm{diag}(e^{-2i\pi\lambda}, \ldots e^{-2i\pi\lambda_n})$, we obtain the equality

$$\begin{aligned}F(\lambda)^{-1}S_1(z-x-c,\lambda)^{-1}B_1^{-1}L_{12}B_1S_1(z-x-c,\lambda)F(\lambda)\\ = S_1(z-x-c,\lambda)^{-1}L_{12}S_1(z-x-c,\lambda).\end{aligned}$$

Applying this to the vector e yields $\mathrm{Ad}F(\lambda)(M(\lambda))(e) = \xi^{-k}M(\lambda)(e)$. This is possible for all λ only if $k \equiv 0 \pmod n$. Hence $X = 1$ and (12) reduces to the equation $D\varphi_2(\lambda) = \varphi_2(\lambda+\gamma h_1)$. In particular $\varphi(\lambda)$ is $\gamma(\omega_i - \omega_j)$-periodic. But by our assumption, $\varphi(\lambda)$ is $(n\omega_i)$-periodic and γ is real and irrational. Therefore $\varphi(\lambda)$ is constant and it is a morphism in the category $\mathcal{C}_B$. □

Now, considering x as a parameter, we obtain:

COROLLARY 1. *The functor $\mathcal{H}^c : \mathcal{C}^x_B \to \mathcal{D}^x_B$ is fully faithful.*

REMARK. Equation (12) shows that $\mathrm{Hom}_{\mathcal{D}_B}(I^c_x, I^c_x \otimes V) = \mathrm{Hom}_{\mathcal{D}_B}(V^*, I^0_{x+c})$. Thus the above proposition states that for any finite-dimensional representation $V \in \mathcal{O}b(\mathcal{C}_F)$ and for all but finitely many x mod $\mathbb{Z} + \tau\mathbb{Z}$, we have $\mathrm{Hom}_{\mathcal{D}_B}(V^*, I^0_x) = \mathrm{Hom}_{\mathcal{C}_B}(V^*, \mathbb{C})$, where the isomorphism is induced by the embedding $\mathbb{C} \subset I^0_x$ (constant functions). However, for finitely many values of x mod $\mathbb{Z} + \tau\mathbb{Z}$, this may not be true: see [11] and [9] where some finite-dimensional subrepresentations of I^0_x are considered.

5. Full faithfullness of the functor $\mathcal{F}_x^c : \mathcal{C}_F \to \mathcal{D}_B$. In this section, we prove the following result:

PROPOSITION 3. *The functor $\mathcal{F}_x^c : \mathcal{C}_F \to \mathcal{D}_B$ is fully faithful.*

Proof. We have to show that for any two objects V, V' in $\mathcal{C}_F$ there is an isomorphism $\mathcal{F}_x^c$: $\mathrm{Hom}_{\mathcal{C}_F}(V, V') \to \mathrm{Hom}_{\mathcal{D}_B}(\mathcal{F}_x^c(V), \mathcal{F}_x^c(V'))$. Since $\mathcal{F}_x^c$ is trivial at the level of morphisms, this map is injective. Now let $V, W \in \mathcal{O}b(\mathcal{C}_F)$ and let $\varphi(\lambda) \in \mathrm{Hom}_{\mathcal{D}_B}(\mathcal{F}_x^c(V), \mathcal{F}_x^c(W))$. By definition, $\varphi(\lambda) : V \to W$ satisfies the relation

$$\begin{aligned}\varphi_2(\lambda)S_1(z-x,\lambda-\gamma h_2)L_{12}^V(z,\lambda)e^{-\gamma D_1}S_1(z-x-c,\lambda)^{-1}\\ = S_1(z-x,\lambda-\gamma h_2)L_{12}^W(z,\lambda)e^{-\gamma D_1}S_1(z-x-c,\lambda)^{-1}\varphi_2(\lambda)\end{aligned}$$

where $L^V(z,\lambda)$ (resp. $L^W(z,\lambda)$) is the L-operator of V (resp. W). This is equivalent to

$$\varphi_2(\lambda)S_1(z-x,\lambda-\gamma h_2)L_{12}^V(z,\lambda) = S_1(z-x,\lambda-\gamma h_2)L_{12}^W(z,\lambda)\varphi_2(\lambda-\gamma h_1) \quad (14)$$

Introduce the following notations: write $W = \bigoplus_\xi W_\xi$, $V = \bigoplus_\mu V_\mu$, $\varphi(\lambda) = \sum_\nu \varphi_\nu(\lambda)$ for the weight decompositions (so that $\varphi_\nu : V_\xi \to W_{\xi+\nu}$). Also let $S(z-x,\lambda) = \sum_{i,j} S^{ij}(z-x,\lambda)E_{ij}$, $L_{12}^V(z,\lambda) = \sum_{i,j} E_{ij} \otimes L_V^{ij}(z,\lambda)$ and use the same notation for $L^W(z,\lambda)$. Applying (14) to $v_i \otimes \zeta_\mu$ for some i and $\zeta_\mu \in V_\mu$ yields

$$\begin{aligned}\sum_{j,k,\nu} S^{kj}(z-x,\lambda-\gamma(\mu+\omega_i-\omega_j))v_k \otimes \varphi_\nu(\lambda)(L_V^{ji}(z,\lambda)\zeta_\mu)\\ = \sum_{l,k,\sigma} S^{kl}(z-x,\lambda-\gamma(\mu+\omega_i-\omega_l+\sigma))v_k \otimes L_W^{li}(z,\lambda)\varphi_\sigma(\lambda-\gamma\omega_i)\zeta_\mu\end{aligned} \quad (15)$$

where we used the weight-zero property of $L^V(z,\lambda)$ and $L^W(z,\lambda)$. Applying v_k^* to (15) and projecting on the weight space $W_{\mu+\omega_i+\xi}$ gives the relation

$$\begin{aligned}\sum_{\substack{\nu,j\\ \nu-\omega_j=\xi}} S^{kj}(z-x,\lambda-\gamma(\mu+\omega_i-\omega_j))\varphi_\nu(\lambda)(L_V^{ji}(z,\lambda)\zeta_\mu)\\ = \sum_{\substack{\sigma,l\\ \sigma-\omega_l=\xi}} S^{kl}(z-x,\lambda-\gamma(\mu+\omega_i-\omega_j+\sigma))L_W^{li}(z,\lambda)(\varphi_\sigma(\lambda-\gamma\omega_i)\zeta_\mu)\end{aligned} \quad (16)$$

for any i, k, ξ and $\zeta_\mu \in V_\mu$. Now let $A = \{\chi \mid \varphi_\chi(\lambda) \neq 0\}$. Fix some j and let $\beta \in A$ be an extremal weight in the direction $-\omega_j$ (i.e $\beta - \omega_j + \omega_k \notin A$ for $k \neq j$). Then (16) for $\xi = \beta - \omega_j$ reduces to

$$\begin{aligned}S^{kj}(z-x,\lambda-\gamma(\mu+\omega_i-\omega_j))\varphi_\beta(\lambda)(L_V^{ji}(z,\lambda)\zeta_\mu)\\ = S^{kj}(z-x,\lambda-\gamma(\mu+\omega_i-\omega_j+\beta))L_W^{ji}(z,\lambda)\varphi_\beta(\lambda-\gamma\omega_i)\zeta_\mu\end{aligned} \quad (17)$$

CLAIM. *There exists $i \in \{1,\ldots n\}$, μ and $\zeta_\mu \in V_\mu$ such that $\varphi_\beta(\lambda)(L_V^{ji}(z,\lambda)\zeta_\mu) \neq 0$ for generic z and λ.*

Proof. Recall the central element $\mathrm{Qdet}(z,\lambda) \in \mathcal{E}_{\tau,\frac{\gamma}{2}}(\mathfrak{gl}_n)$. By definition, its action on V is invertible. Expanding $\mathrm{Qdet}(z,\lambda)$ along the j^{th}-line, we have $\mathrm{Qdet}(z,\lambda) = \sum_i L_V^{ji}(z,\lambda)P_i(z,\lambda)$ for some operators $P_i(z,\lambda) \in \mathrm{End}(V)$. In particular,

$$\sum_i \mathrm{Im}\, L^{ji}(z,\lambda) = V,$$

and the claim follows.

Thus, the ratio $S^{kj}(z-x,\lambda-\gamma(\mu+\omega_i-\omega_j+\beta))/S^{kj}(z-x,\lambda-\gamma(\mu+\omega_i-\omega_j))$ is independent of k. This is possible only if $\beta\in\sum_{r\neq j}\mathbb{C}E^*_{rr}$. Applying this to $j=1,\dots n$, we see that $A=\{0\}$. Hence $\varphi(\lambda)$ is an $\mathfrak{h}$-module map. But then relation (14) reduces to $\varphi_2(\lambda)L^V_{12}(z,\lambda)=L^W_{12}(z,\lambda)\varphi_2(\lambda-\gamma h_1)$, and $\varphi(\lambda)$ is an intertwiner in the category $\mathcal{C}_F$. □

COROLLARY 2. *The functor* $\mathcal{F}^c:\mathcal{C}^x_F\to\mathcal{D}^x_B$ *is fully faithful.*

6. The image of the vector representation. Let us denote

$$\tilde V_F(w)=(\mathbb{C}^n,\chi(w)R^F(w,\lambda)).$$

It is an object of $\mathcal{C}_F$ which equals the tensor product of the vector representation $V_F(w)$ by the one-dimensional representation $(\mathbb{C},\chi(z))$.

PROPOSITION 4. *For any* x,w, $x+c\not\equiv w\ (\mathrm{mod}\ \mathbb{Z}+\tau\mathbb{Z})$, *we have* $\mathcal{F}^c_x(\tilde V_F(w))\simeq\mathcal{H}^c_x(V_B(w))$.

Proof. By definition, we have

$$\mathcal{F}^c_x(\tilde V_F(w))=(\mathbb{C}^n,\chi(z)S_1(z-x,\lambda-\gamma h_2)R^F(z-w,\lambda)e^{-\gamma D_1}\times S_1(z-x-c,\lambda)^{-1}),$$

$$I^c_x\otimes V_B(w)=(\mathbb{C}^n,\chi(z)S_1(z-x,\lambda)e^{-\gamma D_1}S_1(z-x-c,\lambda)R^B(z-w))$$

We claim that the map $\varphi(\lambda)=e^{-\gamma D}(S(w-x-c,\lambda)^{-1})e^{\gamma D}\in\mathrm{End}(\mathbb{C}^n)$ is an intertwiner $\mathcal{H}^c_x(V_B(w))\simeq I^c_x\otimes V_B(w)\xrightarrow{\sim}\mathcal{F}^c_x(\tilde V_F(w))$. Indeed, we have

$$\begin{aligned}
&S_1(z-x,\lambda-\gamma h_2)R^F(z-w,\lambda)e^{-\gamma D_1}S_1(z-x-c,\lambda)^{-1}(1\otimes\varphi(\lambda))\\
&=e^{-\gamma D_2}S_1(z-x,\lambda)e^{\gamma D_2}R^F(z-w,\lambda)e^{-\gamma(D_1+D_2)}\\
&\qquad\qquad S_1(z-x-c,\lambda+\gamma h_2)^{-1}S_2(w-x-c,\lambda)^{-1}e^{\gamma D_2}\\
&=e^{-\gamma D_2}S_1(z-x,\lambda)e^{-\gamma D_1}R^F(z-w,\lambda)\\
&\qquad\qquad S_1(z-x-c,\lambda+\gamma h_2)^{-1}S_2(w-x-c,\lambda)^{-1}e^{\gamma D_2}\\
&=e^{-\gamma D_2}S_1(z-x,\lambda)e^{-\gamma D_1}S_2(w-x-c,\lambda+\gamma h_1)^{-1}\\
&\qquad\qquad S_1(z-x-c,\lambda)^{-1}R^B(z-w)e^{\gamma D_2}\\
&=e^{-\gamma D_2}S_1(z-x,\lambda)S_2(w-x-c,\lambda)e^{-\gamma D_1}S_1(z-x-c,\lambda)^{-1}R^B(z-w)e^{\gamma D_2}\\
&=(1\otimes\varphi(\lambda))S_1(z-x,\lambda)e^{-\gamma D_1}S_1(z-x-c,\lambda)^{-1}R^B(z-w)
\end{aligned}$$

where we used Lemma 2 and the zero-weight property of $R^F(u,\lambda)$. □

LEMMA 4. *Let* $V,V'\in\mathcal{O}b(\mathcal{C}_F)$, $W,W'\in\mathcal{O}b(\mathcal{C}_B)$ *and suppose that* $\mathcal{F}^c_x(V)\simeq\mathcal{H}^c_x(W)$ *and* $\mathcal{F}^c_x(V')\simeq\mathcal{H}^c_x(W')$. *Then* $\mathcal{F}^c_x(V\otimes V')\simeq\mathcal{H}^c_x(W\otimes W')$.

Proof. If $\varphi(\lambda):V\to W$ and $\varphi'(\lambda):V'\to W'$ are intertwiners then it is easy to check using the methods above that $\varphi'_2(\lambda-\gamma h_1)\varphi_1(\lambda):V\otimes V'\to W\otimes W'$ is an intertwiner. □

Applying this to tensor products of the vector representations, we obtain

COROLLARY 3. *For any* $x\in\mathbb{C}$ *and* $w_1,\dots,w_r\in\mathbb{C}\backslash\{x+c+\mathbb{Z}+\tau\mathbb{Z}\}$, *we have*

$$\mathcal{F}^c_x(\tilde V_F(w_1)\otimes\dots\tilde V_F(w_r))\simeq\mathcal{H}^c_x(V_B(w_1)\otimes\dots V_B(w_r)).$$

COROLLARY 4. *For any* $w_1, \ldots, w_r \in \mathbb{C}$*, we have*

$$\mathcal{F}^c(\tilde{V}_F(w_1) \otimes \ldots \tilde{V}_F(w_r)) \simeq \mathcal{H}^c(V_B(w_1) \otimes \ldots V_B(w_r)).$$

Notice that in this case, we have a canonical intertwiner, given by the formula

$$\varphi_{1\ldots r}(\lambda, w_1, \ldots, w_r) = \tilde{S}_r^{-1}(w_r - x - c, \lambda - \gamma \sum_{i=1}^{r-1} h_i) \ldots \tilde{S}_1^{-1}(w_1 - x - c, \lambda),$$

where we set $\tilde{S}(z, \lambda) = e^{-\gamma D} S(z, \lambda) e^{\gamma D}$.

7. Equivalence of subcategories. Let us summarize the results of sections 4-8. By proposition 2, we can identify $\mathcal{C}_B^x$ with a full subcategory $\mathcal{D}_1^x$ of $\mathcal{D}_B^x$. By proposition 3, we can identify $\mathcal{C}_F^x$ with a full subcategory $\mathcal{D}_2^x$ of $\mathcal{D}_B^x$. Moreover, $\mathcal{D}_1^x$ and $\mathcal{D}_2^x$ intersect (at least if we replace $\mathcal{D}_B^x$ by the equivalent category $\widetilde{\mathcal{D}_B^x}$ whose objects are isomorphism classes of objects of $\mathcal{D}_B^x$), and the intersection contains objects of the form $\mathcal{F}^c(\bigotimes_i \tilde{V}_F(w_i)) \simeq \mathcal{H}^c(\bigotimes_i V_B(w_i))$, where $i = 1, \ldots r$ and $w_i \in \mathbb{C}$. Hence,

THEOREM *The abelian subcategory* $\mathcal{V}_B^x$ *of* $\mathcal{C}_B^x$ *generated by objects* $\bigotimes_i V_B(w_i)$ *for* $i = 1, \ldots r$, $r \in \mathbb{N}$ *and* $w_i \in \mathbb{C}$ *and the abelian subcategory* $\mathcal{V}_F^x$ *of* $\mathcal{C}_F^x$ *generated by objects* $\bigotimes_j \tilde{V}_F(w_j)$ *for* $j = 1, \ldots s$, $s \in \mathbb{N}$ *and* $w_j \in \mathbb{C}$ *are equivalent.*

Note that for numerical values of x, $\mathcal{F}_x^c : \mathcal{C}_F \to \mathcal{D}_B$ is always fully faithful, and $\mathcal{F}_x^c(\mathcal{C}_F)$ a full subcategory of $\mathcal{D}_B$, but this is not true of $\mathcal{H}_x^c$, because of the existence of nontrivial finite-dimensional subrepresentations of I_x^0.

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