A LINK BETWEEN TWO ELLIPTIC QUANTUM GROUPS*

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Abstract. We consider the category $C_B$ of meromorphic finite-dimensional representations of the quantum elliptic algebra $B$ constructed via Belavin's R-matrix, and the category $C_F$ of meromorphic finite-dimensional representations of Felder's elliptic quantum group $E_{r,2}(gl_n)$. For any fixed $c \in \mathbb{C}$, we use a version of the Vertex-IRF correspondence to construct two families of (generically) fully faithful functors $\mathcal{H}_c^j : C_B \to D_B$ and $\mathcal{F}_c^j : C_F \to D_B$ where $D_B$ is a certain category of infinite-dimensional representations of $B$ by difference operators. We use this to construct an equivalence between the abelian subcategory of $C_B$ generated by tensor products of vector representations and the abelian subcategory of $C_F$ generated by tensor products of vector representations.

1. Categories of meromorphic representations. In this section, we recall the definitions of various categories of representations of quantum elliptic algebras.

Notations: Let us fix $\tau \in \mathbb{C}$, $\text{Im}(\tau) > 0$, $\gamma \in \mathbb{R}\setminus \mathbb{Q}$ and $n \geq 2$. Denote by $(v_i)_{i=1}^n$ the canonical basis of $\mathbb{C}^n$ and by $(E_{ij})_{i,j=1}^n$ the canonical basis of $\text{End}(\mathbb{C}^n)$, i.e. $E_{ij}v_k = \delta_{jk}v_i$. Let $b = \{\sum_i \lambda_i E_{ii} \mid \sum_i \lambda_i = 0\}$ be the space of diagonal traceless matrices. We have a natural identification $b^* = \{\sum_i \lambda_i E_{ii}^* \mid \sum_i \lambda_i = 0\}$. In particular, the weight of $v_i$ is $\omega_i = E_{ii}^* - \frac{1}{n} \sum_k E_{kk}^*$.

Classical theta functions: The theta function $\theta_{\kappa,\kappa'}(t;\tau)$ with characteristics $\kappa, \kappa' \in \mathbb{R}$ is defined by the formula

$$\theta_{\kappa,\kappa'}(t;\tau) = \sum_{m \in \mathbb{Z}} e^{i\pi(m+\kappa)((m+\kappa)\tau + 2(t+\kappa'))}.$$ 

It is an entire function whose zeros are simple and form the (shifted) lattice $\{\frac{1}{2} - \kappa + (\frac{1}{2} - \kappa')\tau\} + \mathbb{Z} + \tau\mathbb{Z}$.

Theta functions satisfy (and are characterized up to renormalization by) the following fundamental monodromy relations

$$\theta_{\kappa,\kappa'}(t+1;\tau) = e^{2i\pi\kappa}\theta_{\kappa,\kappa'}(t;\tau),$$

$$\theta_{\kappa,\kappa'}(t+\tau;\tau) = e^{-i\pi\tau - 2i\pi(t+\kappa')\theta_{\kappa,\kappa'}(t;\tau)}.$$ 

Theta functions with different characteristics are related to each other by shifts of $t$:

$$\theta_{\kappa_1+\kappa_2,\kappa'_1+\kappa'_2}(t;\tau) = e^{i\pi\kappa_2^2\tau + 2i\pi\kappa_2(t+\kappa'_1+\kappa'_2)}\theta_{\kappa_1,\kappa'_1}(t + \kappa_2\tau + \kappa'_2;\tau).$$

In particular, we set $\theta(t) = \theta_{\frac{1}{2},\frac{1}{2}}(t;\tau)$.

1.1. Meromorphic representations of the Belavin quantum elliptic algebra. Consider the two $n \times n$ matrices

$$A = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \xi & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi^{n-1} \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix}.$$
where $\xi = e^{2i\pi/n}$. We have $A^n = B^n = Id$, $BA = \xi AB$, i.e $A, B$ generate the Heisenberg group. Belavin ([3]) introduced the matrix $R^B(z) \in \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n)$, uniquely determined by the following properties:

1. Unitarity: $R^B(z)R^B_1(-z) = 1$,
2. $R^B(z)$ is meromorphic, with simple poles at $z = \gamma + Z + T Z$,
3. $R^B(0) = P : x \otimes y \mapsto y \otimes x$ for $x, y \in \mathbb{C}^n$ (permutation),
4. Lattice translation properties:

$$R^B(z + 1) = A_1 R^B(z)A_1^{-1} = A_2^{-1} R^B(z)A_2,$$
$$R^B(z + \tau) = e^{-2i\pi n^{-1} \gamma} B_1 R^B(z)B_1^{-1} = e^{-2i\pi n^{-1} \gamma} B_2^{-1} R^B(z)B_2.$$

In particular, $R^B(z)$ commutes with $A \otimes A$ and $B \otimes B$. The matrix $R^B(z)$ satisfies the quantum Yang-Baxter equation with spectral parameters:

$$R^B_{12}(z - w)R^B_{13}(z)R^B_{23}(w) = R^B_{23}(w)R^B_{13}(z)R^B_{12}(z - w).$$

The category $C_B$: Following Faddeev, Reshetikhin, Takhtajan and Semenov-Tian-Shansky, one can define an algebra $B$ from $R^B(z)$, using the RLL formalism—see [4], [12]. However, we will only need to consider a certain category of modules over this algebra, defined as follows.

Let $C_B$ be the category whose objects are pairs $(V, L(z))$ where $V$ is a finite dimensional vector space and $L(z) \in \text{End} \mathbb{C}^n \otimes \text{End}(V)$ is an invertible meromorphic function (the L-operator) such that $L(z + n) = L(z)$ and $L(z + n \tau) = L(z)$, satisfying the following relation in the space $\text{End}(\mathbb{C}^n) \otimes \text{End}(V) \otimes \text{End}(V)$:

$$R^B_{12}(z - w)L_{13}(z)L_{23}(w) = L_{23}(w)L_{13}(z)R^B_{12}(z - w)$$

(as meromorphic functions of $z$ and $w$); morphisms $(V, L(z)) \rightarrow (V', L'(z))$ are linear maps $\varphi : V \rightarrow V'$ such that $(1 \otimes \varphi)L(z) = L'(z)(1 \otimes \varphi)$ in the space $\text{Hom}(\mathbb{C}^n \otimes V, \mathbb{C}^n \otimes V')$. The quantum Yang-Baxter relation for $R^B$ implies that $(\mathbb{C}^n, \chi(z)R^B(z - w)) \in \text{Ob}(C_B)$ for all $w \in \mathbb{C}$, where we set $\chi(z) = \frac{\theta(z - (1 - \frac{1}{n}) \gamma)}{\theta(z)}$. This object is called the vector representation and will be denoted simply by $V_B(w)$.

The category $C_B$ is naturally a tensor category with tensor product

$$(V, L(z)) \otimes (V', L'(z)) = (V \otimes V', L_{12}(z)L_{13}(z))$$

at the level of objects and with the usual tensor product at the level of morphisms.

There is a notion of a dual representation in the category $C_B$: the (right) dual of $(V, L(z))$ is $(V^*, L^*(z))$ where $L^*(z) = L^{-1}(z)\iota_2$ (first apply inversion, then apply the transposition in the second component $\iota_2$). If $V, W \in \text{Ob}(C_B)$ and $\varphi \in \text{Hom}_{C_B}(V, W)$ then $\varphi^* \in \text{Hom}_{C_B}(W^*, V^*)$, and the functor $C_B \rightarrow C_B$, $V \mapsto V^*$ is a contravariant equivalence of categories. Moreover, for $V, W, Z \in \text{Ob}(C_B)$, we have canonical isomorphisms $(V \otimes W)^* \simeq (W^* \otimes V^*)$ and $\text{Hom}_{C_B}(V \otimes W, Z) \simeq \text{Hom}_{C_B}(V, Z \otimes W^*)$.

We will also need an extended category $C_B^e$ defined as follows: objects of $C_B^e$ are objects of $C_B$ but we set

$$\text{Hom}_{C_B^e}(V, V') = \text{Hom}_{C_B}(V, V') \otimes M$$

where $M$ is the field of meromorphic functions of a complex variable $\iota$. In other words, morphisms in $C_B^e$ are meromorphic 1-parameter families of morphisms in $C_B$. 


The category $\mathcal{D}_B$: We now define a difference-operator variant of the categories $\mathcal{C}_B, \mathcal{C}_B^*$. Let us denote by $M_{\mathfrak{h}^*}$ the field of $(n\omega_i)$-periodic meromorphic functions $\mathfrak{h}^* \rightarrow \mathbb{C}$ and by $\mathcal{D}_B$ the $\mathbb{C}$-algebra generated by $M_{\mathfrak{h}^*}$ and shift operators $T_{\mu} : M_{\mathfrak{h}^*} \rightarrow M_{\mathfrak{h}^*}, f(\lambda) \mapsto f(\lambda + \mu)$ for $\mu \in \mathfrak{h}^*$. If $V$ is a finite-dimensional vector space, we set $V_{\mathfrak{h}^*} = M_{\mathfrak{h}^*} \otimes_{\mathbb{C}} V$, and $D(V) = \mathcal{D}_{\mathfrak{h}^*} \otimes_{\mathbb{C}} \text{End}(V)$. Let $\mathcal{D}_B$ be the category whose objects are pairs $(V, L(z))$ where $V$ is a finite-dimensional $\mathbb{C}$-vector space and $L(z) \in \text{End}(\mathbb{C}^n) \otimes D(V)$ is an invertible operator with meromorphic coefficients satisfying (4) in $\text{End}(\mathbb{C}^n) \otimes D(V) \otimes D(V)$; morphisms $(V, L(z)) \rightarrow (V', L'(z))$ are $(n\omega_i)$-periodic meromorphic functions $\phi : \mathfrak{h}^* \rightarrow \text{Hom}(V, V')$ such that $(1 \otimes \phi)L(z) = L(z)(1 \otimes \phi)$ in $\text{Hom}_{\mathbb{C}}(\mathbb{C}^n \otimes V_{\mathfrak{h}^*}, \mathbb{C}^n \otimes V'_{\mathfrak{h}^*})$ (i.e morphisms are $M_{\mathfrak{h}^*}$-linear).

The category $\mathcal{D}_B$ is a right-module category over $\mathcal{C}_B$, i.e, we have a (bi)functor $\otimes : \mathcal{D}_B \times \mathcal{C}_B \rightarrow \mathcal{D}_B$ defined by (5), and for any $V, W \in \text{Ob}(\mathcal{D}_B), Z \in \text{Ob}(\mathcal{C}_B)$, we have a canonical isomorphism $\text{Hom}_{\mathcal{D}_B}(V \otimes Z, W) \simeq \text{Hom}_{\mathcal{D}_B}(V, W \otimes Z^*)$. The category $\mathcal{D}_B^*$ is defined in an analogous way: objects are pairs $(V, L(z, x))$ as in $\mathcal{D}_B$ but the $L$-operator is now a meromorphic function of $z$ and $x$, and morphisms $(V, L(z, x)) \rightarrow (V', L'(z, x))$ are meromorphic maps $\phi(\lambda, x) : \mathfrak{h}^* \times \mathbb{C} \rightarrow \text{Hom}_{\mathbb{C}}(V, V')$ satisfying $(1 \otimes \phi)L(z, x) = L(z, x)(1 \otimes \phi)$.

1.2. Meromorphic representations of the elliptic quantum group $\mathcal{E}_{\tau, \gamma/2}(\mathrm{gl}_n)$.

Felder’s dynamical $R$-matrix: let us consider the functions of two complex variables

$$\alpha(z, l) = \frac{\theta(l + \gamma)\theta(z)}{\theta(l)\theta(z - \gamma)}, \quad \beta(z, l) = \frac{\theta(z - l)\theta(\gamma)}{\theta(l)\theta(z - \gamma)}.$$

As functions of $z, \alpha$ and $\beta$ have simple poles at $z = \gamma + Z + \tau Z$ and satisfy

$$\alpha(z + 1, l) = \alpha(z, l), \quad \alpha(z + \tau, l) = e^{-2i\pi\gamma}\alpha(z, l),$$

$$\beta(z + 1, l) = \beta(z, l), \quad \beta(z + \tau, l) = e^{-2i\pi(\tau - l)}\beta(z, l).$$

Felder introduced in [5] the matrix $R^F(z, \lambda) : \mathbb{C} \times \mathfrak{h}^* \rightarrow \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n)$:

$$R^F(z, \lambda) = \sum_{i} E_{ii} \otimes E_{ii} + \sum_{i \neq j} \alpha(z, \lambda_i - \lambda_j)E_{ii} \otimes E_{jj} + \sum_{i \neq j} \beta(z, \lambda_i - \lambda_j)E_{ji} \otimes E_{jj}$$

where $\lambda = \sum_i \lambda_i E_i^* \in \mathfrak{h}^*$.

This matrix is a solution of the quantum dynamical Yang-Baxter equation with spectral parameters

$$R_{12}^F(z - w, \lambda - \gamma h_3)R_{13}^F(z, \lambda)R_{23}^F(w, \lambda - \gamma h_1) = R_{23}^F(w, \lambda)R_{13}^F(z, \lambda - \gamma h_2)R_{12}^F(z - w, \lambda)$$

where we have used the following convention: if $V_i$ are diagonalizable $\mathfrak{h}$-modules with weight decomposition $V_i = \bigoplus_{\mu} V_i^\mu$ and $a(\lambda) \in \text{End}(\bigotimes_i V_i)$ then

$$a(\lambda - \gamma h_i) \bigotimes_i V_i^\mu = a(\lambda - \gamma \mu).$$

As usual, indices indicate the components of the tensor product on which the operators act.

In addition, $R^F(z, \lambda)$ satisfies the following two conditions:

1. Unitarity: $R^F_{12}(z, \lambda)R^F_{21}(-z, \lambda) = Id,$
2. Weight zero: \( \forall h \in \mathfrak{h}, [h_1 + h_2, R^F(z, \lambda)] = 0 \).

The category \( \mathcal{C}_F \): It is possible to use \( R^F(z, \lambda) \) to define an algebra by the RLL-formalism (see [5]): the elliptic quantum group \( \mathcal{E}_{r, \gamma/2}(\mathfrak{gl}_n(\mathbb{C})) \). However, we will only need the following category of its representations \( \mathcal{C}_F \), introduced by Felder in [5] and studied by Felder and Varchenko in [6]: objects are pairs \( (V, L(z, \lambda)) \) where \( V \) is a finite-dimensional diagonalizable \( \mathfrak{h} \)-module and \( L(z, \lambda) : \mathbb{C} \times \mathfrak{h}^* \rightarrow \text{End}(\mathbb{C}^n) \otimes \text{End}(V) \) is an invertible meromorphic function which is \((n\omega_1)\)-periodic in \( \lambda \) and which satisfies the following two conditions:

\[
[h_1 + h_2, L(z, \lambda)] = 0,
\]

\[
R^F_{12}(z - w, \lambda - \gamma h_3)L_{13}(z, \lambda)L_{23}(w, \lambda - \gamma h_1)
= L_{23}(w, \lambda)L_{13}(z, \lambda - \gamma h_2)R^F_{12}(z - w, \lambda).
\]

Morphisms \( (V, L(z, \lambda)) \rightarrow (V', L'(z, \lambda)) \) are \((n\omega_1)\)-periodic meromorphic weight zero maps \( \varphi(\lambda) : V \rightarrow V' \) such that \( L'(z, \lambda)(1 \otimes \varphi(\lambda - \gamma h_1)) = (1 \otimes \varphi(\lambda))L(z, \lambda) \).

The quantum dynamical Yang-Baxter relation for \( R^F(z, \lambda) \) implies that we have \((\mathbb{C}^n, R^F(z - w, \lambda)) \in \text{Ob}(\mathcal{C}_F) \) for all \( w \in \mathbb{C} \). This is the vector representation and it will be denoted by \( V_F(w) \).

The category \( \mathcal{C}_F \) is naturally equipped with a tensor structure: it is defined on objects by

\[
(V, L(z, \lambda)) \otimes (V', L'(z, \lambda)) = (V \otimes V', L_{12}(z, \lambda - \gamma h_3)L_{13}(z, \lambda)),
\]

and if \( \varphi \in \text{Hom}_{\mathcal{C}_F}(V, W), \varphi' \in \text{Hom}_{\mathcal{C}_F}(V', W') \) then

\[
(\varphi \otimes \varphi')(\lambda) = \varphi(\lambda - \gamma h_2) \otimes \varphi'(\lambda) \in \text{Hom}_{\mathcal{C}_F}(V \otimes V', W \otimes W').
\]

There is a notion of a dual representation in the category \( \mathcal{C}_F \): the (right) dual of \((V, L(z, \lambda))\) is \((V^*, L^*(z, \lambda))\) where \( L^*(z, \lambda) = L^{-1}(z, \lambda + \gamma h_2)^{t_2} \) (apply inversion, shifting and then apply the transposition in the second component \( t_2 \)). If \( V, W \in \text{Ob}(\mathcal{C}_F) \) and \( \varphi(\lambda) \in \text{Hom}_{\mathcal{C}_F}(V, W) \) then \( \varphi^*(\lambda) := \varphi(\lambda + \gamma h_1)^t \in \text{Hom}_{\mathcal{C}_F}(W^*, V^*) \), and the functor \( \mathcal{C}_F \rightarrow \mathcal{C}_F, V \mapsto V^* \) is a contravariant equivalence of categories. Moreover, for any \( V, W \in \text{Ob}(\mathcal{C}_F) \), there is a canonical isomorphism \((V \otimes W)^* \simeq (W^* \otimes V^*)\).

The extended category \( \mathcal{C}_F^\infty \) is defined by \( \text{Ob}(\mathcal{C}_F^\infty) = \text{Ob}(\mathcal{C}_F) \) and

\[
\text{Hom}_{\mathcal{C}_F^\infty}(V, V') = \text{Hom}_{\mathcal{C}_F}(V, V') \otimes M_{\mathbb{C}}
\]

i.e morphisms in \( \mathcal{C}_F^\infty \) are meromorphic 1-parameter families of morphisms in \( \mathcal{C}_F \).

2. The functor \( \mathcal{F}^c_x : \mathcal{C}_F \rightarrow \mathcal{D}_B \). In this section, we define a family of functors from meromorphic (finite-dimensional) representations of \( \mathcal{E}_{r, \gamma/2}(\mathfrak{gl}_n(\mathbb{C})) \) to infinite-dimensional representations of the quantum elliptic algebra \( B \).

2.1. Twists by difference operators. For any finite-dimensional diagonalizable \( \mathfrak{h} \)-module \( V \), let \( e^{\gamma D} \in \text{End}(V_{\mathfrak{h}^*}) \) denote the shift operator: \( e^{\gamma D} \sum f(\mu) v_\mu = \sum f(\lambda + \gamma \mu) v_\mu, v_\mu \in V_\mu \). Now let \((V, L(z, \lambda)) \in \mathcal{C}_F \), and let \( S(z, \lambda), S'(z, \lambda) : \mathbb{C} \times \mathfrak{h}^* \rightarrow \text{End}(\mathbb{C}^n) \) be meromorphic and nondegenerate. Define the difference-twist of \((V, L(z, \lambda))\) to be the pair \((V, L^S, S')(z)\) where

\[
L^S, S'(z) = S_1(z, \lambda - \gamma h_2)L(z, \lambda)e^{-\gamma D_1} S'_1(z, \lambda)^{-1} \in \text{End}(\mathbb{C}^n) \otimes D(V).
\]
This is a difference operator acting on $\mathbb{C}^n \otimes V_0$.

**Lemma 1.** The difference operator $L^S(z, \lambda)$ satisfies the following relation in $\text{End}(\mathbb{C}^n) \otimes D(V) \otimes D(V)$:

$$T_{12}(z, w, \lambda - \gamma h_3)L^{S, S'}_{13}(z)L^{S, S'}_{23}(w) = L^{S, S'}_{23}(w)L^{S, S'}_{13}(z)T_{12}(z, w, \lambda)$$

where

$$T(z, w, \lambda) = S_2(w, \lambda)S_1(z, \lambda - \gamma h_2)R^F_{12}(z - w, \lambda)S_2(w, \lambda - \gamma h_1)^{-1}S_1(z, \lambda)^{-1},$$

$$T'(z, w, \lambda) = S'_1(z, \lambda)S'_2(w, \lambda + \gamma h_1)R^F_{12}(z - w, \lambda)S'_1(z, \lambda + \gamma h_1)^{-1}S'_2(w, \lambda)^{-1}. (9)$$

**Proof.** The proof is straightforward, using relation (6) for $L(z, \lambda)$ and the weight zero property of $R^F(u, \lambda)$ and $L(u, \lambda)$. □

**2.2. The Vertex-IRF transform.** Let $\Phi(u) = e^{2i\pi \left(\frac{i^2}{n} + \frac{\lambda u}{n}\right)}\theta_0(u + i\tau; n\tau)$ for $l = 1, \ldots, n$. Then the vector $\Phi(u) = (\phi_1(u), \ldots, \phi_n(u))$ is, up to renormalization, the unique holomorphic vector in $\mathbb{C}^n$ satisfying the following monodromy relations:

$$\Phi(u + 1) = A\Phi(u),$$

$$\Phi(u + \tau) = e^{-i\pi \frac{u^2}{n} - 2i\pi \frac{\lambda u}{n}}B\Phi(u).$$

Now let $S(z, \lambda) : \mathbb{C} \times \mathbb{I} \to \text{End}(\mathbb{C}^n)$ be the matrix whose columns are $(\Phi_1(z, \lambda), \ldots, \Phi_n(z, \lambda))$ where $\Phi_j(z, \lambda) = \Phi(z - n\lambda_j)$. Using (10)-(11), it is easy to see that we have $\det(S(z, \lambda)) = \text{Const}(\lambda)\theta(z)$ where $\text{Const}(\lambda) \neq 0$ and hence that $S(z, \lambda)$ is invertible for $z \neq 0$ and generic $\lambda$.

**Lemma 2.** We have

$$R^B(z - w)S_1(z, \lambda)S_2(w, \lambda - \gamma h_1) = S_2(w, \lambda)S_1(z, \lambda - \gamma h_2)R^F(z - w, \lambda),$$

$$R^B(z - w)S_2(w, \lambda)S_1(z, \lambda + \gamma h_2) = S_1(z, \lambda)S_2(w, \lambda + \gamma h_1)R^F(z - w, \lambda).$$

**Proof.** The first relation is equivalent to the following identities for $i, j = 1, \ldots, n$:

$$R^B(z - w)\Phi_i(z, \lambda) \otimes \Phi_i(w, \lambda - \gamma \omega_i) = \Phi_i(z, \lambda - \gamma \omega_i) \otimes \Phi_i(w, \lambda),$$

$$R^B(z - w)\Phi_i(z, \lambda) \otimes \Phi_j(w, \lambda - \gamma \omega_i) = \alpha(z - w, \lambda_i - \lambda_j)\Phi_i(z, \lambda - \gamma \omega_j) \otimes \Phi_j(w, \lambda) + \beta(z - w, \lambda_i - \lambda_j)\Phi_j(z, \lambda - \gamma \omega_i) \otimes \Phi_i(w, \lambda).$$

These identities are proved by comparing poles and transformation properties under lattice translations as functions of $z$ and $w$, and using the uniqueness of $\Phi$. The second relation of the lemma is proved in the same way. These identities are essentially the Vertex/Interaction-Round-a-Face transform of statistical mechanics (see [9],[11] and [7] for the case $n = 2$). □

The Vertex-IRF transform first appeared in the work of Baxter [1] and was subsequently generalized to the Belavin R-matrix by Jimbo, Miwa and Okado in [10].

**2.3. Construction of the functor** $\mathcal{F}^c_z : \mathcal{C}_F \to \mathcal{D}_B$. Let us fix some $c \in \mathbb{C}$. We now define the family of functors $\mathcal{F}^c_z : \mathcal{C}_F \to \mathcal{C}_B$ indexed by $x \in \mathbb{C}$: for $(V, L(z, \lambda)) \in \mathcal{C}_F$, set $\mathcal{F}^c_z((V, L(z, \lambda))) = (V, L^{S, S'}(z)S_u(z, \lambda))$ with $S_u(z, \lambda) = S(z - u, \lambda)$ as above and let $\mathcal{F}^c_z$ be trivial at the level of morphisms.

**Proposition 1.** $\mathcal{F}^c_z : \mathcal{C}_F \to \mathcal{D}_B$ is a functor.
Proof. It follows from Lemma 2 that \((V, L^S_\ast, S_{\ast+}(z)) \in \text{Ob} (\mathcal{D}_B)\). Furthermore, if \(\varphi(\lambda) \in \text{Hom}_{C_B}((V, L(z, \lambda)), (V', L'(z, \lambda)))\) then by definition we have \(L'(z, \lambda)(1 \otimes \varphi(\lambda - \gamma h_1)) = (1 \otimes \varphi(\lambda))L(z, \lambda)\), so that
\[
S_1(z - x, \lambda - \gamma h_2)L'(z, \lambda)e^{-\gamma D_1}S_1(z - x - c, \lambda)^{-1}(1 \otimes \varphi(\lambda)) = S_1(z - x, \lambda - \gamma h_2)L'(z, \lambda)(1 \otimes \varphi(\lambda - \gamma h_1))e^{-\gamma D_1}S_1(z - x - c, \lambda)^{-1} = (1 \otimes \varphi(\lambda))S_1(z - x, \lambda - \gamma h_2)L'(z, \lambda)e^{-\gamma D_1}S_1(z - x - c, \lambda)^{-1}
\]
since \(\varphi(\lambda)\) is of weight zero. Thus \(\mathcal{F}_x^c(\varphi(\lambda))\) is an intertwiner in the category \(\mathcal{D}_B\). \(\Box\)

We can also think of the family of functors \(\mathcal{F}_x^c\) as a single functor \(\mathcal{F}^c : C_B^F \rightarrow D_B^B\).

Remark. We can think of the difference-twist and the relations in Lemma 2 as a dynamical analogue of the notion of equivalence of R-matrices due to Drinfeld and Belavin—see [2].

3. The image of the trivial representation and the functor \(\mathcal{H}_x^c : C_B \rightarrow D_B\). Applying the functor \(\mathcal{F}_x^c\) to the trivial representation \((C, \text{Id}) \in \text{Ob}(C_B^F)\) yields
\[
\mathcal{F}_x^c((C, \text{Id})) = (C, S(z - x, \lambda)e^{-\gamma D_1}S(z - x - c, \lambda)^{-1}).
\]
We will denote this object by \(I_x^c\). For instance, when \(n = 2\), we obtain a representation of the Belavin quantum elliptic algebra as difference operators acting on the space of periodic meromorphic functions in one variable \(\lambda\), i.e. given by an \(L\)-operator
\[
L(z) = \begin{pmatrix}
a(z) & b(z) \\
c(z) & d(z)
\end{pmatrix}
\]
where \(a(z), b(z), c(z), d(z)\) are operators of the form \(f(z)T_\gamma + g(z)\) where \(T_\gamma\) is the shift by \(-\gamma\).

Such representations of \(B\) by difference operators already appeared in the work of Krichever, Zabrodin ([11]) (for \(n = 2\)) and Hasegawa ([8],[9]) (for the general case), where they were also derived by some Vertex-IRF correspondence.

Definition. Let \(c \in \mathbb{C}\) and let \(\mathcal{H}_x^c : C_B \rightarrow D_B\) be the functor defined by the assignment \(V \rightarrow I_x^c \otimes V\) and which is trivial at the level of morphisms. The family of functors \(\mathcal{H}_x^c\) gives rise to a functor \(\mathcal{H}^c : C_B^F \rightarrow D_B^B\).

4. Full faithfulness of the functor \(\mathcal{H}_x^c : C_B \rightarrow D_B\). In this section, we prove the following result

**Proposition 2.** Let \(V, V' \in \text{Ob}(C_B)\). Then for all but finitely many values of \(x \mod \mathbb{Z} + \mathbb{Z} \tau\), the map
\[
\mathcal{H}_x^c : \text{Hom}_{C_B}(V, V') \cong \text{Hom}_{D_B}(\mathcal{H}_x^c(V), \mathcal{H}_x^c(V'))
\]
is an isomorphism.

Proof. Since \(\text{Hom}_{C_B}(V, V') \cong \text{Hom}_{C_B}(C, V' \otimes V^*)\), \(\text{Hom}_{D_B}(I_x^c \otimes V, I_x^c \otimes V') \cong \text{Hom}_{D_B}(I_x^c, I_x^c \otimes V' \otimes V^*)\), it is enough to show that the map \(\mathcal{H}_x^c : \text{Hom}_{C_B}(C, W) \rightarrow \text{Hom}_{D_B}(I_x^c, I_x^c \otimes W)\) is an isomorphism for all \(W \in \text{Ob}(C_B)\). Since \(\mathcal{H}_x^c\) is trivial at the level of morphisms, this map is injective. Now let \(W \in \text{Ob}(C_B)\) and let \(\varphi(\lambda) \in \text{Hom}_B(I_x^c, I_x^c \otimes W)\), that is, \(\varphi(\lambda)\) is a \((n\omega_i)\)-periodic meromorphic function \(h^* \rightarrow W\) satisfying the equation
\[
\varphi(\lambda)S_1(z - x, \lambda)e^{-\gamma D_1}S_1(z - x - c, \lambda)^{-1} = S_1(z - x, \lambda)e^{-\gamma D_1}S_1(z - x - c, \lambda)^{-1}L_{12}(z)\varphi(\lambda)
\]
where \( L(z) \) is the L-operator of \( W \). This is equivalent to

\[
L_{12}(z)\varphi_2(\lambda) = S_1(z - x - c, \lambda)\varphi_2(\lambda + \gamma h_1)S_1(z - x - c, \lambda)^{-1}
\]

(12)

Now \( L(z) \) is an elliptic function (of periods \( n \) and \( nr \)) so it is either constant or it has a pole. Restricting \( W \) to the subrepresentation \( \text{Span}(\varphi(A), A \in f) \), we see that the latter case is impossible for generic \( x \) as the RHS of (12) has a pole at \( z = x + c \) only; hence \( L(z) \) is constant. Furthermore, from (12) we see that the matrix

\[
M(\lambda) = S_1(z - x - c, \lambda)^{-1}L_{12}S_1(z - x - c, \lambda)
\]

is independent of \( z \). In particular, setting \( z \mapsto z + 1 \) and using the transformation properties (10) of \( S(z, \lambda) \), we obtain \([A_1, L_{12}] = 0\). This implies that \( L = \sum_i E_{ii} \otimes D_i \) for some \( D_i \in \text{End}(W) \).

**Lemma 3.** Let \( U \) be a finite dimensional vector space, let \( T \in \text{End}(\mathbb{C}^n) \otimes \text{End}(U) \) be an invertible solution of the equation

\[
R^{B}_{12}(z)T_{13}T_{23} = T_{23}T_{13}R^{B}_{12}(z)
\]

(13)
such that \( T = \sum_i E_{ii} \otimes D_i \) for some \( D_i \in \text{End}(U) \). Then \([D_i, D_j] = 0 \) for all \( i, j \) and there exists \( X \in \text{End}(U) \) such that \( X^n = 1 \) and \( D_{i+1} = XD_i \) for all \( i = 1, \ldots, n \).

**Proof.** Let us write \( R^B(z) = \sum_{p,q,r,s} R_{p,q,r,s}(z)E_{pq} \otimes E_{rs} \). Then equation (13) is equivalent to \( R_{p,q,r,s}(z)D_p \times D_q = R_{p,q,r,s}(z)D_sD_r \) for all \( p, q, r, s \). But it follows from the general formula for \( R^B(z) \) that \( R_{p,q,r,s}(z) \neq 0 \) if and only if \( p + q = r + s \) (mod \( n \)). Thus we have \([D_i, D_j] = 0 \) for all \( i, j \) and \( X := D_1D_{i+1}^{-1} \) is independent of \( i \), and satisfies \( X^n = 1 \). \( \square \)

By the above lemma, there exists \( X \in \text{End}(W) \) such that \( X^n = 1 \) and \( D_{i+1} = XD_i \). Suppose that \( X \neq 1 \) and choose \( e \in W \) such that \( X(e) = \xi^k e \) with \( \xi^k \neq 1 \). Now we apply the transformation \( z \mapsto z + \tau \) to the matrix \( M(\lambda) \). Noting that, by (11), \( S(z - x - c + \tau, \lambda) = e^{-i\tau z/2 - 2\pi i(z-x-c)/n}BS(z - x - c, \lambda)F(\lambda) \) where \( F(\lambda) = \text{diag}(e^{-2i\pi \lambda}, \ldots, e^{-2i\pi \lambda}) \), we obtain the equality

\[
F(\lambda)^{-1}S_1(z - x - c, \lambda)^{-1}B_1^{-1}L_{12}B_1S_1(z - x - c, \lambda)F(\lambda)
= S_1(z - x - c, \lambda)^{-1}L_{12}S_1(z - x - c, \lambda).
\]

Applying this to the vector \( e \) yields \( \text{Ad}F(\lambda)(M(\lambda))(e) = \xi^{-k}M(\lambda)(e) \). This is possible for all \( \lambda \) only if \( k \equiv 0 \) (mod \( n \)). Hence \( X = 1 \) and (12) reduces to the equation \( D\varphi_2(\lambda) = \varphi_2(\lambda + \gamma h_1) \). In particular \( \varphi(\lambda) = \gamma(\omega_1 - \omega_j) \)-periodic. But by our assumption, \( \varphi(\lambda) \) is \((n\omega_1)\)-periodic and \( \gamma \) is real and irrational. Therefore \( \varphi(\lambda) \) is constant and it is a morphism in the category \( \mathcal{C}_B \). \( \square \)

**Corollary 1.** The functor \( \mathcal{H}^c : \mathcal{C}_B^c \rightarrow \mathcal{D}_B^c \) is fully faithful.

**Remark.** Equation (12) shows that \( \text{Hom}_{\mathcal{D}_B^c}(I_x^c, I_x^c \otimes V) = \text{Hom}_{\mathcal{D}_B^c}(V^*, I_{x+c}) \). Thus the above proposition states that for any finite-dimensional representation \( V \in \mathcal{O}b(\mathcal{C}_F) \) and for all but finitely many \( x \) mod \( Z + \tau Z \), we have \( \text{Hom}_{\mathcal{D}_B^c}(V^*, I_x^c) = \text{Hom}_{\mathcal{C}_B^c}(V^*, \mathbb{C}) \), where the isomorphism is induced by the embedding \( \mathbb{C} \subset I_x^c \) (constant functions). However, for finitely many values of \( x \) mod \( Z + \tau Z \), this may not be true: see [11] and [9] where some finite-dimensional subrepresentations of \( I_x^c \) are considered.
5. Full faithfulness of the functor $F^x_z : C_F \to D_B$. In this section, we prove the following result:

**Proposition 3.** The functor $F^x_z : C_F \to D_B$ is fully faithful.

**Proof.** We have to show that for any two objects $V, V'$ in $C_F$ there is an isomorphism $F^x_z : \text{Hom}_{C_F}(V, V') \to \text{Hom}_{D_B}(F^x_z(V), F^x_z(V'))$. Since $F^x_z$ is trivial at the level of morphisms, this map is injective. Now let $V, W \in \text{Ob}(C_F)$ and let $\varphi(\lambda) \in \text{Hom}_{D_B}(F^x_z(V), F^x_z(W))$. By definition, $\varphi(\lambda) : V \to W$ satisfies the relation

$$
\varphi(\lambda)S_1(z-x, \lambda-\gamma h_2)L^V(z, \lambda)e^{-\gamma D_1}S_1(z-x-c, \lambda)^{-1} = S_1(z-x, \lambda-\gamma h_2)L^W(z, \lambda)e^{-\gamma D_1}S_1(z-x-c, \lambda)^{-1}\varphi(\lambda)
$$

where $L^V(z, \lambda)$ (resp. $L^W(z, \lambda)$) is the $L$-operator of $V$ (resp. $W$). This is equivalent to

$$
\varphi(\lambda)S_1(z-x, \lambda-\gamma h_2)L^V(z, \lambda) = S_1(z-x, \lambda-\gamma h_2)L^W(z, \lambda)\varphi(\lambda-\gamma h_1)
$$

(14)

Introduce the following notations: write $W = \bigoplus_\xi W_\xi$, $V = \bigoplus_\mu V_\mu$, $\varphi(\lambda) = \sum_\nu \varphi_\nu(\lambda)$ for the weight decompositions (so that $\varphi_\nu : \overline{V} \to W_{\xi+\nu}$). Also let $S(z-x, \lambda) = \sum_{i,j} S^{ij}(z-x, \lambda)E_{ij}$, $L^V_i(z, \lambda) = \sum_{i,j} E_{ij} \otimes L^V_i(z, \lambda)$ and use the same notation for $L^W_i(z, \lambda)$. Applying (14) to $\varphi_\nu \otimes \zeta_\mu$ for some $i$ and $\zeta_\mu \in V_\mu$ yields

$$
\sum_{j,k,v} S^{kj}(z-x, \lambda-\gamma(\mu+\omega_i-\omega_j))v_k \otimes \varphi_\nu(\lambda)(L^{ij}_V(z, \lambda)\zeta_\mu)
$$

$$
= \sum_{i,k,\sigma} S^{kl}(z-x, \lambda-\gamma(\mu+\omega_i-\omega_l+\sigma))v_k \otimes L^{ij}_W(z, \lambda)\varphi_\sigma(\lambda-\gamma\omega_i)\zeta_\mu
$$

(15)

where we used the weight-zero property of $L^V(z, \lambda)$ and $L^W(z, \lambda)$. Applying $v_k^*$ to (15) and projecting on the weight space $W_{\mu+\omega_i+\xi}$ gives the relation

$$
\sum_{\nu,\omega_j=\xi} S^{kj}(z-x, \lambda-\gamma(\mu+\omega_i-\omega_j))\varphi_\nu(\lambda)(L^{ij}_V(z, \lambda)\zeta_\mu)
$$

$$
= \sum_{\sigma,\omega_j=\xi} S^{kl}(z-x, \lambda-\gamma(\mu+\omega_i-\omega_j+\sigma))L^{ij}_W(z, \lambda)(\varphi_\sigma(\lambda-\gamma\omega_i)\zeta_\mu)
$$

(16)

for any $i, k, \xi$ and $\zeta_\mu \in V_\mu$. Now let $A = \{\chi \mid \varphi(\chi) \neq 0\}$. Fix some $j$ and let $\beta \in A$ be an extremal weight in the direction $-\omega_j$ (i.e $\beta - \omega_j + \omega_k \notin A$ for $k \neq j$). Then (16) for $\xi = \beta - \omega_j$ reduces to

$$
S^{kj}(z-x, \lambda-\gamma(\mu+\omega_i-\omega_j))\varphi_\beta(\lambda)(L^{ij}_V(z, \lambda)\zeta_\mu)
$$

$$
= S^{kj}(z-x, \lambda-\gamma(\mu+\omega_i-\omega_j+\beta))L^{ij}_W(z, \lambda)\varphi_\beta(\lambda-\gamma\omega_i)\zeta_\mu
$$

(17)

**Claim.** There exists $i \in \{1, \ldots, n\}$, $\mu$ and $\zeta_\mu \in V_\mu$ such that $\varphi(\lambda)(L^{ij}_V(z, \lambda)\zeta_\mu) \neq 0$ for generic $z$ and $\lambda$.

**Proof.** Recall the central element $Q\det(z, \lambda) \in E_{\frac{1}{2}}(\mathfrak{gl}_n)$. By definition, its action on $V$ is invertible. Expanding $Q\det(z, \lambda)$ along the $j^{th}$-line, we have $Q\det(z, \lambda) = \sum_i L^{ij}_V(z, \lambda)P_i(z, \lambda)$ for some operators $P_i(z, \lambda) \in \text{End}(V)$. In particular,

$$
\sum_i \text{Im } L^{ij}_V(z, \lambda) = V,
$$
and the claim follows.

Thus, the ratio $S^{kj}(z-x, \lambda - \gamma(\mu + \omega_i - \omega_j + \beta))/S^{kj}(z-x, \lambda - \gamma(\mu + \omega_i - \omega_j))$ is independent of $k$. This is possible only if $\beta \in \sum_{r \neq j} \mathbb{C} \mathbb{E}^{x}_{r}$. Applying this to $j = 1, \ldots n$, we see that $A = \{0\}$. Hence $\varphi(\lambda)$ is an $h$-module map. But then relation (14) reduces to $\varphi_{2}(\lambda) L_{12}^{V}(z, \lambda) = L_{12}^{F}(z, \lambda)\varphi_{2}(\lambda - \gamma h_{1})$, and $\varphi(\lambda)$ is an intertwiner in the category $C_{F}$. □

**Corollary 2.** The functor $\mathcal{F}^{c} : C_{F}^{x} \rightarrow D_{B}^{x}$ is fully faithful.

**6. The image of the vector representation.** Let us denote

$$\tilde{V}_{F}(w) = (\mathbb{C}^{n}, \chi(w)R_{F}(w, \lambda)).$$

It is an object of $C_{F}$ which equals the tensor product of the vector representation $V_{F}(w)$ by the one-dimensional representation $(\mathbb{C}, \chi(z))$.

**Proposition 4.** For any $x, w$, $x + c \not\equiv w$ (mod $\mathbb{Z} + \tau\mathbb{Z}$), we have $\mathcal{F}^{c}_{x}(\tilde{V}_{F}(w)) \simeq \mathcal{H}^{x}_{x}(V_{B}(w))$.

**Proof.** By definition, we have

$$\mathcal{F}^{c}_{x}(\tilde{V}_{F}(w)) = (\mathbb{C}^{n}, \chi(z)S_{1}(z-x, \lambda - \gamma h_{2})R_{F}(z-w, \lambda)e^{-\gamma D_{1}} \times S_{1}(z-x-c, \lambda)^{-1}),$$

$$I_{x} \otimes V_{B}(w) = (\mathbb{C}^{n}, \chi(z)S_{1}(z-x, \lambda)e^{-\gamma D_{1}}S_{1}(z-x-c, \lambda)R_{B}(z-w))$$

We claim that the map $\varphi(\lambda) = e^{-\gamma D_{1}(S(w-x-c, \lambda)^{-1})}e^{\gamma D_{1}} \in \text{End}(\mathbb{C}^{n})$ is an intertwiner $\mathcal{H}^{x}_{x}(V_{B}(w)) \simeq I_{x} \otimes V_{B}(w) \simeq \mathcal{F}^{c}_{x}(\tilde{V}_{F}(w))$. Indeed, we have

$$S_{1}(z-x, \lambda - \gamma h_{2})R_{F}(z-w, \lambda)e^{-\gamma D_{1}}S_{1}(z-x-c, \lambda)^{-1}(1 \otimes \varphi(\lambda))$$

$$= e^{-\gamma D_{2}}S_{1}(z-x, \lambda)e^{\gamma D_{2}}R_{F}(z-w, \lambda)e^{-\gamma(D_{1}+D_{2})}$$

$$= e^{-\gamma D_{2}}S_{1}(z-x, \lambda)e^{-\gamma D_{1}}R_{F}(z-w, \lambda)$$

$$= e^{-\gamma D_{2}}S_{1}(z-x-c, \lambda + \gamma h_{2})^{-1}S_{2}(w-x-c, \lambda)^{-1}e^{\gamma D_{2}}$$

$$= e^{-\gamma D_{2}}S_{1}(z-x, \lambda)e^{-\gamma D_{1}}S_{2}(w-x-c, \lambda + \gamma h_{1})^{-1}$$

$$= e^{-\gamma D_{2}}S_{1}(z-x-c, \lambda)R_{B}(z-w)e^{\gamma D_{2}}$$

where we used Lemma 2 and the zero-weight property of $R_{F}(u, \lambda)$. □

**Lemma 4.** Let $V, V' \in \text{Ob}(C_{F})$, $W, W' \in \text{Ob}(C_{B})$ and suppose that $\mathcal{F}^{c}_{x}(V) \simeq \mathcal{H}^{x}_{x}(W)$ and $\mathcal{F}^{c}_{x}(V') \simeq \mathcal{H}^{x}_{x}(W')$. Then $\mathcal{F}^{c}_{x}(V \otimes V') \simeq \mathcal{H}^{x}_{x}(W \otimes W')$.

**Proof.** If $\varphi(\lambda) : V \rightarrow W$ and $\varphi'(\lambda) : V' \rightarrow W'$ are intertwiners then it is easy to check using the methods above that $\varphi_{2}(\lambda - \gamma h_{1}) \varphi_{1}(\lambda) : V \otimes V' \rightarrow W \otimes W'$ is an intertwiner. □

Applying this to tensor products of the vector representations, we obtain

**Corollary 3.** For any $x \in \mathbb{C}$ and $w_{1}, \ldots, w_{r} \in \mathbb{C}\setminus\{x + c + \mathbb{Z} + \tau\mathbb{Z}\}$, we have

$$\mathcal{F}^{c}_{x}(\tilde{V}_{F}(w_{1}) \otimes \ldots \tilde{V}_{F}(w_{r})) \simeq \mathcal{H}^{x}_{x}(V_{B}(w_{1}) \otimes \ldots V_{B}(w_{r})).$$
**Corollary 4.** For any \( w_1, \ldots, w_r \in \mathbb{C} \), we have
\[
\mathcal{F}^c(\tilde{V}_F(w_1) \otimes \cdots \otimes \tilde{V}_F(w_r)) \simeq \mathcal{H}^c(V_B(w_1) \otimes \cdots \otimes V_B(w_r)).
\]

Notice that in this case, we have a canonical intertwiner, given by the formula
\[
\varphi_{1 \ldots r}(\lambda, w_1, \ldots, w_r) = S_r^{-1}(w_r - x - c, \lambda - \gamma \sum_{i=1}^{r-1} h_i) \cdots S_1^{-1}(w_1 - x - c, \lambda),
\]
where we set \( S(z, \lambda) = e^{-\gamma D} S(z, \lambda) e^{\gamma D} \).

**7. Equivalence of subcategories.** Let us summarize the results of sections 4-8. By proposition 2, we can identify \( C^F_B \) with a full subcategory \( \mathcal{D}^F_1 \) of \( \mathcal{D}^F_B \). By proposition 3, we can identify \( C^F_B \) with a full subcategory \( \mathcal{D}^F_2 \) of \( \mathcal{D}^F_B \). Moreover, \( \mathcal{D}^F_1 \) and \( \mathcal{D}^F_2 \) intersect (at least if we replace \( V_B \) by the equivalent category \( \mathcal{D}^F_B \) whose objects are isomorphism classes of objects of \( \mathcal{D}^F_B \), and the intersection contains objects of the form \( \mathcal{F}^c(\otimes_i \tilde{V}_F(w_i)) \simeq \mathcal{H}^c(\otimes_i V_B(w_i)) \), where \( i = 1, \ldots, r \) and \( w_i \in \mathbb{C} \). Hence,

**Theorem** The abelian subcategory \( \mathcal{V}_B^F \) of \( C^F_B \) generated by objects \( \otimes_i V_B(w_i) \) for \( i = 1, \ldots, r \), \( r \in \mathbb{N} \) and \( w_i \in \mathbb{C} \) and the abelian subcategory \( \mathcal{V}_F^F \) of \( C^F_F \) generated by objects \( \otimes_i \tilde{V}_F(w_j) \) for \( j = 1, \ldots, s \), \( s \in \mathbb{N} \) and \( w_j \in \mathbb{C} \) are equivalent.

Note that for numerical values of \( x \), \( \mathcal{F}_x^c : C_F \to \mathcal{D}_B \) is always fully faithful, and \( \mathcal{F}_x^c(C_F) \) a full subcategory of \( \mathcal{D}_B \), but this is not true of \( \mathcal{H}_x^c \), because of the existence of nontrivial finite-dimensional subrepresentations of \( I_x^c \).

**References**


