SOME NEW OBSERVATION ON INVARIANT THEORY OF PLANE QUARTICS

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1. Introduction. Let $S(n, m)$ denote the graded ring of projective invariants of an $n$-ary form (a homogeneous polynomial in $n$ variables) of degree $m$. We are interested in the case $n = 3$ and $m = 4$. A ternary quartic form $F(x_0, x_1, x_2)$ defines a plane curve of genus 3 if it is nonsingular, and conversely any non-hyperelliptic curve of genus 3 can be realized as such a plane quartic via the canonical embedding, which is unique up to projective transformations. Thus the structure of the ring $S(3, 4)$ is closely related to the moduli of genus 3 curves. (For general background of Invariant Theory, see e.g. [4], [13].)

More than thirty years ago ([5, Appendix]), we calculated the generating function (Poincaré series) of $S(3, 4)$ and made a few guess (or conjecture?) about the structure of the graded ring $S(3, 4)$. More recently, Dixmier [2] has proved the existence of a system of parameters for this ring (suggested in [5]) by exhibiting a system of seven explicit projective invariants.

In this paper, we study some close relationship of the ring $S(3, 4)$ of projective invariants to another invariant theory, i.e. to the invariant theory for the Weyl groups $W(E_7)$ and $W(E_8)$ (cf. [1]). We are led to such a connection from the viewpoint of Mordell-Weil lattices ([8], [9]).

2. Formulation of main results. We consider the case of characteristic zero. Taking
\[ F(x_0, x_1, x_2) = \sum_{i_0 + i_1 + i_2 = 4} a_{i_0, i_1, i_2} x_0^{i_0} x_1^{i_1} x_2^{i_2} \]
with variable coefficients $\{a_{i_0, i_1, i_2}\}$, we may regard $S(3, 4)$ as a graded subring of the polynomial ring $\mathbb{C}[a_{i_0, i_1, i_2}]$ graded by the total degree, consisting of those $I = I(F) \in \mathbb{C}[a_{i_0, i_1, i_2}]$ which are invariant under $SL(3)$. Namely, for any $g \in SL(3)$, let $(x'_0, x'_1, x'_2) = (x_0, x_1, x_2)g$ and rewrite $F(x'_0, x'_1, x'_2)$ as a polynomial $F'(x_0, x_1, x_2)$ in $x_0, x_1, x_2$:
\[ F'(x_0, x_1, x_2) = \sum_{i_0 + i_1 + i_2 = 4} a'_{i_0, i_1, i_2} x_0^{i_0} x_1^{i_1} x_2^{i_2}. \]

We set $F^g = F'$. Then, by definition, we have
\[ I \in S(3, 4) \iff I(F^g) = I(F) \quad (\forall g \in SL(3)). \]

For any ternary quartic form $F_0$, we call the map $I \to I(F_0)$ the evaluation map of $S(3, 4)$ at $F_0$. 

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Actually the C-algebra $S(3,4)$ is obtained from the Q-algebra $S(3,4) \cap \mathbb{Q}[a_{i_0, i_1, i_2}]$ by the scalar extension of Q to C. So, in the following, we change the notation so that $S(3,4)$ will denote this Q-subalgebra of $\mathbb{Q}[a_{i_0, i_1, i_2}]$.

Now we recall the following fact on the normal form of a plane quartic with a given flex (cf. [8, §1]). Take the inhomogeneous coordinates $x, t$ such that $(x_0 : x_1 : x_2) = (1 : x : t)$. The normal form of type $E_7$ is

$$f_\lambda = x^3 + x(p_0 + p_1 t + t^3) + q_0 + q_1 t + q_2 t^2 + q_3 t^3 + q_4 t^4$$

with $\lambda = (p_0, p_1, q_0, \ldots, q_4) \in \mathbb{A}^7$, and the normal form of type $E_6$ is

$$f_\lambda = x^3 + x(p_0 + p_1 t + p_2 t^2) + q_0 + q_1 t + q_2 t^2 + t^4$$

with $\lambda = (p_0, p_1, p_2, q_0, q_1, q_2) \in \mathbb{A}^6$. In either case, let $\Gamma_\lambda$ be the plane quartic defined by $f_\lambda = 0$; the flex is given by the point $(x_0 : x_1 : x_2) = (0 : 1 : 0)$. The fact is that every plane quartic with a given flex is isomorphic to $\Gamma_\lambda$ for some $\lambda \in \mathbb{A}^7$ or $\mathbb{A}^6$; the distinction depends on whether the given flex is ordinary or special\(^1\) (i.e. whether the tangent line to the curve at the flex intersects the curve with multiplicity 3 or 4).

It is obvious that the evaluation map $I \to I(f_\lambda)$ gives a ring homomorphism

$$\phi : S(3,4) \to Q[\lambda] = Q[p_i, q_j]$$

for either type of $f_\lambda$. Let us call it the evaluation map of type $E_7$ or $E_6$, and denote it by $\phi_7$ or $\phi_6$ when we need to specify the cases.

The main purpose of this paper is to establish less obvious relationship between the invariant theory of a plane quartic and the invariant theory of the Weyl group $W(E_r)$ ($r = 6, 7$). To formulate the results, note first that the ring of invariants of $W(E_r)$, say $R(E_r)$, can be naturally identified with $Q[\lambda]$ given above (see [1], [6], [7]), which is a graded polynomial ring with the weights of $p_i$ or $q_j$ assigned as follows:

for $E_7$ case: $wt(p_i) = 12 - 4i$, $wt(q_j) = 18 - 4j$.

for $E_6$ case: $wt(p_i) = 8 - 3i$, $wt(q_j) = 12 - 3j$.

On the other hand, let

$$S = S(3,4) = \oplus_m S_m$$

where $S_m$ is the homogeneous part of degree $m$ of $S$. It is known that $S_m \neq 0$ only if $m$ is a multiple of 3 (cf. §3).

**Theorem 1.** (i) The evaluation map of type $E_7$

$$\phi_7 : S(3,4) \to R(E_7) = Q[p_0, p_1, q_0, q_1, q_2, q_3, q_4]$$

is a graded homomorphism from $S(3,4)$ to $R(E_7)$ with weight ratio $3 : 14$ in the sense that $\phi$ sends $S_{3d}$ to $R(E_7)_{14d}$ for all $d$.

(ii) The evaluation map of type $E_6$

$$\phi_6 : S(3,4) \to R(E_6) = Q[p_0, p_1, p_2, q_0, q_1, q_2]$$

\(^1\)see the comments at the end of the paper.
is a graded homomorphism from $S(3,4)$ to $R(E_8)$ with weight ratio $3 : 8$ in a similar sense.

**Theorem 2.** Let $D \in S(3,4)$ denote the discriminant of a plane quartic: its characteristic property is that $D \in S_{27}$ and $D(f) \neq 0$ if and only if $f = 0$ is smooth. Then the image $\phi(D)$ under the evaluation map $\phi$ of type $E_r$ ($r = 7, 6$) is equal, up to a constant, to the “discriminant” $\delta$ of $R(E_r)$ which is defined as the square of the basic anti-invariant of $W(E_r)$; the weight of $\delta$ is 126 or 72 for $r = 7$ or 6.

**Theorem 3.** (i) For $r = 7$, the evaluation map $\phi_7$ is injective.

(ii) For $r = 6$, the evaluation map $\phi_6$ has a nontrivial kernel which contains a projective invariant $J$ of degree 60.²

For a graded integral domain $R$, $F(R)$ will denote the field of fractions of $R$, and $F(R)(0)$ will denote the subfield of homogeneous fractions (i.e. the fractions $a/b$ with $a, b \in R$ of the same weight).

For $S = S(3,4)$, $F(S)(0)$ can be considered as the function field of the moduli space $M_3$ of curves of genus 3.

**Theorem 4.** Let $P = Q[I_1, \ldots, I_6, I_9]$ be the polynomial subring of $S = S(3,4)$ generated by the Dixmier’s system $\{I_d \mid d = 1, \ldots, 6, 9\}$, $I_d$ being a suitable projective invariant of degree $3d$. Then we have the algebraic extensions

$$F(P)(0) \subset F(S)(0) \subset F(Q[\lambda])(0)$$

with the extension degree

$$[F(S)(0) : F(P)(0)] = 50, \quad [F(Q[\lambda])(0) : F(S)(0)] = 24.$$

**Remark.** (1) Note that both

$$F(P)(0) = Q(I_d/I_1^d \mid d = 1, \ldots, 6, 9)$$

and

$$F(Q[\lambda])(0) = Q(p_0/q_4^6, p_1/q_4^4, q_0/q_4^9, q_1/q_4^7, q_2/q_4^5, q_3/q_4^3)$$

are rational fields (i.e. purely transcendental extensions) over $Q$. The famous rationality question of the moduli space $M_3$ of curves of genus 3 is equivalent to asking whether $F(S)(0)$ is a rational field or not. This was answered by Katsylo [3] by a representation-theoretic method. Our approach might be of some use to this question, from a more geometric point of view.

(2) The explicit form of the invariants $I_d$ in the Dixmier’s system is not necessary to prove Theorem 4, but we shall give it in [11] for a possible use in future.

**3. Proof of Theorems.** We keep the notation introduced in the above.

First recall that, for any homogeneous invariant $I \in S = S(3,4)$ of degree $m$ ($I \in S_m$), we have

$$I(F^g) = \det(g)^w I(F) \quad (\forall g \in GL(3))$$

²see the comments at the end of the paper
for some integer \( w \), which is determined by \( 4m = 3w \) (by comparing the degree in generic coefficients of \( g \)). Thus, if \( I \neq 0 \), \( m = 3d \) and \( w = 4d \) for some integer \( d \).

**Proof** of Theorem 1. The key point is the weighted homogeneity of \( f_\lambda \). For the normal form of type \( E_7 \), \( f_\lambda \) is a weighted homogeneous polynomial of total weight 18, if we fix \( wt(x) = 6 \) and \( wt(t) = 4 \). Namely we have

\[
f_\lambda(u^6x, u^4t) = u^{18}f_\lambda(x, t) \quad (\forall u \in G_m)
\]

with \( \lambda' = (u^{12}p_0, u^5p_1, \ldots, u^6q_3, u^2q_4) \).

Let \( g \) be the diagonal matrix \( g = [1, u^6, u^4] \in GL(3) \); note \( \det(g) = u^{10} \). Then we have from the above

\[
(f_\lambda)^g(x, t) = u^{18}f_\lambda(x, t).
\]

For any \( I \in S_{3d} \), we have then

\[
(u^{10})^{4d}I(f_\lambda') = (u^{18})^{3d}I(f_\lambda)
\]

which implies

\[
I(f_\lambda') = u^{14d}I(f_\lambda) \quad (\forall u \in G_m).
\]

This proves that \( \phi_r(I) = I(f_\lambda) \) has weight \( 14d \) for any \( I \in S_{3d} \). Thus part (i) of Theorem 1 is shown.

For the normal form of type \( E_6 \), \( f_\lambda \) is a weighted homogeneous polynomial of total weight 12 by taking \( wt(x) = 4 \) and \( wt(t) = 3 \). The same argument as above shows part (ii) of Theorem 1.

**Proof** of Theorem 2. Since the discriminant \( D \) of a plane quartic has degree 27 (\( D \in S_{27} \)), \( \phi_r(D) \) has weight \( 9 \cdot 14 = 126 \) and \( \phi_6(D) \) has weight \( 9 \cdot 8 = 72 \) by Theorem 1. Hence \( \phi(D) \in \mathbb{Q}[\lambda] \) has the same weight as the discriminant \( \delta \) of \( R(E_r) \) (= the number of the roots in \( E_r \)) for \( r = 7, 6 \).

To prove \( \phi(D) = \delta \) (up to a constant), the simplest would be to assume the knowledge of singularity theory. From this standpoint, note first that the plane quartic \( \Gamma_\lambda \) is smooth at the points at infinity (i.e. on \( z_0 = 0 \)). Thus it will be smooth if and only if the affine curve \( f_\lambda = 0 \) is smooth. By Jacobian criterion, the latter condition is equivalent to the smoothness of the affine surface \( S'_{\lambda} : y^2 = f_\lambda \) (since \( \text{char } \neq 2 \)).

Now the singularity theory tells us that the family \( y^2 = f_\lambda \) parametrized by \( \lambda \in \mathbb{A}^7 \) is a so-called semi-universal deformation of the \( E_r \)-singularity \( y^2 = x^3 + xt^3 \) (\( r = 7 \)) or \( y^2 = x^3 + t^4 \) (\( r = 6 \)) and that \( S'_{\lambda} \) is smooth if and only if \( \delta(\lambda) \neq 0 \).

Therefore we have \( \phi(D) \neq 0 \Leftrightarrow \delta(\lambda) \neq 0 \), proving the assertion.

We give here an alternative proof based on the theory of Mordell-Weil lattices (MWL) (cf. [6], [7], esp. [8, Th.5]). We consider the elliptic curve

\[
E = E_\lambda : y^2 = f_\lambda = x^3 + \cdots
\]

defined over \( K = k(t) \), \( k \) being the algebraic closure of \( \mathbb{Q}(p_i, q_j) \). To fix the idea, suppose \( f_\lambda \) is of type \( E_7 \) and \( \lambda \) is generic over \( \mathbb{Q} \) (i.e. \( p_i, q_j \) are algebraically independent over \( \mathbb{Q} \)). Then the structure of the Mordell-Weil lattice \( E(K) \) is isomorphic to \( E^*_1 \),
the dual lattice of the root lattice \( E_7 \), with the narrow Mordell-Weil lattice \( E(K)^0 \) being isomorphic to \( E_7 \). Corresponding to the 56 minimal vectors of norm \( 3/2 \) in \( E_7^* \), there are 56 \( k(t) \)-rational points \( P = (x, y) \) of the form:

\[
x = at + b, \quad y = ct^2 + dt + e
\]

([6], Lemma 9.1). A nice fact is that the map \( P \mapsto c \) extends to a group homomorphism \( \text{sp} : E(K) \to k \) (the specialization map at \( t = \infty \), up to a constant), which is injective for \( \lambda \) generic.

We can choose \( \{P_1, \ldots, P_7\} \subset E(K) \) such that \( \langle P_i, P_j \rangle = \delta_{ij} + 1/2 \) (see [8], [10]); they generate a subgroup of index 3 in \( E(K) \). Then \( c_i = \text{sp}(P_i) \in k \) \((i = 1, \ldots, 7)\) are algebraically independent over \( Q \), and the Weyl group \( W(E_7) \) acts on the polynomial ring \( Q[c_1, \ldots, c_7] \) in such a way that the ring of invariants is equal to \( Q[p_0, p_1, q_0, \ldots, q_4] \). Moreover the coefficients \( a, b, \ldots, e \) defining \( P = (x, y) \) belong to \( Q[c_1, \ldots, c_7] \) for all \( P \).

The basic anti-invariant in \( Q[c_1] \) is the product of 63 linear forms:

\[
c_i - c_j (i < j), c_i - v, v - c_i - c_j - c_k (i < j < k)
\]

where \( v = (\sum_i c_i)/3 \), which are the image of half of the 126 roots in \( E(K)^0 \simeq E_7 \). The discriminant \( \delta(\lambda) \) is the square of this anti-invariant up to a constant.

Now we consider specializing the generic parameter \( \lambda \) to any \( \lambda' \in A^7 \). If the MWL does not degenerate under this specialization, we have the 126 roots in \( E_{\lambda'}(K)^0 \simeq E_7 \). Recall that a root in \( E_7 \) corresponds to a rational point \( Q = (x, y) \) of the form

\[
x = t^2/u^2 + at + b, \quad y = t^3/u^3 + ct^2 + dt + e
\]

with \( u = \text{sp}(Q) \neq 0 \). Therefore none of the 63 linear forms above corresponding to the roots vanish under the specialization, and we have \( \delta(\lambda') \neq 0 \). In other words, \( \delta(\lambda') = 0 \) implies the degeneration of MWL (this is the MWL-analogue of "vanishing cycles" in the singularity theory).

Further note that the degeneration of MWL occurs if and only if the affine surface \( S_{\lambda'} \) acquires singularities, since both conditions are equivalent to the existence of a reducible fibre in the associated elliptic fibration at \( t \neq \infty \).

Thus we have the implication \( \delta(\lambda') = 0 \Rightarrow D(\lambda') = 0 \). Comparing the degree, we conclude that \( \delta = \phi(D) \) up to a constant.

The case of \( E_6 \) can be treated in a similar way.

Remark. It is also possible to directly verify \( \phi(D) = \delta \) (up to a constant) by means of computer algebra (cf. [11]).

Proof of Theorem 3. The injectivity of the homomorphism \( \phi_7 \) is clear, because a generic plane quartic can be put in the normal form \( \Gamma_\lambda : f_\lambda = 0 \) (over a field of rationality of the curve and a flex) ([8, §1]).

To prove the second part, we use the notation in the above proof of Theorem 2. For each of the 56 \( k(t) \)-rational points \( P = (x, y) \in E_\lambda \), we have the identity in \( t \):

\[
(ct^2 + dt + e)^2 = f_\lambda(at + b, t).
\]
This means that the line \( L : x = at + b \) in \( \mathbb{P}^2 \) is a bitangent to the plane quartic \( \Gamma_\lambda : f_\lambda = 0 \), i.e., we have \( L \cdot \Gamma_\lambda = 2A + 2B \) for the two points \( A, B \in \Gamma_\lambda \), which are determined by the equation \( ct^2 + dt + e = 0 \). In this way, we get all the 28 bitangents to \( \Gamma_\lambda \), since \( \pm P = (x, \pm y) \) give the same bitangent.

Consider the product

\[
J = \prod_{\nu=1}^{28} (d^2_{\nu} - 4c_{\nu}e_{\nu})
\]

which is an element of \( \mathbb{Q}[c_1, \ldots, c_7] \) of weight \( 28 \cdot 10 = 280 \). Since the Weyl group \( W(E_7) \) acts (transitively) on the 56 minimal vectors, \( J \) is an invariant. Hence

\[
J \in \mathbb{Q}[c_1, \ldots, c_7]^{W(E_7)} = \mathbb{Q}[p_0, p_1, q_0, \ldots, q_4].
\]

**Lemma 5.** For the normal form of type \( E_7 \), the vanishing of the invariant \( J \) is equivalent to the existence of a special flex.

**Proof** Assume \( J(\lambda') = 0 \) for \( \lambda' \in A^7 \). Then some factor in the product must be 0, so the two points of contact of a bitangent coincide. In other words, we have

\[
L \cdot \Gamma_{\lambda'} = 4A
\]

for this bitangent \( L \). Then this point \( A \) is a special flex of \( \Gamma_{\lambda'} \) with flex tangent \( L \). The converse is clear. \( \square \)

By the lemma, the vanishing of \( J \) has an invariant meaning in the sense of projective geometry. Hence \( J \) is a projective invariant, or more precisely, we have \( J = \phi_7(I) \) for a unique projective invariant \( I \in S(3,4) \). In view of Theorem 1, \( I \) has degree 60. Finally \( I \) belongs to the kernel of the map \( \phi_6 \), since the normal form of type \( E_6 \) has a special flex by definition.

This complete the proof of Theorem 3. \( \square \)

Presumably the invariant \( I \) constructed above should be the generator of \( \text{Ker}(\phi_6) \), but it is not yet proven.

**Proof** of Theorem 4. By [2], \( P \) is a polynomial subring of \( S \) such that \( S \) is integral over \( P \). Then by [5, Lemma 1], \( F(S) \) is an algebraic extension of \( F(P) \) of degree \( N(1) \) if \( N(T) \) denotes the numerator of the generating function of \( S \). In our case ([5, Appendix]), the generating function is equal to

\[
\frac{N(T)}{\prod_{d=1}^{6}(1 - T^d) \cdot (1 - T^9)} \quad (T = t^3)
\]

with

\[
N(T) = 1 + T^3 + T^4 + T^5 + 2T^6 + 3T^7 + 2T^8 + 3T^9 + 4T^{10} + 3T^{11} + 4T^{12} + 4T^{13} + 3T^{14} + 4T^{15} + 3T^{16} + 2T^{17} + 3T^{18} + 2T^{19} + T^{20} + T^{21} + T^{22} + T^{25}.
\]

Hence we have \( N(1) = 50 \), which shows \( [F(S) : F(P)] = 50 \). It follows easily that we have \( [F(S)(0) : F(P)(0)] = 50 \).

On the other hand, we view \( F(S) \) as a subfield of \( \mathbb{Q}(p_0, p_1, q_0, \ldots, q_4) \) via the injective map \( \phi_7 \) (Theorem 3). Suppose \( \Gamma \) is a generic plane quartic. Then it has
24 flexes, say $\xi_\nu$, which are all ordinary flexes. For each choice of the flex $\xi_\nu$, $\Gamma$ is isomorphic to $\Gamma_\lambda$ for some $\lambda = (p_i, q_j) \in \Lambda^7$, with $\xi_\nu$ mapped to $(0, 1, 0) \in \Gamma_\lambda$; moreover $\lambda = \lambda^{(\nu)}$ is uniquely determined by the condition $q_4 = 1$ for the given pair $(\Gamma, \xi_\nu)$ (see [8, §1]). Thus the 24 values of $\lambda^{(\nu)}$ corresponding to the 24 flexes are mutually conjugate over $F(S)_{(0)}$. It follows that $[F(Q[\lambda])_{(0)}, F(S)_{(0)}] = 24$. □

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(1) about the terminology. A point of undulation is a more standard word for a special flex used in this paper and [8]. See Salmon’s book [12], no. 50, p. 37 and no. 247, p. 218.

(2) about a characterization of undulation. Theorem 3 (ii) is classically known, and is a special case of the following fact. Salmon describes a projective invariant of degree $6(m - 3)(3m - 2)$ for a plane curve of degree $m$ whose vanishing expresses the condition that the curve has a point of undulation ([12], no. 400, p. 362).

REFERENCES
