A REMARK ON DIVISORS OF CALABI-YAU HYPERSURFACES*
LIH-CHUNG WANG†

Abstract. We prove that a non-singular hypersurface of degree \( \geq n+1 \) in \( \mathbb{P}^n \) for \( n \geq 4 \) does not contain a reduced irreducible divisor which admits a desingularization having nef anticanonical bundle.

1. Introduction. In this paper we shall generalize a theorem [1] of Chang and Ran to the higher dimensional case. They proved that a generic hypersurface of degree \( \geq 5 \) in \( \mathbb{P}^3 \) or \( \mathbb{P}^4 \) does not contain a reduced irreducible divisor which admits a desingularization having numerically effective anticanonical bundle. The \( \mathbb{P}^3 \) case is a conjecture of Harris which is first proven by G. Xu [5] with a different method. The natural generalization of their theorem is the nonexistence of a divisor with numerically effective (nef) anticanonical bundle on a generic hypersurface of degree \( \geq n+1 \) in \( \mathbb{P}^n \) for \( n \geq 5 \) (See Corollary 3.3). However, the interesting case is the case of Calabi-Yau hypersurfaces (degree equal to \( n+1 \)) since G. Xu gave a geometric genus bound for divisors on generic hypersurfaces of general type. In fact, our setup in this paper is a little more general. We prove that a non-singular complete intersection in Grassmannian with a similar degree assumption does not contain a reduced irreducible divisor which admits a desingularization having numerically effective anticanonical bundle.

Let us fix notations in this paper. Thus let \( X \) be a non-singular complete intersection of type \( (m_1, m_2, \ldots, m_k) \) in Grassmann variety \( G(r, n+1) \) such that \( \dim X \geq 3 \) and \( m = m_1 + m_2 + \cdots + m_k \geq n+1 \), and suppose \( D \subset X \) is an irreducible and reduced divisor. Let \( f: D \rightarrow \bar{D} \subset X \) be a desingularization, \( I \) denote the dimension of \( D \) and \( L \) denote \( f^*O_G(1) \). Obviously, \( L \) is nef and big. Let \( K_D \) be the canonical bundle of \( D \). Let \( S \) and \( Q \) be the universal subbundle and universal quotient bundle on \( G(r, n+1) \). \( Q^\vee \) denotes the dual of \( Q \).

The main technical statement we are going to prove is the following.

**Proposition 1.1.** A non-singular complete intersection \( X \) of type \( (m_1, m_2, \ldots, m_k) \) in Grassmann variety \( G(r, n+1) \) such that \( m = m_1 + m_2 + \cdots + m_k \geq n+1 \) does not contain a reduced irreducible divisor which admits a desingularization having \( H^0(K_D \otimes f^*Q^\vee) = 0 \) and \( H^1(K_D - m_iL) = 0 \) for \( i = 1, \ldots, k \).

Here we review the definition and some basic properties of reflexive sheaves (See [3]). Let \( F^{\vee\vee} \) be the double dual of \( F \). A coherent sheaf \( F \) is reflexive if the natural map \( F \rightarrow F^{\vee\vee} \) is an isomorphism. Define the singularity set of \( F \) to be the locus where the \( F \) is not free over the local ring.

It is well-known that the singularity set of a torsion-free sheaf on \( D \) is at least 2-codimensional. Moreover, the singularity set of a reflexive sheaf on \( D \) is at least 3-codimensional. It is also well-known that, in general, any reflexive rank 1 sheaf on an integral and locally factorial scheme is invertible.

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†Department of Applied Mathematics, National Donghwa University Shoufeng, Hualien 974, R.O.C. (lcwang@server.am.ndhu.edu.tw). Paper partially supported by National Science Council (NSC-88-2115-M-259-001).
2. Proof of Proposition 1.1. The proof is by contradiction. Assume such divisors $D$ exist.

First, consider the dual tautological sequence.

\[ 0 \to Q^\vee \to \bigoplus_{(n+1) \text{ copies}} \mathcal{O}_G \to S^\vee \to 0. \]  

We pull back the dual tautological sequence tensoring with $f^*Q$.

\[ 0 \to f^*Q \otimes f^*Q^\vee \to \bigoplus_{(n+1) \text{ copies}} f^*Q \to f^*T_G \to 0. \]

The top cohomology group $h^1(f^*Q) = h^0(K_D \otimes f^*Q^\vee) = 0$ makes

\[ H^1(f^*T_G) = 0. \]

Second, we pull back the defining sequence of normal bundle of $X$.

\[ 0 \to f^*TX \to f^*T_G \to \bigoplus_m m_iL \to 0. \]

Note that we need the smoothness of $X$ to get the above sequence. Then we have $h^{l-1}(m_iL) = h^1(K_D - m_iL) = 0$ which implies

\[ H^l(f^*TX) = 0. \]

Third, consider the defining sequence of normal sheaf $N_f$.

\[ 0 \to T_D \to f^*TX \to N_f \to 0. \]

With the above three sequences, we obtain

\[ H^l(N_f) = 0 \]

and

\[ c_1(N_f) = K_D + (n + 1 - m)L. \]

Let $N_f^{\vee\vee}$ be the double dual of $N_f$. $N_f^{\vee\vee}$ is a reflexive sheaf of rank 1 so it is invertible. The image of $N_f$ in $N_f^{\vee\vee}$ under the canonical map is torsion-free since $N_f^{\vee\vee}$ is torsion-free. The singularity set of a torsion-free sheaf is at least 2-codimensional. Therefore, we have an exact sequence

\[ 0 \to \tau \to N_f \to N_f^{\vee\vee} \to \phi \to 0 \]

with support of $\phi$ at least 2-codimensional. Divide the above sequence into two short exact sequences.

\[ 0 \to \tau \to N_f \to \psi \to 0, \]

\[ 0 \to \psi \to N_f^{\vee\vee} \to \phi \to 0. \]

Then $H^l(N_f) = 0$ implies

\[ H^l(N_f^{\vee\vee}) = 0. \]

On the other hand, we have $c_1(N_f^{\vee\vee}) = K_D + (n + 1 - m)L - c_1(\tau)$. Note that the first chern class of a torsion sheaf is always effective. ([4] V.6.14) Therefore,

\[ h^l(N_f^{\vee\vee}) = h^0(K_D - N_f^{\vee\vee}) = h^0((m - n - 1)L + c_1(\tau)) > 0 \]

gives a contradiction.
3. Main Theorems. For $r = 1$, we identify $G(1, n + 1)$ with $\mathbb{P}^n$.

**Proposition 3.1.** A non-singular complete intersection $X$ of type $(m_1, m_2, \cdots, m_k)$ in $\mathbb{P}^n$ for $n \geq 4$ such that $m = m_1 + m_2 + \cdots + m_k \geq n + 1$ does not contain a reduced irreducible divisor which admits a desingularization having $H^0(K_D - L) = 0$ and $H^1(K_D - m_i L) = 0$ for $i = 1, \cdots, k$.

**Proof.** Replace the dual tautological sequence in the proof of Proposition 1.1 with the Euler sequence.

\[
0 \longrightarrow \mathcal{O}_D \longrightarrow \bigoplus_{(n+1) \text{ copies}} L \longrightarrow f^*\mathbb{T}_{\mathbb{P}^n} \longrightarrow 0.
\]

$h^1(L) = h^0(K_D - L) = 0$ concludes

\[H^1(f^*\mathbb{T}_{\mathbb{P}^n}) = 0.\]

and the remaining proof is the same as the proof in Proposition 1.1. □

Note that we can get the above proposition immediately from Proposition 1.1 if we identify $\mathbb{P}^n$ with $G(n, n + 1)$.

**Theorem 3.1.** A non-singular complete intersection $X$ of type $(m_1, m_2, \cdots, m_k)$ in $\mathbb{P}^n$ such that $\dim X \geq 3$ and $m = m_1 + m_2 + \cdots + m_k \geq n + 1$ does not contain a reduced irreducible divisor which admits a desingularization having nef anticanonical bundle.

**Proof.** If $-K_D$ is nef, $-K_D + L$ and $-K_D + m_i L$ are nef and big. With the Kawamata-Viehweg Vanishing Theorem, we obtain that $H^0(K_D - L) = 0$ and $H^1(K_D - m_i L) = 0$ for $i = 1, \cdots, k$ (Note that $\dim D = \dim X - 1 \geq 2$). Hence the theorem follows. □

**Corollary 3.2.** A non-singular hypersurface of degree $\geq n + 1$ in $\mathbb{P}^n$ for $n \geq 4$ does not contain a reduced irreducible divisor which admits a desingularization having nef anticanonical bundle.

For $n = 3$, a hypersurface $X$ of degree $d = 4$ in $\mathbb{P}^3$ is a K3 surface. The divisor $D$ becomes a curve. Therefore, $h^1(K_D - dL) = h^0(dL)$ is never zero. Hence, our proof doesn't work for this case. By the way, it is well-known that K3 surfaces have rational curves.

Now assume that $r \geq 2$. We can get a similar result.

**Theorem 3.2.** A non-singular complete intersection $X$ of type $(m_1, m_2, \cdots, m_k)$ in Grassmann variety $G(r, n + 1)$ such that $m = m_1 + m_2 + \cdots + m_k \geq n + 1$ and $(k + 1) + (n + 1 - r) \leq \dim G(r, n + 1)$ does not contain a reduced irreducible divisor which admits a desingularization having nef anticanonical bundle.

**Proof.** If $-K_D$ is nef, $-K_D + m_i L$ is nef and big. With the Kawamata-Viehweg Vanishing Theorem, we obtain that $H^1(K_D - m_i L) = 0$ for $i = 1, \cdots, k$. In order to get $H^0(K_D \otimes f^*Q^\vee) = 0$, we need to prove that $H^0(f^*Q^\vee) = 0$.

If $H^0(f^*Q^\vee)$ is non-trivial, from the the pull back of the dual tautological sequence, the non-trivial section of $H^0(f^*Q^\vee)$ gives a linear form $F$ and $D$ is contained in $(F)_0$, the zero locus of $F$. We may identify $(F)_0$ with the Schubert cycle $\sigma_{1,\cdots,1}$. Since $X$ is a complete intersection, we also can identify $X$ with the intersection of Schubert cycles $(\Pi m_i)\sigma_{1,0,\cdots,0}^k$. $\tilde{D}$ is a divisor of $X$ so it is also a complete intersection. Hence we may identify $\tilde{D}$ with a multiple of $\sigma_{1,0,\cdots,0}^{k+1}$. Now consider a Schubert cycle
\(\sigma_{(n+1)-r,0,\ldots,0}\), which does not intersect \(\sigma_1,\ldots,1\). On the other hand, \(\sigma_{(n+1)-r,0,\ldots,0}\) does intersect \(\sigma_{1,0,\ldots,0}\) if \((k+1) + (n+1-r) \leq \dim G(r, n+1)\). We get a contradiction. Therefore \(H^0(f^*Q^\vee) = 0\).

If \(K_D\) is trivial, then we get \(H^0(K_D \otimes f^*Q^\vee) = H^0(f^*Q^\vee) = 0\). If \(K_D\) is not trivial, \(H^0(K_D) = 0\). By the injectivity of

\[
\begin{array}{c}
0 \longrightarrow K_D \otimes Q^\vee \longrightarrow \bigoplus_{(n+1) \text{ copies}} K_D,
\end{array}
\]

we also get \(H^0(K_D \otimes f^*Q^\vee) = 0\). □

REFERENCES