SPIN\(^c\) MANIFOLDS AND RIGIDITY THEOREMS IN K-THEORY*

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Abstract. We extend our family rigidity and vanishing theorems in [LiuMaZ] to the Spin\(^c\) case. In particular, we prove a K-theory version of the main results of [H], [Liu1, Theorem B] for a family of almost complex manifolds.

0. Introduction. Let \(M, B\) be two compact smooth manifolds, and \(\pi : M \to B\) be a smooth fibration with compact fibre \(X\). Let \(TX\) be the relative tangent bundle. Assume that a compact Lie group \(G\) acts fiberwise on \(M\), that is, the action preserves each fiber of \(\pi\). Let \(P\) be a family of \(G\)-equivariant elliptic operators along the fiber \(X\). Then the family index of \(P\), \(\text{Ind}(P)\), is a well-defined element in \(K(B)\) (cf. [AS]) and is a virtual \(G\)-representation (cf. [LiuMa1]). We denote by \((\text{Ind}(P))^G\) the \(G\)-invariant part of \(\text{Ind}(P)\).

A family of elliptic operator \(P\) is said to be rigid on the equivariant Chern character level with respect to this \(G\)-action, if the equivariant Chern character \(\text{ch}_g(\text{Ind}(P)) \in H^*(B)\) is independent of \(g \in G\). If \(\text{ch}_g(\text{Ind}(P))\) is identically zero for any \(g\), then we say \(P\) has vanishing property on the equivariant Chern character level. More generally, we say that \(P\) is rigid on the equivariant K-theory level, if \(\text{Ind}(P) = (\text{Ind}(P))^G\). If this index is identically zero in \(K_G(B)\), then we say that \(P\) has vanishing property on the equivariant K-theory level. To study rigidity and vanishing, we only need to restrict to the case where \(G = S^1\). From now on we assume \(G = S^1\).

As was remarked in [LiuMaZ], the rigidity and vanishing properties on the K-theory level are more subtle than that on the Chern character level. The reason is that the Chern character can kill the torsion elements involved in the index bundle.

In [LiuMaZ], we proved several rigidity and vanishing theorems on the equivariant K-theory level for elliptic genera. In this paper, we apply the method in [LiuMaZ] to prove rigidity and vanishing theorems on the equivariant K-theory level for Spin\(^c\) manifolds, as well as for almost complex manifolds. To prove the main results of this paper, to be stated in Section 2.1, we will introduce some shift operators on certain vector bundles over the fixed point set of the circle action, and compare the index bundles after the shift operation. Then we get a recursive relation of these index bundles which will in turn lead us to the final result (cf. [LiuMaZ]).

Let us state some of our main results in this paper more explicitly. As was remarked in [LiuMaZ], our method is inspired by the ideas of Taubes [T] and Bott-Taubes [BT].

For a complex (resp. real) vector bundle \(E\) over \(M\), let

\[
\text{Sym}_t(E) = 1 + tE + t^2\text{Sym}^2E + \cdots,
\]

\[
\Lambda_t(E) = 1 + tE + t^2\Lambda^2E + \cdots
\]

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be the symmetric and exterior power operations of $E$ (resp. $E \otimes \mathbb{R} C$) in $K(M)[[t]]$ respectively.

We assume that $TX$ has an $S^1$-invariant almost complex structure $J$. Then we can construct canonically the Spin$^c$ Dirac operator $D^X$ on $\Lambda^* (T^{(0,1)} X)$ along the fiber $X$. Let $W$ be an $S^1$-equivariant complex vector bundle over $M$. We denote by $K_W = \det W$ and $K_X = \det (T^{(1,0)} X)$ the determinant line bundles of $W$ and $T^{(1,0)} X$ respectively. Let

$$Q_1(W) = \bigotimes_{n=0}^{\infty} \Lambda_{-q^n}(\overline{W}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^n}(W).$$

For $N \in \mathbb{N}, N \geq 2$, let $y = e^{2\pi i/N} \in \mathbb{C}$. Let $G_y$ be the multiplicative group generated by $y$. Following Witten [W], we consider the fiberwise action $G_y$ on $W$ and $\overline{W}$ by sending $y \in G_y$ to $y$ on $W$ and $y^{-1}$ on $\overline{W}$. Then $G_y$ acts naturally on $Q_1(W)$. We define $Q_1(T^{(1,0)} X)$ and the action $G_y$ on it in the above way.

The following theorem generalizes the result in [H] to the family case.

**Theorem 0.1.** Assume $c_1(T^{(1,0)} X) = 0 \mod(N)$, the family of $G_y \times S^1$ equivariant Spin$^c$ Dirac operators $D^X \otimes_{n=1}^{\infty} \text{Sym}_{q^n} (TX \otimes \mathbb{R} C) \otimes Q_1(T^{(1,0)} X)$ is rigid on the equivariant $K$-theory level, for the $S^1$ action.

The following family rigidity and vanishing theorem generalizes [Liu1, Theorem B] to the family case.

**Theorem 0.2.** Assume $\omega_2(TX - W)_{S^1} = 0$, $\frac{1}{2} p_1 (TX - W)_{S^1} = e \pi^* u^2$ ($e \in \mathbb{Z}$) in $H^2_{S^1}(M, \mathbb{Z})$, and $c_1(W) = 0 \mod(N)$. Consider the family of $G_y \times S^1$ equivariant Spin$^c$ Dirac operators

$$D^X \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes_{n=1}^{\infty} \text{Sym}_{q^n} (TX \otimes \mathbb{R} C) \otimes Q_1(W).$$

i) If $e = 0$, then these operators are rigid on the equivariant $K$-theory level for the $S^1$ action.

ii) If $e < 0$, then the index bundles of these operators are zero in $K_{G_y \times S^1}(B)$. In particular, these index bundles are zero in $K_{G_y}(B)$.

We refer to Section 2 for more details on the notation in Theorem 0.2. Actually, our main result, Theorem 2.2, holds on a family of Spin$^c$-manifolds with Theorem 0.2 being one of its special cases.

This paper is organized as follows. In Section 1, we recall a $K$-Theory version of the equivariant family index theorem for the circle action case [LiuMaZ, Theorem 1.2]. As an immediate corollary, we get a $K$-theory version of the vanishing theorem of Hattori for a family of almost complex manifolds. In Section 2, we prove the rigidity and vanishing theorem for elliptic genera in the Spin$^c$ case, on the equivariant $K$-theory level. The proof of the main results in Section 2 is based on two intermediate results which will be proved in Sections 3 and 4 respectively.

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**1. A $K$-theory version of the equivariant family index theorem.** In this section, we recall a $K$-theory version of the equivariant family index theorem [LiuMaZ, Theorem 1.2] for $S^1$-actions, which will play a crucial role in the following sections.
This section is organized as follows: In Section 1.1, we recall the K-theory version of the equivariant family index theorem for $S^1$-actions on a family of Spin$^c$ manifolds. In Section 1.2, as a simple application of Theorem 1.1, we obtain a K-theory version of the vanishing theorem of Hattori [Ha] for the case of almost complex manifolds.

1.1. A K-theory version of the equivariant family index theorem. Let $M$, $B$ be two compact manifolds, let $\pi : M \to B$ be a fibration with compact fibre $X$ such that $\dim X = 2l$ and that $S^1$ acts fiberwise on $M$. Let $h^{TX}$ be a metric on $TX$. We assume that $TX$ is oriented. Let $(W, h^W)$ be a Hermitian complex vector bundle over $M$.

Let $V$ be a $2p$ dimensional oriented real vector bundle over $M$. Let $L$ be a complex line bundle over $M$ with the property that the vector bundle $U = TX \oplus V$ obeys $\omega_2(U) = c_1(L) \mod (2)$. Then the vector bundle $U$ has a Spin$^c$-structure. Let $h^V$, $h^L$ be the corresponding metrics on $V$, $L$. Let $S(U, L)$ be the fundamental complex spinor bundle for $(U, L)$ [LaM, Appendix D.9] which locally may be written as

(1.1) \[ S(U, L) = S_0(U) \otimes L^{1/2}, \]

where $S_0(U)$ is the fundamental spinor bundle for the (possibly non-existent) spin structure on $U$, and where $L^{1/2}$ is the (possibly non-existent) square root of $L$.

Assume that the $S^1$-action on $M$ lifts to $V$, $L$ and $W$, and assume the metrics $h^{TX}$, $h^V$, $h^L$, $h^W$ are $S^1$-invariant. Also assume that the $S^1$-actions on $TX$, $V$, $L$ lift to $S(U, L)$.

Let $\nabla^{TX}$ be the Levi-Civita connection on $(TX, h^{TX})$ along the fibre $X$. Let $\nabla^V$, $\nabla^L$ and $\nabla^W$ be the $S^1$-invariant and metric-compatible connections on $(V, h^V)$, $(L, h^L)$ and $(W, h^W)$ respectively. Let $\nabla^{S(U, L)}$ be the Hermitian connection on $S(U, L)$ induced by $\nabla^{TX} \oplus \nabla^V$ and $\nabla^L$ (cf. [LaM, Appendix D], [LiuMaZ, §1.1]). Let $\nabla^{S(U, L) \otimes W}$ be the tensor product connection on $S(U, L) \otimes W$ induced by $\nabla^{S(U, L)}$ and $\nabla^W$,

(1.2) \[ \nabla^{S(U, L) \otimes W} = \nabla^{S(U, L)} \otimes 1 + 1 \otimes \nabla^W. \]

Let $\{e_i\}_{i=1}^{2l}$ (resp. $\{f_j\}_{j=1}^{2p}$) be an oriented orthonormal basis of $(TX, h^{TX})$ (resp. $(V, h^V)$). We denote by $c(\cdot)$ the Clifford action of $TX \oplus V$ on $S(U, L)$. Let $D^X \otimes W$ be the family Spin$^c$-Dirac operator on the fiber $X$ defined by

(1.3) \[ D^X \otimes W = \sum_{i=1}^{2l} c(e_i) \nabla^{S(U, L) \otimes W}_{e_i}. \]

There are two canonical ways to consider $S(U, L)$ as a $\mathbb{Z}_2$-graded vector bundle. Let

(1.4) \[ \tau_s = i^l c(e_1) \cdots c(e_{2l}), \tau_e = i^{1+l} p c(e_1) \cdots c(e_{2l}) c(f_1) \cdots c(f_{2p}) \]

be two involutions of $S(U, L)$. Then $\tau_s^2 = \tau_e^2 = 1$. We decompose $S(U, L) = S^+(U, L) \oplus S^-(U, L)$ corresponding to $\tau_s$ (resp. $\tau_e$) such that $\tau_s|_{S^\pm(U, L)} = \pm 1$ (resp. $\tau_e|_{S^\pm(U, L)} = \pm 1$).

For $\tau = \tau_s$ or $\tau_e$, by [LiuMa1, Proposition 1.1], the index bundle $\text{Ind}_\tau(D^X)$ over $B$ is well-defined in the equivariant $K$-group $K_{S^1}(B)$. 

Let $F = \{ F_a \}$ be the fixed point set of the circle action on $M$. Then $\pi : F_a \to B$ (resp. $\pi : F \to B$) is a smooth fibration with fibre $Y_a$ (resp. $Y$). Let $\bar{\pi} : N \to F$ denote the normal bundle to $F$ in $M$. Then $N = TX/TY$. We identify $N$ as the orthogonal complement of $TY$ in $TX|_F$. Let $h^{TY}$, $h^N$ be the corresponding metrics on $TY$ and $N$ induced by $h^{TX}$. Then, we have the following $S^1$-equivariant decomposition of $TX$ over $F$,

$$TX|_F = N_{m_1} \oplus \cdots \oplus N_{m_i} \oplus TY,$$

where each $N_i$ is a complex vector bundle such that $g \in S^1$ acts on it by $g^i$. To simplify the notation, we will write simply that

$$(1.5) \quad TX|_F = \oplus_{v \neq 0} N_v \oplus TY,$$

where $N_v$ is a complex vector bundle such that $g \in S^1$ acts on it by $g^v$ with $v \in \mathbb{Z}$. Clearly, $N = \oplus_{v \neq 0} N_v$. We will denote by $N$ a complex vector bundle, and $N_{\mathbb{R}}$ the underlying real vector bundle of $N$.

Similarly let

$$(1.6) \quad W|_F = \oplus_v W_v$$

be the $S^1$-equivariant decomposition of the restriction of $W$ over $F$. Here $W_v$ ($v \in \mathbb{Z}$) is a complex vector bundle over $F$ on which $g \in S^1$ acts by $g^v$.

We also have the following $S^1$-equivariant decomposition of $V$ restricted to $F$,

$$(1.7) \quad V|_F = \oplus_{v \neq 0} V_v \oplus V_0^R,$$

where $V_v$ is a complex vector bundle such that $g$ acts on it by $g^v$, and $V_0^R$ is the real subbundle of $V$ such that $S^1$ acts as identity. For $v \neq 0$, let $V_{v,\mathbb{R}}$ denote the underlying real vector bundle of $V_v$. Denote by $2p' = \dim V_0^R$ and $2l' = \dim Y$.

Let us write

$$(1.8) \quad L_F = L \otimes \left( \bigotimes_{v \neq 0} \det N_v \bigotimes_{v \neq 0} \det V_v \right)^{-1}.$$ 

Then $TY \oplus V_0^R$ has a Spin$^c$ structure as $\omega_2(TY \oplus V_0^R) = c_1(L_F) \mod (2)$. Let $S(TY \oplus V_0^R, L_F)$ be the fundamental spinor bundle for $(TY \oplus V_0^R, L_F)$ [LaM, Appendix D, pp. 397].

Let $D^Y$, $D^Y_a$ be the families of Spin$^c$ Dirac operators acting on $S(TY \oplus V_0^R, L_F)$ over $F$, $F_a$ as (1.3). If $R$ is an Hermitian complex vector bundle equipped with an Hermitian connection over $F$, let $D^Y \otimes R$, $D^Y_a \otimes R$ denote the twisted Spin$^c$ Dirac operators on $S(TY \oplus V_0^R, L_F) \otimes R$ and on $S(TY_a \oplus V_0^R, L_F) \otimes R$ respectively.

Recall that $N_{v,\mathbb{R}}$ and $V_{v,\mathbb{R}}$ are canonically oriented by their complex structures. The decompositions (1.5), (1.7) induce the orientations on $TY$ and $V_0^R$ respectively. Let $\{ e_1 \}_{i=1}^{2l'}$, $\{ f_j \}_{j=1}^{2p'}$ be the corresponding oriented orthonormal basis of $(TY, h^{TY})$ and $(V_0^R, h^{VR})$. There are two canonical ways to consider $S(TY \oplus V_0^R, L_F)$ as a $\mathbb{Z}_2$-graded vector bundle. Let

$$(1.9) \quad \tau_a = i^{l'} c(e_1) \cdots c(e_{2l'}),$$

$$\tau_c = i^{l'+p'} c(e_1) \cdots c(e_{2l'}) c(f_1) \cdots c(f_{2p'}).$$
be two involutions of $S(TY \oplus V_0^R, L_F)$. Then $\tau_s^2 = \tau_e^2 = 1$. We decompose $S(TY \oplus V_0^R, L_F) = S^+(TY \oplus V_0^R, L_F) \oplus S^-(TY \oplus V_0^R, L_F)$ corresponding to $\tau_s$ (resp. $\tau_e$) such that $\tau_s |_{S^\pm(TY \oplus V_0^R, L_F)} = \pm 1$ (resp. $\tau_e |_{S^\pm(TY \oplus V_0^R, L_F)} = \pm 1$).

Upon restriction to $F$, one has the following isomorphism of $\mathbb{Z}_2$-graded Clifford modules over $F$,

$$(1.10) \quad S(U, L) \simeq S(TY \oplus V_0^R, L_F) \bigotimes_{v \neq 0} \Lambda N_v \bigotimes_{v \neq 0} \Lambda V_v.$$

We denote by $\text{Ind}_{rs}$, $\text{Ind}_{re}$ the index bundles corresponding to the involutions $\tau_s$, $\tau_e$ respectively.

Let $S^1$ act on $L$ by sending $g \in S^1$ to $g^{l_c}$ ($l_c \in \mathbb{Z}$) on $F$. Then $l_c$ is locally constant on $F$. We define the following elements in $K(F)[[q^{1/2}]]$,

$$(1.11) \quad R_\pm(q) = q^{\frac{1}{2} \dim N_v - \frac{1}{2} \dim V_c + \frac{1}{2} l_c \otimes 0 < v \otimes \text{Sym}_{q^v}(N_v) \otimes \det N_v),$$

$$R'_\pm(q) = q^{-\frac{1}{2} \dim N_v - \frac{1}{2} \dim V_c + \frac{1}{2} l_c \otimes 0 < v \otimes \text{Sym}_{q^v}(N_v) \otimes \text{Sym}^{q^v}(V_c) \otimes q^v W_v = \sum_n R'_{\pm,n} q^n.$$

The following result was proved in [LiuMaZ, Theorem 1.2]:

**Theorem 1.1.** For $n \in \mathbb{Z}$, we have the following identity in $K(B)$,

$$\text{Ind}_{rs}(D^X \otimes W, n) = \sum \alpha (-1)^{\dim N_v} \text{Ind}_{rs}(D^{Y_\alpha} \otimes R_{+,n}),$$

$$\text{Ind}_{re}(D^X \otimes W, n) = \sum \alpha (-1)^{\dim N_v} \text{Ind}_{re}(D^{Y_\alpha} \otimes R_{-,n}),$$

$$\text{Ind}_{re}(D^X \otimes W, n) = \sum \alpha (-1)^{\dim N_v} \text{Ind}_{re}(D^{Y_\alpha} \otimes R_{-,n}).$$

**Remark 1.1.** If $TX$ has an $S^1$-equivariant Spin structure, by setting $V = 0, L = C$, we get [LiuMaZ, Theorem 1.1].

**1.2. K-theory version of the vanishing theorem of Hattori.** In this subsection, we assume that $TX$ has an $S^1$-equivariant almost complex structure $J$. Then one has the canonical splitting

$$(1.13) \quad TX \otimes_R C = T^{(1,0)}X \oplus T^{(0,1)}X,$$

where

$$T^{(1,0)}X = \{z \in TX \otimes_R C, Jz = \sqrt{-1}z\},$$

$$T^{(0,1)}X = \{z \in TX \otimes_R C, Jz = -\sqrt{-1}z\}.$$

Let $K_X = \text{det}(T^{(1,0)}X)$ be the determinant line bundle of $T^{(1,0)}X$ over $M$. Then the complex spinor bundle $S(TX, K_X)$ for $(TX, K_X)$ is $\Lambda(T^{(0,1)}X)$. In this case, the almost complex structure $J$ on $TX$ induces an almost complex structure on $TY$. Then we can rewrite (1.5) as,

$$(1.14) \quad T^{(1,0)}X = \oplus_{v \neq 0} N_v \oplus T^{(1,0)}Y,$$

where $N_v$ are complex vector subbundles of $T^{(1,0)}X$ on which $g \in S^1$ acts by multiplication by $g^v$. 
We suppose that $c_1(T^{(1,0)}X) = 0 \mod(N)$ ($N \in \mathbb{Z}, N \geq 2$). Then the complex line bundle $K_{X}^{1/N}$ is well defined over $M$. After replacing the $S^1$ action by its $N$-fold action, we can always assume that $S^1$ acts on $K_{X}^{1/N}$. For $s \in \mathbb{Z}$, let $D^X \otimes K_{X}^{s/N}$ be the twisted Dirac operator on $\Lambda(T^{(0,1)\ast}X) \otimes K_{X}^{s/N}$ defined as in (1.3).

The following result generalizes the main result of [Ha] to the family case.

**Theorem 1.2.** We assume that $M$ is connected and that the $S^1$ action is non-trivial. If $c_1(T^{(1,0)}X) = 0 \mod(N)$ ($N \in \mathbb{Z}, N \geq 2$), then for $s \in \mathbb{Z}$, $-N < s < 0$,

$$\text{ind}(D^X \otimes K_{X}^{s/N}) = 0 \text{ in } K_{S^1}(B).$$

**Proof.** Consider $R_+ (q)$, $R'_+ (q)$ of (1.11) with $V = 0, W = K_{X}^{s/N}$. We know

$$R_{r,n} = 0 \text{ if } n < a_1 = \inf \alpha \left(\frac{1}{2} \sum v |v| \dim N_v + \left(\frac{1}{2} + \frac{N}{2}\right) \sum v \dim N_v \right),
R'_r,n = 0 \text{ if } n > a_2 = \sup \alpha \left(-\frac{1}{2} \sum v |v| \dim N_v + \left(\frac{1}{2} + \frac{N}{2}\right) \sum v \dim N_v \right).$$

As $-N < s < 0$, by (1.16), we know that $a_1 \geq 0$, $a_2 \leq 0$, with $a_1$ or $a_2$ equal to zero iff $\sum_v |v| \dim N_v = 0$ for all $\alpha$, which means that the $S^1$ action does not have fixed points.

From Theorem 1.1 (cf. [Z, Theorem A.1]) and the above discussion, we get Theorem 1.2. □

**Remark 1.2.** From the proof of Theorem 1.2, one also deduces that $D^X \otimes K_{X}^{-1}$, $D^X$ are rigid on the equivariant $K$-theory level (cf. [Z, (2.17)]).

2. Rigidity and vanishing theorems in $K$-Theory. The purpose of this section is to establish the main results of this paper: the rigidity and vanishing theorems on the equivariant $K$-theory level for a family of Spin$^c$ manifolds. The results in this section refine some of the results in [LiuMa2] to the $K$-theory level.

This section is organized as follows: In Section 2.1, we state our main results, the rigidity and vanishing theorems on the equivariant $K$-theory level for a family of Spin$^c$ manifolds. In Section 2.2, we state two intermediate results which will be used to prove our main results stated in Section 2.1. In Section 2.3, we prove the family rigidity and vanishing theorems.

Throughout this section, we keep the notations of Section 1.1.

2.1. Family rigidity and vanishing Theorem. Let $\pi : M \to B$ be a fibration of compact manifolds with fiber $X$ and $\dim X = 2l$. We assume that $S^1$ acts fiberwise on $M$, and $TX$ has an $S^1$-invariant Spin$^c$ structure. Let $V$ be an even dimensional real vector bundle over $M$. We assume that $V$ has an $S^1$-invariant spin structure. Let $W$ be an $S^1$-equivariant complex vector bundle of rank $r$ over $M$. Let $K_W = \det(W)$ be the determinant line bundle of $W$.

Let $K_X$ be the $S^1$-equivariant complex line bundle over $M$ which is induced by the $S^1$-invariant Spin$^c$ structure of $TX$. Its equivariant first Chern class $c_1(K_X)_{S^1}$ may also be written as $c_1(TX)_{S^1}$.

Let $S(TX, K_X)$ be the complex spinor bundle of $(TX, K_X)$ as in Section 1.1. Let $S(V) = S^+(V) \oplus S^-(V)$ be the spinor bundle of $V$. 

We define the following elements in \( K(M)[[q^{1/2}]] \):

\[
\begin{align*}
Q_1(W) &= \bigotimes_{n=0}^{\infty} \Lambda_{-q^n}(\overline{W}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{-q^n}(W), \\
R_1(V) &= (S^+(V) + S^-(V)) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(V), \\
R_2(V) &= (S^+(V) - S^-(V)) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(V), \\
R_3(V) &= \bigotimes_{n=1}^{\infty} \Lambda_{-q^{n-1/2}}(V), \\
R_4(V) &= \bigotimes_{n=1}^{\infty} \Lambda_{q^{n-1/2}}(V).
\end{align*}
\]

(2.1)

For \( N \in \mathbb{N}, N \geq 2 \), let \( y = e^{2\pi i/N} \in \mathbb{C} \). Let \( G_y \) be the multiplicative group generated by \( y \). Following Witten [W], we consider the fiberwise action \( G_y \) on \( W \) and \( \overline{W} \) by sending \( y \in G_y \) to \( y \) on \( W \) and \( y^{-1} \) on \( \overline{W} \). Then \( G_y \) acts naturally on \( Q_1(W) \).

Recall that the equivariant cohomology group \( H^*_S(M, \mathbb{Z}) \) of \( M \) is defined by

\[
H^*_S(M, \mathbb{Z}) = H^*(M \times S^1, ES^1, \mathbb{Z}),
\]

where \( ES^1 \) is the usual universal \( S^1 \)-principal bundle over the classifying space \( BS^1 \) of \( S^1 \). So \( H^*_S(M, \mathbb{Z}) \) is a module over \( H^*(BS^1, \mathbb{Z}) \) induced by the projection \( \overline{\pi} : M \times S^1 ES^1 \to BS^1 \). Let \( p_1(V)_{S^1}, p_1(TX)_{S^1} \in H^*_S(M, \mathbb{Z}) \) be the \( S^1 \)-equivariant first Pontrjagin classes of \( V \) and \( TX \) respectively. As \( V \times S^1 ES^1 \) is spin over \( M \times S^1 ES^1 \), one knows that \( \frac{1}{2}p_1(V)_{S^1} \) is well-defined in \( H^*_S(M, \mathbb{Z}) \) (cf. [T, pp. 456-457]). Also recall that

\[
H^*(BS^1, \mathbb{Z}) = \mathbb{Z}[u],
\]

with \( u \) a generator of degree 2.

In the following, we denote by \( D^X \otimes R \) the family of Dirac operators acting fiberwise on \( S(TX, K_X) \otimes R \) as was defined in Section 1.1.

We can now state the main results of this paper as follows.

**Theorem 2.1.** If \( \omega_2(W)_{S^1} = \omega_2(TX)_{S^1}, \frac{1}{2}p_1(V + W - TX)_{S^1} = e \cdot \overline{\pi}^*u^2 \ (n \in \mathbb{Z}) \) in \( H^*_S(M, \mathbb{Z}) \), and \( c_1(W) = 0 \mod(N) \). For \( i = 1, 2, 3, 4 \), consider the family of \( G_y \times S^1 \)-equivariant elliptic operators

\[
D^X \otimes (K_W \otimes K^{-1}_X)^{1/2} \otimes \bigotimes_{n=1}^{\infty} \text{Sym}_{q^n}(TX) \otimes Q_1(W) \otimes R_i(V).
\]

i) If \( e = 0 \), then these operators are rigid on the equivariant K-theory level for the \( S^1 \) action.

ii) If \( e < 0 \), then the index bundles of these operators are zero in \( K_{G_y \times S^1}(B) \). In particular, these index bundles are zero in \( K_{G_y}(B) \).

**Remark 2.1.** As \( \omega_2(W)_{S^1} = \omega_2(TX)_{S^1}, \frac{1}{2}p_1(W - TX)_{S^1} \in H^*_S(M, \mathbb{Z}) \) is well-defined. The condition \( \omega_2(W)_{S^1} = \omega_2(TX)_{S^1} \) also means \( c_1(K_W \otimes K^{-1}_X)^{1/2} \) is zero modulo(2), by [HaY, Corollary 1.2], the \( S^1 \)-action on \( M \) can be lifted to \( (K_W \otimes K^{-1}_X)^{1/2} \) and is compatible with the \( S^1 \) action on \( K_W \otimes K^{-1}_X \).

**Remark 2.2.** If we assume \( c_1(W)_{S^1} = c_1(TX)_{S^1} \in H^*_S(M, \mathbb{Z}) \) instead of \( \omega_2(W)_{S^1} = \omega_2(TX)_{S^1} \) in Theorem 2.1, then \( K_W \otimes K^{-1}_X \) is a trivial line bundle over \( M \), and \( S^1 \) acts trivially on it. In this case, Theorem 2.1 gives the family version of the results of [De].

**Remark 2.3.** The interested reader can apply our method to get various rigidity and vanishing theorems, for example, to get a generalization of Theorem 1.2 for the elements [W, (65)].
Actually, as in [LiuMaZ], our proof of these theorems works under the following slightly weaker hypothesis. Let us first explain some notations.

For each \( n > 1 \), consider \( \mathbb{Z}_n \subset S^1 \), the cyclic subgroup of order \( n \). We have the \( \mathbb{Z}_n \) equivariant cohomology of \( M \) defined by \( H^*_\mathbb{Z}_n (M, \mathbb{Z}) = H^*(M \times \mathbb{Z}_n, ES^1, \mathbb{Z}) \), and there is a natural “forgetful” map \( \alpha(S^1, \mathbb{Z}_n) : M \times \mathbb{Z}_n, ES^1 \to M \times S^1, ES^1 \) which induces a pullback \( \alpha(S^1, \mathbb{Z}_n)* : H^*_S^1 (M, \mathbb{Z}) \to H^*_S^1 (M, \mathbb{Z}) \). The arrow which forgets the \( S^1 \) action altogether we denote by \( \alpha(S^1, 1) \). Thus \( \alpha(S^1, 1)* : H^*_S^1 (M, \mathbb{Z}) \to H^*_S^1 (M, \mathbb{Z}) \) is induced by the inclusion of \( M \) into \( M \times S^1, ES^1 \) as a fiber over \( BS^1 \).

Finally, note that if \( \mathbb{Z}_n \) acts trivially on a space \( Y \), then there is a new arrow \( t^* : H^*(Y, \mathbb{Z}) \to H^*_S^1 (Y, \mathbb{Z}) \) induced by the projection \( Y \times \mathbb{Z}_n, ES^1 = Y \times B\mathbb{Z}_n \).

We let \( S_p^1 \). For each \( 1 < n \leq +\infty \), let \( i : M(n) \to M \) be the inclusion of the fixed point set of \( \mathbb{Z}_n \subset S^1 \) in \( M \) and so \( i \) induces \( i_S^1 : M(n) \times S^1, ES^1 \to M \times S^1, ES^1 \).

In the rest of this paper, we suppose that there exists some integer \( e \in \mathbb{Z} \) such that for \( 1 < n \leq +\infty \),

\[
\alpha(S^1, \mathbb{Z}_n)^* \circ i_S^1 \left( \frac{1}{2} p_1 (V + W - TX)_{S^1} - e \cdot \bar{\pi} u^2 \right) = t^* \circ \alpha(S^1, 1)^* \circ i_S^1 \left( \frac{1}{2} p_1 (V + W - TX)_{S^1} \right). \tag{2.4}
\]

**Remark 2.4.** The relation (2.4) clearly follows from the hypotheses of Theorem 2.1 by pulling back and forgetting. Thus it is weaker.

We can now state a slightly more general version of Theorem 2.1.

**Theorem 2.2.** Under the hypothesis (2.4), we have

i) If \( e = 0 \), then the index bundles of the elliptic operators in Theorem 2.1 are rigid on the equivariant K-theory level for the \( S^1 \)-action.

ii) If \( e < 0 \), then the index bundles of the elliptic operators in Theorem 2.1 are zero as elements in \( K_{G_y \times S^1} (B) \). In particular, these index bundles are zero in \( K_{G_y} (B) \).

The rest of this section is devoted to a proof of Theorem 2.2.

**2.2. Two intermediate results.** Let \( F = \{ F_a \} \) be the fixed point set of the circle action. Then \( \pi : F \to B \) is a fibration with compact fibre denoted by \( Y = \{ Y_a \} \).

As in [LiuMaZ, §2], we may and will assume that

\[
TX\mid_F = TY \oplus \bigoplus_{0 < v} N_v, \\
TX\mid_F \otimes \mathbb{C} = TY \otimes \mathbb{C} \bigoplus_{0 < v} (N_v \oplus \overline{N}_v), \tag{2.5}
\]

where \( N_v \) is the complex vector bundle on which \( S^1 \) acts by sending \( g \) to \( g^v \) (Here \( N_v \) can be zero). We also assume that

\[
V\mid_F = V_0^\mathbb{R} \oplus \bigoplus_{0 < v} V_v, \\
W\mid_F = \oplus_v W_v, \tag{2.6}
\]

where \( V_v, W_v \) are complex vector bundles on which \( S^1 \) acts by sending \( g \) to \( g^v \), and \( V_0^\mathbb{R} \) is a real vector bundle on which \( S^1 \) acts as identity.

By (2.5), as in (1.10), there is a natural isomorphism between the \( \mathbb{Z}_2 \)-graded \( C(TX) \)-Clifford modules over \( F \),

\[
S(TY, K_X \otimes_{0 < v} (\det N_v)^{-1}) \hat{\otimes}_{0 < v} \Lambda N_v \simeq S(TX, K_X)\mid_F. \tag{2.7}
\]
For $R$ a complex vector bundle over $F$, let $D^Y \otimes R, D^Y \otimes R$ be the twisted Spin$^c$ Dirac operator on $S(TY, K_X \otimes O_Y \otimes (\det N_v)^{-1}) \otimes R$ on $F, F_0$, respectively.

On $F$, we write

\begin{align}
\frac{e}{e}(N) &= \sum_{0<v} v^2 \dim N_v, & d'(N) &= \sum_{0<v} v \dim N_v, \\
\frac{e}{e}(V) &= \sum_{0<v} v^2 \dim V_v, & d'(V) &= \sum_{0<v} v \dim V_v, \\
\frac{e}{e}(W) &= \sum_{0<v} v^2 \dim W_v, & d'(W) &= \sum_{0<v} v \dim W_v.
\end{align}

Then $e(N), e(V), e(W), d'(N), d'(V)$ and $d'(W)$ are locally constant functions on $F$.

By [H, §8], we have the following property,

**Lemma 2.1.** If $c_1(W) = 0 \mod(N)$, then $d'(W) \mod(N)$ is constant on each connected component of $M$.

**Proof.** As $c_1(W) = 0 \mod(N)$, $(K_W)^{1/N}$ is well defined. Consider the $N$-fold covering $S^1 \to S^1$, with $\mu \to \lambda = \mu^N$, then $\mu$ acts on $M$ and $K_W$ through $\lambda$. This action can be lift to $(K_W)^{1/N}$. On $F$, $\mu$ acts on $(K_W)^{1/N}$ by multiplication by $\mu^{d'(W)}$. However, if $\mu = \zeta = e^{2\pi i/N}$, then it operates trivially on $M$. So the action of $\zeta$ in each fibre of $L$ is by multiplication by $\zeta^a$, and a mod$(N)$ is constant on each connected component of $M$.

The proof of Lemma 2.1 is complete. □

We denote the Chern roots of $N_v$ by $\{x_v^i\}$ (resp. $V_v$ by $u_v^i$ and $W_v$ by $w_v^i$), and the Chern roots of $TY \otimes R C$ by $\{\pm y_j\}$ (resp. $V_0 = V_0 \otimes R C$ by $\{\pm u_0^i\}$). Then if we take $Z_\infty = S^1$ in (2.4), we get

\begin{align}
\frac{1}{2}(\Sigma_{v,j}(u_v^j + vu)^2 + \Sigma_{v,j}(w_v^j + vu)^2 - \Sigma_{j}(y_j)^2 - \Sigma_{v,j}(x_v^j + vu)^2) - eu^2 = \\
\frac{1}{2}(\Sigma_{v,j}(u_v^j)^2 + \Sigma_{v,j}(w_v^j)^2 - \Sigma_{j}(y_j)^2 - \Sigma_{v,j}(x_v^j)^2).
\end{align}

By (2.3), (2.10), we get

\begin{align}
c_1(L) &= \Sigma_{v,j} vu^j + \Sigma_{v,j} uv^j - \Sigma_{v,j} vx^j = 0, \\
\frac{e}{e}(V) + e(N) &= \sum_{0<v} v^2 \dim V_v + \sum_{0<v} v^2 \dim W_v - \sum_{0<v} v^2 \dim N_v = 2e,
\end{align}

which does not depends on the connected components of $F$. This means $L$ is a trivial complex line bundle over each component $F_0$ of $F$, and $S^1$ acts on $L$ by sending $g$ to $g^{2e}$, and $G_y$ acts on $L$ by sending $y$ to $y^{d'(W)}$. By Lemma 2.1, we can extend $L$ to a trivial complex line bundle over $M$, and we extend the $S^1$-action on it by sending $g$ on the canonical section 1 to $L$ to $g^{2e} \cdot 1$, and $G_y$ acts on $L$ by sending $y$ to $y^{d'(W)}$.

The line bundles in (2.9) will play important roles in the next two sections which consist of the proof of Theorems 2.3, 2.4 to be stated below.

In what follows, if $R(q) = \sum_{m \in \mathbb{Z}} R_m q^m \in K_{S^1}(M)[[q^{1/2}]]$, we will also denote $\text{Ind}(D^X \otimes R_m)$ by $\text{Ind}(D^X \otimes R(q), m, h)$. For $k = 1, 2, 3, 4$, set

\begin{align}
R_{1k} &= (K_W \otimes K_X^{-1})^{1/2} \otimes Q_1(W) \otimes R_k(V).
\end{align}
We first state a result which expresses the global equivariant family index via the family indices on the fixed point set.

**Proposition 2.1.** For \( m \in \frac{1}{2} \mathbb{Z}, h \in \mathbb{Z}, 1 \leq k \leq 4, \) we have the following identity in \( K_{G_v}(B), \)
\[
\text{Ind}(D^X \otimes_{n=1}^{\infty} \text{Sym}_q^n(TX) \otimes R_{1k}, m, h) = \sum_{\alpha} (-1)^{\Sigma_{0<v} \dim N_v} \text{Ind}(D^{Y_{\alpha}} \otimes_{n=1}^{\infty} \text{Sym}_q^n(TX) \otimes R_{1k} \otimes \text{Sym}(\oplus_{0<v} N_v) \otimes_{0<v} \det N_v, m, h). \tag{2.13}
\]

**Proof.** This follows directly from Theorem 1.1 and (2.7). □

For \( p \in \mathbb{N}, \) we define the following elements in \( K_{s^1}(F)[[q]]: \)
\[
\mathcal{F}_p(X) = \bigotimes_{0<v} \text{Sym}_q^n(N_v) \otimes_{n>pv} \text{Sym}_q^n(N_v) \otimes_{n=1}^{\infty} \text{Sym}_q^n(TY),
\]
\[
\mathcal{F}^{-p}(X) = \mathcal{F}_p(X) \otimes \mathcal{F}_p(X). \tag{2.14}
\]

Then, from (2.5), over \( F, \) we have
\[
\mathcal{F}^0(X) = \bigotimes_{n=1}^{\infty} \text{Sym}_q^n(TX) \otimes \text{Sym}(\oplus_{0<v} N_v) \otimes_{0<v} \det N_v. \tag{2.15}
\]

We now state two intermediate results on the relations between the family indices on the fixed point set. They will be used in the next subsection to prove Theorem 2.2.

**Theorem 2.3.** For \( 1 \leq k \leq 4, h, p \in \mathbb{Z}, p > 0, m \in \frac{1}{2} \mathbb{Z}, \) we have the following identity in \( K_{G_v}(B), \)
\[
\sum_{\alpha} (-1)^{\Sigma_{0<v} \dim N_v} \text{Ind}(D^{Y_{\alpha}} \otimes \mathcal{F}^0(X) \otimes R_{1k}, m, h) = \sum_{\alpha} (-1)^{pd(N)+\Sigma_{0<v} \dim N_v} \text{Ind}(D^{Y_{\alpha}} \otimes \mathcal{F}^{-p}(X) \otimes R_{1k}, m + \frac{1}{2}pd(N) + \frac{1}{2}pd(N), h). \tag{2.16}
\]

**Theorem 2.4.** For each \( \alpha, 1 \leq k \leq 4, h, p \in \mathbb{Z}, p > 0, m \in \frac{1}{2} \mathbb{Z}, \) we have the following identity in \( K_{G_v}(B), \)
\[
\text{Ind}(D^{Y_{\alpha}} \otimes \mathcal{F}^{-p}(X) \otimes R_{1k}, m + \frac{1}{2}p^2e(N) + \frac{1}{2}pd(N), h) = (-1)^{pd(W)} \text{Ind}(D^{Y_{\alpha}} \otimes \mathcal{F}^0(X) \otimes R_{1k} \otimes L^{-p}, m + ph + p^2e, h). \tag{2.17}
\]

Theorem 2.3 is a direct consequence of Theorem 2.5 to be stated below, which will be proved in Section 4, while Theorem 2.4 will be proved in Section 3.

To state Theorem 2.5, let \( J = \{v \in \mathbb{N} \mid \text{There exists } \alpha \text{ such that } N_v \neq 0 \text{ on } F_\alpha \} \) and
\[
\Phi = \{\beta \in [0, 1] \mid \text{There exists } v \in J \text{ such that } \beta v \in \mathbb{Z} \}. \tag{2.18}
\]
\[\footnote{Here by \( K_{s^1}(F) \) we also mean the direct sum of the form \( \oplus_{n \in \mathbb{Z}} E_n \) with each \( E_n \) a finite dimensional vector bundle over \( F \) of weight \( n \) under the \( S^1 \)-action.} \]
We order the elements in $\Phi$ so that $\Phi = \{\beta_i | 1 \leq i \leq J_0, J_0 \in \mathbb{N} \text{ and } \beta_i < \beta_{i+1}\}$. Then for any integer $1 \leq i \leq J_0$, there exist $p_i, n_i \in \mathbb{N}$, $0 < p_i \leq n_i$, with $(p_i, n_i) = 1$ such that

$$
\beta_i = p_i / n_i.
$$

Clearly, $\beta_{J_0} = 1$. We also set $p_0 = 0$ and $\beta_0 = 0$.

For $1 \leq j \leq J_0$, $p \in \mathbb{N}^*$, we write

$$
I^p_0 = \phi, \text{ the empty set,}
$$

$$
I^p_j = \{(v, n) | v \in J, (p - 1)v < n \leq pv, n_v = p - 1 + \frac{n_j}{n_j}\},
$$

$$
\bar{I}^p_j = \{(v, n) | v \in J, (p - 1)v < n \leq pv, n_v > p - 1 + \frac{n_j}{n_j}\}.
$$

For $0 \leq j \leq J_0$, set

$$
\mathcal{F}_{p,j}(X) = \mathcal{F}_p(X) \otimes \mathcal{F}_{p-1}(X) \bigotimes_{(v,n) \in \bigcup_{i=1}^j I^p_j} \left( \text{Sym}_{q-n}(N_v) \otimes \det N_v \right) \bigotimes_{(v,n) \in \bar{I}^p_j} \text{Sym}_{q-n}(\tilde{N}_v).
$$

Then

$$
\mathcal{F}_{p,0}(X) = \mathcal{F}^{-p+1}(X),
$$

$$
\mathcal{F}_{p,J_0}(X) = \mathcal{F}^{-p}(X).
$$

For $s \in \mathbb{R}$, let $[s]$ denote the greatest integer which is less than or equal to the given number $s$. For $0 \leq j \leq J_0$, denote by

$$
e(p, \beta_j, N) = \frac{1}{2} \sum_{0 < v} (\dim N_v) \left( (p - 1)v + \left\lfloor \frac{\beta_j}{n_j}v \right\rfloor \right) \left( (p - 1)v + \left\lfloor \frac{\beta_j}{n_j}v \right\rfloor + 1 \right),
$$

$$d'(p, \beta_j, N) = \sum_{0 < v} (\dim N_v) \left( \left\lfloor \frac{\beta_j}{n_j}v \right\rfloor + (p - 1)v \right).
$$

Then $e(p, \beta_j, N) \text{ and } d'(p, \beta_j, N)$ are locally constant functions on $F$. And

$$
e(p, \beta_0, N) = \frac{1}{2} (p - 1)^2 e(N) + \frac{1}{2} (p - 1)d'(N),
$$

$$e(p, \beta_0, N) = \frac{1}{2} p^2 e(N) + \frac{1}{2} p d'(N),
$$

$$d'(p, \beta_0, N) = d'(p + 1, \beta_0, N) = p d'(N).
$$

**THEOREM 2.5.** For $1 \leq k \leq 4$, $1 \leq j \leq J_0$, $p \in \mathbb{N}^*$, $h \in \mathbb{Z}$, $m \in \frac{1}{2} \mathbb{Z}$, we have the following identity in $K_{G_e}(B)$,

$$
\sum_{\alpha} (-1)^d'(p, \beta_{j-1}, N) + \Sigma_{0 < v} \dim N_v \text{Ind}(D_{Y_\alpha} \otimes \mathcal{F}_{p,j-1}(X) \otimes R_{1k}, m + e(p, \beta_{j-1}, N), h),
$$

$$\sum_{\alpha} (-1)^{d'(p, \beta_j, N)} + \Sigma_{0 < v} \dim N_v \text{Ind}(D_{Y_\alpha} \otimes \mathcal{F}_{p,j}(X) \otimes R_{1k}, m + e(p, \beta_j, N), h).
$$

**Proof.** The proof is delayed to Section 4. \(\square\)

**Proof of Theorem 2.3.** From (2.22), (2.24), and Theorem 2.5, for $1 \leq k \leq 4$, $h \in \mathbb{Z}$, $p \in \mathbb{N}^*$ and $m \in \frac{1}{2} \mathbb{Z}$, we have the following identity in $K_{G_e}(B)$:

$$
\sum_{\alpha} (-1)^{d'(p, \beta_0, N)} + \Sigma_{0 < v} \dim N_v \text{Ind}(D_{Y_\alpha} \otimes \mathcal{F}^{-p}(X) \otimes R_{1k}, m + \frac{1}{2} p^2 e(N) + \frac{1}{2} pd'(N), h),
$$

$$\sum_{\alpha} (-1)^{d'(p, \beta_0, N)} + \Sigma_{0 < v} \dim N_v \text{Ind}(D_{Y_\alpha} \otimes \mathcal{F}^{-p+1}(X) \otimes R_{1k}, m + \frac{1}{2} (p - 1)^2 e(N) + \frac{1}{2} (p - 1)d'(N), h).
$$

From (2.24), (2.26), we get Theorem 2.3. \(\square\)
2.3. Proof of Theorem 2.2. As \( p_1(TX - W)_{s^1} \in H^5_s(M, \mathbb{Z}) \) is well defined, by (2.8), and (2.10),

\[
(2.27) \quad d'(N) + d'(W) = 0 \mod(2).
\]

From Proposition 2.1, Theorems 2.3, 2.4, (2.23), (2.27), for \( 1 \leq k \leq 4, h, p \in \mathbb{Z}, p > 0, m \in \frac{1}{2} \mathbb{Z} \), we get the following identity in \( K_{G_y}(B) \),

\[
(2.28) \quad \text{Ind}(D^X \otimes_{n=1}^\infty \text{Sym}_{q^n}(TX) \otimes R_{1k}, m, h) = \text{Ind}(D^X \otimes_{n=1}^\infty \text{Sym}_{q^n}(TX) \otimes R_{1k} \otimes L^{-p}, m', h),
\]

with

\[
(2.29) \quad m' = m + ph + p^2 e.
\]

Note that from (2.1), (2.12), if \( m < 0 \), or \( m' < 0 \), then two side of (2.28) are zero in \( K_{G_y}(B) \). Also recall that \( y \in G_y \) acts on the trivial line bundle \( L \) by sending \( y \) to \( y^{d'(W)} \).

i) Assume that \( e = 0 \). Let \( h \in \mathbb{Z}, m_0 \in \frac{1}{2} \mathbb{Z}, h \neq 0 \) be fixed. If \( h > 0 \), we take \( m' = m_0 \), then for \( p \) big enough, we get \( m < 0 \) in (2.29). If \( h < 0 \), we take \( m = m_0 \), then for \( p \) big enough, we get \( m' < 0 \) in (2.29).

So for \( h \neq 0, m_0 \in \frac{1}{2} \mathbb{Z}, 1 \leq k \leq 4 \), we get

\[
(2.30) \quad \text{Ind}(D^X \otimes_{n=1}^\infty \text{Sym}_{q^n}(TX) \otimes R_{1k}, m_0, h) = 0 \quad \text{in} \quad K_{G_y}(B).
\]

ii) Assume that \( e < 0 \). For \( h \in \mathbb{Z}, m_0 \in \frac{1}{2} \mathbb{Z} \), we take \( m = m_0 \), then for \( p \) big enough, we get \( m' < 0 \) in (2.29), which again gives us (2.30).

The proof of Theorem 2.2 is complete. \( \square \)

REMARK 2.5. Under the condition of Theorem 2.2 i), if \( d'(W) \neq 0 \mod(N) \), we can't deduce these index bundles are zero in \( K_{G_y}(B) \). If in addition, \( M \) is connected, by (2.28), for \( 1 \leq k \leq 4 \), in \( K_{G_y}(B) \), we get

\[
(2.31) \quad \text{Ind}(D^X \otimes_{n=1}^\infty \text{Sym}_{q^n}(TX) \otimes R_{1k}) = \text{Ind}(D^X \otimes_{n=1}^\infty \text{Sym}_{q^n}(TX) \otimes R_{1k} \otimes [d'(W)]).
\]

Here we denote by \([d'(W)]\) the one dimensional complex vector space on which \( y \in G_y \) acts by multiplication by \( y^{d'(W)} \). In particular, if \( B \) is a point, by (2.31), we get the vanishing theorem analogue to the result of [H, §10].

REMARK 2.6. If we replace \( c_1(W) = 0 \mod(N), y = e^{2\pi i/N} \) by \( c_1(W) = 0, y = e^{2\pi c i} \), with \( c \in \mathbb{R} \setminus \mathbb{Q} \) in Theorem 2.2, then by Lemma 2.1, \( d'(W) \) is constant on each connected component of \( M \). In this case, we still have Theorem 2.2. In fact, we only use \( c_1(W) = 0 \mod(N) \) to insure the action \( G_y \) on \( L \) is well defined. So we also generalize the main result of [K] to family case.

3. Proof of Theorem 2.4. This section is organized as follows: In Section 3.1, we introduce some notations. In Section 3.2, we prove Theorem 2.4 by introducing some shift operators as in [LiuMaZ, §3].

Throughout this section, we keep the notations of Section 2.
3.1. Reformulation of Theorem 2.4. To simplify the notations, we introduce some new notations in this subsection. For \( n_0 \in \mathbb{N}^* \), we define a number operator \( P \) on \( K_{S^1}(M)[[q^{n_0}]] \) in the following way: if \( R(q) = \oplus_{n \in \mathbb{N}} q^n R_n \in K_{S^1}(M)[[q^{n_0}]] \), then \( P \) acts on \( R(q) \) by multiplication by \( n \) on \( R_n \). From now on, we simply denote \( \text{Sym}^n(TX), A_q^n(V) \) by \( \text{Sym}(TX_n), A(V)_n \) respectively. In this way, \( P \) acts on \( TX_n, V_n \) by multiplication by \( n \), and the action \( P \) on \( \text{Sym}(TX_n), A(V)_n \) is naturally induced by the corresponding action of \( P \) on \( TX_n, V_n \). So the eigenspace of \( P = n \) is just given by the coefficient of \( q^n \) of the corresponding element \( R(q) \). For \( R(q) = \oplus_{n \in \mathbb{N}} q^n R_n \in K_{S^1}(M)[[q^{n_0}]] \), we will also denote

\[
(3.1) \quad \text{Ind}(D^X \otimes R(q), m, h) = \text{Ind}(D^X \otimes R_m, h).
\]

Let \( H \) be the canonical basis of \( \text{Lie}(S^1) = \mathbb{R} \), i.e., \( \exp(tH) = \exp(2\pi it) \) for \( t \in \mathbb{R} \). If \( E \) is an \( S^1 \)-equivariant vector bundle over \( M \), on the fixed point set \( F \), let \( J_H \) be the representation of \( \text{Lie}(S^1) \) on \( E|_F \). Then the weight of \( S^1 \) action on \( \Gamma(F, E|_F) \) is given by the action

\[
(3.2) \quad J_H = \frac{-1}{2\pi}\sqrt{-1} J_H.
\]

Recall that the \( \mathbb{Z}_2 \) grading on \( S(TX, K_X) \otimes_{n=1}^{\infty} \text{Sym}(TX_n) \) (resp. \( S(TY, K_X \otimes \otimes_{0<v} (\det N_v)^{-1}) \otimes F^{-p}(X) \)) is induced by the \( \mathbb{Z}_2 \)-grading on \( S(TX, K_X) \) (resp. \( S(TY, K_X \otimes \otimes_{0<v} (\det N_v)^{-1}) \)). Let

\[
F^1 = S(V) \otimes_{n=1}^{\infty} \Lambda(V_n),
F^2 = \otimes_{n \in \mathbb{N}+\frac{1}{2}} \Lambda(V_n),
Q(W) = \otimes_{n=1}^{\infty} \Lambda(W_n) \otimes_{n=1}^{\infty} \Lambda(W_n).
\]

There are two natural \( \mathbb{Z}_2 \) gradings on \( F^1, F^2 \) (resp. \( Q(W) \)). The first grading is induced by the \( \mathbb{Z}_2 \)-grading of \( S(V) \) and the forms of homogeneous degree in \( \otimes_{n=1}^{\infty} \Lambda(V_n), \otimes_{n \in \mathbb{N}+\frac{1}{2}} \Lambda(V_n) \) (resp. \( Q(W) \)). We define \( \tau \mid Q(W) = \pm1 \) to be the involution defined by this \( \mathbb{Z}_2 \)-grading. The second grading is the one for which \( F^i \) (\( i = 1, 2 \)) are purely even, i.e., \( F^{i+1} = F^i \). We denote by \( \tau_s = \text{Id} \) the involution defined by this \( \mathbb{Z}_2 \) grading. Then the coefficient of \( q^n \) (\( n \in \frac{1}{2} \mathbb{Z} \)) in (2.1) of \( R_1(V) \) or \( R_2(V) \) (resp. \( R_3(V) \), \( R_4(V) \), or \( Q_1(W) \)) is exactly the \( \mathbb{Z}_2 \)-graded vector subbundle of \( (F^1|_V, \tau_s) \) or \( (F^2|_V, \tau_s) \) (resp. \( (F^2|_V, \tau_s) \), \( (F^2|_V, \tau_s) \), or \( (Q(W), \tau_1) \)), on which \( P \) acts by multiplication by \( n \).

We denote by \( \tau_e \) (resp. \( \tau_s \)) the \( \mathbb{Z}_2 \)-grading on \( S(TX, K_X) \otimes_{n=1}^{\infty} \text{Sym}(TX_n) \otimes F^k \) (\( k = 1, 2 \)) induced by the above \( \mathbb{Z}_2 \)-gradings. We will denote by \( \tau_{e1} \) (resp. \( \tau_{s1} \)) the \( \mathbb{Z}_2 \)-gradings on \( S(TX, K_X) \otimes_{n=1}^{\infty} \text{Sym}(TX_n) \otimes F^k \otimes Q(W) \) defined by

\[
(3.4) \quad \tau_{e1} = \tau_e \otimes 1 + 1 \otimes \tau_1,
\tau_{s1} = \tau_s \otimes 1 + 1 \otimes \tau_1.
\]

Let \( h^V \) be the metric on \( V \) induced by the metric \( h^V \) on \( V \). In the following, we identify \( AV \) with \( \overline{AV}^* \) by using the Hermitian metric \( h^V \) on \( V \). By (2.6), as in (1.10), there is a natural isomorphism between \( \mathbb{Z}_2 \)-graded \( C(V) \)-Clifford modules over \( F \),

\[
(3.5) \quad S(V_0^R, \otimes_{0<v} (\det V_v)^{-1}) \otimes_{0<v} \Lambda V_v \simeq S(V)|_F.
\]
By using the above notations, we rewrite (2.14), on the fixed point set \( F \), for \( p \in \mathbb{N} \),
\[
\mathcal{F}_p(X) = \bigotimes_{0 \leq n \leq p} \left( \bigotimes_{n=1}^{\infty} \text{Sym}(N_{v,n}) \otimes \text{Sym}(\overline{N}_{v,n}) \right) \bigotimes_{n=1}^{\infty} \text{Sym}(TY_n),
\]
(3.6)
\[
\mathcal{F}_{-p}(X) = \bigotimes_{0 \leq n \leq p} \left( \bigotimes_{n=1}^{\infty} \text{Sym}(N_{v,n}) \otimes \det N_v \right),
\]
\[
\mathcal{F}^0(X) = \bigotimes_{0 < n \leq p} \left( \bigotimes_{n=1}^{\infty} \text{Sym}(N_{v,n}) \otimes \det N_v \right).
\]
(3.7)

Let \( V_0 = V_0^R \otimes \mathbb{R} C \). From (2.5), (3.5), we get
\[
\mathcal{F}^0(X) = \bigotimes_{n=1}^{\infty} \text{Sym} \left( \bigoplus_{0 < v} (N_{v,n} \oplus \overline{N}_{v,n}) \right) \bigotimes_{n=1}^{\infty} \text{Sym}(TY_n)
\]
\[
F_V^1 = \bigotimes_{0 < n \in Z_{1/2}} \Lambda \left( \bigoplus_{0 < v} (V_{v,n} \oplus \overline{V}_{v,n}) \oplus V_0 \right)
\]
\[
\otimes S(V_0^R, \bigotimes_{0 < v} (\det V_v)^{-1}) \otimes \bigoplus_{0 < v} \Lambda (V_v, 0),
\]
\[
F_V^2 = \bigotimes_{0 < n \in Z_{1/2}} \Lambda \left( \bigoplus_{0 < v} (V_{v,n} \oplus \overline{V}_{v,n}) \oplus V_0 \right),
\]
\[
Q(W) = \bigotimes_{n=0}^{\infty} \Lambda (\bigoplus_{0 < v} W_{v,n}) \bigotimes_{n=1}^{\infty} \Lambda (\bigoplus_{0 < v} W_{v,n}).
\]

Now we can reformulate Theorem 2.4 as follows.

**Theorem 3.1.** For each \( a, h, p \in \mathbb{Z} \), \( p > 0 \), \( m \in \mathbb{Z} \), \( i = 1, 2 \), \( \tau = \tau_{e1} \) or \( \tau_{e1} \), we have the following identity in \( K_{G_y}(B) \),
\[
\text{Ind}_{\tau} (D^{Y_\alpha} \otimes (K_V \otimes K_Y^{-1})^{1/2} \otimes \mathcal{F}^{-p}(X) \otimes F_V^i \otimes Q(W),
\]
\[
m + \frac{1}{2} p^2 e(N) + \frac{1}{2} p d(N), h)
\]
(3.8)
\[
= (-1)^{pd(W)} \text{Ind}_{\tau} (D^{Y_\alpha} \otimes (K_V \otimes K_Y^{-1})^{1/2} \otimes \mathcal{F}^0(X) \otimes F_V^i
\]
\[
\otimes Q(W) \otimes L^{-p}, m + ph + p^2 e, h).
\]

**Proof.** The rest of this section is devoted to a proof of Theorem 3.1. \( \Box \)

**3.2. Proof of Theorem 3.1.** Inspired by [T, §7], as in [LiuMaZ, §3], for \( p \in \mathbb{N}^* \), we define the shift operators,
\[
r_* : N_{v,n} \rightarrow N_{v,n+p}, \quad r_* : \overline{N}_{v,n} \rightarrow \overline{N}_{v,n+p},
\]
(3.9)
\[
r_* : W_{v,n} \rightarrow W_{v,n+p}, \quad r_* : \overline{W}_{v,n} \rightarrow \overline{W}_{v,n+p},
\]
\[
r_* : V_{v,n} \rightarrow V_{v,n+p}, \quad r_* : \overline{V}_{v,n} \rightarrow \overline{V}_{v,n+p}.
\]

Recall that \( L(N), L(W), L(V) \) are the complex line bundles over \( F \) defined by (2.9). Recall also that \( L = L(N)^{-1} \otimes L(W) \otimes L(V) \) is a trivial complex line bundle over \( F \), and \( g \in S^1 \) acts on it by multiplication by \( g^{2e} \).

**Proposition 3.1.** For \( p \in \mathbb{Z} \), \( p > 0 \), \( i = 1, 2 \), there are natural isomorphisms of vector bundles over \( F \),
\[
r_* (\mathcal{F}^{-p}(X)) \simeq F^0(X) \otimes L(N)^p,
\]
(3.10)
\[
r_* (F_V^i) \simeq F_V^i \otimes L(V)^{-p}.
\]

For any \( p \in \mathbb{Z} \), \( p > 0 \), there is a natural \( G_y \times S^1 \)-equivariant isomorphism of vector bundles over \( F \),
\[
r_* (Q(W)) \simeq Q(W) \otimes L(W)^{-p}.
\]
(3.11)
Proof. The equation (3.10) was proved in [LiuMaZ, Prop. 3.1]. To prove (3.11), we only need to consider the shift operator on the following elements,

(3.12) \[ Q_W = \bigotimes_{n=0}^{\infty} \Lambda(\oplus_{v \neq 0} W_{v,n}) \otimes_{n=1}^{\infty} \Lambda(\oplus_{v \neq 0} W_{v,n}). \]

We compute easily that

(3.13) \[ r_* Q_W = \bigotimes_{n=0}^{\infty} \Lambda(\oplus_{v \neq 0} W_{v,n-pv}) \otimes_{n=1}^{\infty} \Lambda(\oplus_{v \neq 0} W_{v,n+pv}). \]

Let \( h^W \) be a Hermitian metric on \( W \). Let \( h^W_v \) be the metric on \( W_v \) induced by \( h^W \). As in [LiuMaZ, §3], the hermitian metric \( h^W_v \) on \( W_v \) induces a natural isomorphism of complex vector bundles over \( F \),

(3.14) \[ \Lambda^iW_v \simeq \Lambda^\dim W_v-iW_v \otimes \det W_v. \]

- If \( v > 0 \), for \( n \in \mathbb{N} \), \( 0 \leq n < pv \), \( 0 \leq i \leq \dim W_v \), (3.14) induces a natural \( G_y \times S^1 \)-equivariant isomorphism of complex vector bundles

(3.15) \[ \Lambda^iW_{v,n-pv} \simeq \Lambda^\dim W_{v,-n+pv} \otimes \det W_v. \]

- If \( v < 0 \), for \( n \in \mathbb{N} \), \( 0 < n \leq -pv \), \( 0 \leq i \leq \dim W_v \), (3.14) induces a natural \( G_y \times S^1 \)-equivariant isomorphism of complex vector bundles

(3.16) \[ \Lambda^iW_{v,n+pv} \simeq \Lambda^\dim W_{v,-n-pv} \otimes (\det W_v)^{-1}. \]

From (2.9), (3.15) and (3.16), we have

(3.17) \[ \bigotimes_{n \in \mathbb{N}, v > 0, 0 \leq n < pv} \Lambda^iW_{v,n-pv} \bigotimes_{n \in \mathbb{N}, v < 0, 0 < n \leq -pv} \Lambda^iW_{v,n+pv} \bigotimes_{n \in \mathbb{N}, v > 0, 0 \leq n < pv} \Lambda^\dim W_{v-n-pv} \rightleftharpoons L(W)^{-p}. \]

From (3.13), (3.17), we get (3.11).

The proof of Proposition 3.1 is complete. □

Proposition 3.2. For \( p \in \mathbb{Z} \), \( p > 0 \), \( i = 1, 2 \), the \( G_y \)-equivariant bundle isomorphism induced by (3.10) and (3.11),

(3.18) \[ r_* : S(TY, K_X \otimes_{0<v} (\det N_v)^{-1}) \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes F^{-p}(X) \otimes F_Y \otimes Q(W) \rightarrow S(TY, K_X \otimes_{0<v} (\det N_v)^{-1}) \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes F^0(X) \otimes F_Y \otimes Q(W) \otimes L^{-p}, \]

verifies the following identities

(3.19) \[ r_*^{-1} J_H \cdot r_* = J_H, \]

(3.20) \[ r_*^{-1} P \cdot r_* = P + pJ_H + p^2e - \frac{1}{2}p^2e(N) - \frac{p}{2}d'(N). \]

For the \( \mathbb{Z}_2 \)-gradings, we have

(3.21) \[ r_*^{-1} r_1 r_* = r_1, \quad r_*^{-1} r_2 r_* = r_2, \]

(3.22) \[ r_*^{-1} r_1 r_* = (-1)^{pd(W)} r_1. \]
Proof. We divide the argument into several steps.
1) The first equation of (3.19) is obvious.
2) a) From [LiuMaZ, (3.23)] and (2.8), for \( i = 1, 2 \), on \( F^i \), we have

\[
(3.21) \quad r^{-1}_* P_* = P + pJ_H + \frac{1}{2} p^2 e(V).
\]

b) Note that on \( \otimes_{0 < v, 0 \leq n \leq pv} \det N_v, J_H \) acts as \( pe(N) + d'(N) \). On \( S(TY, K_X \otimes \det((\otimes_{0 < v} N_v)^{-1}) \otimes (K_W \otimes K_X^{-1})^{1/2}) J_H \) acts as \( -\frac{1}{2} d'(N) + \frac{1}{2} d'(W) \). From (2.8), (3.6), on \( S(TY, K_X \otimes \det((\otimes_{0 < v} N_v)^{-1}) \otimes (K_W \otimes K_X^{-1})^{1/2}) \mathcal{F} = P \),

\[
(3.22) \quad r^{-1}_* P_* = P + pJ_H - p^2 e(N) - \frac{1}{2} p(d'(N) + d'(W)).
\]

c) From (2.8), (3.17), on \( \otimes_{n \in \mathbb{N}, v > 0} \Lambda^i_n \overline{W}_{v,n} \otimes_{n \in \mathbb{N}, v < 0} \Lambda^i_n W_{v,n} \), one has

\[
(3.23) \quad r^{-1}_* P_* = \sum_{n \in \mathbb{N}, v > 0, 0 < n < pv} (\dim W_v - i_n)(-n + pv) + \sum_{n \in \mathbb{N}, v < 0, 0 < n < pv}(\dim W_v - i'_n)(-n - pv)
\]

\[
= P + pJ_H + \sum_{n \in \mathbb{N}, v > 0, 0 < n < pv} (\dim W_v)(-n + pv) + \sum_{n \in \mathbb{N}, v < 0, 0 < n < pv}(\dim W_v)(-n - pv)
\]

\[
= P + pJ_H + \frac{1}{2} p^2 e(W) + \frac{1}{2} p d'(W).
\]

From (2.11), (3.21), (3.22) and (3.23), we get the second equality of (3.19).
3) The first two identities of (3.20) were proved in [LiuMaZ, Proposition 3.2].
For the \( \mathbb{Z}_2 \)-grading \( \tau_1 \), it changes only on \( \otimes_{n \in \mathbb{N}, v > 0} \Lambda^i_n \overline{W}_{v,n} \otimes_{n \in \mathbb{N}, v < 0} \Lambda^i_n W_{v,n} \).

From (2.8), (3.17), we get the last equality of (3.20).

The proof of Proposition 3.2 is complete. \( \square \)

Proof of Theorem 3.1. From (2.11), (3.4) and Propositions 3.2, we easily obtain Theorem 3.1. \( \square \)

4. Proof of Theorem 2.5. In this section, we prove Theorem 2.5. As in [LiuMaZ, §4], we will construct a family twisted Dirac operator on \( M(n_j) \), the fixed point set of the induced \( \mathbb{Z}_n \) action on \( M \). By applying our \( K \)-theory version of the equivariant family index theorem to this operator, we prove Theorem 2.5.

This section is organized as follows: In Section 4.1, we construct a family Dirac operator on \( M(n_j) \). In Section 4.2, by introducing a shift operator, we will relate both sides of equation (2.25) to the index bundle of the family Dirac operator on \( M(n_j) \). In Section 4.3, we prove Theorem 2.5.

In this section, we make the same assumptions and use the same notations as in Sections 2, 3.

4.1. The Spin\(^c\) Dirac operator on \( M(n_j) \). Let \( \pi : M \to B \) be a fibration of compact manifolds with fiber \( X \) and \( \dim_X X = 2l \). We assume that \( S^1 \) acts fiberwise on \( M \), and \( TX \) has an \( S^1 \)-invariant Spin\(^c\) structure. Let \( F = \{ F_0 \} \) be the fixed point set of the \( S^1 \)-action on \( M \). Then \( \pi : F \to B \) is a fibration with compact fiber \( Y \). For \( n \in \mathbb{N}, n > 0 \), let \( Z_n \subset S^1 \) denote the cyclic subgroup of order \( n \).

Let \( V \) be a real even dimensional vector bundle over \( M \) with an \( S^1 \)-invariant spin structure. Let \( W \) be an \( S^1 \)-equivariant complex vector bundle over \( M \).
For $n_j \in \mathbb{N}$, $n_j > 0$, let $M(n_j)$ be the fixed point set of the induced $\mathbb{Z}_{n_j}$-action on $M$. Then $\pi : M(n_j) \to B$ is a fibration with compact fiber $X(n_j)$. Let $N(n_j) \to M(n_j)$ be the normal bundle to $M(n_j)$ in $M$. As in [LiuMaZ, §4.1], we see that $N(n_j)$ and $V$ can be decomposed, as real vector bundles over $M(n_j)$, to

$$N(n_j) \simeq \bigoplus_{0 < v < n_j/2} N(n_j)_v \oplus N(n_j)_R^{\mathbb{R}},$$

$V|_{M(n_j)} \simeq V(n_j)_0^R \bigoplus_{0 < v < n_j/2} V(n_j)_v \oplus V(n_j)_R^{\mathbb{R}}$

respectively. In (4.1), the last term is understood to be zero when $n_j$ is odd. We also denote by $V(n_j)_0$, $V(n_j)_0^R$, $N(n_j)_0^{\mathbb{R}}$ the corresponding complexification of the real vector bundles $V(n_j)^R$, $V(n_j)^R_{\mathbb{R}}$, and $N(n_j)^R_{\mathbb{R}}$ on $M(n_j)$. Then $N(n_j)_v$, $V(n_j)_v$'s are complex vector bundles over $M(n_j)$ with $g \in \mathbb{Z}_{n_j}$ acting by $g^v$ on it.

Similarly, we also have the following $\mathbb{Z}_{n_j}$-equivariant decomposition of $W$ on $M(n_j)$,

$$W = \oplus_{0 \leq v < n_j} W(n_j)_v.$$

Here $W(n_j)_v$ is a complex vector bundle over $M(n_j)$ with $g \in \mathbb{Z}_{n_j}$ acting by $g^v$ on it.

It is essential for us to know that the vector bundles $TX(n_j)$ and $V(n_j)^R$ are orientable. For this we have the following lemma which generalizes [BT, Lemmas 9.4, 10.1] (See also [O]).

**Lemma 4.1.** Let $R$ be a real, even dimensional orientable vector bundle over a manifold $M$. Let $G$ be a compact Lie group. We assume that $G$ acts on $M$, and lifts to $R$. We assume that $R$ has a $G$-invariant Spin$^c$ structure. For $g \in G$, let $M^g$ be the fixed point set of $g$ on $M$. Let $R_0$ be the subbundle of $R$ over $M^g$ on which $g$ acts trivially. Then $R_0$ is even dimensional and orientable.

**Proof.** Let $h^R$ be the metric on $R$ which is induced from the Spin$^c$ structure on $R$. As $g$ preserves the Spin$^c$ structure of $R$, $g$ is an isometry on $R$ and preserves the orientation of $R$. On $M^g$, we have the following decomposition of real vector bundles,

$$R = R_0 \oplus R_1.$$

Since the only possible real eigenvalue of $g$ on $R_1$ is $-1$, and $\det(g|_{R_1}) = 1$ on $M^g$, we know that $\dim_R R_1 = \dim_R R - \dim_R R_0$ must be even. So $\dim_R R_0$ is even.

Let $K_R$ be the $G$-equivariant complex line bundle over $M$ which is induced by the Spin$^c$ structure of $R$. Then $E = R \oplus K_R$ has an $G$-invariant spin structure. On $M^g$, we have the decomposition of vector bundles $E = E_1 \oplus E_0$, here $E_0$ is the subbundle of $E$ on which $g$ acts trivially. Now the action of $g$ on the fiber of the spinor bundle $S(E)$ at $x \in M^g$ gives an element $\tilde{g} \in \text{Spin}(E_x) \subset C(E_x)$, here $C(E_x)$ is the Clifford algebra of $E_x$. Let $\rho : \text{Spin}(E_x) \to SO(E_x)$ be the standard representation of $\text{Spin}(E_x)$, then $\rho(\tilde{g}) = g$. So $\tilde{g}c(a) = c(ga)\tilde{g}$ for $a \in E_x$. Here we denote by $c(\cdot)$ the Clifford action. This means that $\tilde{g}$ commutes $c(a)$ for $a \in E_{0x}$, so $\tilde{g} \in \text{Spin}(E_{1x})$.

Let $e_1, \ldots, e_{2k}$ be an orthonormal basis of $E_{1x}$, then $e_{i_1} \cdots e_{i_j}$ $(1 \leq i_1 < \cdots < i_j \leq 2k)$ is an orthonormal basis of the vector space $C(E_{1x})$. We define $\sigma : C(E_{1x}) \to \det(E_{1x})$ by

$$\sigma(e_{i_1} \cdots e_{i_j}) = \begin{cases} e_1 \wedge \cdots \wedge e_{2k} & \text{if } j = 2k = \dim_R E_1, \\ 0 & \text{otherwise.} \end{cases}$$
By [BGV, Lemma 3.22],
\[
(4.3) \quad |\sigma(\overline{g})| = \det^{1/2}(1 - g_{|E_1})/2.
\]
So \(\sigma(\overline{g})\) is a nonvanishing section of \(\det(E_1)\), \(\det(E_1)\) is a trivial line bundle on \(M^g\). But \(E_1\) is equal \(R_1\) or \(R_1 \oplus K_R\), this means \(R_1\) is orientable. So \(R_0\) is orientable.

This completes the proof of Lemma 4.1. \(\square\)

By Lemma 4.1, \(TX(n_j)\) and \(V(n_j)_0^R\) are even dimensional and orientable over \(M(n_j)\). Thus \(N(n_j)\) is orientable over \(M(n_j)\). By (4.1), \(N(n_j)_{R_1}^R\) and \(V(n_j)_{R_1}^R\) are also even dimensional and orientable over \(M(n_j)\). In the following, we fix the orientations of \(N(n_j)_{R_1}^R\) and \(V(n_j)_{R_1}^R\) over \(M(n_j)\). We also fix the orientations of \(TX(n_j)\) and \(V(n_j)_0^R\) which are induced by (4.1) and the orientations on \(TX, V, N(n_j)_{R_1}^R\) and \(V(n_j)_0^R\).

Let
\[
(4.4) \quad r(n_j) = \frac{1}{2}(1 + (-1)^{n_j}).
\]

**Lemma 4.2.** Assume that (2.4) holds. Let
\[
L(n_j) = \bigotimes_{0 < v < n_j/2} \left( \det(N(n_j)_v) \otimes \det(\overline{V(n_j)_v}) \otimes \det(W(n_j)_v) \otimes \det(W(n_j)_{n_j - v}) \right)^{r(n_j)/n_j}
\]
be the complex line bundle over \(M(n_j)\). Then we have
i) \(L(n_j)\) has an \(n_j\)th root over \(M(n_j)\).
ii) Let
\[
L_1 = K_X \bigotimes_{0 < v < n_j/2} \left( \det(N(n_j)_v) \otimes \det(\overline{V(n_j)_v}) \otimes \det(W(n_j)_v) \otimes \det(W(n_j)_{n_j - v}) \right)^{r(n_j)/n_j},
\]
\[
L_2 = K_X \bigotimes_{0 < v < n_j/2} \left( \det(N(n_j)_v) \otimes \det(\overline{V(n_j)_v}) \otimes \det(W(n_j)_v) \otimes \det(W(n_j)_{n_j - v}) \right)^{r(n_j)/n_j}.
\]
Let \(U_1 = TX(n_j) \oplus V(n_j)_0^R\) and \(U_2 = TX(n_j) \oplus V(n_j)_{R_1}^R\). Then \(U_1\) (resp. \(U_2\)) has a \(\text{Spin}^c\) structure defined by \(L_1\) (resp. \(L_2\)).

**Proof.** Both statements follow from the proof of [BT, Lemmas 11.3 and 11.4]. \(\square\)

Lemma 4.2 allows us, as we are going to see, to apply the constructions and results in Section 1.1 to the fibration \(M(n_j) \to B\), which is the main concern of this section.

For \(p_j \in \mathbb{N}, p_j < n_j, (p_j, n_j) = 1, \beta_j = \frac{p_j}{n_j}\), let us write
\[
F_1^+ (\beta_j) = \bigotimes_{0 < n \in \mathbb{Z}} \Sym(TX(n_j)_n) \bigotimes_{0 < v < n_j/2} \Sym \left( \bigoplus_{0 < n \in \mathbb{Z} + p_jv/n_j} N(n_j)_v, n \right) \bigotimes_{0 < n \in \mathbb{Z} - p_jv/n_j} N(n_j)_v, n + \frac{1}{2} \Sym(N(n_j)_{R_1}^R), n,
\]
\[
F_1^- (\beta_j) = \Lambda \left( \bigotimes_{0 < n \in \mathbb{Z}} V(n_j)_n \oplus_{0 < v < n_j/2} \left( \bigoplus_{0 < n \in \mathbb{Z} + p_jv/n_j} V(n_j)_v, n \right) \bigotimes_{0 < n \in \mathbb{Z} - p_jv/n_j} \overline{V(n_j)_v, n} \bigotimes_{0 < n \in \mathbb{Z} + \frac{1}{2}} V(n_j)_{R_1}^R, n \right),
\]
\[
F_2^+ (\beta_j) = \Lambda \left( \bigotimes_{0 < n \in \mathbb{Z}} V(n_j)_n \oplus_{0 < v < n_j/2} \left( \bigoplus_{0 < n \in \mathbb{Z} + p_jv/n_j + \frac{1}{2}} V(n_j)_v, n \right) \bigotimes_{0 < n \in \mathbb{Z} - p_jv/n_j + \frac{1}{2}} \overline{V(n_j)_v, n} \bigotimes_{0 < n \in \mathbb{Z} + \frac{1}{2}} V(n_j)_n \right),
\]
\[
QW (\beta_j) = \Lambda \left( \bigoplus_{0 < n_j} \left( \bigoplus_{0 < n \in \mathbb{Z} + p_jv/n_j} W(n_j)_v, n \bigoplus_{0 < n \in \mathbb{Z} - p_jv/n_j} \overline{W(n_j)_v, n} \right) \right).
We denote by $D^{X(n_j)}$ the $S^1$-equivariant Spin$^c$-Dirac operator on $S(U_1, L_1)$ or $S(U_2, L_2)$ along the fiber $X(n_j)$ defined as in Section 1.1. We denote by $D^{X(n_j)} \otimes F'(\beta_j) \otimes F'_y(\beta_j) \otimes Q'(\beta_j)$ (for $i = 1, 2$) the corresponding twisted Spin$^c$ Dirac operator on $S(U_i, L_i) \otimes F'(\beta_j) \otimes F'_y(\beta_j) \otimes Q'(\beta_j)$ along the fiber $X(n_j).$

**Remark 4.1.** In fact, to define an $S^1$ (resp. $G_y$)-action on $L(n_j)^{r(n_j)/n_j}$, one must replace the $S^1$-action by its $n_j$-fold action (resp. the $G_y$-action by $G_y/n_j$-action). Here by abusing notation, we still say an $S^1$ (resp. $G_y$)-action without causing any confusion.

In the rest of this subsection, we will reinterpret all of the above objects when we restrict ourselves to $F$, the fixed point set of the $S^1$ action. We will use the notation of Sections 1.1 and 2.

Let $N_{F/M(n_j)}$ be the normal bundle to $F$ in $M(n_j)$. Then by (2.5),

$$N_{F/M(n_j)} = \bigoplus_{0 < v' \equiv v \mod(n_j)} N_v, \quad TX(n_j) \otimes \mathbb{R} C = TY \otimes \mathbb{R} C \oplus \bigoplus_{0 < v, v' \equiv n_j \in \mathbb{Z}} (N_v \oplus \overline{N}_v).$$

By (2.5), (2.6) and (4.1), the restriction to $F$ of $N(n_j)_v, V(n_j)_v (1 \leq v \leq n_j/2)$ is given by

$$N(n_j)_v = \bigoplus_{0 < v' : v' = v \mod(n_j)} N_{v'}, \quad 0 < v' : v' = -v \mod(n_j)
V(n_j)_v = \bigoplus_{0 < v' : v' = v \mod(n_j)} V_{v'}, \quad 0 < v' : v' = -v \mod(n_j)$$

And

$$V(n_j)_0 = V_0^R \otimes \mathbb{R} C \bigoplus_{0 < v, v = 0 \mod(n_j)} (V_v \oplus \overline{V}_v).$$

By (4.8)-(4.10), we have the following identifications of real vector bundles over $F$,

$$N(n_j)^R = \bigoplus_{0 < v, v = \frac{n_j}{2} \mod(n_j)} N_v, \quad TX(n_j) = TY \bigoplus_{0 < v, v = 0 \mod(n_j)} N_v,
V(n_j)_0^R = V_0^R \bigoplus_{0 < v, v = 0 \mod(n_j)} V_v,
V(n_j)^R = \bigoplus_{0 < v, v = \frac{n_j}{2} \mod(n_j)} V_v.$$

By (2.6) and (4.2), the restriction to $F$ of $W(n_j)_v (0 \leq v < n_j)$ is given by

$$W(n_j)_v = \oplus_{v': v = v \mod(n_j)} W_{v'}.$$

We denote by $V_0 = V_0^R \otimes \mathbb{R} C$ the complexification of $V_0^R$ over $F$. As $(p_j, n_j) = 1$, we know that for $v \in \mathbb{Z}, p_j v/n_j \in \mathbb{Z}$ iff $v/n_j \in \mathbb{Z}$. Also, $p_j v/n_j \in \mathbb{Z} + \frac{1}{2}$ iff $v/n_j \in \mathbb{Z} + \frac{1}{2}$. From (4.8)-(4.12), we then get

$$\mathcal{F}(\beta_j) = \bigotimes_{0 < n \in \mathbb{Z}} \text{Sym}(TY_n) \otimes \bigoplus_{0 < v, v = 0, \frac{n_j}{2}} \text{mod}(n_j) \bigotimes_{0 < n \in \mathbb{Z} + \frac{p_j v}{n_j}} \text{Sym}(N_{v,n} \oplus \overline{N}_{v,n})$$

$$\bigotimes_{0 < v', n_j/2} \text{Sym} \left( \bigoplus_{v = v'} \text{mod}(n_j) \left( \bigoplus_{0 < n \in \mathbb{Z} + \frac{p_j v}{n_j}} N_{v,n} \oplus \bigoplus_{0 < n \in \mathbb{Z} - \frac{p_j v}{n_j}} \overline{N}_{v,n} \right) \right).$$
$F^j_\gamma(\beta_j) = \Lambda \left[ \bigoplus_{0 < n \in \mathbb{Z}} V_{0,n} \bigoplus_{0 < v,v=0} \mathbb{Z}^j \mod(n_j) \right] \left( \bigoplus_{0 < n \in \mathbb{Z}^j} V_{v,v} \bigoplus_{0 < n \in \mathbb{Z}^j} \mathbb{Z}^j \mod(n_j) \right)$

$F^j_\gamma(\beta_j) = \Lambda \left[ \bigoplus_{0 < n \in \mathbb{Z}^j} V_{0,n} \bigoplus_{0 < v,v=0} \mathbb{Z}^j \mod(n_j) \right] \left( \bigoplus_{0 < n \in \mathbb{Z}^j} V_{v,v} \bigoplus_{0 < n \in \mathbb{Z}^j} \mathbb{Z}^j \mod(n_j) \right)$

$QW(\beta_j) = \Lambda \left( \bigoplus_v \left( \bigoplus_{0 < n \in \mathbb{Z}^j + p_j/v \mod(n_j)} W_{v,v} \bigoplus_{0 < n \in \mathbb{Z}^j - p_j/v \mod(n_j)} \mathbb{Z}^j \mod(n_j) \right) \right)$

Now, we want to compare the spinor bundles over $F$. From (4.5), (4.6), (4.9) and (4.12), we get that over $F$ we have the identities

$L(n_j) = \bigoplus_{0 < v' < n_j/2} \left( \bigoplus_{v' = v' \mod(n_j)} \det N_v \otimes \det V_v \otimes \det W_v \right)^{r(n_j)/n_j}$

$L_1 = K \otimes L(n_j)^{r(n_j)/n_j} \bigoplus_{0 < v' < n_j/2} \left( \bigoplus_{v' = v' \mod(n_j)} \det N_v \otimes \det V_v \otimes \det W_v \right)^{-1}$

$L_2 = K \otimes L(n_j)^{r(n_j)/n_j} \bigoplus_{0 < v' < n_j/2} \left( \bigoplus_{v' = v' \mod(n_j)} \det N_v \otimes \det V_v \otimes \det W_v \right)^{-1}$

Recall that the Spin$^c$ vector bundles $U_1, U_2$ have been defined in Lemma 4.2. Denote by

$S(U_1, L_1)' = S(TY \oplus V_0^R, L_1, \bigotimes_{v=0 \mod(n_j)} \det N_v \otimes \det V_v)^{-1} \otimes \Lambda V_v$

$S(U_2, L_2)' = S(TY, L_2, \bigotimes_{v=0 \mod(n_j)} \det N_v)^{-1} \otimes \Lambda V_v$

Then from (1.10) and (4.16), for $i = 1, 2$, we have the following isomorphism of Clifford modules over $F$,

$S(U_i, L_i) \simeq S(U_i, L_i)' \otimes \Lambda \left( \bigoplus_{0 < v,v=0} \mathbb{Z}^j \mod(n_j) \right) V_v$.

We define the $\mathbb{Z}_2$ gradings on $S(U_i, L_i)'$ ($i = 1, 2$) induced by the $\mathbb{Z}_2$-gradings on $S(U_i, L_i)$ ($i = 1, 2$) and on $\Lambda \left( \bigoplus_{0 < v,v=0} \mathbb{Z}^j \mod(n_j) \right) V_v$ such that the isomorphism (4.17) preserves the $\mathbb{Z}_2$-grading.

We introduce formally the following complex line bundles over $F$,

$L_1 = \left[ L_1^{-1} \bigotimes_{v=0 \mod(n_j)} \det N_v \otimes \det V_v \otimes \det V_v \right]^{1/2}$

$L_2 = \left[ L_2^{-1} \bigotimes_{v=0 \mod(n_j)} \det V_v \otimes \det V_v \right]^{1/2}$. 
From (1.10), Lemma 4.2 and the assumption that $V$ is spin, one verifies easily that $c_1(L_i^2) = 0 \mod(2)$ for $i = 1, 2$. Thus $L_1, L_2$ are well defined complex line bundles over $F$. For the later use, we also write down the following expressions of $L_i$ ($i = 1, 2$) which can be deduced from (4.14):

\[
L_1 = \left[ (L(n_j)/n_j) \otimes \delta_{v,v' \equiv n_j^2 \mod(n_j)} \left( \det N_v \otimes \det \overline{W}_v \right) \right]^{1/2}
\]

(4.18)

\[
L_2 = \left[ (L(n_j)/n_j) \otimes \delta_{v,v' \equiv n_j^2 \mod(n_j)} \left( \det N_v \otimes \det W_v \right) \right]^{1/2}
\]

\[
\otimes_{0 < v \leq n_j^2 \mod(n_j)} \left( \det N_v \right)^{-1}.
\]

From (4.14), (4.16), and the definition of $L_i$ ($i = 1, 2$), we get the following identifications of Clifford modules over $F$,

\[
S(U_1, L_1') \otimes L_1' = S(TY, KX \otimes \delta_{v,v' \equiv 0 \mod(n_j)} \left( \det N_v \right)^{-1}) \otimes \Lambda(\oplus_{v,v' \equiv 0 \mod(n_j)} \left( \det N_v \right)^{-1})
\]

\[
S(U_2, L_2') \otimes L_2' = S(TY, KX \otimes \delta_{v,v' \equiv 0 \mod(n_j)} \left( \det N_v \right)^{-1}) \otimes \Lambda(\oplus_{v,v' \equiv n_j^2 \mod(n_j)} \left( \det N_v \right)^{-1})
\]

(4.19)

Let

\[
\Delta(n_j, N) = \sum_{n_j^2 < v < n_j^2} \sum_{0 < v' \equiv \text{mod}(n_j)} \dim N_v + o(N(n_j)^{R_j^2})
\]

(4.20)

\[
\Delta(n_j, V) = \sum_{n_j^2 < v < n_j^2} \sum_{0 < v' \equiv \text{mod}(n_j)} \dim V_v + o(V(n_j)^{R_j^2})
\]

with $o(N(n_j)^{R_j^2}) = 0$ or $1$ (resp. $o(V(n_j)^{R_j^2}) = 0$ or $1$), depending on whether the given orientation on $N(n_j)^{R_j^2}$ (resp. $V(n_j)^{R_j^2}$) agrees or disagrees with the complex orientation of $\oplus_{v,v', \equiv \text{mod}(n_j)} N_v$ (resp. $\oplus_{v,v', \equiv \text{mod}(n_j)} V_v$).

By [LiuMaZ, §4.1], (4.12) and (4.17), for the $Z_2$-gradings induced by $\tau_s$, the difference of the $Z_2$-gradings of (4.19) is $(-1)^{\Delta(n_j, N)}$; for the $Z_2$-gradings induced by $\tau_e$, the difference of the $Z_2$-gradings of the first (resp. second) equation of (4.19) is $(-1)^{\Delta(n_j, N) + \Delta(n_j, V)}$ (resp. $(-1)^{\Delta(n_j, N) + o(V(n_j)^{R_j^2})}$).

4.2. The Shift operators. Let $p \in \mathbb{N}^*$ be fixed. For any $1 \leq j \leq J_0$, inspired by [T, §9], as in [LiuMaZ, §4], we define the following shift operators $r_{j*}$:

\[
r_{j*}: N_{v,n} \to N_{v,n+(p-1)v+p_j v/n_j}, \quad r_{j*}: \overline{N}_{v,n} \to \overline{N}_{v,n-(p-1)v-p_j v/n_j},
\]

\[
r_{j*}: W_{v,n} \to W_{v,n+(p-1)v+p_j v/n_j}, \quad r_{j*}: \overline{W}_{v,n} \to \overline{W}_{v,n-(p-1)v-p_j v/n_j},
\]

\[
r_{j*}: V_{v,n} \to V_{v,n+(p-1)v+p_j v/n_j}, \quad r_{j*}: \overline{V}_{v,n} \to \overline{V}_{v,n-(p-1)v-p_j v/n_j}.
\]

If $E$ is a combination of the above bundles, we denote by $r_{j*}E$ the bundle on which the action of $P$ is changed in the above way.

Recall that the vector bundles $F_{V_i}^i$ ($i = 1, 2$) have been defined in (3.7). From (2.21), we see that

\[
F_{p,j}(X) = F_p(X) \otimes F_{p-1}(X) \otimes \left( \Sym(N_{v,-n}) \otimes \det N_v \right) \otimes \left( \Sym(N_{v,n}) \right)
\]

(4.22)
PROPOSITION 4.1. There are natural isomorphisms of vector bundles over $F$,

$$
\begin{align*}
\mathcal{F}_{p,j-1}(X) & \simeq \mathcal{F}(\beta_j) \otimes_{0<v,v=0 \mod(n_j)} \text{Sym}(\mathcal{N}_v,0) \\
& \otimes_{0<v} (\det N_v)^{(p-1)v+1} \otimes_{0<v,v=0 \mod(n_j)} (\det N_v)^{-1}, \\
\mathcal{F}_{p,j}(X) & \simeq \mathcal{F}(\beta_j) \otimes_{0<v,v=0 \mod(n_j)} \text{Sym}(\mathcal{N}_v,0) \otimes_{0<v} (\det N_v)^{(p-1)v+1}, \\
\mathcal{F}_V & \simeq S(V_0^R, \otimes_{0<v} (\det V_v)^{-1}) \otimes \mathcal{F}(\beta_j) \otimes_{0<v,v=0 \mod(n_j)} \Lambda(V_v,0) \\
& \otimes_{0<v} (\det V_v)^{(p-1)v}, \\
\mathcal{F}_V & \simeq \mathcal{F}(\beta_j) \otimes_{0<v,v=0 \mod(n_j)} \Lambda(V_v,0) \otimes_{0<v} (\det V_v)^{(p-1)v+1} + (p-1)v, \\
\mathcal{Q}(W) & \simeq \mathcal{Q}(\beta_j) \otimes_{0<v} (\det W_v)^{(p-1)v+1} \otimes_{0<v,v=0 \mod(n_j)} (\det W_v)^{-1} \\
& \otimes_{v<0} (\det W_v)^{-1}.
\end{align*}
$$

(4.23)

Proof. The proof is similar to the proof of Proposition 3.1.

Note that, by (2.19), for $v \in J = \{v \in \mathbb{N} | \text{There exists } \alpha \text{ such that } N_v \neq 0 \text{ on } F_\alpha \}$, there are no integer in $\left[\frac{p_j-1}{n_j-1}\right]_{\text{n}}^{\frac{p_j}{n_j}}$. So for $v \in J$, the elements $(v, n) \in \bigcup_{i=1}^{n_j} I_i^p$ are $(v, (p-1)v+1), \cdots, (v, (p-1)v+\left[\frac{p_jv}{n_j}\right])$ for $i_0 = j-1$, $j$. Furthermore,

$$
\begin{align*}
\left[\frac{p_j-1}{n_j-1}\right] & = \left[\frac{p_j}{n_j}\right] - 1 \quad \text{if} \quad v = 0 \mod(n_j), \\
\left[\frac{p_j-1}{n_j-1}\right] & = \left[\frac{p_j}{n_j}\right] \quad \text{if} \quad v \neq 0 \mod(n_j).
\end{align*}
$$

(4.24)

By using (3.7), (4.21), (4.22), (4.24), we can prove the first four equalities of (4.23) as in the proof of [LiuMaZ, Proposition 4.1].

From (3.14), we have the natural $G_y \times S^1$-equivariant isomorphisms of complex vector bundles over $F$,

(4.25)

$$
\begin{align*}
\otimes_{n \in \mathbb{N}, v>0} \Lambda^{\dim W_{v,n-(p-1)v}\frac{p_jv}{n_j}} \simeq \otimes_{n \in \mathbb{N}, v>0} \Lambda^{\dim W_{v,-n+(p-1)v+\frac{p_jv}{n_j}}} & \\
\otimes_{0<v} (\det W_v)^{(p-1)v+1} & \otimes_{0<v} (\det W_v)^{-1}, \\
\otimes_{n \in \mathbb{N}, v<0} \Lambda^{\dim W_{v,n+(p-1)v+\frac{p_jv}{n_j}}} & \otimes_{n \in \mathbb{N}, v<0} \Lambda^{\dim W_{v,-n-(p-1)v-\frac{p_jv}{n_j}}}
\end{align*}
$$

(4.26)

From (3.7), (4.13), (4.25), we get the last equation of (4.23).

The proof of Proposition 4.1 is complete. □
Lemma 4.3. Let us write

\[ L(\beta_j)_1 = L'_1 \otimes_{0<v<(\det N_v)}^{p_j v} \otimes_{0<v<(\det V_v)}^{p_j v} \]
\[ \otimes_{0<v,v=0 \mod (n_j)}^{p_j v} \otimes_{0<v<(\det W_v)}^{-(p_j v)} \]
\[ \otimes_{0<v,v=0 \mod (n_j)}^{p_j v} \otimes_{0<v<(\det W_v)}^{(p_j v)^{-1}} \]
\[ L(\beta_j)_2 = L'_2 \otimes_{0<v<(\det N_v)}^{p_j v} \otimes_{0<v<(\det V_v)}^{p_j v} \]
\[ \otimes_{0<v,v=0 \mod (n_j)}^{p_j v} \otimes_{0<v<(\det W_v)}^{-(p_j v)} \]
\[ \otimes_{0<v,v=0 \mod (n_j)}^{p_j v} \otimes_{0<v<(\det W_v)}^{(p_j v)^{-1}} \]
(4.26)

Then \( L(\beta_j)_1, L(\beta_j)_2 \) can be extended naturally to \( G_y \times S^1 \)-equivariant complex line bundles which we will still denote by \( L(\beta_j)_1, L(\beta_j)_2 \) respectively over \( M(n_j) \).

Proof. Write

\[ \left[ \frac{p_j v}{n_j} \right] = \frac{p_j v}{n_j} - \frac{\omega(v)}{n_j} \]
(4.27)

Note that for \( v = \frac{n_j}{2} \mod (n_j) \), \( \frac{\omega(v)}{n_j} = \frac{1}{2} \).

We introduce the following line bundle over \( M(n_j) \),

\[ L^\omega(\beta_j) = \otimes_{0<v<\frac{n_j}{2}} \left( \det(N(n_j)v) \otimes \det(V(n_j)v) \right)^{-\omega(v)-r(n_j)v} \]
(4.28)

As in [LiuMaZ, (4.38)], Lemma 4.2 implies \( L^\omega(\beta_j)^{1/n_j} \) is well defined over \( M(n_j) \).

The contributions of \( N \) and \( V \) in \( L(\beta_j)_1, L(\beta_j)_2 \) are the same as given in [LiuMaZ, Lemma 4.2], we only need to calculate the contribution of \( W \) in \( L(\beta_j)_1, L(\beta_j)_2 \). Actually from [LiuMaZ, (4.37), (4.44)], (2.9), (4.12), (4.18), (4.26), (4.27) and (4.28), we get

\[ L(\beta_j)_1 = L^{-(p-1)-p_j/n_j} \otimes L^\omega(\beta_j)^{1/n_j} \otimes \det(W(n_j)_v) \]
\[ L(\beta_j)_2 = L^{-(p-1)-p_j/n_j} \otimes L^\omega(\beta_j)^{1/n_j} \otimes \det(W(n_j)_v) \]
(4.29)

The proof of Lemma 4.3 is complete. □

Let us write

\[ \varepsilon(W) = -\frac{1}{2} \sum_{0<v<\dim W_v} \left( \left( \frac{p_j v}{n_j} + (p-1)v \right) \left( \frac{p_j v}{n_j} + (p-1)v + 1 \right) \right) \]
\[ -\left( \frac{p_j v}{n_j} + (p-1)v \right) \left( 2 \left( \frac{p_j v}{n_j} + (p-1)v \right) + 1 \right) \]
\[ -\frac{1}{2} \sum_{v<0} \left( \left( \frac{p_j v}{n_j} - (p-1)v \right) \left( \frac{p_j v}{n_j} - (p-1)v + 1 \right) \right) \]
\[ + \left( \frac{p_j v}{n_j} + (p-1)v \right) \left( 2 \left( \frac{p_j v}{n_j} - (p-1)v \right) + 1 \right) \]
(4.30)
\[ \varepsilon_1 = \frac{1}{2} \sum_{0 < \nu} (\dim N_{\nu} - \dim V_{\nu}) \left[ \left( \frac{P_{\nu}}{n_j} \right) + (p - 1)\nu \left( \frac{P_{\nu}}{n_j} \right) + (p - 1)\nu + 1 \right] \\
- \left( \frac{P_{\nu}}{n_j} \right) + (p - 1)\nu \left( 2 \left( \frac{P_{\nu}}{n_j} \right) + (p - 1)\nu + 1 \right). \]

\[ \varepsilon_2 = \frac{1}{2} \sum_{0 < \nu} (\dim N_{\nu}) \left[ \left( \frac{P_{\nu}}{n_j} + (p - 1)\nu \right) \left( \frac{P_{\nu}}{n_j} \right) + (p - 1)\nu + 1 \right] \\
- \left( \frac{P_{\nu}}{n_j} + (p - 1)\nu \right) \left( 2 \left( \frac{P_{\nu}}{n_j} \right) + (p - 1)\nu + 1 \right) \\
- \frac{1}{2} \sum_{0 < \nu} (\dim V_{\nu}) \left[ \left( \frac{P_{\nu}}{n_j} + \frac{1}{2} \right) + (p - 1)\nu \right] \\
- 2 \left( \frac{P_{\nu}}{n_j} + (p - 1)\nu \right) \left[ \left( \frac{P_{\nu}}{n_j} + \frac{1}{2} \right) + (p - 1)\nu \right]. \]

Then \( \varepsilon(W), \varepsilon_1, \varepsilon_2 \) are locally constant functions on \( F \).

Recall that the involutions \( \tau_{\gamma}, \tau_s, \) and \( \tau_1 \) were defined in Section 3.1. Also recall that if \( E \) is an \( S^1 \)-equivariant vector bundle over \( M \), then the weight of the \( S^1 \)-action on \( \Gamma(F, E) \) is given by the action \( J_H \) (cf. \$3.1).

**Proposition 4.2.** For \( i = 1, 2 \), the \( G_y \)-equivariant isomorphisms induced by (4.19) and (4.23),

\[ r_{i1} : S(TY, K_X \otimes_{0 < \nu} (\det N_{\nu})^{-1}) \otimes (K_W \otimes K_X^{-1})^{1/2} \]
\[ \otimes \mathcal{F}_{p, j-1}(X) \otimes F_{V, j} \otimes Q(W) \rightarrow \]
\[ S(U_i, L_i) \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes \mathcal{F}(\beta_j) \otimes F_{V, j}(\beta_j) \]
\[ \otimes Q_W(\beta_j) \otimes L(\beta_j) \otimes_{0 < \nu, v = 0 \text{mod}(n_j)} \text{Sym}(N_{\nu, 0}), \]

(4.31)

have the following properties: 1) for \( i = 1, 2, \gamma = 1, 2 \),

\[ r_{\gamma}^{-1}J_H r_{\gamma} = J_H, \]
\[ r_{\gamma}^{-1}P r_{\gamma} = P + \left( \frac{P_{\nu}}{n_j} + (p - 1)\right) J_H + \varepsilon_{\gamma}, \]

where

\[ \varepsilon_{i1} = \varepsilon_i + \varepsilon(W) - e(p, \beta_{i-1}, N), \]
\[ \varepsilon_{i2} = \varepsilon_i + \varepsilon(W) - e(p, \beta_j, N). \]

2) Recall that \( o(V(n_j)_{2}) \) was defined in (4.20). Let

\[ \mu_1 = - \sum_{0 < \nu} \left( \frac{P_{\nu}}{n_j} \right) \dim V_{\nu} + \Delta(n_j, N) + \Delta(n_j, V) \mod(2), \]
\[ \mu_2 = - \sum_{0 < \nu} \left( \frac{P_{\nu}}{n_j} + \frac{1}{2} \right) \dim V_{\nu} + \Delta(n_j, N) + o(V(n_j)_{2}) \mod(2), \]
\[ \mu_3 = \Delta(n_j, N) \mod(2), \]
\[ \mu_4 = \sum_{0 < \nu} (\dim W_{\nu}) \left( \frac{P_{\nu}}{n_j} \right) + (p - 1)\nu + \dim W + \dim W(n_j) \mod(2). \]

Then for \( i = 1, 2; \gamma = 1, 2 \),

\[ r_{\gamma}^{-1}r_{\gamma}r_{\gamma} = (-1)^{\mu_1} r_{\gamma}, \]
\[ r_{\gamma}^{-1}r_{\gamma}r_{\gamma} = (-1)^{\mu_3} r_{\gamma}, \]
\[ r_{\gamma}^{-1}r_{\gamma}r_{\gamma} = (-1)^{\mu_4} r_{\gamma}. \]

**Proof.** The first equality of (4.32) is trivial. From (2.23) and (4.24), one has

\[ e(p, \beta_j, N) = e(p, \beta_{j-1}, N) + \sum_{0 < \nu, v = 0 \mod(n_j)} ((p - 1)\nu + \left( \frac{P_{\nu}}{n_j} \right) \dim N_{\nu}). \]
Denote by $\epsilon_i(V)$ ($i = 1, 2$) the contribution of $\dim V$ in $\epsilon_i$ ($i = 1, 2$) respectively. Then from [LiuMaZ, (4.52), (4.53)], on $F^*_i$, we have

$$r_{j^*}^{-1}Pr_{j^*} = P + ((p - 1) + \frac{p_j}{n_j})J_H + \epsilon_i(V).$$

From (4.25), as in (3.23), on $Q(W)$, we get

$$r_{j^*}^{-1}Pr_{j^*} = P + ((p - 1) + \frac{p_j}{n_j})J_H + \frac{1}{2}((p - 1) + \frac{p_j}{n_j})d'(W).$$

From (4.36), (4.37), (4.38), and by proceeding as in the proof of Proposition 3.2, as in [LiuMaZ, Proposition 4.2], one deduces easily the second equation of (4.32).

Finally from the discussion following (4.20), and [LiuMaZ, (4.50)], we get the first two equations of (4.35). By (4.12) and (4.25), we get the last equation of (4.35).

The proof of Proposition 4.2 is complete. $\Box$

**Lemma 4.4.** For each connected component $M'$ of $M(n_j)$, $\epsilon_1 + \epsilon(W)$, $\epsilon_2 + \epsilon(W)$ are independent on the connected component of $F$ in $M'$.

*Proof.* From (2.11), (4.10), (4.12), (4.27) and (4.30), we have

$$\epsilon_1 = \frac{1}{2} \sum_{0 \leq v' < n_j} \sum_{v \equiv v' \mod(n_j)} (\dim N_v - \dim V_v - \dim W_v)$$

$$\left[ - \frac{(p_j v)}{n_j} + (p - 1)v^2 - \frac{\omega(v')(n_j - \omega(v'))}{n_j^2} \right]$$

$$= (p - 1 + \frac{p_j}{n_j})^2 \frac{v}{16} \left( \dim_R N(n_j)^{\frac{R}{x_j}} - \dim_R V(n_j)^{\frac{R}{x_j}} - 2 \dim W(n_j)^{\frac{R}{x_j}} \right)$$

$$- \frac{1}{2} \sum_{0 \leq v' < n_j/2} \left( \dim N(n_j)v - \dim V(n_j)v - \dim W(n_j)v \right)$$

$$- \frac{\omega(v')(n_j - \omega(v'))}{n_j^2}.$$  

By (4.30), $\epsilon_2 - \epsilon_1$ was given in [LiuMaZ, (4.49)], it is independent on the connected component of $F$ in $M'$.

The proof of Lemma 4.4 is complete. $\Box$

The following Lemma was proved in [BT, Lemma 9.3] and [T, Lemma 9.6] (cf. [LiuMaZ, Lemma 4.6]).

**Lemma 4.5.** Let $M$ be a smooth manifold on which $S^1$ acts. Let $M'$ be a connected component of $M(n_j)$, the fixed point set of the subgroup $Z_{n_j}$ of $S^1$ on $M$. Let $F$ be the fixed point set of the $S^1$-action on $M$. Let $V \to M$ be a real, oriented, even dimensional vector bundle to which the $S^1$-action on $M$ lifts. Assume that $V$ is Spin over $M$. Let $p_j \in [0, n_j[\), $p_j \in \mathbb{N}$ and $(p_j, n_j) = 1$, then

$$\sum_{0 < v}(\dim V_v)[\frac{P_{jv}}{n_j} + \Delta(n_j, V, V) \mod(2),$$

$$\sum_{0 < v}(\dim V_v)[\frac{P_{jv}}{n_j} + \frac{1}{2}] + o(V(n_j)^{R_{x_j/2}}) \mod(2)$$

are independent on the connected components of $F$ in $M'$.

Recall that the number $d'(p, \beta_j, N)$ has been defined in (2.23).

**Lemma 4.6.** For each connected component $M'$ of $M(n_j)$, $d'(p, \beta_j, N) + \mu_i + \mu_4 \mod(2)$ ($i = 1, 2, 3$) are independent on the connected component of $F$ in $M'$.
Proof. By (4.34), and Lemma 4.5, to prove Lemma 4.6, we only need to prove
\[
\sum_{0<v}(\dim N_v)[\frac{P_{jv}}{n_j}] + (p-1)v + \Delta(n_j, N) + \mu_4 \mod(2)
\]
is independent on the connected components of \(F\) in \(M'\). But by [BT, Lemma 9.3], as \(\omega_2(TX \oplus W)_{S^1} = 0\), we know that, \(\mod(2)\),
\[
(4.41) \sum_{0<v}(\dim N_v)[\frac{P_{jv}}{n_j}] + \Delta(n_j, N) + \sum_{v}(\dim W_v)[\frac{P_{jv}}{n_j}]
\]
is independent on the connected components of \(F\) in \(M'\). From (2.23), (2.27), (4.41), we get Lemma 4.6.

The proof of Lemma 4.6 is complete. □

4.3. Proof of Theorem 2.5. From (2.23), (4.9), (4.12) and (4.24), we see that
\[
(4.42) \sum \dim N_v = \sum_{0<v<\frac{n_j}{2}} \dim N_v(n_j,v) + \frac{1}{2} \dim_{\mathbb{R}} N(n_j)^{\mathbb{R}}_{n_j/2} + \sum_{0<v,v=0 \mod(n_j)} \dim N_v,
\]
\[
d'(p, \beta_j, N) = d'(p, \beta_{j-1}, N) + \sum_{0<v, v=0 \mod(n_j)} \dim N_v.
\]

By Lemma 4.6, (4.42), \(d'(p, \beta_{j-1}, N) + \sum_{0<v} \dim N_v + \mu_i + \mu_4 \mod(2) (i = 1, 2, 3)\) are constant functions on each connected component of \(M(n_j)\).

From Lemma 4.3, one knows that the Dirac operator \(D^X(n_j) \otimes F(\beta_j) \otimes F^i(\beta_j) \otimes Q_W(\beta_j) \otimes L(\beta_j)i (i = 1, 2)\) is well-defined on \(M(n_j)\). Thus, by using Proposition 4.2, Lemma 4.4, (4.17) and (4.42), for \(i = 1, 2, h \in \mathbb{Z}, m \in \frac{1}{2} \mathbb{Z}, \tau = \tau_{e_1} \) or \(\tau_{s_1}\), and by applying both the first and the second equations of Theorem 1.1 to each connected component of \(M(n_j)\) separately, we get the following identity in \(K_{G_q}(B)\),
\[
(4.43) \sum_{\alpha}(-1)^d'(p, \beta_{j-1}, N) + \sum_{0<v} \dim N_v \text{Ind}_{\tau}(D^Y_v \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes F_{p,j-1}(X) \\
\otimes F^2 \otimes Q(W), m + e(p, \beta_{j-1}, N), h) = \sum_{\beta}(-1)^d'(p, \beta_{j-1}, N) + \sum_{0<v} \dim N_v + \mu \text{Ind}_{\tau}(D^X(n_j) \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes F(\beta_j) \\
\otimes F^i(\beta_j) \otimes Q_W(\beta_j) \otimes L(\beta_j)i, m + \varepsilon_i + e(W) + (\frac{P_{j,+}}{n_j} + (p-1))h, h) = \sum_{\alpha}(-1)^d'(p, \beta_{j}, N) + \sum_{0<v} \dim N_v \text{Ind}_{\tau}(D^Y_v \otimes (K_W \otimes K_X^{-1})^{1/2} \otimes F_{p,j}(X) \\
\otimes F^2 \otimes Q(W), m + e(p, \beta_{j}, N), h).
\]

Here \(\sum_{\beta}\) means the sum over all connected components of \(M(n_j)\). In (4.43), if \(\tau = \tau_{s_1}\), then \(\mu = \mu_3 + \mu_4\); if \(\tau = \tau_{e_1}\), then \(\mu = \mu_i + \mu_4\).

The proof of Theorem 2.5 is complete. □

REFERENCES


