GENERIC MODULES FOR EXTENSION ALGEBRAS*

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Abstract. Let $A$ be a tame hereditary algebra (finite-dimensional over an algebraically closed field), $R_A^m (m \geq 1)$ the extension algebra of $A$. A generic $R$-module $M$ over an arbitrary ring $R$ is by definition an indecomposable $R$-module of infinite length, such that $M$ considered as an $\text{End}(M)$-module, is of finite length (its endolength). In this paper we investigate the generic modules of $\hat{A}$ (the repetitive algebra of $A$) and $R_A^m$. It is proved that $R_A^m$ has at least $2m$ generic modules.

Introduction. The notion of generic module was introduced in [1] by Crawley-Boevey. The concept seems to be quite natural and important. The generic modules even have a dominating position in the category of modules. In [2], it was shown that whether a finite-dimensional algebra over an algebraically closed field is tame or wild is determined completely by the behaviour of the generic modules for that algebra.

In [3], Aronszajn and Fixman gave the concept of a divisible module for the Kronecker algebra and showed that for the Kronecker algebra there exists a unique indecomposable torsion-free divisible module. In [4], Ringel generalized the work of Aronszajn and Fixman and proved the same result for a tame hereditary algebra. Ringel's work, in fact, showed that for a tame hereditary algebra, there exists a unique generic module. In [6], we solved the existence and uniqueness of generic module for the tilted algebra determined by a tame hereditary algebra.

Following [1], A generic $R$-module $M$ over an arbitrary ring $R$ is by definition an indecomposable $R$-module of infinite length, such that $M$ considered as an $\text{End}(M)$-module, is of finite length (its endolength). Of course, the generic modules with endomorphism ring a division ring just, form the vertices of the (Cohn) spectrum of $R$. By [1], the endomorphism ring of a generic module always is a local ring.

Our purpose here is to investigate the generic module of the extension algebra $R_A^m$ (defined below) for a tame hereditary algebra $A$. In section 1, we investigate the $\nu$-orbits of generic modules for a repetitive algebra. we shall prove that $\text{Mod} \hat{A}$ has at least two $\nu$-orbits of generic $\hat{A}$-modules (Theorem 1.2). In section 2, we shall prove our main result on generic modules of $R_A^m$: $R_A^m$ has at least $2m$ generic modules (Theorem 2.4 and Corollary 2.5).

Throughout this paper, we denote by $k$ an algebraically closed field. An algebra means basic, connected and finite-dimensional $k$-algebra. For an algebra $A$ we denote by $\text{Mod} A$ the category of all right $A$-modules, by $\text{mod} A$ the full subcategory of $\text{Mod} A$ consisting of all finitely generated right $A$-modules and by $\text{mod} A$ the corresponding stable category. We shall use freely properties of the Auslander-Reiten sequences, irreducible maps, Auslander-Reiten translation $\tau = D\text{Tr}$ and $\tau^{-1} = \text{Tr}D$, and the Auslander-Reiten quiver $\Gamma_A$ of an algebra $B$, for which we refer to [5].

1. $\nu$-orbits of Generic Modules of Repetitive Algebras. Let $A = k\hat{A}$ be a tame hereditary algebra over an $k$. We denote by $D^b(A)$ the derived category $D^b(\text{mod} A)$ of bounded complexes over $\text{mod} A$. For the definition of derived category we refer to [7]. By $DA$ we denote the minimal injective cogenerator of $A$, where $D = \text{Hom}_k(-, k)$ is the usual dual functor. Consider the repetitive algebra $[7]:$

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with $A_i = A$ on the main diagonal, $DA_i = DA$, and zeros elsewhere. The elements are all matrices with only a finite number of nonzero entries, and multiplication is given by the canonical bimodule structure of $DA$ and the zero map $DA \otimes_A DA \to 0$. It is a Frobenius algebra and always infinite-dimensional. We know that the identity maps $A_i \to A_{i+1}$ and $DA_i \to DA_{i+1}$ induce an automorphism $\nu$ (Nakayama automorphism) of $\tilde{A}$, and also an automorphism of $\text{mod} \, A$. Since $A$ is of finite global dimension, we have $\text{mod} \, \tilde{A} \cong D^b(A)$ [7]. We may identity $\text{mod} \, \tilde{A}$ with $D^b(A)$. By [7], $\Gamma_{D^b(A)}$ is the union:

\[
\cdots \lor T[i] \lor P[i] \lor T[i+1] \lor Q[i+1] \lor P[i+1] \lor T[i+2] \lor \cdots
\]

Set $C_i = Q[i] \lor P[i+1], i \in \mathbb{Z}$. Let $\Delta_0$ and $\Delta_1$ be the complete sections of $C_0$ and $C_1$ respectively. Let $S_i = \text{add}_A \Delta_i$, that is, $S_i$ are the module classes in $\text{mod} \, \tilde{A}$ generated by $\Delta_i(i = 0, 1)$. $S_j + 2m = \nu^m S_j, m \in \mathbb{Z}, j = 0, 1$. The support of $S_i$, denote by $\text{Supp} \, S_i$ is the set:

\[
\{P(x) | P(x) \text{is indecomposable projective } \tilde{A} \text{ - module with } \text{Hom}(P(x), S_i) \neq 0\}.
\]

Let, for each $i \in \mathbb{Z}, A_i$ be the support algebra of $S_i$, i.e.,

\[
A_i = \text{End}(\bigoplus_{P(x) \in \text{Supp} \, S_i} P(x)).
\]

Then

\[
A_i = \tilde{A} / (P(x) \notin \text{Supp} \, S_i)
\]

and $A_i$ are tilted algebras of Euclidean type. Since the main purpose of the paper is to investigate the generic modules of $R^n_A$, we may assume that each $A_i$ is representation-infinite. By [6], for each $i \in \mathbb{Z}$, there exists a unique generic $A_i$-module $M_i$. Of course, all $M_i$ are also the generic $\tilde{A}$-modules.

**Lemma 1.1.** For each $i \in \mathbb{Z}$ we have

\[
\text{Supp}(\nu S_i) = \nu \text{Supp} \, S_i).
\]

**Proof.** Let $P(x) \in \text{Supp} \, (\nu S_i)$. Then there exists an $\tilde{A}$-module $S \in S_i$ such that $\text{Hom}_{\tilde{A}}(P(x), S) \neq 0$. Hence we get that $\text{Hom}_{\tilde{A}}(\nu^{-1}P(x), S) \neq 0$. This means $\nu^{-1}P(x) \in \text{Supp} \, (\nu S_i)$ and hence $P(x) \in \nu \text{Supp}(S_i)$. So, $\text{Supp}(\nu S_i) \subseteq \nu \text{Supp}(S_i)$. Similarly, we have $\nu \text{Supp}(S_i) \subseteq \text{Supp}(\nu S_i)$.

The main result of this section is the follow

**Theorem 1.2.** Let $A$ be a tame hereditary algebra, then $\text{Mod} \, \tilde{A}$ has at least two $\nu$-orbits of generic $\tilde{A}$-modules.

**Proof.** Suppose $A_i$ and $M_i (i \in \mathbb{Z})$ are as before. Then for each $i \in \mathbb{Z}$, $M_i$ is also a generic $\tilde{A}$-module.

For each $P(x) \notin \text{Supp} \, (S_{i+2})$, we show $\text{Hom}_{\tilde{A}}(P(x), \nu M_i) = 0$. If not, we have $\text{Hom}_{\tilde{A}}(\nu^{-1}P(x), M_i) \neq 0$. This gives $\nu^{-1}P(x) \in \text{Supp} \, (M_i) \subseteq \text{Supp} \, S_i$. By Lemma 1.1,
$P(x) \in \nu\text{Supp}(S_i) = \text{Supp}(\nu S_i) = \text{Supp}(S_i + 2)$. This is a contradiction. Thus $\nu M_i$ is a $A_{i+2}$-module. Since $\nu M_i$ is a generic $A_{i+2}$-module and $A_{i+2}$ has a unique generic module, we get $\nu M_i = M_{i+2}$. In general, we have $\nu^m M_i = M_{i+2m}$, $i \in \mathbb{Z}$, $m \in \mathbb{Z}$. By the structure of $\text{Mod} \hat{A}$ we know that $M_i \neq M_i (i \neq j)$ as $\hat{A}$-modules. We get two distinct $\nu$-orbits $O_1$ and $O_2$ of generic $\hat{A}$-modules:

\[ O_0 = \{ \nu^m M_0 | m \in \mathbb{Z} \}, O_1 = \{ \nu^n M_1 | n \in \mathbb{Z} \}. \]

2. Generic Modules for Extension Algebra $R^m_A$. Let $A = k\tilde{A}$ be a tame hereditary algebra over $k$. $\hat{A}$ the repetitive algebra of $A$. We consider, for each $m \geq 1$, the algebra $R^m_A$:

\[
R^m_A = \left\{ \begin{array}{c}
\lambda_1 & x_1 \\
\vdots & \vdots \\
\lambda_m & x_m \\
\end{array} \right| \lambda_i \in A, x_i \in DA \right\}.
\]

As above, the multiplication is given by the bimodule structure of $DA$ and zero map $DA \otimes_A DA \to 0$. In particular, $R^1_A$ is the trivial extension $A \ltimes DA$. The category $R^m_A$ is just the quotient category $\hat{A}/(\nu^m)$.

For a fixed $m \geq 1$, we consider the canonical Galois covering functor $F^m : \hat{A} \to \hat{A}/(\nu^m) = R^m_A$, and the associated pushdown functor $F^m : \text{Mod} \hat{A} \to \text{Mod} R^m_A$ and the pull-up functor $F^m : \text{Mod} R^m_A \to \text{Mod} \hat{A}$ [8].

From now on we fix some $m$. In this section we show that $R^m_A$ has at least $2m$ generic modules.

We first prove some lemmas

**Lemma 2.1 [8]**. For each $N \in \text{Mod} \hat{A}$ and each $r \in \mathbb{Z}$, we have

\[ F^m(\nu^r N) \cong F^m_\lambda(N). \]

**Lemma 2.2**. Let $M$ be a generic $\hat{A}$-module, $N$ an indecomposable $\hat{A}$-module. If $F^m_\lambda N \cong F^m_\lambda M$, then $N \cong \nu^r M$ for some $r \in \mathbb{Z}$.

**Proof**. Assume that $F^m_\lambda N \cong F^m_\lambda M$. Then by [8], we have

\[ \bigoplus_{r \in \mathbb{Z}} \nu^r N \cong F^m_\lambda F^m_\lambda N \cong F^m_\lambda F^m_\lambda M \cong \bigoplus_{l \in \mathbb{Z}} \nu^l M. \]

Since $M$ is a generic $\hat{A}$-module, $\nu^s M (s \in \mathbb{Z})$ are also generic $\hat{A}$-modules, it follows from [1] that every ring $\text{End}(\nu^s M)$ is local, we infer that $N = \nu^r M$ for some $r \in \mathbb{Z}$.

**Lemma 2.3**. Suppose that $M$ is a generic $\hat{A}$-module. Then $F^m_\lambda M$, as a left $\text{End}_\hat{A}(M)$-module, is of finite length.

**Proof**. Since we have an imbeding map $\text{End}_\hat{A}(M) \to \text{End}_{R^m_A}(F^m_\lambda M)$, we infer that $F^m_\lambda M$ is also a left $\text{End}_\hat{A}(M)$-module. Suppose that

\[ (*) \quad 0 = N_0 \leq N_1 \leq N_2 \leq \cdots \leq N_i \leq \cdots \]

be a composition series for left $\text{End}_\hat{A}(M)$-module $F^m_\lambda M$. Since $F^m_\lambda M = M$ as a $k$ vector space, every $N_i$ in $(*)$ is a subspace of $M$. For each $f \in \text{End}_\hat{A}(M)$, we have $fN_i \subseteq N_i$ and hence each $N_i$ is a left $\text{End}_\hat{A}(M)$-submodule of $M$. Hence we
may regard (*) as a composition series for left $\text{End}_A(M)$-module $M$. Since $M$ is a
generic $\widehat{A}$-module, it follows that (*) has only finite terms. Therefore $F^m_A M$, as a left
$\text{End}_A(M)$-module, is of finite length.

We can now prove our main result

**Theorem 2.4.** Let $M$ be a generic $\widehat{A}$-module. Then $F^m_A M$ is a generic $R^m_A$
module.

**Proof.** Since $F^m_A$ is left adjoint to $F^m$, it follows from [8] that

$$
\text{End}_{R^m_A}(F^m_A M) = \text{Hom}_{R^m_A}(F^m_A M, F^m_A M) \\
\cong \text{Hom}_{\widehat{A}}(M, F^m F^m_A M) \\
\cong \text{Hom}_{\widehat{A}}(M, \bigoplus_{i \in \mathbb{Z}} \nu^m M).
$$

If $m = 1$, then, for $s \neq 0, 1$, we have $\text{Hom}_{\widehat{A}}(M, \nu^s M) = 0$, and hence

$$
\text{End}_{R^1_A}(F^1_A M) \cong \text{Hom}_{\widehat{A}}(M, M \oplus \nu M) = \text{End}_{\widehat{A}}(M) \oplus \text{Hom}_{\widehat{A}}(M, \nu M).
$$

Since $\text{Hom}_{\widehat{A}}(M, \nu M)$ is an $\text{End}_{\widehat{A}}(M)$-bimodule: $g \circ f = gf$ is the ordinary composition
and $(f \circ g)(x) = \nu f(g(x))$ for $f \in \text{End}_{\widehat{A}}(M), g \in \text{Hom}_{\widehat{A}}(M, \nu M)$ and $x \in M$. It follows
from [9] that we have the following ring isomorphism

$$
\text{End}_{R^1_A}(F^1_A M) \cong \text{End}_{\widehat{A}}(M) \ltimes \text{Hom}_{\widehat{A}}(M, \nu M).
$$

From the definition of trivial extension of algebra we know

$$
\text{End}_{R^1_A}(F^1_A M)/\text{rad} \text{End}_{R^1_A}(F^1_A M) \cong \text{End}_{\widehat{A}}(M)/\text{rad} \text{End}_{\widehat{A}}(M).
$$

Suppose that $m \geq 2$. By the structure of $\text{Mod} \widehat{A}$ we have that $\text{Hom}_{\widehat{A}}(M, \nu^m M) = 0$
for $s \neq 0$. Hence

$$
\text{End}_{R^m_A}(F^m_A M)/\text{rad} \text{End}_{R^m_A}(F^m_A M) \cong \text{End}_{\widehat{A}}(M)/\text{rad} \text{End}_{\widehat{A}}(M).
$$

Since $M$ is a generic $\widehat{A}$-module, we infer that $\text{End}_{\widehat{A}}(M)/\text{rad} \text{End}_{\widehat{A}}(M)$ is a division ring
and hence $\text{End}_{R^m_A}(F^m_A M)$ is local for $m \geq 1$. Therefore, $F^m_A M$ is an indecomposable
$R^m_A$-module.

Write $C = \text{End}_{\widehat{A}}(M), D = \text{End}_{R^m_A}(F^m_A M), \overline{C} = C/\text{rad} C, \overline{D} = D/\text{rad} D$.
Let $l_A(M)$ denote the length of $A$-module $M$. We have

$$
l_D(F^m_A M) = l_D(F^m_A M/(\text{rad} D) F^m_A M) + l_D((\text{rad} D) F^m_A M/(\text{rad}^2 D) F^m_A M) + \ldots \\
+ l_D((\text{rad}^i D) F^m_A M/(\text{rad}^{i+1} D) F^m_A M) + \ldots
$$

Since each $(\text{rad}^i D) F^m_A M/(\text{rad}^{i+1} D) F^m_A M$ is a $\overline{D}$-module and

$$
l_D((\text{rad}^i D) F^m_A M/(\text{rad}^{i+1} D) F^m_A M) = l_{\overline{D}}((\text{rad}^i D) F^m_A M/(\text{rad}^{i+1} D) F^m_A M).
$$
We have

\[ l_D(F^m_\lambda M) = l_D(F^m_\lambda M/(\text{rad} D) F^m_\lambda M) + l_D((\text{rad} D) F^m_\lambda M/(\text{rad}^2 D) F^m_\lambda M) + \cdots + l_D((\text{rad}^i D) F^m_\lambda M/(\text{rad}^{i+1} D) F^m_\lambda M) + \cdots \]

\[ \overset{i=0}{\overset{\infty}{\sum}} l_C(F^m_\lambda M/(\text{rad} D) F^m_\lambda M) + l_C((\text{rad} D) F^m_\lambda M/(\text{rad}^2 D) F^m_\lambda M) + \cdots + l_C((\text{rad}^i D) F^m_\lambda M/(\text{rad}^{i+1} D) F^m_\lambda M) + \cdots \]

\[ = l_C(F^m_\lambda M/(\text{rad} D) F^m_\lambda M) + l_C((\text{rad} D) F^m_\lambda M/(\text{rad}^2 D) F^m_\lambda M) + \cdots + l_C((\text{rad}^i D) F^m_\lambda M/(\text{rad}^{i+1} D) F^m_\lambda M) + \cdots \]

\[ = l_C(F^m_\lambda M). \]

By lemma 2.3, we have \( l_C(F^m_\lambda M) \leq \infty \) and hence \( l_D(F^m_\lambda M) \leq \infty \). \( F^m_\lambda M \) is clearly of infinite -dimension since \( M \) is so. Therefore \( F^m_\lambda M \) is a generic \( R^m_A \)-module.

**Corollary 2.5.** \( R^m_A \) has at least 2m generic modules.

**Proof.** By Theorem 1.2, \( \text{Mod} \tilde{A} \) has two \( \nu \)-orbits \( O_0 \) and \( O_1 \) of generic \( \tilde{A} \)-modules.

\[ O_0 = \{ \nu^m M_0 \mid m \in \mathbb{Z} \}, \quad O_1 = \{ \nu^n M_1 \mid n \in \mathbb{Z} \}. \]

From Lemma 2.1 and 2.2, it is easy to know that \( F^m_\lambda (\nu^l N), \ F^m_\lambda (\nu^t N) \ (l, t = 0, 1, 2, \ldots, m - 1) \) are different generic \( R^m_A \)-modules.

**REFERENCES**


