REGULARITY OF BUTTERWORTH REFINABLE FUNCTIONS*

AIHUA FAN† AND QIYU SUN‡

Abstract. Let $\Psi_N$ be the refinable function with Butterworth filter $\cos^{2N} \frac{\xi}{2} (\cos^{2N} \frac{\xi}{2} + \sin^{2N} \frac{\xi}{2})^{-1}$ and let $s_p(\Psi_N)$ be the Fourier exponent of $\Psi_N$ of order $p$ ($0 < p \leq \infty$). It is proved that

$$0 \leq s_\infty(\Psi_N) - N \frac{\ln 3}{\ln 2} \leq \frac{\ln(1 + 3^{-N})}{\ln 2}$$

and for $0 < p < \infty$

$$-\frac{\ln(1 + r_0^N)}{p \ln 2} \leq s_p(\Psi_N) - N \frac{\ln 3}{\ln 2} \leq \frac{\ln(1 + 3^{-N})}{\ln 2}$$

where $r_0 \in (0, 1)$ is independent of $p$ and $N$.

1. Introduction and Result. In this paper we study the solutions of some refinement equations of the form

$$\phi(x) = \sum_{j \in \mathbb{Z}} c_j \phi(2x - j) \quad (x \in \mathbb{R})$$

where the coefficients $c_j$ are supposed to satisfy the arithmetic condition $\sum_{j \in \mathbb{Z}} c_j = 2$ and the exponential decay condition $|c_j| \leq C e^{-\beta |j|}$ ($C, \beta > 0$ constants). Solutions of a refinement equation are called refinable functions. The $2\pi$-periodic function

$$m(\xi) = \frac{1}{2} \sum_{j \in \mathbb{Z}} c_j e^{-ij\xi}$$

is called the filter of the refinement equation (1.1). A continuous function $\phi$ is called a cardinal interpolant if $\phi(0) = 1$ and $\phi(k) = 0$ for all nonzero integer $k$. It is known that there is an important class of refinable functions which are cardinal interpolants and whose filters satisfy

$$m(\xi) + m(\xi + \pi) = 1. \quad (1.2)$$

Such a filter $m(\xi)$ can be put into the factorized form

$$m(\xi) = \left( \frac{1 + e^{-i\xi}}{2} \right)^N R(\xi)$$

(1.3)

where $N$ is a strictly positive integer and $R(\xi)$ is a $2\pi$-periodic function whose Fourier coefficients decay exponentially. The minimal degree solution of (1.2) having the factorized form (1.3) is given by

$$m_N(\xi) = \cos^{2N} \frac{\xi}{2} \sum_{s=0}^{N-1} \binom{N-1+s}{s} \sin^{2s} \frac{\xi}{2}.$$
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The corresponding refinable functions, denoted by $\Phi_N$, are the self-convolution of Daubechies' scaling functions, and they are cardinal interpolants (see [3, 4, 5]). We will study the solution of the equation (1.2) whose filter has a simpler factorized form (1.3) given by

$$\hat{m}_N(\xi) = \cos^{2N} \frac{\xi}{2} \left( \cos^{2N} \frac{\xi}{2} + \sin^{2N} \frac{\xi}{2} \right)^{-1}. $$

These filters are well known in signal processing as the transfer functions of the “Butterworth filter” (see [8] for a detailed review). The corresponding refinable functions, denoted by $\Psi_N$, are said to be Butterworth refinable functions, which are also cardinal interpolants. Denote by $\hat{f}$ the Fourier transform of an integrable function or a tempered distribution $f$. In the form of Fourier transform, the equation (1.1) becomes $\hat{\phi}(\xi) = m(\xi/2)\hat{\phi}(\xi/2)$. Hence we get the useful formula

$$\hat{\Psi}_N(\xi) = \left( \frac{\sin \xi/2}{\xi/2} \right)^{2N} \prod_{n=1}^{\infty} \left( \cos^{2N} 2^{-n-1} \xi + \sin^{2N} 2^{-n-1} \xi \right)^{-1}. $$

The aim of this paper is to study the regularity of $\Psi_N$. The regularity of refinable functions is of central importance in the theory of wavelets. A usual approach is to study the Fourier exponents, which are also called Sobolev exponents in the literature. For a tempered distribution $f$ with measurable Fourier transform, define its Fourier exponents $s_p(f)$ by

$$s_p(f) = \sup \left\{ s : \int_{\mathbb{R}} |\hat{f}(\xi)|^p (1 + |\xi|)^{ps} d\xi < \infty \right\} \quad (0 < p < \infty)$$

$$s_\infty(f) = \sup \left\{ s : \hat{f}(\xi)(1 + |\xi|)^s = O(1) \quad |\xi| \to \infty \right\}. $$

In [1], Cohen and Daubechies studied the regularity of refinable functions $\Psi_N$ and gave some numerical results on the Fourier exponents $s_p(\Psi_N)$ for $p = 1/2, 1, 2, 4$ and $N = 1, 2, \cdots, 19$. They noticed that for large value of $N$ the Fourier exponent $s_p(\Psi_N)$ reveals a linear asymptotic behavior and the limit ratio $s_p(\Psi_N)/N$ indicates that the worst decay of $\hat{\Psi}_N$ occurs at the points $2^{j+1} \pi/3$. In this paper, we confirm the above observation by proving

**Theorem 1.** Let $\Psi_N$ be defined as above. Then

$$0 \leq s_\infty(\Psi_N) - \frac{N \ln 3}{\ln 2} \leq \frac{\ln(1 + 3^-N)}{\ln 2}$$

for all $N \geq 1$, and

$$-\frac{\ln(1 + r_0^{Np})}{p \ln 2} \leq s_p(\Psi_N) - \frac{N \ln 3}{\ln 2} \leq \frac{\ln(1 + 3^-N)}{\ln 2}$$

for all $N \geq 1$ and $0 < p < \infty$, where $r_0 \in (0, 1)$ is a constant independent of $p$ and $N$.

As a consequence of Theorem 1, we have

**Corollary 1.** Let $\Psi_N$ be defined as above. Then

$$\lim_{N \to \infty} \frac{s_p(\Psi_N)}{N} = \frac{\ln 3}{\ln 2} \quad (0 < p \leq \infty).$$
and
\[ \lim_{N \to \infty} (s_p(\Psi_N) - s_q(\Psi_N)) = 0 \quad (0 < p, q \leq \infty). \]

2. Proof. To get the lower bound estimate of \( s_p(\Psi_N) \), we introduce an auxiliary \( \pi \)-periodic even function defined by
\[ h(\xi) = \max\{|\cos \xi/2|, |\sin \xi/2|\}. \]
It is clear that \( h(\xi) = \cos \xi/2 \) if \( |\xi| \leq \pi/2 \) and \( h(\xi) = |\sin \xi/2| \) if \( \pi/2 \leq |\xi| \leq \pi \). Furthermore, we have

**Lemma 1.** Let \( h(\xi) \) be the function defined by (2.1). Then
\[
\begin{cases}
    h(\xi) \geq \frac{\sqrt{3}}{2}, & \xi \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right] + \pi\mathbb{Z}, \\
    h(\xi)h(2\xi) \geq \frac{3}{4}, & \xi \in \left[-\frac{5\pi}{12}, -\frac{\pi}{3}\right] \cup \left[\frac{\pi}{3}, \frac{5\pi}{12}\right] + \pi\mathbb{Z}, \\
    h(\xi)h(2\xi)h(4\xi) > \left(\frac{\sqrt{3}}{2}\right)^3, & \xi \in \left[-\frac{\pi}{2}, -\frac{5\pi}{12}\right] \cup \left[\frac{5\pi}{12}, \frac{\pi}{2}\right] + \pi\mathbb{Z}.
\end{cases}
\]

**Proof.** For simplicity, we write \( H_2(\xi) = h(\xi)h(2\xi) \) and \( H_3(\xi) = h(\xi)h(2\xi)h(4\xi) \). Since \( h \) is an even function with period \( \pi \), it suffices to prove (2.2) for \( \xi \in [0, \pi/2] \). The first inequality of (2.2) follows from the facts that \( h(\xi) \) decreases on \([0, \pi/2]\) and that \( h(\pi/3) = \sqrt{3}/2 \).

Let \( t = \cos^2 \xi/2 \). By a simple calculation, we obtain that
\[ H_2(\xi)^2 = \cos^2 \frac{\xi}{2} \sin^2 \xi \xi = 4t^2(1 - t) \]
and that \( t \in [\frac{5\pi}{12}, 3/4] \) for any \( \xi \in [\pi/3, 5\pi/12] \). Observe that
\[
\frac{d}{dt}(t^2(1 - t)) = 3t(2/3 - t).
\]
This, together with (2.3), implies that \( H_2(\xi) \) increases on the interval \( \left[\frac{\pi}{3}, 2 \arccos \frac{\sqrt{3}}{2}\right] \) and decreases on the interval \([2 \arccos \sqrt{3}/2, 3\pi/2] \). Thus,
\[ H_2(\xi) \geq \min\{H_2(\pi/3), H_2(5\pi/12)\} = H_2(\pi/3) = 3/4, \quad \forall \xi \in [\pi/3, 5\pi/12]. \]

It is the second inequality of (2.2).

If \( \xi \in [5\pi/12, \pi/2] \), we have \( 2\xi \in [5\pi/6, \pi] \) and \( 4\xi \in [5\pi/3, 2\pi] = [-\pi/3, 0] + 2\pi \). Therefore
\[ H_3(\xi)^2 = \cos^2 \frac{\xi}{2} \sin^2 \xi \cos^2 2\xi = 4t^2(1 - t)(8t^2 - 8t + 1)^2 \]
where \( t = \cos^2 \xi/2 \in [1/2, (2 + \sqrt{2 - \sqrt{3}})/4] \). Let
\[ g_1(t) = t^2(1 - t)(8t^2 - 8t + 1)^2, \quad g_2(t) = 56t^3 - 88t^2 + 35t - 2. \]
Notice that
\[
\frac{d}{dt}g_1(t) = t(-8t^2 + 8t - 1)g_2(t), \quad \frac{d}{dt}g_2(t) = 168t^2 - 176t + 35.
\]
It follows that \(\frac{d}{dt}g_2(t) < 0\) on \([1/2, (2 + \sqrt{2 - \sqrt{3}})/4]\). On the other hand, \(g_2(1/2) = 1/2 > 0\) and
\[
g_2((2 + \sqrt{2 - \sqrt{3}})/4) \leq g_2(5/8) = -53/64 < 0.
\]
Therefore there exists \(t_0 \in [1/2, (2 + \sqrt{2 - \sqrt{3}})/4]\) such that \(g_2(t) > 0\) on \([1/2, t_0]\) and \(g_2(t) < 0\) on \([t_0, (2 + \sqrt{2 - \sqrt{3}})/4]\). Observe that \(-8t^2 + 8t - 1 = -\cos 2\xi > 0\). Thus \(H_3(\xi)\) increases on \([5\pi/12, 2 \arccos \sqrt{t_0}]\) and decreases on \([2 \arccos \sqrt{t_0}, \pi/2]\). Hence
\[
H_3(\xi) \geq \min\{H_3(5\pi/12), H_3(\pi/2)\} = H_3(5\pi/12) > (\sqrt{3}/2)^3.
\]
Thus we have proved the third inequality of (2.2). \(\square\)

For \(N \geq 1\), let
\[
(2.4) \quad R_N(\xi) = \left(\cos^{2N} \frac{\xi}{2} + \sin^{2N} \frac{\xi}{2}\right)^{-1}.
\]
Clearly \(R_N\) is a \(\pi\)-periodic function and
\[
m_N(\xi) = \cos^{2N} \frac{\xi}{2} R_N(\xi)
\]
(see (1.4)). Note that \(R_N(\xi) \leq h(\xi)^{-2N}\). Therefore, by Lemma 1 and the strict monotonocity of \(h(\xi), h(\xi)h(2\xi), h(\xi)h(2\xi)h(4\xi)\) on their respective intervals, we have

**Lemma 2.** Let \(R_N\) be defined as above and let \(q = (4/3)^N\). Then for any \(0 < \delta \leq \frac{\pi}{24}\), there exists \(0 < r = r(\delta) < 1\) such that
\[
\begin{cases}
R_N(\xi) \leq q, & \xi \in \left[\frac{\pi}{3}, \frac{\pi}{3}\right] + \pi\mathbb{Z} \\
R_N(\xi)R_N(2\xi) \leq q^2, & \xi \in \left[\left(-\frac{5\pi}{12}, -\frac{\pi}{3}\right) \cup \left(\frac{\pi}{3}, \frac{5\pi}{12}\right)\right] + \pi\mathbb{Z} \\
R_N(\xi)R_N(2\xi)R_N(4\xi) \leq q^3, & \xi \in \left[\left(-\frac{\pi}{2}, \frac{5\pi}{12}\right) \cup \left(\frac{5\pi}{12}, \frac{\pi}{2}\right)\right] + \pi\mathbb{Z}
\end{cases}
\]
and
\[
\begin{cases}
R_N(\xi) \leq r^N q, & \xi \in \left[-\frac{\pi}{3} + \delta, \frac{\pi}{3} - \delta\right] + \pi\mathbb{Z} \\
R_N(\xi)R_N(2\xi) \leq r^N q^2, & \xi \in \left[\left(-\frac{5\pi}{12}, \frac{\pi}{3} - \delta\right) \cup \left[\frac{\pi}{3} + \delta, \frac{5\pi}{12}\right]\right] + \pi\mathbb{Z} \\
R_N(\xi)R_N(2\xi)R_N(4\xi) \leq r^{2N} q^3, & \xi \in \left[\left(-\frac{\pi}{2}, -\frac{5\pi}{12}\right) \cup \left(\frac{5\pi}{12}, \frac{\pi}{2}\right)\right] + \pi\mathbb{Z}.
\end{cases}
\]
In particular, \(r\) can be chosen as
\[
\max \left\{\frac{3}{4} \left(h\left(\frac{\pi}{3} - \delta\right)\right)^{-2}, \left(\frac{3}{4}\right)^2 \left(H_2\left(\frac{\pi}{3} + \delta\right)\right)^{-2}, \left(\frac{3}{4}\right)^{3/2} \left(H_3\left(\frac{5\pi}{12}\right)\right)^{-2}\right\}.
\]
Define
\[
I_k(\xi) = \{j : 1 \leq j \leq k, 2^j \xi \in \cup_{m \in \mathbb{Z}}[-\pi/4, \pi/4] + m\pi\}
and let \( i_k(\xi) \) be the cardinality of the set \( I_k(\xi) \).

**Lemma 3.** Let \( R_N \) be defined as above and let \( q = (4/3)^N \). Then there exists a positive constant \( C_N \) such that for any \( k \geq 1 \)

\[
(2.5) \quad \prod_{j=1}^{k} R_N(2^j \xi) \leq C_N r_0^{N i_k(\xi)} q^k
\]

where \( r_0 = r(\pi/24) \) is defined in Lemma 2.

**Proof.** The idea of proof was used in [7]. It is clear that the assertion in Lemma 3 holds for \( k = 1, 2, 3 \) if \( C_N \) is chosen large enough. We assume that (2.5) holds for all \( k < l \) with \( l \geq 3 \). For \( k = l \), we distinguish five cases.

(i) If \( 2\xi \in [-\pi/4, \pi/4] + \pi Z \), then \( i_k(\xi) = i_{k-1}(2\xi) + 1 \). Write

\[
\prod_{j=1}^{k} R_N(2^j \xi) = R_N(2\xi) \prod_{j=1}^{k-1} R_N(2^j(2\xi)).
\]

Thus (2.5) holds by using Lemma 2 and the induction hypothesis.

(ii) If \( 2\xi \) or \(-2\xi \in (\pi/4, \pi/3] + \pi Z \), then \( i_k(\xi) = i_{k-1}(2\xi) \). Again the induction hypothesis together with Lemma 2 implies (2.5).

(iii) If \( 2\xi \) or \(-2\xi \in (\pi/3, 3\pi/8] + \pi Z \), then \( i_k(\xi) = i_{k-2}(4\xi) \). It suffices to write

\[
\prod_{j=1}^{k} R_N(2^j \xi) = R_N(2\xi) R_N(4\xi) \prod_{j=1}^{k-2} R_N(2^j(4\xi))
\]

and then to apply Lemma 2 and the induction hypothesis.

(iv) If \( 2\xi \) or \(-2\xi \in [3\pi/8, 5\pi/12] + \pi Z \), then \( i_k(\xi) \leq i_{k-2}(4\xi) + 1 \). By using the induction hypothesis and Lemma 2, we have

\[
\prod_{j=1}^{k} R_N(2^j \xi) \leq R_N^q \left[ C_{NT}^{N i_{k-2}(4\xi)} q^{k-2} \right]
\]

\[
\leq C_{NT}^{N i_k(\xi)} q^k.
\]

(v) If \( 2\xi \) or \(-2\xi \in (5\pi/12, \pi/2] + \pi Z \), then \( i_k(\xi) \leq i_{k-3}(8\xi) + 2 \). Hence

\[
\prod_{j=1}^{k} R_N(2^j \xi) = R_N(2\xi) R_N(4\xi) R_N(8\xi) \prod_{j=1}^{k-3} R_N(2^j(8\xi))
\]

\[
\leq r_0^q \left[ C_{NT}^{N i_{k-3}(8\xi)} q^{k-3} \right]
\]

\[
\leq C_{NT}^{N i_k(\xi)} q^k.
\]

□

Let \( k \geq 2 \). For \( (\epsilon_1, \cdots, \epsilon_k) \in \{0,1\}^k \), let

\[ Q(\epsilon_1, \cdots, \epsilon_k) = \{ i : \epsilon_i = \epsilon_{i+1} \} \]

and \( q(\epsilon_1, \cdots, \epsilon_k) \) be the cardinality of the set \( Q(\epsilon_1, \cdots, \epsilon_k) \). For \( 0 \leq q \leq k - 1 \), let

\[ G_q = \{ (\epsilon_1, \cdots, \epsilon_k) \in \{0,1\}^k : q(\epsilon_1, \cdots, \epsilon_k) = q \} \].
Then, for any \((\epsilon_1, \ldots, \epsilon_k) \in G_{q,k}\) there exist unique integers \(1 < i_1 < i_2 < \ldots < i_q < k - 1\) such that \(\epsilon_s = \epsilon_{i_s+1}\) for all \(1 \leq s \leq q\). On the other hand, given any \(\epsilon_1 \in \{0,1\}\) and integers \(1 < i_1 < i_2 < \ldots < i_q < k - 1\), we may find one and only one \((\epsilon_1, \ldots, \epsilon_k) \in G_{q,k}\) such that \(\epsilon_s = \epsilon_{i_s+1}\) for any \(1 \leq s \leq q\). Therefore, the cardinality of \(G_{q,k}\) is \(2^{(k-1)}\) for any \(0 \leq q \leq k - 1\).

**Lemma 4.** Let \(k \geq 2, \xi \in [0, \pi)\) and let \(i_k(\xi)\) and \(q(\epsilon_1, \ldots, \epsilon_k)\) be defined as above. Write \(\xi / \pi = \sum_{i=1}^{k} \epsilon_i 2^{-i} + \eta\) with \(0 \leq \eta < 2^{-k}\) and \(\epsilon_i \in \{0,1\}\) for \(1 \leq i \leq k\). Then \(i_k(\xi) \geq q(\epsilon_1, \ldots, \epsilon_k) - 1\).

**Proof.** For any \(i \in Q(\epsilon_1, \ldots, \epsilon_k)\) and \(i \geq 2\), we have \(\epsilon_i = \epsilon_{i+1}\) and

\[
2^{i-1} \xi = \frac{3}{4} \epsilon_i \pi + \eta' \pi + m \pi
\]

with \(0 \leq \eta' < \frac{1}{4}\) and \(m \in \mathbb{Z}\). Therefore \(2^{i-1} \xi \in \left[0, \frac{\pi}{4}\right] + \pi \mathbb{Z}\) if \(\epsilon_i = 0\) and \(2^{i-1} \xi \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] + \pi \mathbb{Z}\) if \(\epsilon_i = 1\). This implies that \(i - 1 \geq I_k(\xi)\). Thus \(i_k(\xi) \geq q(\epsilon_1, \ldots, \epsilon_k) - 1\).

**Proof of Theorem 1.** The upper bound estimate of \(s_p(\Psi_N)\) will be proved by a modification of the method used in [2]. (The method is also used in [7].) By (1.5) and \(R_N(2\pi/3) = 2^N(1 + 3^N)^{-1}\), we have \(\tilde{\Psi}_N(2\pi/3) \neq 0\) and

\[
\tilde{\Psi}_N(2^{k+1}\pi/3) = (1 + 3^N)^{-k} \tilde{\Psi}_N(2\pi/3) \quad \forall k \geq 1.
\]

This implies that \(s_\infty(\Psi_N) \leq \ln(1 + 3^N)/\ln 2\).

By the continuity of \(\tilde{\Psi}_N\) and \(R_N\), for any \(\epsilon > 0\) there exists \(0 < \delta < 1\) such that for all \(\xi \in [-\delta, \delta]\) we have

\[
|R_N(2\pi/3 + \xi)| = |R_N(-2\pi/3 - \xi)| \geq (1 - \epsilon)2^N(1 + 3^N)^{-1}
\]

and

\[
|\tilde{\Psi}_N(2\pi/3 + \xi)| \geq (1 - \epsilon)|\tilde{\Psi}_N(2\pi/3)| > 0.
\]

This together with (1.5) implies that for all \(\xi \in [-\delta, \delta]\) and \(k \geq 1\),

\[
\tilde{\Psi}_N(2^{k+1}\pi/3 + \xi) = \prod_{j=1}^{k} \tilde{m}_N(2^{k-j+1}\pi/3 + 2^{-j}\xi) \tilde{\Psi}_N(2\pi/3 + 2^{-k}\xi)
\]

\[
\geq C(1 + 3^N)^{-k}(1 - \epsilon)^k
\]

where \(C\) is a positive constant independent of \(k\). Therefore for \(0 < p < \infty\) and \(k \geq 1\), we have

\[
\int_{2^{k-1}\pi/3}^{2^{k+1}\pi/3+1} |\tilde{\Psi}_N(\xi)|^p d\xi \geq C_1 \int_{-\delta}^{\delta} |\tilde{\Psi}_N(2^{k+1}\pi/3 + \xi)|^p d\xi \geq C_2 \delta(1 + 3^N)^{-kp}(1 - \epsilon)^{kp}
\]

where \(C_1\) and \(C_2\) are positive constants independent of \(k\). This gives the desired upper bound estimate of \(s_p(\Psi_N)\) for \(0 < p < \infty\).

For \(k \geq 1\) and \(2^{k-1}\pi \leq |\xi| \leq 2^k\pi\), it follows from (1.5) and Lemma 3 that

\[
|\tilde{\Psi}_N(\xi)| \leq C_1 \prod_{j=1}^{k} |\tilde{m}_N(2^{-j}\xi)| \leq C_2 |\xi|^{-2N} \prod_{j=1}^{k} |R_N(2^{j}(2^{-k}\xi))| \leq C_3 3^{-Nk}
\]
where $C_1, C_2$ and $C_3$ are positive constants independent of $k$. This leads to the desired lower bound estimate of $s_{\infty}(\Psi_N)$.

Let $r_0 = r(\pi/24)$. Then for any $k \geq 1$ and $0 < p < \infty$, there exist positive constants $C_i$ ($1 \leq i \leq 4$) independent of $k$ such that

$$\int_{2^{k-1} \pi \leq |\xi| \leq 2^k \pi} |\hat{\Psi}_N(\xi)|^p d\xi = 2 \int_{2^{k-1} \pi}^{2^k \pi} |\hat{\Psi}_N(\xi)|^p d\xi$$

$$\leq C_1 3^{-kNp} \int_{2^{k-1} \pi}^{2^k \pi} r_0^{Nq} (2^{-k} \xi) d\xi$$

$$\leq C_2 3^{-kNp} \sum_{(\epsilon_1, \ldots, \epsilon_k) \in \{0,1\}^k} \int_{\sum_{j=1}^{k} 2^{-j(\epsilon_j+\pi)}}^{\sum_{j=1}^{k} 2^{-j\epsilon_j+\pi}} r_0^{Npq(\epsilon_1, \ldots, \epsilon_k)} d\xi$$

$$\leq C_3 3^{-kNp} \sum_{q=0}^{k-1} r_0^{Npq} \sum_{q(\epsilon_1, \ldots, \epsilon_k) = q} 1$$

$$\leq C_4 3^{-kNp}(1 + r_0^{Np})^k$$

where we have used (5) and Lemma 3 in the first inequality, Lemma 4 in the second one, the fact that the cardinality of $G_{q,k}$ is $2^{(k-1)}$ in the last one. Hence we obtain the desired lower bound estimate of $s_p(\Psi_N)$ for $0 < p < \infty$. □

3. Remarks. From the above proof, we see that $r_0$ in the theorem can be chosen to be 0.9787028. When $N$ is large, $s_p(\Psi_N)$ is well approximated by $N \ln 3/\ln 2$. Let us compare the numerical results obtained in [1] for $p = 1/2, 1, 4$ and the approximation given by $N \ln 3/\ln 2$ (see Table 1). We point out that the differences between the last two columns are small and that when $N \geq 20$ we can use $N \log_2 3$ to get rather precise approximation for $s_p(\Psi_N)$.

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Table 1: Fourier exponent $s_{2p}(\Psi_N) = \frac{1}{2} s_p(\Psi_N)$ and its approximation $\frac{1}{2} N \log_2 3$.
Let \( \Psi_N \) be the refinable function with corresponding filter \((\frac{1+e^{-i\xi}}{2})^N (\cos^2 \xi/2 + \sin^2 \xi/2)^{-1/2}\). Then \( \hat{\Psi}_N(\xi) = |\hat{\Psi}_N(\xi)|^2 \) and \( s_p(\Psi_N) = 2s_{p/2}(\hat{\Psi}_N) \). In fact, the original numerical results in [1] is about the Fourier exponents \( s_p(\Psi_N) \) with \( p = 1, 2, 4, 8 \) and \( N = 1, 2, \ldots, 19 \).

For the Daubechies scaling functions \( \Phi_N \), there are many papers devoted to the estimates of \( s_p(\Phi_N) \) (see [1, 6, 7, 9] and references therein). In [7], Lau and Sun proved that

\[
\frac{-C}{N} \leq s_p(\Phi_N) - 2N + \frac{\ln P_N(3/4)}{\ln 2} \leq 0
\]

for \( 0 < p < \infty \) and

\[
s_\infty(\Phi_N) = 2N - \frac{\ln P_N(3/4)}{\ln 2}
\]

where \( C \) is a positive constant independent of \( N \) and

\[
P_N(t) = \sum_{s=0}^{N-1} \binom{N+k-1}{k} t^s.
\]

By the idea we used in the proof of the theorem, the term \( -\frac{C}{N} \) in the above lower estimate can be improved to be \( -Cr_0^N \) for some \( 0 < r_0 < 1 \).

REFERENCES


