A MODEL OF BRILL-NOETHER THEORY FOR RANK TWO VECTOR BUNDLES AND ITS PETRI MAP *

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Abstract. We study here the Brill-Noether theory for rank two vector bundles. First we construct a parameter space \( H_d \) for all base point free rank two vector bundles of degree \( d \) which generated by its sections. Then for each \( E \in H_d \), we define a \( 2g \times d \) matrix \( W_E \), which we call the Brill-Noether matrix of \( E \), it shares the same properties as the Brill-Noether matrix \( W_D \) for effective divisor \( D \). By using \( W_E \), the Brill-Noether variety \( C_d^r \) could be given by \( C_d^r = \{ E \in H_d | \dim H^0(C, E) \geq r + 1 \} \), so \( C_d^r \) is a determinant variety, we get its expected dimension is \( 4(g-1)+1-(r+1)(2(g-1)-d+r+1)+2r+1 \). On the other hand, by using \( W_E \), we define the Petri map to be \( P : H^0(C, K(-E)) \otimes \text{Im}(H^0(C, E) \hookrightarrow H^0(C, [D])) \twoheadrightarrow H^0(C, K(D)(-E)) \), we show that \( C_d^r \) has the expected dimension if and only if the Petri map is injective.

1. Introduction. Let \( C \) be a smooth irreducible complex projective curve of genus \( g \) (a compact Riemann surface), \( L \) a line bundle on \( C \). We also use \( L \) to denote the sheaf of holomorphic sections of \( L \). The Brill-Noether theory for line bundles is to study those bundles \( L \) for which both \( H^0(C, L) \) and \( H^1(C, L) \) are non-zero (\( L \) is then called special line bundle).

Let \( C_d \) be the \( d \)-fold symmetric product of \( C \), \( C_d \) is a \( d \)-dimensional complex manifold. It is the space of all effective divisors of degree \( d \). Since each line bundle \( L \) with \( H^0(C, L) \neq 0 \) is defined by an effective divisor, so \( C_d \) could be considered as a parameter space for all line bundles \( L \) with \( \deg(L) = d \) and \( H^0(C, L) \neq 0 \).

Define on \( C_d \) the Brill-Noether variety \( C_d^r \) to be

\[
C_d^r = \{ D \in C_d | \dim H^0(C, [D]) \geq r + 1 \}.
\]

Where \( [D] \) is the line bundle defined by divisor \( D \).

\( C_d^r \) could be considered as a parameter space for line bundles \( L \) with \( \deg(L) = d \) and \( \dim H^0(C, L) \geq r + 1 \). The key tool to study \( C_d^r \) is the Brill-Noether matrix.

Let \( D = n_1p_1 + \cdots + n_kp_k \) be a given effective divisor with \( d = \deg(D) = n_1 + \cdots + n_k \). For \( i = 1, \ldots, k \), let \( z_i \) be a local coordinate at \( p_i \) with \( z_i(p_i) = 0 \). Let \( \{w_1, \ldots, w_g\} \) be a linear basis of the space of all holomorphic forms on \( C \), for each \( i \) assume at \( p_i, w_t(z_i) = f_{ti}(z_i)dz_i \) for \( t = 1, \ldots, g \), let \( W_D \) be the matrix of the restrictions of \( \{w_1, \ldots, w_g\} \) on \( D \), that is

\[
W_D = \left[ \begin{array}{c}
w_1 |_D \\
w_2 |_D \\
\vdots \\
w_g |_D \\
f_{11}(p_1) & \cdots & \frac{1}{(n_1-1)!}f_{11}^{(n_1-1)}(p_1) & \cdots & \frac{1}{(n_2-1)!}f_{12}^{(n_2-1)}(p_2) & \cdots \\
\vdots & \cdots & \vdots & \cdots & \vdots & \cdots \\
f_{g1}(p_1) & \cdots & \frac{1}{(n_k-1)!}f_{g1}^{(n_k-1)}(p_1) & \cdots & \frac{1}{(n_2-1)!}f_{g2}^{(n_2-1)}(p_2) & \cdots \\
\end{array} \right].
\]

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For a collection of Laurent tails \( \mu = \{ \mu_i = \sum_{k=-n_i}^{-1} b_{ik} z_i^k \} \), we denote it as a \( d \)-dimensional vector

\[
\mu = (b_{1-1}, b_{1-2}, \ldots, b_{1-n_1}, b_{2-1}, \ldots, b_{2-n_2}, \ldots) \in \mathbb{C}^d.
\]

Then \( \mu \) is the Laurent of a global meromorphic function if and only if \( W_D \cdot \mu^t = 0 \). From this one can get Riemann-Roch theorem easily.

The matrix \( W_D \) is called the Brill-Noether matrix of \( D \).

Now let \( [D]|_D \) be the skyscraper sheaf of the restriction of \( [D] \) on \( D \), then what we have above could be represented as

\[
\ker(W_D) = \{ \mu | \mu \in \mathbb{C}^d, W_D \cdot \mu^t = 0 \} \cong \text{Im}\{H^0(X, [D]) \to H^0(X, [D]|_D)\}
\]

and in particular, we get

\[
\dim H^0(X, [D]) = \deg(D) - \text{rank}(W_D) + 1.
\]

so \( C^r_d \) could be defined by

\[
C^r_d = \{ D \in \mathbb{C}^d | \text{Rank}(W_D) \leq d - r \}.
\]

It is a subvariety of \( C_d \) which locally is defined by the simultaneously vanishing of all \((d - r + 1) \times (d - r + 1)\) minors of \( W_D \) (Ref [ACGH] p159).

Now let \( M(m, n) = M \) be the variety of all \( m \times n \) complex matrices, and for \( 0 \leq k \leq \min\{m, n\} \), denote by \( M_k(m, n) = M_k \) the locus of matrices of rank at most \( k \), that is

\[
M_k = \{ E \in M(m, n) | \text{Rank}(E) \leq k \}.
\]

\( M_k \) is an irreducible subvariety of \( M(m, n) \), and \( \text{codim}(M_k) = (n - k)(m - k) \) (Ref [ACGH] p67).

By using the Brill-Noether matrix, locally we have a holomorphic map \( BN : C_d \to M(m, n) \) with \( BN(D) = W_D \) for each \( D \in C_d \). \( C^r_d \) is then could be given by \( C^r_d = BN^{-1}(M_{d-r}) \).

From the Theory of determinant variety, we get that if \( C^r_d \neq \emptyset \), then \( \text{codim}(C^r_d) \leq \text{codim}(M_{d-r}) = (g - (d - r))(d - (d - r)) \). So if \( C^r_d \neq \emptyset \), then

\[
\dim C^r_d \geq d - r(g - d + r) = g - (r + 1)(g - d + 1) = \rho(g, d, r) + r.
\]

where \( \rho(g, d, r) = g - (r + 1)(g - d + 1) \) is the Brill-Noether number for line bundles. (Ref [ACGH] p215).

It was conjectured by Brill-Noether and Proved by Griffiths-Harris [GH] that for generic \( C, C^r_d \) do have the expected dimension \( \rho(g, d, r) + r \).

On the other hand, by study the tangent map of \( BN : C_D \to M(m, n), D \mapsto W_D \), Petri got that the variety \( C^r_d \) is smooth and has the "expected dimension" \( \rho(g, d, r) + r \) at \( D \in C^r_d - C^r_d \) if and only if the cup product homomorphism

\[
\mu : H^0(C, [D]) \otimes H^0(C, K[-D]) \mapsto H^0(C, K)
\]

is injective, where, \( K \) is the canonical line bundle of \( C \) (Ref [ACGH] p163).

The map \( \mu \) is called the Petri map. Again, it was proved by Gieseker[G] that for generic \( C \), the cup product homomorphism \( \mu \) is indeed injective. This gives another prove of the result of Griffiths-Harris.

In this paper, we are trying to generalize those ideals to the study of rank two vector bundles.

First we will define a parameter space \( H_d \) for all base point free rank two vector bundles of degree \( d \) which generated by its sections (we called such vector bundles the effective vector bundles). \( H_d \) is a \( d \)-dimensional holomorphic vector bundle on \( C_d \), so it is a \( 2d \)-dimensional complex manifold.
For each $E \in H_d$, we construct a $2g \times d$ matrix $W_E$ for $E$ which we call it the Brill-Noether matrix of $E$, it shares the same properties for $E$ as the Brill-Noether matrix $W_D$ for line bundle $[D]$. In particular, we have

$$\dim H^0(C, E) = d - \text{Rank}(W_E) + 2.$$  

From this, the Brill-Noether variety of rank two vector bundles

$$C^r_{2,d} = \{ E \in H_d \mid \dim H^0(C, E) \geq r + 1 \}$$

could be given by

$$C^r_{2,d} = \{ E \in H_d \mid \text{Rank}(W_E) \leq d - r + 1 \}.$$  

This defines $C^r_{2,d}$ as a subvariety of $H_d$.  

Also by using $W_E$, locally we get a holomorphic map

$$BN : H_d \hookrightarrow M(2d, g); \ BN(E) = W_E,$$

so $C^r_{2,d} = BN^{-1}(M_{d-r+1})$, and from the theory of determinant variety, we get that if $C^r_{2,d} \neq \emptyset$ then

$$\text{codim} C^r_{2,d} \leq (2g - (d - r + 1))(d - (d - r + 1))$$

so if $C^r_{2,d} \neq \emptyset$, then

$$\dim C^r_{2,d} \geq 2d - (2g - (d - r + 1))(d - (d - r + 1)) = 2d - (r + 1)(2g - 1) - d + r + 1 =$$

$$2d - (r + 1)(2g - 1) - d + r + 1 + 2(2g - 1) - d + r + 1 =$$

$$4(g - 1) + 1 - (r + 1)(2g - 1) - d + r + 1 + 2r + 1 = \rho_2(g, d, r) + 2r + 1$$

here $\rho_2(g, d, r) = 4(g - 1) + 1 - (r + 1)(2g - 1) - d + r + 1$ is the Brill-Noether number for rank two vector bundles.  

Also, by studying the tangent map of $BN : H_d \hookrightarrow M(2g, d)$, we generalize the Petri map to rank two vector bundles. This is for each $E \in C^r_{2,d}$, we define a cup product homomorphism

$$P : H^0(C, K(-E)) \otimes \text{Im}\{ H^0(C, E) \hookrightarrow H^0(C, [D]) \} \hookrightarrow H^0(C, K[D](-E)).$$

Here $[D] = E/I$ is the quotient bundle of $E$ with respect to the trivial line bundle $I$. We call $P$ the Petri map for rank two vector bundles, and we show that $C^r_{2,d}$ has the "expected dimension" $\rho_2(g, d, r) + 2r + 1$ if and only if the Petri map $P$ is injective.

2. The parameter space $H_d$.  

**Definition 1.** A point $p \in C$ is called a base point of vector bundle $E$ if $s(p) = 0$ for all $s \in H^0(C, E)$. $E$ is said to be base point free if $E$ don't have base point.

**Definition 2 [A].** A rank two vector bundle $E$ is said to be generated by its sections, if $E$ has a splitting

$$0 \hookrightarrow L_1 \hookrightarrow E \hookrightarrow L_2 \hookrightarrow 0$$

such that both $H^0(C, L_1)$ and $\text{Im}\{ H^0(C, E) \hookrightarrow H^0(C, L_2) \}$ are not zero. Where $L_1$ is a line sub-bundle of $E$, and $L_2 = E/L_1.$
The Brill-Noether theory for rank two vector bundles is to study those bundles \( E \) with both \( H^0(C, E) \) and \( H^1(C, E) \) are non-zero. \( E \) is then called special rank two vector bundle. If \( E \) has a base point \( p \), then \( E \otimes [-p] \) is also special and we have \( \dim H^0(C, E \otimes [-p]) = \dim H^0(C, E) \), \( \dim H^1(C, E \otimes [-p]) = \dim H^1(C, E) + 2 \) and \( \deg(E \otimes [-p]) = \deg(E) - 2 \). We can reduce the degree of \( E \). If \( E \) is not generated by its sections, since \( H^0(C, E) \neq 0 \), let \( s \in H^0(C, E) \) with \( s \neq 0 \), let \( L_1 \) be the line sub-bundle of \( E \) which generated by \( s \), \( L_2 = E/L_1 \). Since \( E \) is not generated by its sections, so \( H^0(C, E) = H^0(C, L_1) \), the study of \( H^0(C, E) \) could be reduced to the study of \( H^0(C, L_1) \), that is reduced to the study of Brill-Noether for line bundles. So to study the Brill-Noether for rank two vector bundles, we can restrict ourself to the study of base point free vector bundles which generated by its sections.

**Lemma 1.** If \( E \) is a base point free rank two vector bundle which generated by its sections, then the trivial line bundle \( I \) is a line sub-bundle of \( E \).

**Proof.** This is a special case of Lemma 1.1 of [TE].

Let \( E \) be a base point free rank two vector bundles which generated by its sections, assume \( \deg(E) = d \), by our Lemma, \( I \) is a line sub-bundle of \( E \), so \( E \) has a splitting

\[
0 \rightarrow I \rightarrow E \rightarrow L \rightarrow 0
\]

where \( L = E/I \). Since \( E \) is generated by its sections, we have \( \text{Im}\{H^0(C, E) \rightarrow H^0(C, L)\} \neq 0 \). Choose \( s \in \text{Im}\{H^0(C, E) \rightarrow H^0(C, L)\} \) with \( s \neq 0 \), let \( D = \text{div}(s) \), then \( D \geq 0 \), and \( L = [D] \). \( E \) is then an extension of \([D]\) by \( I \), it is determined by an element \( e \in H^1(C, [-D]) \).

Since \( s \in H^0(C, [D]) \) can be lift to a section of \( E \), we get in particular that \( s \cdot e = 0 \), and from sequence

\[
0 \rightarrow [D] \rightarrow \ast I \rightarrow I \mid_D \rightarrow 0 \quad (***)
\]

we get an exact sequence

\[
0 \rightarrow H^0(C, [D]) \rightarrow H^0(C, I) \rightarrow H^0(C, I \mid_D) \rightarrow H^1(C, [-D]) \rightarrow \cdots
\]

\( s \cdot e = 0 \) if and only if \( e \in \text{Im}\{H^0(C, I \mid_D) \rightarrow H^1(C, [-D])\} \). Let \( e \) be the image of some \( f \in H^0(C, I \mid_D) \), \( f \) is then determined uniquely up to a constant. So from \( E \) we get a triple \( \{I, D, f\} \).

Conversely, if we have a triple \( \{I, D, f\} \), where \( D \) is an effective divisor of degree \( d \), and \( f \in H^0(C, I \mid_D) \), then let \( e \in H^1(C, [-D]) \) be the image of \( f \) in the map \( H^0(C, I \mid_D) \rightarrow H^1(C, [-D]) \) which induced from sequence \((***)\), let \( E \) be the extension of \([D]\) by \( I \) which determined by \( e \), then \( E \) has a splitting \( 0 \rightarrow I \rightarrow E \rightarrow [D] \rightarrow 0 \), and \( s \in \text{Im}\{H^0(C, E) \rightarrow H^0(C, [D])\} \), where \( s \) is the canonical section of \( D \) \( (s \in H^0(C, [D]) \), with \( \text{div}(s) = D \) \). We get a base point free rank two vector bundle \( E \) of degree \( d \) which generated by its sections.

So to give a base point free rank two vector bundle of degree \( d \) which generated by its sections will be the same as to give a triple \( \{I, D, f\} \), here \( D \subseteq C_d \) and \( f \in H^0(C, [D] \mid_D) \), or the same the set of all base point free rank two vector bundle of degree \( d \) which generated by its sections could be represented by the set of all triples \( \{I, D, f\} \). We will denote this as \( E = \{I, D, f\} \).

Now let \( H_d \) be the vector bundle on \( C_d \) which for each \( D \in C_d \), \( H_d \mid_D = H^0(C, I \mid_D) \), by using local coordinate, it is easy to see that \( H_d \) is a holomorphic vector bundle of dimension \( d \) on \( C_d \).

Each point of \( H_d \) could be represented as a triple \( E = \{I, D, f\} \), and each triple \( E = \{I, D, f\} \) could be represented as a point in \( H_d \), so \( H_d \) could be considered as a parameter space for the set of all base point free rank two vector bundles of degree \( d \) which generated by its sections.
3. Brill-Neother matrix for $E = \{I, D, f\}$. Let $L$ be a line bundle, $D = n_1p_1 + \cdots + n_kp_k \geq 0$ be a given effective divisor of degree $d$. For $i = 1, \ldots, k$, let $z_i$ be a local coordinate at $p_i$ with $z_i(p_i) = 0$. Then each $f \in H^0(C, L | D)$ could be represented as a set of polynomials $f = \{f_i(z_i)\}_{i=1}^k$, where $f_i(z_i) = a_0^i z_i + a_1^i z_i^2 + \cdots + a_{n_i-1}^i z_i^{n_i-1}$ is a polynomial of $z_i$ of degree less than $n_i$. So $f$ could also be denoted as a $d$-dimensional vector $f = (a_0^1, a_1^1, \ldots, a_{n_1-1}^1; a_0^2, a_1^2, \ldots, a_{n_2-1}^2; \cdots)$. This gives $H^0(C, L | D) \cong \mathbb{C}^d$, where $d = \text{deg}(d)$.

**Definition 3.** Let $L_1, L_2$ be two line bundles, $D = n_1p_1 + \cdots + n_kp_k \geq 0$ be a given effective divisor. For $f = \{f_i(z_i)\}_{i=1}^k \in H^0(C, L_1 | D)$ and $g = \{g_i(z_i)\}_{i=1}^k \in H^0(C, L_2 | D)$, we define $f \ast g \in H^0(C, L_1 \otimes L_2 | D)$ to be

$$f \ast g = \{f_i(z_i)g_i(z_i)(\text{mod}(z_i^{n_i}))\}_{i=1}^k.$$

**Lemma 2.** $f \ast g = g \ast f$, and $(f \ast g) \ast h = f \ast (g \ast h)$.

**Proof.** Trivial.

**Lemma 3.** For $E = \{I, D, f\}$, a section $s \in H^0(C, [D])$ could be lift to be a section of $H^0(C, E)$ (which means $s \in \text{Im} \{H^0(C, E) \hookrightarrow H^0(C, [D])\}$), if and only if

$$s |_{D} \ast f \in \text{Im} \{H^0(C, [D]) \hookrightarrow H^0(C, [D] | D)\}.$$  

**Proof.** See [T].

Now let $(w_1, \cdots, w_g)$ be a linear basis of $H^0(C, K)$ of the space of all holomorphic forms on $C$. then for effective divisor $D$, the Brill-Noether matrix $W_D$ for $D$ could be defined by

$$W_D = \begin{bmatrix} w_1 | D \\ w_2 | D \\ \vdots \\ w_g | D \end{bmatrix}.$$  

An element $t \in H^0(C, [D] | D)$ is in the image of map $H^0(C, [D]) \hookrightarrow H^0(C, [D] | D)$, if and only if

$$W_D \ast t = \begin{bmatrix} w_1 | D \ast t \\ w_2 | D \ast t \\ \vdots \\ w_g | D \ast t \end{bmatrix} = 0.$$  

That is $\text{Im} \{H^0(C, [D]) \hookrightarrow H^0(C, [D] | D)\} = \text{Ker} \{W_D\}$.

Now for $E = \{I, D, f\}$, we define its Brill-Noether matrix $W_E$ to be

$$W_E = \begin{bmatrix} w_1 | D \\ w_2 | D \\ \vdots \\ w_g | D \\ w_1 | D \ast f \\ w_2 | D \ast f \\ \vdots \\ w_g | D \ast f \end{bmatrix} = \begin{bmatrix} W_D \\ W_D \ast f \end{bmatrix}.$$  

**Theorem 1.** $\text{Ker} \{W_E\} = \{v \in \mathbb{C}^d | W_E \cdot v = 0\} \cong \text{Im} \{H^0(C, E) \hookrightarrow H^0(C, [D]) \hookrightarrow H^0(C, [D] | D)\}.$
Proof. By $H^0(C, [D]) \cong C^d$, each $v \in C^d$ could be identified to an element $v \in H^0(C, [D]) |_D$, let $W_D$ be the Brill-Noether matrix for $D$, then $W_D \cdot v = W_D * v$, and $(W_D * f) * v = W_D * (f * v)$. So $W_D \cdot v = 0$ if and only if $W_D * v = 0$ and $W_D * (f * v) = 0$. From $W_D * v = 0$, we get that $v \in Im \{H^0(C, [D]) \rightarrow H^0(C, [D])|_D\}$. Let it be the image of some $s \in H^0(C, [D])$, this is $v = s |_D$. Then from $(W_D * f) * v = 0$, we get $(W_D * f) * s |_D = W_D * (f * s |_D) = 0$. That means $f * s |_D \in Im \{H^0(C, [D]) \rightarrow H^0(C, [D])|_D\}$. By our Lemma 3, $s$ is then can be lift to a section of $E$.

Conversely, if $v \in Im \{H^0(C, [D]) \rightarrow H^0(C, [D])|_D\}$, let it be the image of some $s \in H^0(C, [D])$, so $W_D \cdot v = 0$, and since $s$ can be lift to a section of $E$, by our Lemma 3, $f * v \in Im \{H^0(C, [D]) \rightarrow H^0(C, [D])|_D\}$, so $W_D * f * v = 0$, we get $W_E \cdot v = 0$. This completes the proof.

Now from the exact sequence

$$0 \rightarrow I \rightarrow E \rightarrow [D] \rightarrow 0$$

we get exact sequence

$$0 \rightarrow H^0(C, I) \rightarrow H^0(C, E) \rightarrow H^0(C, [D]) \rightarrow H^1(C, I) \rightarrow \cdots.$$ 

Since $\dim H^0(C, I) = 1$, so

$$\dim H^0(C, E) = \dim Im \{H^0(C, E) \rightarrow H^0(C, [D])\} + 1 =$$

$$\dim Im \{H^0(C, E) \rightarrow H^0(C, [D]) \rightarrow H^0(C, [D])|_D\} + 2 =$$

$$\dim \ker (W_E) + 2 = d - \text{rank}(W_E) + 2.$$ 

That is

**Theorem 2.** Let $E = \{I, D, F\}$ and $W_E$ be its Brill-Noether matrix, then we have $\ker (W_E) \cong Im \{H^0(C, E) \rightarrow H^0(C, [D]) \rightarrow H^0(C, [D])|_D\}$, and in particular $\dim H^0(C, E) = d - \text{rank}(W_E) + 2$.

Now we define the Brill-Noether variety $C^r_{2, d}$ for rank two vector bundles to be

$$C^r_{2, d} = \{E \in H_d \mid \dim H^0(C, E) \geq r + 1\}.$$ 

By Theorem 2, $C^r_{2, d}$ could also be given by

$$C^r_{2, d} = \{E \in H_d \mid \text{rank}(W_E) \leq d - r + 1\}.$$ 

This gives $C^r_{2, d}$ as a subvariety of $H_d$ which $C^r_{2, d}$ is defined locally by the simultaneously vanishing of all $(d - r + 2) \times (d - r + 2)$ minors of $W_E$.

By using the Brill-Noether matrix $W_E$, locally, we get a holomorphic map $BN : H_d \rightarrow M(2g, d)$ with $BN(E) = W_E$ for each $E \in H_d$, where $M(2g, d)$ is the variety of all $2g \times d$ complex matrices. Let

$$M_{d-r+1} = \{E \in M(2g, d) \mid \text{rank}(E) \leq d - r + 1\}.$$ 

Then $M_{d-r+1}$ is a subvariety of $M(2g, d)$, and $\text{codim}(M_{d-r+1}) = (2g - (d - r + 1)) \times (d - (d - r + 1))$. By definition, we have $C^r_{2, d} = BN^{-1}(M_{d-r+1})$. So from the Theory of determinant variety, we get that if $C^r_{2, d} \neq \emptyset$, then

$$\text{codim} C^r_{2, d} \leq (2g - (d - r + 1)) \times (d - (d - r + 1)).$$ 

This is

$$\dim C^r_{2, d} \geq 2d - (2g - (d - r + 1)) \times (d - (d - r + 1)) =$$
Here \( \rho_2(g, d, r) = 4(g - 1) + 1 - (r + 1)(2(g - 1) - d + r + 1) \) is the Brill-Noether number for rank two vector bundles. Same as the case of line bundles, we get that the expected dimension of \( C_{2,d} \) is \( \rho_2(g, d, r) + 2r + 1 \), this is

**Theorem 3.** If \( C_{2,d} \neq \emptyset \), then each component of \( C_{2,d} \) will have dimension at least \( \rho_2(g, d, r) + 2r + 1 \).

4. The Petri map. Since \( C_d = \text{BN}^{-1}(M_d-r+1) \), to get the dimension of \( C_d \), analogous to the case of line bundles, we should consider the tangent map

\[ \text{BN}^*: T_E \to T_{\text{BN}(E)} \]

for each \( E = \{I, f, D\} \in H_d \). Here \( T_E \) and \( T_{\text{BN}(E)} \) are the tangent space of \( E \) and \( \text{BN}(E) \) in \( H_d \) and \( M(2g, d) \).

Now let \( E = \{I, D, f\} \), then

\[ \text{BN}(E) = W_E = \begin{bmatrix} W_D \\ W_D * f \end{bmatrix} \]

Since for each \( D \in C_d \), the tangent space of \( C_d \) at \( D \) is \( T_D = H^0(C, [D] \mid D) \) (Ref [ACGH] P160), so by definition we get that the tangent space of \( H_d \) at \( E \) is \( T_E = H^0(C, [D] \mid D) \oplus H^0(C, I \mid D) \).

Now let \( t = (-v, u) \in T_E = H^0(C, [D] \mid D) \oplus H^0(C, I \mid D) \), then by direct calculation, we have

\[ \text{BN}^*(t) = \begin{bmatrix} W_D * (-v) \\ W_D * (-v) * f + W_D * u \end{bmatrix} \]

Where \( W_D \) means the differential of \( W_D \) with respect to the local coordinates, and \( f = I \).

To get the dimension of \( C_{2,d} \), we need to get the dimension of the space \( V = \{t \in T_E \mid \text{BN}^*(t) \in T_{\text{BN}(E)}(M_d-r+1)\} \). But from the theory of determinant variety (Ref [ACGH] p69), we know that \( \text{BN}^*(t) \in T_{\text{BN}(E)}(M_d-r+1) \) if and only if \( \text{Ker}(W_E) \cdot \text{BN}^*(t) \subset \text{Im}(W_E) = C^d \cdot W_E \). Here \( \text{Ker}(W_E) = \{(b, e) = (b_1, \ldots, b_g; e_1, \ldots, e_g) \in C^d \mid (b, e)W_E = 0\} \).

Now let \( (b, e) = (b_1, \ldots, b_g; e_1, \ldots, e_g) \in \text{Ker}(W_E) \), this is \( (b, e) \cdot W_E = b \cdot W_D + e \cdot W_D * f = 0 \). Choose an open cover \( \{U_a\}_{a=1}^k \) of \( C \), let \( s = \{s_\alpha\}_{\alpha=1}^k \in H^0(C, [D]) \) be the canonical section of \( [D] \), this is \( s \in H^0(C, [D]) \) and \( \text{div}(s) = D \). For the linear basis \( \{w_1, \ldots, w_g\} \) of the holomorphic forms, let \( w_i \) be given with respect to the open cover by \( w_i = \{w_{i1}\} \), let \( bw = b_1w_1 + \cdots + b_gw_g = \{b_1w_{i1} + \cdots + b_gw_{i1}\} = \{bw_{i1}\} \in H^0(C, K) \), and \( ew = e_1w_1 + \cdots + e_gw_g = \{e_1w_{i1} + \cdots + e_gw_{i1}\} = \{ew_{i1}\} \in H^0(C, K) \), let \( f = \{f_\alpha\} \) be a given representation for \( f \in H^0(C, I \mid D) \), where \( f_\alpha \) is a holomorphic function on \( U_\alpha \).

**Lemma 4.** \( (b, e) \in \text{Ker}(W_E) \) if and only if

\[ F = \{F_\alpha = \begin{bmatrix} e \cdot w_\alpha \\ -(b \cdot w_\alpha + e \cdot w_\alpha * f_\alpha)/s_\alpha \end{bmatrix} \} \in H^0(C, K(-E)). \]

Here \( (-E) \) is the dual vector bundle of \( E \).

**Proof.** For later using and also for making our notations easy to understand, we will give a proof of this Lemma in detail, and we will also use the proof to give a proof of Riemann-Roch Theorem for rank two vector bundles.

Let \( \{U_\alpha\}_{\alpha=1}^k \) be the open cover of \( C \). Then on \( U_\alpha \cap U_\beta \), the transition matrix of \( E = \{I, f, D\} \) can be given by

\[ E_{\alpha\beta} = \begin{bmatrix} 1 & (f_\alpha - f_\beta)/s_\beta \\ 0 & s_\alpha/s_\beta \end{bmatrix} \]
where $e = \{e_{\alpha\beta} = (f_{\alpha} - f_{\beta})/s_{\beta}\}$ is a representation of $e \in H^1(C, [-D])$.

From $E_{\alpha\beta}$, and by the definition of dual vector bundle, the transition matrix of $K(-E)$ can be given on $U_{\alpha} \cap U_{\beta}$ by

$$(K(-E))_{\alpha\beta} = \begin{bmatrix} k_{\alpha\beta} & 0 \\ -k_{\alpha\beta}(f_{\alpha} - f_{\beta})/s_{\beta} & k_{\alpha\beta}s_{\beta}/s_{\alpha} \end{bmatrix}$$

where $\{k_{\alpha\beta}\}$ is the transition function of the canonical line bundle $K$.

By definition, $K(-E)$ is an extension of $K$ by $K[-D]$, which determined also by $f \in H^0(C, I_D)$.

Now let $(b, e) \in Ker(W_E)$, that is $b \cdot W_D + e \cdot W_D \ast f = 0$, let $ew = e_1w_1 + \cdots + e_gw_g \in H^0(C, K)$, $bw = b_1w_1 + \cdots + b_gw_g \in H^0(C, K)$, then $b \cdot W_D + e \cdot W_D \ast f = 0$ means $ew \mid_D \ast f = -bw \mid_D$, by our Lemma 3 (also Ref [T]), that means, $ew$ can be lift to a section of $K(-E)$ and

$$F = \{F_{\alpha} = [e \cdot w_{\alpha} - (b \cdot w_{\alpha} + e \cdot w_{\alpha} \ast f_{\alpha})/s_{\alpha}] \} \in H^0(C, K(-E)).$$

is one of the lift. This can also be proved by direct computation that $F_{\alpha} = K(-E)_{\alpha\beta} \cdot F_{\beta}$.

Conversely, let

$$F = \{F_{\alpha} = [e \cdot w_{\alpha} / v_{\alpha}] \} \in H^0(C, K(-E)).$$

then $ew = e_1w_1 + \cdots + e_gw_g \in \{ew_{\alpha} = e_1w_1 \mid v_{\alpha} + \cdots + e_gw_g \mid v_{\alpha}\}$, is a section of $K$, here $e = (e_1, \cdots, e_g)$, and $F$ is a lift of $ew$. $ew \in H^0(C, K)$ can be lift to a section of $H^0(C, K(-E))$, by our Lemma 3, there exists an $bw = b_1w_1 + \cdots + b_gw_g \in H^0(C, K)$, such that $ew \mid_D \ast f = -bw \mid_D$, or the same, $ew \mid_D \ast f + bw \mid_D = 0$, that is $(b, e) \cdot W_E = 0$, so $(b, e) \in Ker(W_E)$.

Now if $ew = 0$, that is $e = 0$, then $F = \{F_{\alpha} = [0 \mid v_{\alpha}] \} \in H^0(C, K(-E))$ means $v = \{v_{\alpha}\} \in H^0(C, K \otimes [-D])$, but we know that $H^0(C, K \otimes [-D]) = \{w \in H^0(C, K) \mid w \mid_D = 0\}$. Assume $v = b_1w_1 + \cdots + b_gw_g = bw$, here $b = (b_1, \cdots, b_g)$, then $bw \mid_D = 0$ means $bW_D = 0$, so $(b, 0)W_E = 0$, this is $(b, 0) \in Ker(W_E)$. That completes the proof.

From the proof, we get

**COROLLARY 1.** $H^0(C, K(-E)) \cong Ker(W_E)$, and in particular

$$\dim H^0(C, K(-E)) = 2g - \text{rank}(W_E).$$

But from the definition of $W_E$, we know that

$$\dim H^0(C, E) = d - \text{rank}(W_E) + 2.$$
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\[(b,e)BN^*(t) = (b,e) \left[ W_D * (-v) \right. \]

\[\left. W_D * (-v) * f + W_D * u \right] \subseteq \text{Im}(W_E).\]

for all \((b,e) \in \text{Ker}(W_E)\). For this, we will first define a short exact sequence of sheaves.

Let \(V\) be a vector bundle on \(C\), we will use \(V\) itself to denote the sheaf of holomorphic sections of \(V\). For \(E = \{I, f, D\}\), let \(\{U_\alpha\}_{\alpha=1}^k\) be the given open cover of \(C\), and \(s = \{s_\alpha\}_{\alpha=1}^k \in H^0(C, [D])\) be the canonical section of \([D]\), this is \(s \in H^0(C, [D])\) and \(\text{div}(s) = D\).

Let \(f = \{f_\alpha\}\) be a given representation for \(f \in H^0(C, I_{|D})\), where \(f_\alpha\) is a holomorphic function on \(U_\alpha\). Then by using the transition matrix \(E_{\alpha\beta}\) given in the proof of Lemma 4, one can check directly that

\[F = \{F_\alpha = \begin{bmatrix} f_\alpha \\ s_\alpha \end{bmatrix} \} \in H^0(C, E).\]

is the lift of the canonical section \(s\). Now let \(P_1 : K(-E) \twoheadrightarrow K\) be the projective map which induced from sequence \(0 \twoheadrightarrow K[-D] \twoheadrightarrow K \otimes [-E] \twoheadrightarrow K \twoheadrightarrow 0\), then from \(F\) and \(P_1\), we define a map of sheaves \(K(-E) \twoheadrightarrow K \oplus K\) by

\[x \mapsto (P_1(x), -(x, F))\]

here \(x \in K(-E)\), and \((\ , \ ) : (K(-E) \otimes E) \twoheadrightarrow K\) is the duality map. We also define a map of sheaves \(K \oplus K \twoheadrightarrow K_{|D}\) to be \((s, t) \mapsto (s_{|D} * f + t_{|D})\) for \((s, t) \in K \oplus K\).

Locally, let \(\{U_\alpha\}\) be the given open cover of \(C\), if \(G \subseteq K(-E) \cap U_\alpha\), then \(K(-E) \twoheadrightarrow K\) is defined by \(\begin{bmatrix} a \\ b \end{bmatrix} \mapsto (a, -a f_\alpha - b s_\alpha)\), and the map \(K \oplus K \twoheadrightarrow K_{|D}\) could be given by \((c, d) \mapsto (c_{|D} * f + d_{|D})\).

**Lemma 5.** The sequence \(0 \twoheadrightarrow K(-E) \twoheadrightarrow K \oplus K \twoheadrightarrow K_{|D} \twoheadrightarrow 0\) is a short exact sequence of sheaves on \(C\).

**Proof.** We will use the local representation to give the proof.

If \(\begin{bmatrix} a \\ b \end{bmatrix} \in K(-E),\) and \(\begin{bmatrix} a \\ b \end{bmatrix} \mapsto (a, -(a f - b s)) = 0\), then \(a = 0\), and since \(s \neq 0\) so \(bs = 0\) means \(b = 0\), the map \(K(-E) \twoheadrightarrow K \oplus K\) is injective.

If \((c, d) \in K \oplus K\), and \((c, d) \mapsto (c_{|D} * f + d_{|D}) = 0\), we then get \(c_{|D} * f = -d_{|D}\), by our Lemma 3, \(c\) can be lift locally to section of \(K(-E)\) and same as Lemma 4, \(\begin{bmatrix} c \\ -(c f + d)/s \end{bmatrix} \in K(-E)\) is one of the lift. But \(\begin{bmatrix} c \\ -(c f + d)/s \end{bmatrix} \mapsto (c, -c f + (c f + d)) = (c, d)\).

This shows that the sequence is exact at \(K \oplus K\).

Also it is easy to see that the map \(K \oplus K \twoheadrightarrow K_{|D}\) is an onto map. This completes the proof.

From this short exact sequence, we get a long exact sequence

\[0 \twoheadrightarrow \text{H}^0(C, K(-E)) \twoheadrightarrow \text{H}^0(C, K \oplus K) \twoheadrightarrow \text{H}^0(C, K_{|D}) \twoheadrightarrow \text{H}^1(C, K(-E)) \twoheadrightarrow \cdots\]

\(a \in \text{H}^0(C, K_{|D})\) is in the image of map \(\text{H}^0(C, K \oplus K) = \text{H}^0(C, K) \oplus \text{H}^0(C, K) \twoheadrightarrow \text{H}^0(C, K_{|D})\) if and only if \(\delta(a) = 0\), here \(\delta : \text{H}^0(C, K_{|D}) \twoheadrightarrow \text{H}^1(C, K(-E))\) is the co-boundary map. But from Serra duality, we know that for \(\delta(a) \in \text{H}^1(C, K(-E))\), \(\delta(a) = 0\) if and only if for any \(f \in \text{H}^0(C, E)\), we have \(\langle \delta(a), f \rangle = 0\). Here \((\ , \ ) : \text{H}^1(C, K(-E)) \otimes \text{H}^0(C, E) \twoheadrightarrow \text{H}^1(C, K)\) is the duality map.

Now assume, for open cover \(\{U_\alpha\}\), \(a\) is given by \(a = \{a_\alpha\}\), where \(a_\alpha \in \text{H}^0(U_\alpha, K_{|U_\alpha})\) and \(a_\alpha |_{D \cup U_\alpha} = a |_{D \cup U_\alpha}\). Then by direct calculation, we get \(\delta(a) \in \text{H}^1(C, K(-E))\), could be represented as

\(\delta(a) = \left\{ \begin{bmatrix} k_{\alpha\beta}(-a_\alpha + a_\beta)/s_\alpha \end{bmatrix} \right\} = \begin{bmatrix} 0 \\ \tilde{\delta}(a) \end{bmatrix}\),
where \( \delta : H^0(C, K \mid_D) \rightarrow H^1(C, K[-D]) \) is the co-boundary map from the following sequence

\[
0 \rightarrow H^0(C, K[-D]) \rightarrow^\ast H^0(C, K) \rightarrow H^0(C, K \mid_D) \rightarrow H^1(C, K[-D]) \rightarrow \cdots.
\]

So for any \( f = \begin{bmatrix} y_\alpha \\ x_\alpha \end{bmatrix} \in H^0(C, E) \), the dual map could be given by

\[
(\delta(a), f) = \left( \begin{bmatrix} 0 \\ (-a_\alpha + a_\beta)/s_\alpha \end{bmatrix}, \begin{bmatrix} y_\alpha \\ x_\alpha \end{bmatrix} \right)
= \left( (-a_\alpha + a_\beta)/s_\alpha \cdot x_\alpha \right) = (\tilde{\delta}(a), \{x_\alpha\}).
\]

but \( \delta(a) = 0 \) if and only if \( (\delta(a), f) = 0 \) for all \( f \in H^0(C, E) \), from what we get above, this is same that \( \delta(a) = 0 \) if and only if for any \( x = \{x_\alpha\} \in \text{Im}(H^0(C, E) \rightarrow H^0(C, [D])) \), \( (\tilde{\delta}(a), x) = 0 \). We get the following Lemma.

**Lemma 6.** For \( a \in H^0(C, K \mid_D) \), \( \delta(a) \in H^1(C, K(-E)) \), with \( \delta(a) = 0 \) if and only if for any \( x = \{x_\alpha\} \in \text{Im}(H^0(C, E) \rightarrow H^0(C, [D])) \), \( (\delta(a), x) = 0 \).

Now go back to the tangent map of \( BN : H_d \mapsto M(2g, d) \).

For \( E = \{I, f, D\} \in C_{2, d}^r \), we know

\[
BN(E) = WE = \begin{bmatrix} WD \\ WD \ast f \end{bmatrix}.
\]

if \( t = (u, -v) \in T_E = H^0(C, [D] \mid_D) \oplus H^0(C, I \mid_D) \), then

\[
BN^*(t) = \begin{bmatrix} W_D \ast (-v) \\ W_D \ast (-v) \ast f + W_D \ast u \end{bmatrix}.
\]

But we know that \( BN^*(t) \in T_{BN(E)}(M_{d-r+1}) \) if and only if \( Ker(W_E) \cdot BN^*(t) \in \text{Im}(W_E) \). Since \( \text{Im}(W_E) = C^{2g} \cdot W_E = \{(c, d) \begin{bmatrix} W_D \\ W_D \ast f \end{bmatrix} | (c, d) \in C^{2g}\} \). If we identify \( C^{2g} = C^g \oplus C^g \cong H^0(C, K) \oplus H^0(C, C) \), then we get

\[
\text{Im}(W_E) = \text{Im}(H^0(C, K) \oplus H^0(C, K) \mapsto H^0(C, K \mid_D)).
\]

Where the map \( H^0(C, K) \oplus H^0(C, K) \mapsto H^0(C, K \mid_D) \) is induced from above exact sequence.

From this we get \( BN^*(t) \in T_{BN(E)}(M_{d-r+1}) \) if and only if for any \( (b, e) \in Ker(W_E) \), \( (b, e)BN^*(t) \in \text{Im}(W_E) \). This is \( \delta((b, e)BN^*(t)) = 0 \). By Lemma 6, we get

**Lemma 7.** let \( t \in T_E \), then \( BN^*(t) \in T_{BN(E)}(M_{d-r+1}) \) if and only if for any \( (b, e) \in Ker(W_E) \), we have \( (\tilde{\delta}(b, e)BN^*(t), x) = 0 \) for all \( x \in \text{Im}(H^0(C, E) \mapsto H^0(C, [D])) \).

But by direct calculation, we get

\[
(b, e)BN^*(t) = (b, e) \begin{bmatrix} \hat{W}_D \ast (-u) \\ \hat{W}_D \ast (-u) \ast f + \hat{W}_D \ast u \end{bmatrix} = b\hat{W}_D \ast u + e\hat{W}_D \ast u \ast f + eW_D \ast v
= \begin{bmatrix} eW_D \\ -(b\hat{W}_D + eW_D \ast f) \end{bmatrix} \ast \begin{bmatrix} u \\ -v \end{bmatrix}.
\]

Notice that by using local coordinate, it is easy to see that

\[
\begin{bmatrix} eW_D \\ -(b\hat{W}_D + eW_D \ast f) \end{bmatrix} = \begin{bmatrix} eW_D \\ -(b\hat{W}_D + eW_D \ast f)/s \end{bmatrix}.
\]

Since \( (b, e) \in Ker(W_E) \), by Lemma 4, we get

\[
\begin{bmatrix} eW_D \\ -(b\hat{W}_D + eW_D \ast f)/s \end{bmatrix} \in \text{Im}(H^0(C, K(-E)) \mapsto H^0(C, K(-E) \mid_D)).
\]
let it be the image of some \( F \in H^0(C, K(-E)) \). Now notice that \( E |_D = I |_D \oplus [D] |_D = T_E \) and \( K(-E) |_D = K |_D \oplus [D] |_D = (I |_D \oplus [D] |_D)^* = T_E^* \) then follow the proof of Lemma 1.5 p162 [ACGH] step by step, for \( x \in \text{Im}\{H^0(C, E) \hookrightarrow H^0(C, [D])\} \), we have

\[
\delta ((b, e)BN^*(t)), x) = (\delta (F * t), x) = (\delta_1(t), (F \otimes x)) = (t, (F \otimes x) |_D)
\]

Where \( \delta_1 : (I |_D \oplus [D] |_D) \rightarrow H^1(C, E[-D]) \) is the co-boundary map follow from sequence \( 0 \rightarrow E[-D] \rightarrow E \rightarrow E |_D \rightarrow 0 \). So \( t \in V = \{ t \in T_E \mid BN^*(t) \in T_{BN(E)}(M_{d-\tau +1}) \} \) if and only if for any \( F \in H^0(C, K(-E)) \) and \( x \in \text{Im}\{H^0(C, E) \hookrightarrow H^0(C, [D])\} \), we have \( \langle t, (F \otimes x) |_D \rangle = 0 \).

\[ \text{LEMMA 8.} \quad t \in V = \{ t \in T_E \mid BN^*(t) \in T_{BN(E)}(M_{d-\tau +1}) \} \text{ if and only if} \]

\[ t \in \{ \text{Im}\{H^0(C, K(-E)) \otimes \text{Im}\{H^0(C, E) \hookrightarrow H^0(C, [D])\}\} \]

\[ \hookrightarrow H^0(C, K(-E)[D]) \hookrightarrow H^0(C, K(-E)[D] |_D) \} \]

\[ \rightarrow H^0(C, K(-E)[D]) \hookrightarrow H^0(C, K(-E)[D] |_D) \} \]

Now assume \( E \in C_{2,d} - C_{2,d}^{r+1} \). From what we get above, the expected dimension of \( C_{2,d}^r \) at \( E \) could be given by

\[
\text{dim}(C_{2,d}^r) = \text{dim}(V) = 2d - \text{dim}\{\text{Im}\{H^0(C, K(-E)) \otimes \text{Im}\{H^0(C, E) \hookrightarrow H^0(C, [D])\}\} \}
\]

\[ \rightarrow H^0(C, K(-E)[D]) \hookrightarrow H^0(C, K(-E)[D] |_D) \} \]

\[ = 2d - (2(g - 1) - d + r + 1)r + 2(g - 1) - d + r + 1 + \text{dim}W. \]

where \( (2(g - 1) - d + r + 1)r = \text{dim}\{H^0(C, K(-E)) \otimes \text{Im}\{H^0(C, E) \hookrightarrow H^0(C, [D])\}\} = \text{dim}H^0(C, K(-E)) \times \text{dim}\{H^0(C, E) \hookrightarrow H^0(C, [D])\} \}, \quad 2(g - 1) - d + r + 1 = \text{dim}\{H^0(C, K(-E)[D]) \hookrightarrow H^0(C, K(-E)[D] |_D) \}
\]

We then get

\[
\text{dim}(C_{2,d}^r) = 4(g - 1) + 1 - (r + 1)(2(g - 1) - d + r + 1) + 2r + 1 + \text{dim}W
\]

\[ = \rho(2, d, r) + 2r + 1 + \text{dim}W. \]

\[ \text{THEOREM 3.} \quad C_{2,d}^r \text{ has the expected dimension } \rho(2, d, r) + 2r + 1 \text{ at } E \in C_{2,d} - C_{2,d}^{r+1}, \text{ if and only if for all } E \in C_{2,d}^r, W = \{0\}. \]

This is the same that \( C_{2,d}^r \) has the expected dimension \( \rho(2, d, r) + 2r + 1 \), if and only if for all \( E \in C_{2,d}^r \), the map

\[
H^0(C, K(-E)) \otimes \text{Im}\{H^0(C, E) \hookrightarrow H^0(C, [D])\} \hookrightarrow H^0(C, K(-E)[D])
\]

is injective.

Compare with the case of line bundles, we then called the map

\[
H^0(C, K(-E)) \otimes \text{Im}\{H^0(C, E) \hookrightarrow H^0(C, [D])\} \hookrightarrow H^0(C, K(-E)[D])
\]

the Petri map for rank two vector bundles. We have

\[ \text{THEOREM 4.} \quad C_{2,d}^r \text{ has the expected dimension } \rho(2, d, r) + 2r + 1, \text{ if and only if for all } E \in C_{2,d}^r, \text{ the Petri map is injective.} \]

This is a generalization of Lemma 1.6 of [ACGH] P163.
REFERENCES


