ROUGH SINGULAR INTEGRALS WITH KERNELS SUPPORTED
BY SUBMANIFOLDS OF FINITE TYPE

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1. Introduction. Let $n \geq 2$, $\mathbb{R}^n$ be the $n$-dimensional Euclidean space, and $S^{n-1}$ denote the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue measure $d\sigma$. For $d \in \mathbb{N}$, let $B(0,1)$ be the unit ball centered at the origin in $\mathbb{R}^n$ and $\Phi : B(0,1) \to \mathbb{R}^d$ be a $C^\infty$ mapping. Define the singular integral operator $T_\Phi$ and the related maximal operator $M_\Phi$ by

$$T_\Phi f(x) = \text{p.v.} \int_{B(0,1)} f(x - \Phi(y)) \frac{\Omega(y)}{|y|} dy,$$  \hspace{1cm} (1.1)

$$M_\Phi f(x) = \sup_{0 < r \leq 1} \frac{1}{r^n} \int_{|y| \leq r} |f(x - \Phi(y))| \Omega(y) dy$$  \hspace{1cm} (1.2)

for $f \in S(\mathbb{R}^d)$. Here $\Omega$ is a homogeneous function of degree 0, integrable over $S^{n-1}$ and satisfies the vanishing condition

$$\int_{S^{n-1}} \Omega(u) d\sigma(u) = 0. \hspace{1cm} (1.3)$$

The corresponding maximal truncated singular integral operator $T^*_\Phi$ is defined by

$$T^*_\Phi f(x) = \sup_{\varepsilon > 0} \left| \int_{\varepsilon \leq |y| < 1} f(x - \Phi(y)) \frac{\Omega(y)}{|y| > \varepsilon} dy \right|.$$  \hspace{1cm} (1.4)

When $\Phi(y) \equiv y$, $T_\Phi$ is simply the localized version of a classical Calderón-Zygmund operator and we shall denote it by $T$. Our point of departure is the following $L^p$ boundedness result from [St].

**Theorem 1.1.** Let $T_\Phi$ and $M_\Phi$ be given as in (1.1)-(1.3). Assume that:

(i) $\Phi$ is of finite type at 0;

(ii) $\Omega \in C^1(S^{n-1})$.

Then for $1 < p < \infty$ there exists a constant $C_p > 0$ such that

$$\|T_\Phi f\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}$$  \hspace{1cm} (1.5)

and

$$\|M_\Phi f\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}$$  \hspace{1cm} (1.6)
for every $f \in L^p(\mathbb{R}^d)$.

Recently, the results in Theorem 1.1 were improved by Fan, Guo, and Pan in [FGP] who showed that the $L^p$ boundedness of $T_\Phi$ and $M_\Phi$ continues to hold if the condition $\Omega \in C^1(S^{n-1})$ is replaced by the weaker condition $\Omega \in L^q(S^{n-1})$ for some $q > 1$. Also, the authors of [FGP] were able to establish the $L^p$ boundedness of the maximal operator $T_\Phi$ under the condition $\Omega \in L^q(S^{n-1})$ for some $q > 1$.

The main purpose of this paper is to present further improvements of the above results in which the condition $\Omega \in L^q(S^{n-1})$ is replaced by a weaker condition $\Omega \in B^{0,0}_q(S^{n-1})$. It is worth pointing out that the authors of this paper were able in [AqAsP] to show that the condition $\Omega \in B^{0,0}_q(S^{n-1})$ is the best possible for the $L^p$ boundedness of the classical operator $T$ to hold. Namely, the $L^p$ boundedness of $T$ may fail for any $p$ if it is replaced by a weaker condition $\Omega \in B^{0,v}_q(S^{n-1})$ for any $-1 < v < 0$ and $q > 1$.

The definition of the block spaces $B^{0,v}_q(S^{n-1})$ on the sphere will be recalled in Section 2.

Our main results can be stated as follows.

**Theorem 1.2.** Let $T_\Phi$ and $M_\Phi$ be given as in (1.1)-(1.3). Assume that:

(i) $\Phi$ is of finite type at 0;

(ii) $\Omega \in B^{0,0}_q(S^{n-1})$ for some $q > 1$.

Then

$$\|T_\Phi f\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}$$

(1.7)

and

$$\|M_\Phi f\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}$$

(1.8)

hold for all $1 < p < \infty$ and $f \in L^p(\mathbb{R}^d)$.

**Theorem 1.3.** Let $\Omega$ and $T_\Phi$ be given as in (1.3)-(1.4). Assume that:

(i) $\Phi$ is of finite type at 0;

(ii) $\Omega \in B^{0,0}_q(S^{n-1})$ for some $q > 1$.

Then for $1 < p < \infty$ there exists a constant $C_p > 0$ such that

$$\|T_\Phi f\|_{L^p(\mathbb{R}^d)} \leq C_p \|f\|_{L^p(\mathbb{R}^d)}$$

(1.9)

for every $f \in L^p(\mathbb{R}^d)$.

2. Preliminaries. Let us begin with the definition of block functions on $S^{n-1}$.

**Definition 2.1.** (1) For $x_0' \in S^{n-1}$ and $0 < \theta_0 \leq 2$, the set

$$B(x_0', \theta_0) = \{x' \in S^{n-1} : |x' - x_0'| < \theta_0\}$$

is called a cap on $S^{n-1}$.

(2) For $1 < q \leq \infty$, a measurable function $b$ is called a $q-$block on $S^{n-1}$ if $b$ is a function supported on some cap $I = B(x_0', \theta_0)$ with $\|b\|_{L^q} \leq \frac{1}{|I|^{1/q'}}$ where $|I| = \sigma(I)$ and $1/q + 1/q' = 1$.

(3) $B^{0,v}_q(S^{n-1}) = \{\Omega \in L^1(S^{n-1}) : \Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu \text{ where each } c_\mu \text{ is a complex number; each } b_\mu \text{ is a } q-$block supported on a cap } I_\mu \text{ on } S^{n-1}; \text{ and } M^{v}_q(\Omega) = \sum_{\mu=1}^{\infty} |c_\mu| (1 + \phi_{\kappa,v}(|I_\mu|)) < \infty\}$, where

$$\phi_{\kappa,v}(t) = \chi_{(0,1)}(t) \int_t^1 u^{-1-\kappa \log^v (u^{-1})} du.$$  

(2.1)
One observes that
\[ \phi_{\kappa,v}(t) \sim t^{-\kappa} \log^v(t^{-1}) \text{ as } t \to 0 \text{ for } \kappa > 0, v \in \mathbb{R}, \]
\[ \phi_{0,v}(t) \sim \log^{v+1}(t^{-1}) \text{ as } t \to 0 \text{ for } v > -1. \]

The following properties of \( B_{q}^{\kappa,v} \) can be found in [KS]:

(i) \( B_{q}^{\kappa,v_2} \subset B_{q}^{\kappa,v_1} \) if \( v_2 > v_1 > -1 \) and \( \kappa \geq 0 \); \hspace{1cm} (2.2)

(ii) \( B_{q}^{\kappa_2,v_2} \subset B_{q}^{\kappa_1,v_1} \) if \( v_1, v_2 > -1 \) and \( 0 \leq \kappa_1 < \kappa_2 \); \hspace{1cm} (2.3)

(iii) \( B_{q_2}^{\kappa,v} \subset B_{q_1}^{\kappa,v} \) if \( 1 < q_1 < q_2 \); \hspace{1cm} (2.4)

(iv) \( L^q(S^{n-1}) \subset B_{q}^{\kappa,v}(S^{n-1}) \) for \( v > -1 \) and \( \kappa \geq 0 \). \hspace{1cm} (2.5)

In their investigations of block spaces, Keitoku and Sato showed in [KS] that these spaces enjoy the following properties:

**Lemma 2.2.** (i) If \( 1 < p \leq q \leq \infty \), then for \( \kappa > \frac{1}{p'} \) we have

\[ B_{q}^{\kappa,v}(S^{n-1}) \subseteq L^p(S^{n-1}) \text{ for any } v > -1; \]

(ii)

\[ B_{q}^{\kappa,v}(S^{n-1}) = L^q(S^{n-1}) \text{ if and only if } \kappa \geq \frac{1}{q} \text{ and } v \geq 0; \]

(iii) for any \( v > -1 \), we have

\[ \bigcup_{q > 1} B_{q}^{0,v}(S^{n-1}) \not\subset \bigcup_{q > 1} L^q(S^{n-1}). \]

For a \( q \)-block function \( b \) on \( S^{n-1} \) supported in an interval with \( q > 1 \) and \( \|b\|_q \leq |I|^{-1/q'} \), \( 1/q + 1/q' = 1 \), we define the function \( \tilde{b} \) on \( S^{n-1} \) by

\[ \tilde{b}(x) = b(x) - \int_{S^{n-1}} b(u) d\sigma(u). \] \hspace{1cm} (2.6)

Then one can easily see that \( \tilde{b} \) enjoys the following properties:

\[ \int_{S^{n-1}} \tilde{b}(u) d\sigma(u) = 0; \] \hspace{1cm} (2.7)

\[ \|\tilde{b}\|_q \leq 2 |I|^{-1/q'}; \] \hspace{1cm} (2.8)

\[ \|\tilde{b}\|_1 \leq 2. \] \hspace{1cm} (2.9)

To simplify matters, we shall call the function \( \tilde{b} \) the blocklike function corresponding to the block function \( b \).

We shall need the following two lemmas from [FGP].

**Lemma 2.3.** Let \( \Phi : B(0,1) \to \mathbb{R}^d \) be a smooth mapping and \( \Omega \) be a homogeneous function of degree 0. Suppose that \( \Phi \) is of finite type at 0 and \( \Omega \in L^q(S^{n-1}) \) for some \( q > 1 \). Then there are \( N \in \mathbb{N}, \delta \in (0,1], C > 0 \) and \( j_0 \in \mathbb{Z}_- \) such that

\[ \left| \int_{2^{-1} \leq |y| < 2^j} e^{-i\xi \cdot \Phi(y)} \frac{\Omega(y)}{|y|^n} dy \right| \leq C \|\Omega\|_q (2^{Nj} |\xi|)^{-\delta} \] \hspace{1cm} (2.10)
for all \( j \leq j_0 \) and \( \xi \in \mathbb{R}^d \).

**Lemma 2.4.** Let \( m \in \mathbb{N} \) and \( R(\cdot) \) be a real-valued polynomial on \( \mathbb{R}^n \) with \( \deg(R) \leq m - 1 \). Suppose that

\[
P(y) = \sum_{|\alpha|=m} a_\alpha y^\alpha + R(y),
\]

\( \Omega \) is a homogeneous function of degree zero, and \( \Omega \in L^q(S^{n-1}) \) for some \( q > 1 \). Then there exists a constant \( C = C(m, n) > 0 \) such that

\[
\left| \int_{|y| \leq \omega} e^{-iP(y)} \frac{\Omega(y)}{|y|^2} dy \right| \leq C \|\Omega\|_q (2^{mj} \sum_{|\alpha|=m} |a_\alpha|)^{-\frac{1}{2m}}
\]

holds for all \( j \in \mathbb{Z} \) and \( a_\alpha \in \mathbb{R} \).

The proofs of our results will rely heavily on the following lemma from [AqP] which is an extension of earlier results of Duoandikoetxea-Rubio de Francia in [DR] and Fan-Pan in [FP].

**Lemma 2.5.** Let \( N \in \mathbb{N} \) and \( \{\sigma_k^{(l)} : k \in \mathbb{Z}, 0 \leq l \leq N\} \) be a family of Borel measures on \( \mathbb{R}^n \) with \( \sigma_k^{(0)} = 0 \) for every \( k \in \mathbb{Z} \). Let \( \{a_l : 1 \leq l \leq N\} \subseteq \mathbb{R}^+/(0, 2) \), \( \{m_l : 1 \leq l \leq N\} \subseteq \mathbb{N} \), \( \{\alpha_l : 1 \leq l \leq N\} \subseteq \mathbb{R}^+ \), and let \( L_l : \mathbb{R}^n \to \mathbb{R}^{m_l} \) be linear transformations for \( 1 \leq l \leq N \). Suppose that for all \( k \in \mathbb{Z}, 1 \leq l \leq N \), for all \( \xi \in \mathbb{R}^n \) and for some \( C > 0 \), \( A > 1, p_0 \in (2, \infty) \) we have the following:

\[
\begin{align*}
(i) \quad & \|\sigma_k^{(l)}\| \leq CA; \\
(ii) \quad & |\hat{\sigma}_k^{(l)}(\xi)| \leq CA |a_l^k A L_l(\xi)|^{-\frac{1}{4}}; \\
(iii) \quad & |\hat{\sigma}_k^{(l)}(\xi) - \hat{\sigma}_k^{(l-1)}(\xi)| \leq CA |a_l^k A L_l(\xi)|^{-\frac{1}{4}}; \\
(iv) \quad & \left\|\left(\sum_{k \in \mathbb{Z}} |\sigma_k^{(l)} * g_k|^2\right)^{\frac{1}{2}}\right\|_{p_0} \leq CA \left\|\sum_{k \in \mathbb{Z}} |g_k|^2\right\|_{p_0} \quad \text{(2.11)}
\end{align*}
\]

holds for all functions \( \{g_k\} \) on \( \mathbb{R}^n \).

Then for \( p_0 < p < p_0 \) there exists a positive constant \( C_p \) such that

\[
\left\|\sum_{k \in \mathbb{Z}} \sigma_k^{(N)} * f \right\|_{L^p(\mathbb{R}^n)} \leq C_p A \|f\|_{L^p(\mathbb{R}^n)} \quad \text{(2.12)}
\]

\[
\left\|\left(\sum_{k \in \mathbb{Z}} \sigma_k^{(N)} * f\right)^2\right\|_{L^p(\mathbb{R}^n)} \leq C_p A \|f\|_{L^p(\mathbb{R}^n)} \quad \text{(2.13)}
\]

hold for all \( f \) in \( L^p(\mathbb{R}^n) \). The constant \( C_p \) is independent of the linear transformations \( \{L_l\}_{l=1}^N \).

We shall also need the following result from [DR] (see also [AqP]):

**Lemma 2.6.** Let \( \{\lambda_j : j \in \mathbb{Z}\} \) be a sequence of Borel measures in \( \mathbb{R}^n \) and let
\[ \lambda^* (f) = \sup_{j \in \mathbb{Z}} |\lambda_j | * f |. \]

Assume that

\[ \| \lambda^* (f) \|_q \leq B \| f \|_q \quad \text{for some } q > 1 \text{ and } B > 1. \] (2.14)

Then, for arbitrary functions \( \{g_j\} \) on \( \mathbb{R}^n \) and \( \frac{1}{p_0} - \frac{1}{2} = \frac{1}{2q} \), the following inequality holds

\[ \left\| \left( \sum_{k \in \mathbb{Z}} |\lambda_k * g_k|^2 \right)^{\frac{1}{2}} \right\|_{p_0} \leq (B \sup_{k \in \mathbb{Z}} \|\lambda_k\|) \left\| \left( \sum_{k \in \mathbb{Z}} |g_k|^2 \right)^{\frac{1}{2}} \right\|_{p_0}. \] (2.15)

3. \( L^p \) boundedness of certain maximal functions. For given sequences \( \{\mu_k\} \) and \( \{\tau_k\} \) of nonnegative Borel measures on \( \mathbb{R}^n \) we define the maximal functions \( \mu^* \) and \( \tau^* \) by

\[ \mu^* (f) = \sup_{k \in \mathbb{Z}} |\mu_k * f | \quad \text{and} \quad \tau^* (f) = \sup_{k \in \mathbb{Z}} |\tau_k * f |. \]

We have the following lemma.

**Lemma 3.1.** Let \( \{\mu_k\} \) and \( \{\tau_k\} \) be sequences of nonnegative Borel measures on \( \mathbb{R}^n \). Let \( L: \mathbb{R}^a \to \mathbb{R}^m \) be a linear transformation. Suppose that for all \( k \in \mathbb{Z}, \xi \in \mathbb{R}^n \), for some \( a \geq 2, \alpha, C > 0 \) and for some constant \( B > 1 \) we have

(i) \( \|\mu_k\| \leq B; \|\tau_k\| \leq B; \)

(ii) \( |\hat{\mu}_k(\xi)| \leq CB(a^{kB} L(\xi))^{-\frac{\alpha}{n}}; \)

(iii) \( |\hat{\mu}_k(\xi) - \hat{\tau}_k(\xi)| \leq CB(a^{kB} L(\xi))^{\frac{\alpha}{n}}; \)

(iv) \( \|\tau^* (f)\|_p \leq B \| f \|_p \quad \text{for all } 1 < p \leq \infty \) and \( f \in L^p(\mathbb{R}^n). \) (3.1)

Then the inequality

\[ \|\mu^* (f)\|_p \leq C_p B \| f \|_p \] (3.2)

holds for all \( 1 < p \leq \infty \) and \( f \) in \( L^p(\mathbb{R}^n) \) with a constant \( C_p \) independent of \( B \) and \( L \).

**Proof.** By the arguments in the proof of Lemma 6.2 in [FP], we may assume that \( m \leq n \) and \( L \xi = \pi_{m,n} \xi = (\xi_1, \ldots, \xi_m) \) for \( \xi = (\xi_1, \ldots, \xi_n) \). Now, choose and fix a \( \theta \in S(\mathbb{R}^m) \) such that \( \hat{\theta}(\xi) = 1 \) for \( |\xi| \leq 1 \) and \( \hat{\theta}(\xi) = 0 \) for \( |\xi| \geq 2 \). For each \( k \in \mathbb{Z} \), let \( (\theta_k)(\xi) = \hat{\theta}(a^{kB} \xi) \), and define the sequence of measures \( \{\Upsilon_k\} \) by

\[ \Upsilon_k (\xi) = \hat{\mu}_k(\xi) - (\theta_k(\pi_{m,n}) \hat{\tau}_k(\xi). \] (3.3)

By (i)-(iii) we get

\[ \left| \Upsilon_k (\xi) \right| \leq CB(a^{kB} |\pi_{m,n}|)^{\frac{\alpha}{n}} \] (3.4)

for \( \xi \in \mathbb{R}^n \). Let

\[ S_\xi (f) (x) = \left( \sum_{k \in \mathbb{Z}} |\Upsilon_k * f(x)|^2 \right)^{\frac{1}{2}} \]

and \( \Upsilon^* (f) = \sup_{k \in \mathbb{Z}} \|\Upsilon_k * f| \).
Then by using (3.3) we have
\[
\mu^*(f)(x) \leq S_r(f)(x) + C(M_{R^d} \otimes id_{R^{n-m}}) (\tau^*(f)(x)) \tag{3.5}
\]
\[
\mathcal{T}^*(f)(x) \leq S_r(f)(x) + 2C[M_{R^n} \otimes id_{R^{n-m}}] (\tau^*(f)(x)) \tag{3.6}
\]
where $M_{R^d}$ is the classical Hardy-Littlewood maximal function on $R^d$.

By (3.4) and Plancherel’s theorem we obtain
\[
\| S_r(f) \|_2 \leq CB \| f \|_2 \tag{3.7}
\]
which when combined with the $L^p$ boundedness of $M_{R^n}$, (3.1), and (3.6)-(3.7) gives that
\[
\| \mathcal{T}^*(f) \|_2 \leq CB \| f \|_2 \tag{3.8}
\]
with $C$ independent of $B$. By using the fact $\|Y_k\| \leq CB$ together with Lemma 2.6 (for $q = 2$) we get
\[
\left\| \sum_{k \in Z} (|Y_k * g_k|^2)^{\frac{1}{2}} \right\|_{p_0} \leq C_{p_0} B \left\| \sum_{k \in Z} |g_k|^2 \right\|_{p_0} \tag{3.9}
\]
if $1/4 = |1/p_0 - 1/2|$. Now, by (3.4), (3.9) and applying Lemma 2.5 we get
\[
\| S_r(f) \|_p \leq C_{p_0} B \| f \|_p \text{ for } p \in (\frac{4}{3}, 4). \tag{3.10}
\]
Again, the $L^p$ boundedness of $M_{R^d}$, (3.1), (3.6) and (3.10) imply that
\[
\| \mathcal{T}^*(f) \|_p \leq CB \| f \|_p \text{ for } p \in (\frac{4}{3}, 4). \tag{3.11}
\]
Reasoning as above, (3.4), (3.11), Lemma 2.5 and Lemma 2.6 provide
\[
\| S_r(f) \|_p \leq C_{p_0} B \| f \|_p \text{ for } p \in (\frac{8}{7}, 8). \tag{3.12}
\]
By successive application of the above argument we ultimately obtain that
\[
\| S_r(f) \|_p \leq C_{p_0} B \| f \|_p \text{ for } p \in (1, \infty). \tag{3.13}
\]
Therefore, by the $L^p$ boundedness of $M_{R^d}$, (3.1), (3.5) and (3.13) we conclude that
\[
\| \mu^*(f) \|_p \leq C_{p_0} B \| f \|_p \text{ for } p \in (1, \infty). \tag{3.14}
\]
Finally, the inequality (3.2) holds trivially for $p = \infty$. This concludes the proof of our lemma.

**Definition 3.2.** Let $\tilde{b}(\cdot)$ be a blocklike function defined as in (2.2) and $\Gamma$ be an arbitrary function on $R^n$. Define the measures $\{\sigma_{\Gamma, \tilde{b}, j} : j \in Z\}$ and the maximal operator $\sigma_{\Gamma, \tilde{b}}^*$ on $R^n$ by
\[
\int_{R^d} f \, d\sigma_{\Gamma, \tilde{b}, j} = \int_{2^{j-1} \leq |u| < 2^j} f (\Gamma(u)) \frac{\tilde{b}(u)}{|u|^n} du; \tag{3.15}
\]
\[
\sigma_{\Gamma, \tilde{b}}^*(f) = \sup_{j \in Z} \left| \sigma_{\Gamma, \tilde{b}, j} * f \right|. \tag{3.16}
\]
These measures will be useful only in the case \(|I| \geq e^{-2}\) where \(I\) is the support of \(b\). On the other hand, for the case \(|I| < e^{-2}\) we need to define the following measures.

**Definition 3.3.** Let \(\tilde{b}(\cdot)\) be a \(q\)-blocklike function defined as in (2.2) and \(\Gamma\) be an arbitrary function on \(\mathbb{R}^n\). We define the measures \(\{\lambda_{\Gamma, \tilde{b}, j} : j \in \mathbb{Z}\}\) and the maximal operators \(\lambda_{\Gamma, \tilde{b}}^*\) on \(\mathbb{R}^n\) by

\[
\int_{\mathbb{R}^d} f \, d\lambda_{\Gamma, \tilde{b}, j} = \int_{\omega j^{-1} \leq |u| < \omega j} f(\Gamma(u)) \frac{\tilde{b}(u)}{|u|} \, du;
\]

\[
\lambda_{\Gamma, \tilde{b}}^* f(x) = \sup_{j \in \mathbb{Z}} \left| \lambda_{\Gamma, \tilde{b}, j} * f(x) \right|
\]

where \(\omega = 2^{[\log(|I|^{-1})]}\), \(|I| < e^{-2}\) and \([\cdot]\) denotes the greatest integer function.

**Lemma 3.4.** Let \(\Phi : B(0,1) \rightarrow \mathbb{R}^d\) be a smooth mapping and for \(q > 1\) let \(\tilde{b}\) be a \(q\)-blocklike function defined as in (2.2). Suppose that \(\Phi\) is of finite type at 0. If \(|I| < e^{-2}\), then there are \(N \in \mathbb{N}, \delta \in (0,1], C > 0\) and \(j_0 \in \mathbb{Z}^+\) such that

\[
\left| \lambda_{\Phi, \tilde{b}, j}(\xi) \right| \leq C[\log(|I|)](\omega^N j |\xi|)^{1 - \frac{1}{[\log(|I|^{-1})]}}
\]

for all \(j \leq j_0, \xi \in \mathbb{R}^d\) with \(C\) independent of \(j\) and \([\log(|I|^{-1})]\).

**Proof.** By (2.4), Lemma 2.3 and the definition of \(\lambda_{\Phi, \tilde{b}, j}\) we get

\[
\left| \lambda_{\Phi, \tilde{b}, j}(\xi) \right| \leq \sum_{s=0}^{[\log(|I|^{-1})]-1} \left| \int_{\omega^{(j-s)2^s} \leq |y| < \omega^{(j-s)2^{s+1}}} e^{-i\xi \cdot \Phi(y)} \tilde{b}(y) \frac{dy}{|y|} \right|
\]

\[
\leq \sum_{s=0}^{[\log(|I|^{-1})]-1} C |I|^{-\frac{s}{2}} (\omega^N (j-s)^{2^{s+1}} |\xi|)^{-\delta}
\]

\[
\leq C |I|^{-\frac{1}{2}} \omega^{\delta N} (\omega^N j |\xi|)^{-\delta} \left( \frac{1 - \omega^{-\delta N}}{1 - 2^{\delta N}} \right)
\]

\[
\leq C \omega^{\delta N} |I|^{-\frac{1}{2}} (\omega^N j |\xi|)^{-\delta}.
\]

By interpolating between this estimate and the trivial estimate

\[
\left| \lambda_{\Phi, \tilde{b}, j}(\xi) \right| \leq C[\log(|I|^{-1})]
\]

we get the estimate in (3.19). This concludes the proof of our lemma.

By Lemma 2.4 and the argument used in the proof of Lemma 3.4 we get the following:

**Lemma 3.5.** Let \(m \in \mathbb{N}, \tilde{b}\) be a \(q\)-blocklike function (for \(q > 1\)) defined as in (2.2) and \(R(\cdot)\) be a real-valued polynomial on \(\mathbb{R}^n\) with \(\deg(R) \leq m - 1\). Suppose

\[
P(y) = \sum_{|\alpha| = m} a_\alpha y^\alpha + R(y),
\]
and $|I| < e^{-2}$. Then there exists a constant $C = C(m, n) > 0$ such that
\[ \left| \int_{\omega^{m-1} \leq |y| < \omega} e^{-ip(y) b(y)} \frac{dy}{|y|^m} \right| \leq C \left[ \log( |I|^{-1} ) \right] (\omega^m \sum_{|\alpha|=m} |a_\alpha|)^{-\frac{1}{2m} \log( |I|^{-1} )} \]
holds for all $j \in \mathbb{Z}$ and $a_\alpha \in \mathbb{R}$.

By Proposition 1 on page 477 of [St] it is easy to see that the following result holds.

**Lemma 3.6.** Let $\mathcal{P} = (P_1, \ldots, P_d)$ be a polynomial mapping from $\mathbb{R}^n$ into $\mathbb{R}^d$. Let $\deg(\mathcal{P}) = \max_{1 \leq j \leq d} \deg(P_j)$. Suppose that $b(\cdot)$ is a blocklike function defined as in (2.2) and $\sigma_{\mathcal{P}, \Omega}^*$ be given as in (2.16). Then for every $1 < p \leq \infty$, there exists a constant $C_p$ independent of $b$ and the coefficients of $\mathcal{P}$ such that
\[ \| \sigma_{\mathcal{P}, b}^* (f) \|_p \leq C_p \| f \|_p \]
for $f \in L^p(\mathbb{R}^d)$.

By the above lemma and the proof of Lemma 3.4 we obtain the following:

**Lemma 3.7.** Let $\mathcal{P} = (P_1, \ldots, P_d)$ be a polynomial mapping from $\mathbb{R}^n$ into $\mathbb{R}^d$ and $\tilde{b}$ be a $q$-blocklike function defined as in (2.2). Let $\deg(\mathcal{P}) = \max_{1 \leq j \leq d} \deg(P_j)$. Suppose that $|I| < e^{-2}$. Then for every $1 < p \leq \infty$, there exists a constant $C_p$ independent of $\tilde{b}$ and the coefficients of $\mathcal{P}$ such that
\[ \| \lambda_{\mathcal{P}, \tilde{b}}^* (f) \|_p \leq C_p [\log( |I|^{-1} )] \| f \|_p \]
for $f \in L^p(\mathbb{R}^d)$.

Our next step is to prove the following result on maximal functions:

**Theorem 3.8.** Let $\Phi : B(0, 1) \to \mathbb{R}^d$ be a smooth mapping and for $q > 1$ let $\tilde{b}$ be a $q$-blocklike function defined as in (2.2). Suppose that $\Phi$ is of finite type at 0. Then for $1 < p \leq \infty$ and $f \in L^p(\mathbb{R}^d)$ there exists a positive constant $C_p$ which is independent of $\tilde{b}$ such that
\[ \| \lambda_{\Phi, \tilde{b}}^* (f) \|_{L^p(\mathbb{R}^d)} \leq C_p [\log( |I|^{-1} )] \| f \|_{L^p(\mathbb{R}^d)} \text{ if } |I| < e^{-2}; \tag{3.20} \]
\[ \| \sigma_{\Phi, \tilde{b}}^* (f) \|_{L^p(\mathbb{R}^d)} \leq C_p \| f \|_{L^p(\mathbb{R}^d)} \text{ if } |I| \geq e^{-2}. \tag{3.21} \]

**Proof.** Assume first that $|I| < e^{-2}$. Without loss of generality we may assume that $\tilde{b} \geq 0$. By Lemma 3.4, there are $N \in \mathbb{N}$, $\delta \in (0, 1]$, $C > 0$ and $k_0 \in \mathbb{Z}_+$ such that
\[ \left| \lambda_{\Phi, \tilde{b}, k}^*(\xi) \right| \leq C [\log( |I|^{-1} )] (\omega^N k |\xi|)^{-\frac{1}{2m} \log( |I|^{-1} )} \tag{3.22} \]
for all \( k \leq k_0, \xi \in \mathbb{R}^d \) with \( C \) independent of \( k \) and \( \|\log(\|I\|^{-1})\| \) where \( \omega = 2^{\|\log(\|I\|^{-1})\|} \). For \( \Phi = (\Phi_1, \ldots, \Phi_d) \) we let \( \mathcal{P} = (P_1, \ldots, P_d) \) where

\[
P_j(y) = \sum_{|\beta| \leq N-1} \frac{1}{\beta!} \frac{\partial^\beta \Phi_j}{\partial y^\beta}(0) y^\beta, \quad 1 \leq j \leq d.
\]

Then we have

\[
|\hat{\lambda}_{\Phi, \mathbf{b}, k}(\xi) - \hat{\lambda}_{\mathcal{P}, \mathbf{b}, k}(\xi)| \leq C [\log(\|I\|^{-1})] \omega^{-N} (\omega^N |\xi|).
\] (3.23)

By (2.5) we have

\[
|\hat{\lambda}_{\Phi, \mathbf{b}, k}(\xi) - \hat{\lambda}_{\mathcal{P}, \mathbf{b}, k}(\xi)| \leq C [\log(\|I\|^{-1})].
\] (3.24)

By interpolating between this estimate and (3.23) we get

\[
|\hat{\lambda}_{\Phi, \mathbf{b}, k}(\xi) - \hat{\lambda}_{\mathcal{P}, \mathbf{b}, k}(\xi)| \leq C [\log(\|I\|^{-1})] \omega^{-N} (\omega^N |\xi|). \] (3.25)

Therefore, (3.20) follows from (3.22), (3.25), Lemma 3.1 and Lemma 3.7. The proof of the inequality (3.21) will be much easier. In fact, it follows from (2.4)-(2.5), Lemma 2.3, 3.1, and 3.6. We omit the details.

4. Proofs of the theorems. By assumption, \( \Omega \) can be written as \( \Omega = \sum_{\mu=1}^\infty c_\mu b_\mu \) where \( c_\mu \in \mathbb{C} \), \( b_\mu \) is a \( q \)-block with support on an \( n \) cap \( I_\mu \) on \( S^{n-1} \) and

\[
M_{q,0}^0(\{c_\mu\}, \{I_\mu\}) = \sum_{\mu=1}^\infty |c_\mu| \left(1 + (\log |I_\mu|^{-1}) \right) < \infty.
\] (4.1)

For each \( \mu = 1, 2, \ldots \), let \( \tilde{b}_\mu \) be the blocklike function corresponding to \( b_\mu \). By the vanishing condition on \( \Omega \) we have

\[
\Omega = \sum_{\mu=1}^\infty c_\mu \tilde{b}_\mu
\] (4.2)

and hence

\[
\|T_\Phi f\|_p \leq \sum_{\mu=1}^\infty |c_\mu| \|T_{\Phi, \tilde{b}_\mu} f\|_p; \]
(4.3)

where

\[
T_{\Phi, \tilde{b}_\mu} f(x) = \text{p.v.} \int_{B(0,1)} f(x - \Phi(u)) \tilde{b}_\mu \frac{u'}{|u|^n} du.
\]

Let \( \delta, N, \mathcal{P} \) be given as in the proof of Theorem 3.8. For \( 1 \leq j \leq d \), let \( a_{j, \beta} = \frac{1}{|\beta|} \frac{\partial^\beta \Phi_j}{\partial y^\beta}(0) \). For \( 0 \leq l \leq N - 1 \) we define \( Q^l = (Q^l_1, \ldots, Q^l_d) \) by

\[
Q^l_j(y) = \sum_{|\beta| \leq l} a_{j, \beta} y^\beta, \quad j = 1, \ldots, d \]
(4.4)
when $0 \leq l \leq N - 1$ and $Q^N = \Phi$. For each $0 \leq l \leq N$, let $\lambda_{b_{\mu,k}}^{(l)} = \lambda_{Q^l,\mu,k}$ and $\sigma_{b_{\mu,k}}^{(l)} = \sigma_{Q^l,\mu,k}$. Then by (2.3)-(2.5), Lemma 2.4 we have

$$\left\| \sigma_{b_{\mu,k}}^{(l)} \right\| \leq C; \quad (4.5)$$

$$\left\| \sigma_{b_{\mu,k}}^{(l)}(\xi) - \sigma_{b_{\mu,k}}^{(l)}(\xi) \right\| \leq C(2^k \sum_{|j|=l} \sum_{\beta} a_{j,\beta} |\xi_j|) \frac{1}{\kappa_{h_{\mu,k}}}; \quad (4.6)$$

$$\left\| \sigma_{b_{\mu,k}}^{(l)}(\xi) - \sigma_{b_{\mu,k}}^{(l-1)}(\xi) \right\| \leq C(2^{Nk} |\xi|); \quad (4.7)$$

$$\left\| \sigma_{b_{\mu,k}}^{(l)}(\xi) - \sigma_{b_{\mu,k}}^{(l-1)}(\xi) \right\| \leq C(2^k \sum_{|j|=l} \sum_{\beta} a_{j,\beta} |\xi_j|) \quad (4.8)$$

for $|I_\mu| \geq e^{-2}, \mu = 1, 2, \ldots, 0 \leq l \leq N - 1$, and $k \leq k_0$. Also, by (2.3)-(2.5), Lemma 3.5, and the same argument as in the proof (3.25) we have

$$\left\| \lambda_{b_{\mu,k}}^{(l)} \right\| \leq CA_{\mu}; \quad (4.9)$$

$$\left\| \lambda_{b_{\mu,k}}^{(l)}(\xi) \right\| \leq CA_{\mu} (2^k A_{\mu} \sum_{|j|=l} \sum_{\beta} a_{j,\beta} |\xi_j|) \frac{1}{\kappa_{h_{\mu,k}}}; \quad (4.10)$$

$$\left\| \lambda_{b_{\mu,k}}^{(l)}(\xi) - \lambda_{b_{\mu,k}}^{(l-1)}(\xi) \right\| \leq CA_{\mu} (2^k A_{\mu} \sum_{|j|=l} \sum_{\beta} a_{j,\beta} |\xi_j|) \frac{1}{\kappa_{h_{\mu,k}}}; \quad (4.11)$$

where $A_{\mu} = \log(|I_\mu|^{-1}), |I_\mu| < e^{-2}, \mu = 1, 2, \ldots, k \leq k_0, 0 \leq l \leq N - 1$.

By (3.20)-(3.22), (3.25), (4.5)-(4.11), Theorem 3.8, Lemmas 2.5-2.6, and 3.6-3.7 we get

$$\left\| T_{\Phi,\mu} f \right\|_p = \left\| \sum_{j \in \mathbb{Z}_{-}} \lambda_{b_{\mu,k}}^{(N)} * f \right\|_p \leq C_p A_{\mu} \left\| f \right\|_p \text{ if } |I_\mu| < e^{-2}; \quad (4.12)$$

$$\left\| T_{\Phi,\mu} f \right\|_p = \left\| \sum_{j \in \mathbb{Z}_{-}} \sigma_{b_{\mu,k}}^{(N)} * f \right\|_p \leq C_p \left\| f \right\|_p \text{ if } |I_\mu| \geq e^{-2}, \quad (4.13)$$

for every $f \in L^p(\mathbb{R}^d), \mu = 1, 2, \ldots, \text{ and for all } p, 1 < p < \infty$. Hence, (1.7) follows from (4.1), (4.3) and (4.12)-(4.13). On the other hand, (1.8) follows from (3.20)-(3.21), (4.2) and the following inequality

$$\mathcal{M}_f(x) \leq 4 \sum_{\mu=1}^{\infty} |c_\mu| \sigma_{\Phi,\mu}^*(|f|)(x) \leq 4 \sum_{\mu=1}^{\infty} |c_\mu| \sigma_{\Phi,\mu}^*(|f|)(x) + 8 \sum_{\mu=1}^{\infty} |c_\mu| \lambda_{\Phi,\mu}^*(|f|)(x). \quad (4.14)$$

This concludes the proof of Theorem 1.2.

Finally, the proof of Theorem 1.3 follows from the above estimates and the techniques in [AqP]. We omit the details.
REFERENCES


