A UNIVERSAL METRIC FOR THE CANONICAL BUNDLE OF A HOLOMORPHIC FAMILY OF PROJECTIVE ALGEBRAIC MANIFOLDS∗

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Dedicated to M. Salah Baouendi on the occasion of his 70th birthday

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1. Introduction. In his celebrated work [S-98, S-02], Siu proved that the plurigenera of any algebraic manifold are invariant in families. More precisely, let $\pi: \mathcal{X} \to \mathbb{D}$ be a holomorphic submersion (i.e., $d\pi$ is nowhere zero) from a complex manifold $\mathcal{X}$ to the unit disk $\mathbb{D}$, and assume that every fiber $\mathcal{X}_t := \pi^{-1}(t)$ is a compact projective manifold. Then for every $m \in \mathbb{N}$, the function $P_m : \mathbb{D} \to \mathbb{N}$ defined by $P_m(t) := h^0(\mathcal{X}_t, mK_{\mathcal{X}_t})$ is constant.

Siu’s approach to the problem begins with the observation that the function $P_m$ is upper semi-continuous. Thus in order to prove that $P_m$ is continuous (hence constant) it suffices to show that given a global holomorphic section $s$ of $mK_{\mathcal{X}_0}$, there is a family of global holomorphic sections $s_t$ of $\mathcal{X}_t$, for all $t$ in a neighborhood of 0, that varies holomorphically with $t$ and satisfies $s_0 = s$.

To prove such an extension theorem, Siu establishes a generalization of the Ohsawa-Takegoshi Extension Theorem to the setting of complex submanifolds of a Kahler manifold having codimension 1 and cut out by a single, bounded holomorphic function. This theorem, which we will discuss below, requires the existence of a singular Hermitian metric on the ambient manifold having non-negative curvature current, with respect to which the section to be extended is $L^2$. Thus in the presence of the extension theorem, the approach reduces to construction of such a metric.

The case where the fibers $\mathcal{X}_t$ of our holomorphic family are of general type was treated in [S-98]. In this setting, Siu produced a single singular Hermitian metric $e^{-\kappa}$ for $K_{\mathcal{X}}$ so that every $m$-canonical section is $L^2$ with respect to $e^{-(m-1)\kappa}$.

However, in the case where the fibers $\mathcal{X}_t$ of our holomorphic family are assumed only to be algebraic, and not necessarily of general type, Siu’s proof in [S-02] does not construct a single metric as in the case of general type. Instead, Siu constructs for every section $s$ of $mK_{\mathcal{X}_0}$ a singular Hermitian metric for $mK_{\mathcal{X}}$ of non-negative curvature so that $s$ is $L^2$ with respect to this metric.

DEFINITION. Let $\mathcal{X} \to \Delta$ be a holomorphic family of complex manifolds and $\mathcal{X}_0$ the central fiber of $\mathcal{X}$. A universal canonical metric for the pair ($\mathcal{X}$, $\mathcal{X}_0$) is a singular Hermitian metric $e^{-\kappa}$ for the canonical bundle $K_{\mathcal{X}}$ of $\mathcal{X}$ such that for every global holomorphic section $s \in H^0(\mathcal{X}_0, mK_{\mathcal{X}_0})$,

$$\int_{\mathcal{X}_0} |s|^2 e^{-(m-1)\kappa} < +\infty.$$
The goal of this paper is to prove that for any holomorphic family \( \mathcal{X} \rightarrow \Delta \) of compact complex algebraic manifolds with central fiber \( \mathcal{X}_0 \), the pair \((\mathcal{X}, \mathcal{X}_0)\) has a universal canonical metric having non-negative curvature current. To this end, our main theorem is the following result.

**Theorem 1.** Let \( X \) be a complex manifold admitting a positive line bundle \( A \rightarrow X \), and \( Z \subset X \) a smooth compact complex submanifold of codimension 1. Assume there is a subvariety \( V \subset X \) not containing \( Z \) such that \( X - V \) is a Stein manifold. Let \( T \in H^0(X, Z) \) be a holomorphic section of the line bundle associated to \( Z \), thought of as a divisor. Let \( E \rightarrow X \) be a holomorphic line bundle and denote by \( K_X \) the canonical bundle of \( X \). Assume we are given singular metrics \( e^{-\varphi_E} \) for \( E \) and \( e^{-\varphi_Z} \) for the line bundle associated to \( Z \).

Suppose in addition that the above data satisfy the following assumptions.

1. **(R)** The metrics \( e^{-\varphi_E} \) and \( e^{-\varphi_Z} \) restrict to singular metrics on \( Z \).
2. **(B)** \[
\sup_X |T|^2 e^{-\varphi_Z} < +\infty.
\]
3. **(G)** For each \( m > 0 \), the line bundles \( p(K_X + Z + E) + A, 0 \leq p \leq m-1 \), are globally generated, in the sense that a finite number of sections of \( H^0(X, p(K_X + Z + E) + A) \) generate the sheaf \( O_X(p(K_X + Z + E) + A) \).
4. **(P)** \( \sqrt{-1} \partial \bar{\partial} \varphi_E \geq 0 \) and there exists a constant \( \mu \) such that \( \mu \sqrt{-1} \partial \bar{\partial} \varphi_E \geq \sqrt{-1} \partial \bar{\partial} \varphi_Z \).
5. **(T)** The singular metric \( e^{-(\varphi_Z + \varphi_E)}|Z \) has trivial multiplier ideal: \( \mathcal{J}(Z, e^{-(\varphi_Z + \varphi_E)}|Z) = O_Z \).

Then there is a metric \( e^{-\kappa} \) for \( K_X + Z + E \) with the following properties:

1. **(C)** \( \sqrt{-1} \partial \bar{\partial} \kappa \geq 0 \).
2. **(L)** For every \( m > 0 \) and every section \( s \in H^0(Z, m(K_Z + E)|Z) \), \( |s|^2 e^{-(m-1)\kappa + \varphi_E + \varphi_Z} \) is locally integrable.
3. **(I)** For every integer \( m > 0 \) and every section \( s \in H^0(Z, m(K_Z + E)) \),
   \[
   \int_Z |s|^2 e^{-(m-1)\kappa + \varphi_E} < +\infty.
   \]

**Remarks.**

(i) For the ambient manifold \( X \), we have in mind the following two examples: either \( X \) is compact complex projective (in which case the variety \( V \) could be taken to be a hyperplane section of some embedding of \( X \)) or else \( X \) is a family of compact complex algebraic manifolds. In the former case, it is well-known [S-98] that the hypothesis (G) holds for any sufficiently ample \( A \), while in the latter case, one might have to shrink \( X \) a little to obtain (G). Of course, there are many other examples of such \( X \).

(ii) Note that in condition (L), the local functions \( |s|^2 e^{-(m-1)\kappa + \varphi_E + \varphi_Z} \) depend on the local trivializations of the line bundles in question. However, the local integrability condition is independent of these choices.

Together with a variant of the Ohsawa-Takegoshi Theorem (Theorem 4 below), Theorem 1 implies a generalization of Siu’s extension theorem to the case where the normal bundle of the submanifold \( Z \) is not necessarily trivial. The first extension
Theorem of this type was established by Takayama [Ta-05, Theorem 4.1] under some additional hypotheses. The general case was done in [V-06], where Theorem 4 was also established. (In the case where \( Z \) is a fiber in a smooth family, the result in [V-06] was also proved by Claudon in [C-06].) The argument here is related to that of [V-06], but the focus is on construction of the metric rather than on the extension theorem.

As a result of Theorem 1, we have the following corollary, which is our stated goal.

**Corollary 2.** For every holomorphic family \( \mathcal{X} \to \Delta \) of smooth projective varieties with central fiber \( \mathcal{X}_0 \), the pair \((\mathcal{X}, \mathcal{X}_0)\) has, perhaps after slightly shrinking the family, a universal canonical metric having non-negative curvature current.

**Proof.** Let \( \mathcal{X} \) be a family of compact projective manifolds \( \pi: \mathcal{X} \to \Delta \), and \( Z = \mathcal{X}_0 \) the central fiber. Take \( T = \pi, E = \mathcal{O}_{\mathcal{X}} \) and \( \varphi_E \equiv 0 \). Since \( \mathcal{X}_0 \) is cut out by a single holomorphic function, the line bundle associated to \( \mathcal{X}_0 \) is trivial. Take \( \varphi_Z \equiv 0 \). Then the hypotheses of Theorem 1 are satisfied, perhaps after shrinking the family, and we obtain a metric \( e^{-\kappa} \) for \( K_{\mathcal{X}} \) such that \( \sqrt{-1} \partial \bar{\partial} \kappa \geq 0 \) and \( |s|^2 e^{-(m-1)\kappa} \) is integrable for every integer \( m > 0 \) and every section \( s \in H^0(\mathcal{X}_0, mK_{\mathcal{X}_0}) \).

**Remark.** Note that in the setting of families, the constant \( \mu \) is not needed, and the hypotheses (L) and (I) are the same.

**Remark.** In his paper [Ts-02], Tsuji has claimed the existence of a metric with the properties stated in Corollary 2. As in our approach, Tsuji’s proof makes use of an infinite process. It seems that convergence of Tsuji’s process was not checked; in fact, it is demonstrated in [S-02] that Tsuji’s process, as well as any reasonable modification of it, diverges.

**Proposition 3.** For each integer \( m > 0 \), fix a basis \( s_1^{(m)},...,s_{N_m}^{(m)} \) of \( H^0(X, m(K_Z + E|Z)) \). Choose constants \( \varepsilon_m \) such that the metric

\[
\kappa_0 := \log \left( \sum_{m=1}^{\infty} \varepsilon_m \left( \sum_{\ell=1}^{N_m} |s_{\ell}^{(m)}|^2 \right)^{1/m} \right)
\]

is convergent. Suppose \( e^{-\varphi_E} \) is locally integrable. Then for each \( m > 0 \) and every \( s \in H^0(X, m(K_Z + E|Z)) \),

\[
\int_Z |s|^2 e^{-(m-1)\kappa_0 + \varphi_E} < +\infty.
\]

**Proof.** Fix \( s \in H^0(X, m(K_Z + E|Z)) \), and let \( \kappa_{0,m} = \log \left( \sum_{\ell=1}^{N_m} |s_{\ell}^{(m)}|^2 \right)^{1/m} \).
Note that $e^{-\kappa_0} \lesssim e^{-\kappa_0 \cdot \varphi_E}$, and thus we have
\[
\int_Z |s|^2 e^{-(m-1)\kappa_0 + \varphi_E} \lesssim \int_Z |s|^2 e^{-(m-1)\kappa_0, m + \varphi_E}
\]
\[
= \int_Z |s|^{2/m} \left( \frac{|s|^2}{|s_1^{(m)}|^2 + \cdots + |s_{N_m}^{(m)}|^2} \right)^{(m-1)/m} e^{\varphi_E - \varphi_E} e^{-\gamma E}
\]
\[
\lesssim \int_Z |s|^{2/m} e^{\gamma E - \varphi_E} e^{-\gamma E}
\]
\[
\lesssim \left( \int_Z |s|^2 e^{\gamma E - \varphi_E} e^{-m\gamma E} \omega^{-(n-1)(m-1)} \right)^{1/m} \left( \int_Z e^{\gamma E - \varphi_E} \omega^{m-1} \right)^{1/m},
\]
where $\omega$ is a fixed Kähler form for $Z$ and $e^{-\gamma E}$ is a smooth metric for $E|Z$. The last inequality is a consequence of Hölder’s Inequality. Since $e^{-\varphi_E}$ is locally integrable, we are done. \[\]

A calculation similar to the proof of Proposition 3 shows that $|s|^2 e^{-(m-1)\kappa_0 + \varphi_E + \varphi_F}$ is locally integrable on $Z$. Thus in view of Proposition 3, Theorem 1 follows if we construct a metric $e^{-\kappa}$ with non-negative curvature current such that $e^{-\kappa}|Z = e^{-\kappa_0}$. This is precisely what we do. We employ a technical simplification, due to Paun [P-05], of Siu’s original idea of extending metrics using an Ohsawa-Takegoshi-type extension theorem for sections. Paun’s simplification allows one to get rid of a rather difficult part of Siu’s original proof; the use (and proof) of an effective version of global generation of multiplier ideal sheaves. As a consequence of Paun’s methods, the present paper is also substantially shortened.

2. The Ohsawa-Takegoshi Extension theorem. Let $Y$ be a Kähler manifold of complex dimension $n$. Assume there exists an analytic hypersurface $V \subset Y$ such that $Y - V$ is Stein. Examples of such manifolds are Stein manifolds (where $V$ is empty) and projective algebraic manifolds (where one can take $V$ to be the intersection of $Y$ with a projective hyperplane in some projective space in which $Y$ is embedded).

Fix a smooth hypersurface $Z \subset Y$ such that $Z \not\subset V$. In [V-06] we proved the following generalization of the Ohsawa-Takegoshi Extension Theorem.

**Theorem 4.** Suppose given a holomorphic line bundle $H \to Y$ with a singular Hermitian metric $e^{-\varphi}$, and a singular Hermitian metric $e^{-\varphi_Z}$ for the line bundle associated to the divisor $Z$, such that the following properties hold.

(i) The restrictions $e^{-\varphi}|Z$ and $e^{-\varphi_Z}|Z$ are singular metrics.

(ii) There is a global holomorphic section $T \in H^0(Y, Z)$ such that $Z = \{ T = 0 \}$ and $\sup_Y |T|^2 e^{-\varphi_Z} = 1$.

(iii) $\sqrt{-1} \partial \bar{\partial} \varphi \geq 0$ and there is an integer $\mu > 0$ such that $\mu \sqrt{-1} \partial \bar{\partial} \varphi \geq \sqrt{-1} \partial \bar{\partial} \varphi_Z$. Then for every $s \in H^0(Z, K_Z + H)$ such that

\[
\int_Z |s|^2 e^{-\varphi} < +\infty \quad \text{and} \quad s \wedge dT \in J(e^{-\varphi_Z})|Z,
\]

there exists a section $S \in H^0(Y, K_Y + Z + H)$ such that

\[
S|Z = s \wedge dT \quad \text{and} \quad \int_Y |S|^2 e^{-(\varphi_Z + \psi)} \leq 40\pi \mu \int_Z |s|^2 e^{-\varphi}.
\]
Remark. The list of $L^2$ extension theorems is by now rather long. For a large collection of such results and additional references, see [MV-05].

3. Inductive construction of certain sections by extension. In this section we use the method of Paun [P-05] mentioned in the introduction. Fix a holomorphic line bundle $A \to X$ such that the property (G) in Theorem 1 holds.

Let us fix bases

$$\{ \tilde{\sigma}_j^{(m,0,p)} ; 1 \leq j \leq M_p \}$$

of $H^0(X, p(K_X + Z + E) + A)$. We let $\sigma_j^{(m,0,p)} \in H^0(Z, p(K_Z + E|Z) + A|Z)$ be such that

$$\tilde{\sigma}_j^{(m,0,p)}|Z = \sigma_j^{(m,0,p)} \wedge (dT)^{\otimes p}.$$

We also fix smooth metrics $e^{-\gamma_Z}$ and $e^{-\gamma_E}$ for $Z \to X$, and $E \to X$ respectively. Finally, let us fix bases

$$s_1^{(m)}, ..., s_{N_m}^{(m)}$$

for $H^0(X, m(K + E|Z))$, $m = 1, 2, ..., \infty$.

orthonormal with respect to the singular metric $(\omega^{-(n-1)} e^{-\gamma_Z} e^{-\gamma_E})$ for $(m - 1)K_Z + mE|Z$. (Since $e^{-\gamma_E}$ is locally integrable, every holomorphic section is integrable with respect to this metric.)

**Proposition 5.** For each $m = 1, 2, ...$ there exist a constant $C_m < +\infty$ and sections

$$\tilde{\sigma}_j^{(m,k,p)} \in H^0(X, (km + p)(K_X + Z + E) + A)$$

where $p = 1, 2, ..., m - 1$, $1 \leq j \leq M_p$, $1 \leq \ell \leq N_m$ and $k = 1, 2, ..., m - 1$, with the following properties.

(a) $\tilde{\sigma}_j^{(m,k,p)}|Z = (s_\ell^{(m)})^{\otimes k} \otimes \sigma_j^{(m,0,p)} \wedge (dT)^{(km+p)}$

(b) If $k \geq 1$,

$$\int_X \sum_{j=1}^{M_0} |\tilde{\sigma}_j^{(m,k,0)}|_2 e^{-(\gamma_Z + \gamma_E)} \sum_{j=1}^{M_{m-1}} |\tilde{\sigma}_j^{(m,k-1,m-1)}|_2 \leq C_m.$$

(c) For $1 \leq p \leq m - 1$,

$$\int_X \sum_{j=1}^{M_0} |\tilde{\sigma}_j^{(m,k,p)}|_2 e^{-(\gamma_Z + \gamma_E)} \sum_{j=1}^{M_{p-1}} |\tilde{\sigma}_j^{(m,k,p-1)}|_2 \leq C_m.$$

**Proof.** (Double induction on $k$ and $p$.) Fix a constant $\tilde{C}_m$ such that the

$$\sup_X \sum_{j=1}^{M_0} |\tilde{\sigma}_j^{(m,0,0)}|_2 \omega^{n(m-1)} e^{(m-1)(\gamma_Z + \gamma_E)} \sum_{j=1}^{M_{m-1}} |\tilde{\sigma}_j^{(m,0,m-1)}|_2 \leq \tilde{C}_m.$$
and
\[
\sup_Z \left( \sum_{j=1}^{M_0} \sigma_j^{(m,0,0)} \right)^2 \omega^{(n-1)(m-1)} e^{\gamma_E} \leq \hat{C}_m,
\]
and for all \(0 \leq p \leq m-2\),
\[
\sup_X \left( \sum_{j=1}^{M_p} \sigma_j^{(m,p+1)} \right)^2 \omega^{-n} e^{-(\gamma_E+\gamma_E)} \leq \hat{C}_m,
\]
and
\[
\sup_Z \left( \sum_{j=1}^{M_p} \sigma_j^{(m,p+1)} \right)^2 \omega^{-n} e^{-(\gamma_E+\gamma_E)} \leq \hat{C}_m.
\]

\((k = 0)\) We set \(\tilde{\sigma}_j^{(m,0,p)} := \sigma_j^{(m,0,p)}\) and simply observe that
\[
\int_X \left( \sum_{j=1}^{M_p} |\tilde{\sigma}_j^{(m,p+1)}| \omega^{-n} e^{-(\gamma_E+\gamma_E)} \right) \leq \hat{C}_m \int_X \omega^n.
\]

\((k \geq 1)\) Assume the result has been proved for \(k-1\).

\((p = 0)\): Consider the sections \((s_\ell^{(m)}) \otimes \sigma_j^{(m,0,0)}\), and define the semi-positively curved metric
\[
\psi_{k,0} := \log \sum_{j=1}^{M_{m-1}} |\tilde{\sigma}_j^{(m,k-1,m-1)}|^2
\]
for the line bundle \((mk-1)(K_X + Z + E) + A\). Observe that locally on \(Z\),
\[
|s_\ell^{(m)} \wedge dT^m| \otimes \sigma_j^{(m,0,0)} e^{-(\varphi_Z + \psi_{k,0} + \varphi_E)} = |s_\ell^{(m)} \wedge dT^m| \left( \sum_{j=1}^{M_{m-1}} |\tilde{\sigma}_j^{(m,0,m-1)}| \right) e^{-(\varphi_Z + \varphi_E)} \leq |s_\ell^{(m)}|^2 e^{-(\varphi_Z + \varphi_E)}.
\]
Moreover, we have
\[
\sqrt{-1} \partial \bar{\partial} (\psi_{k,0} + \varphi_E) \geq 0 \quad \text{and} \quad \mu \sqrt{-1} \partial \bar{\partial} (\psi_{k,0} + \varphi_E) \geq \sqrt{-1} \partial \bar{\partial} \varphi_Z.
\]
Finally,
\[
\int_Z |s_\ell^{(m)}| \otimes \sigma_j^{(m,0,0)} e^{-(\psi_{k,0} + \varphi_E)} = \int_Z |s_\ell^{(m)}| \left( \sum_{j=1}^{M_{m-1}} |\tilde{\sigma}_j^{(m,0,m-1)}| \right) < +\infty.
\]
We may thus apply Theorem 4 to obtain sections
\[
\tilde{\sigma}_j^{(m,k,0)} \in H^0(X, mk(K_X + Z + E) + A), \quad 1 \leq j \leq M_0, \; 1 \leq \ell \leq N_m,
\]
such that
\[
\hat{\sigma}^{(m,k,0)}_{j,\ell}|_{Z} = (s^{(m)}_{\ell})^{\otimes k} \otimes \sigma^{(m,0,0)}_{j,\ell}\wedge (dT)^{\otimes km}, \quad 1 \leq j \leq M_0, \ 1 \leq \ell \leq N_m,
\]
and
\[
\int_X |\hat{\sigma}^{(m,k,0)}_{j,\ell}|^2 e^{-(\varphi_E + \varphi E)} \leq 40\pi\mu \int_Z |s^{(m)}_{\ell}|^2 \frac{\hat{\sigma}^{(m,0,0)}_{j,\ell}|^2 e^{-(\varphi_E + \varphi_E)}}{\sum_{j=1}^{N_{m-1}} |\hat{\sigma}^{(m-1,0,0)}_{j}|^2}.
\]
Summing over \( j \), we obtain
\[
\int_X \sum_{j=1}^{M_0} |\hat{\sigma}^{(m,k,0)}_{j,\ell}|^2 e^{-(\gamma_E + \gamma_E)} \leq \sup_X e^{|\varphi_E| + \varphi_E - \gamma_E - \gamma_E} \int_X \sum_{j=1}^{M_0} |\hat{\sigma}^{(m,k,0)}_{j,\ell}|^2 e^{-(\varphi_E + \varphi_E)} \leq 40\pi \sup_X e^{|\varphi_E| + \varphi_E - \gamma_E - \gamma_E} \int_Z |s^{(m)}_{\ell}|^2 \frac{\sum_{j=1}^{M_0} |\hat{\sigma}^{(m,0,0)}_{j,\ell}|^2 e^{-\kappa}}{\sum_{j=1}^{N_{m-1}} |\hat{\sigma}^{(m,0,0)}_{j}|^2} \leq 40\pi \hat{C}_m \sup_X e^{|\varphi_E| + \varphi_E - \gamma_E - \gamma_E} \int_Z |s^{(m)}_{\ell}|^2 \frac{\sum_{j=1}^{M_{p-1}} |\hat{\sigma}^{(m,p-1)}_{j,\ell}|^2 e^{-\kappa}}{\sum_{j=1}^{N_{m-1}} |\hat{\sigma}^{(m,0,0)}_{j}|^2} \leq 40\pi \hat{C}_m \sup_X e^{|\varphi_E| + \varphi_E - \gamma_E - \gamma_E}.
\]

((1 \leq p \leq m - 1)): Assume that we have obtained the sections \( \hat{\sigma}^{(m,k,p-1)}_{j,\ell} \), \( 1 \leq j \leq M_{p-1}, \ 1 \leq \ell \leq N_m \). Consider the non-negatively curved singular metric
\[
\psi_{k,\ell,p} := \log \sum_{j=1}^{M_{p-1}} |\hat{\sigma}^{(m,k,p-1)}_{j,\ell}|^2
\]
for \( (km + p - 1)(K_X + Z + E) + A \). We have
\[
|(s^{(m)}_{\ell})^k \otimes \sigma^{(m,0,p)}_{j}|^2 e^{-(\varphi_E + \psi_{k,\ell,p} + \varphi E)} = \frac{|\sigma^{(m,0,p)}_{j}|^2 e^{-(\varphi_E + \varphi E)}}{\sum_{j=1}^{M_{p-1}} |\sigma^{(m,0,p-1)}_{j}|^2} \leq e^{-(\varphi_E + \varphi E)},
\]
which is locally integrable on \( Z \) by the hypothesis (T). Next,
\[
\int_Z |(s^{(m)}_{\ell})^k \otimes \sigma^{(m,0,p)}_{j}|^2 e^{-(\varphi_E + \psi_{k,\ell,p} + \varphi E)} = \int_Z |\sigma^{(m,0,p)}_{j}|^2 e^{-\varphi E} \leq C^* \int_Z e^{\gamma_E} \frac{|\sigma^{(m,0,p)}_{j}|^2 e^{-(\varphi_E + \varphi E)}}{\sum_{j=1}^{M_{p-1}} |\sigma^{(m,0,p-1)}_{j}|^2} < +\infty,
\]
where
\[
C^* := \sup_Z e^{\gamma_E - \gamma_E}.
\]
Moreover,
\[
\sqrt{-1} \partial \bar{\partial} (\psi_{k,\ell,p} + \varphi E) \geq 0 \quad \text{and} \quad \sqrt{-1} \partial \bar{\partial} (\psi_{k,\ell,p} + \varphi E) \geq \sqrt{-1} \partial \bar{\partial} \varphi E.
\]
By Theorem 4 there exist sections

$$\tilde{\sigma}_{j,\ell}^{(m,k,p)} \in H^0(X, (mk + p)(K_X + Z + E) + A), \quad 1 \leq j \leq M_0$$

such that

$$\tilde{\sigma}_{j,\ell}^{(m,k,p)} \mid Z = (\tilde{\sigma}_{\ell}^{(m)})^{\otimes k} \otimes \tilde{\sigma}_{j,\ell}^{(m,0,p)} \wedge (dT)^{\otimes km + p}, \quad 1 \leq j \leq M_p,$$

and

$$\int_X |\tilde{\sigma}_{j,\ell}^{(m,k,p)}|^2 e^{- (\psi_{k,\ell,p} + \varphi_Z + \varphi_E)} \leq 40 \pi \mu \int_Z \frac{|\sigma_j^{(m,0,p)}|^2 e^{-\varphi_E}}{\sum_{j=1}^{M_{p-1}} |\sigma_j^{(m,0,p-1)}|^2}.$$

Summing over \(j\), we obtain

$$\int_X \sum_{j=1}^{M_p} |\tilde{\sigma}_{j,\ell}^{(m,k,p)}|^2 e^{- (\gamma_Z + \gamma_E)} \leq 40 \pi \mu \sup_X e^{\varphi_Z + \varphi_E + \gamma_Z - \gamma_E} \hat{C}_m \int_Z e^{-\varphi_E} \omega^{n-1}.$$

Letting

$$C_m := 40 \pi \hat{C}_m \max \left( \int_X \omega^n, \sup_X e^{\varphi_Z + \varphi_E + \gamma_Z - \gamma_E}, \sup_X e^{\varphi_Z + \varphi_E - \gamma_Z - \gamma_E} \int_Z e^{-\varphi_E} \omega^{n-1} \right)$$

completes the proof. \(\square\)

4. Construction of the metric. This part of the proof follows the ideas of Siu set forth in [S-02].

4.1. A metric associated to \(m(K_X + Z + E)\). Fix a smooth metric \(e^{-\psi}\) for \(A \to X\). Consider the functions

$$\lambda^{(m)}_{\ell,N} := \log \sum_{j=1}^{M_p} |\tilde{\sigma}_{j,\ell}^{(m,k,p)}|^2 e^{-n(mk + p)} e^{-(km + \gamma_Z + \gamma_E)} \psi,$$

where \(N = mk + p\). Set

$$\lambda^{(m)}_N := \log \sum_{\ell=1}^{N_m} e^{\lambda^{(m)}_{\ell,N}}.$$

**Lemma 6.** For any non-empty open subset \(V \subset X\) and any smooth function \(f : V \to \mathbb{R}_+\),

$$\frac{1}{f \omega^n} \int_V (\lambda^{(m)}_N - \lambda^{(m)}_{N-1}) f \omega^n \leq \log \left( \frac{N_mC_m \sup_V f}{f \omega^n} \right).$$

**Proof.** Observe that by Proposition 5, there exists a constant \(C_m\) such that for any open subset \(V \subset X\),

$$\int_V (e^{\lambda^{(m)}_{\ell,N}} - e^{\lambda^{(m)}_{\ell,N-1}}) f \omega^n \leq C_m \sup_V f,$$
and thus
\[ \int_V (e^{\lambda_N^{(m)} - \lambda_{N-1}^{(m)}}) f \omega^n = \sum_{i=1}^{N_m} \int_V (e^{\lambda_{i,N}^{(m)} - \lambda_{i,N-1}^{(m)}}) f \omega^n \leq N_m C_m \sup_V f. \]

An application of (the concave version of) Jensen’s inequality to the concave function log then gives
\[ \frac{1}{|V|} \int_V (\lambda_N^{(m)} - \lambda_{N-1}^{(m)}) f \omega^n \leq \log \left( \frac{N_m C_m \sup_V f}{\int_V f \omega^n} \right). \]

The proof is complete. \( \Box \)

Consider the function
\[ \Lambda_k^{(m)} = \frac{1}{k} \lambda_{mk}. \]

Note that \( \Lambda_k^{(m)} \) is locally the sum of a plurisubharmonic function and a smooth function. By applying Lemma 6 and using the telescoping property, we see that for any open set \( V \subset X \) and any smooth function \( f : V \rightarrow \mathbb{R}^+ \),
\[ \int_V \Lambda_k^{(m)} f \omega^n \leq m \log \left( \frac{N_m C_m \sup_V f}{\int_V f \omega^n} \right). \]

**Proposition 7.** There exists a constant \( C_\alpha^{(m)} \) such that
\[ \Lambda_k^{(m)}(x) \leq C_\alpha^{(m)}, \quad x \in X. \]

**Proof.** Let us cover \( X \) by coordinate charts \( V_1, ..., V_N \) such that for each \( j \) there is a biholomorphic map \( F_j \) from \( V_j \) to the ball \( B(0,2) \) of radius 2 centered at the origin in \( \mathbb{C}^n \), and such that if \( U_j = F_j^{-1}(B(0,1)) \), then \( U_1, ..., U_N \) is also an open cover. Let \( W_j = V_j \setminus F_j^{-1}(B(0,3/2)) \).

Now, on each \( V_j \), \( \Lambda_k^{(m)} \) is the sum of a plurisubharmonic function and a smooth function. Say \( \Lambda_k^{(m)} = h + g \) on \( V_j \), where \( h \) is plurisubharmonic and \( g \) is smooth. Then for constant \( A_j \) we have
\[ \sup_{U_j} \Lambda_k^{(m)} \leq \sup_{U_j} g + \sup_{U_j} h \]
\[ \leq \sup_{U_j} g + A_j \int_{W_j} h \cdot F_j \cdot dV \]
\[ \leq \sup_{U_j} g - A_j \int_{W_j} g \cdot F_j \cdot dV + A_j \int_{W_j} \Lambda_k^{(m)} \cdot F_j \cdot dV \]

Let
\[ C_j^{(m)} := \sup_{U_j} g - A_j \int_{W_j} g \cdot F_j \cdot dV \]
and define the smooth function \( f_j \) by
\[ f_j \omega^n = F_j \cdot dV. \]
Then by (1) applied with $V = W_j$ and $f = f_j$, we have

$$\sup_{U_j} \Lambda_k^{(m)} \leq C_j^{(m)} + mA_j \log \left( \frac{N_mC_m \sup W_j f_j}{\int_{W_j} f_j \omega_n^j} \right) \int_{W_j} f_j \omega_n^j.$$ 

Letting

$$C_o^{(m)} := \max_{1 \leq j \leq N} \left\{ C_j^{(m)} + mA_j \log \left( \frac{N_mC_m \sup W_j f_j}{\int_{W_j} f_j \omega_n^j} \right) \int_{W_j} f_j \omega_n^j \right\}$$

completes the proof. $\blacksquare$

Since the upper regularization of the lim sup of a uniformly bounded sequence of plurisubharmonic functions is plurisubharmonic (see, e.g., [H-90, Theorem 1.6.2]), we essentially have the following corollary.

**Corollary 8.** The function

$$\Lambda^{(m)}(x) := \limsup_{y \to x} \limsup_{k \to \infty} \Lambda_k^{(m)}(y)$$

is locally the sum of a plurisubharmonic function and a smooth function.

**Proof.** One need only observe that the function $\Lambda_k$ is obtained from a singular metric on the line bundle $m(K_X + Z + E)$ (this singular metric $e^{-\kappa_k^{(m)}}$ will be described shortly) by multiplying by a fixed smooth metric of the dual line bundle. $\blacksquare$

Consider the singular Hermitian metric $e^{-\kappa^{(m)}}$ for $m(K_X + Z + E)$ defined by

$$e^{-\kappa^{(m)}} = e^{-\Lambda^{(m)}} \omega^{-nm} e^{-m(\gamma_Z + \gamma_E)}.$$ 

This singular metric is given by the formula

$$e^{-\kappa^{(m)}}(x) = \exp \left( - \limsup_{y \to x} \limsup_{k \to \infty} \kappa_k^{(m)}(y) \right),$$

where

$$e^{-\kappa_k^{(m)}} = e^{-\Lambda_k^{(m)}} \omega^{-nm} e^{-m(\gamma_Z + \gamma_E)}.$$ 

The curvature of $e^{-\kappa_k^{(m)}}$ is thus

$$\sqrt{-1} \partial \bar{\partial} \kappa_k^{(m)} = \frac{1}{k} \partial \bar{\partial} \log \sum_{\ell=1}^{N_m} \sum_{j=1}^{N_0} |\bar{\omega}_{j,\ell}^{(m,k,0)}|^2 \geq \frac{1}{k} \sqrt{-1} \partial \bar{\partial} \psi.$$ 

We claim next that the curvature of $e^{-\kappa}$ is non-negative. To see this, it suffices to work locally. Then we have that the functions

$$\kappa_k^{(m)} + \frac{1}{k} \psi$$

are plurisubharmonic. But

$$\limsup_{y \to x} \limsup_{k \to \infty} \kappa_k^{(m)} + \frac{1}{k} \psi = \limsup_{y \to x} \limsup_{k \to \infty} \kappa_k^{(m)} = \kappa^{(m)}.$$ 

It follows that $\kappa^{(m)}$ is plurisubharmonic, as desired.
4.2. The metric for $K_X + Z + E$; Proof of Theorem 1. Let $\varepsilon_m$ be constants, chosen so $\varepsilon_m \to 0$ sufficiently rapidly that the sum

$$e^\kappa := \sum_{m=1}^\infty \varepsilon_m e^{\frac{1}{m}\kappa(m)} = \sum_{m=1}^\infty \exp(\frac{1}{m}\kappa(m) + \log \varepsilon_m),$$

converges everywhere on $X$ (to a metric for $-(K_X + Z + E)$). It is possible to find such constants since, by Proposition 7, each $\kappa(m)$ is locally uniformly bounded from above. (The lower bound $e^{\kappa(m)} \geq 0$ is trivial.) Moreover, by elementary properties of plurisubharmonic functions, $\kappa$ is plurisubharmonic. Indeed, for any $r \in \mathbb{N}$, the function

$$\psi_r := \log \sum_{m=1}^r \exp(\frac{1}{m}\kappa(m) + \log \varepsilon_m)$$

is plurisubharmonic, and $\psi_r \not< \kappa$. It follows that $\kappa = \sup_r \psi_r$ is plurisubharmonic. (Again, see [H-90, Theorem 1.6.2].) Thus $e^{-\kappa}$ is a singular Hermitian metric for $K_X + Z + E$ with non-negative curvature current.

Observe that, after identifying $K_Z$ with $(K_X + Z)|Z$ by dividing by $dT$,

$$\kappa_k(m)|Z = \log \left( \sum_{\ell=1}^{N_m} |s_{\ell}^{(m)}|^2 \right) + \frac{1}{k} \log \sum_{j=1}^{M_0} |\sigma_j^{(m,0,0)}|^2.$$

Thus we obtain $e^{-\kappa}|Z = \left( \sum_{\ell=1}^{N_m} |s_{\ell}^{(m)}|^2 \right)^{-1}$. It follows that

$$e^{-\kappa}|Z = \frac{1}{\sum_{m=1}^\infty \varepsilon_m \left( \sum_{\ell=1}^{N_m} |s_{\ell}^{(m)}|^2 \right)^{2/m}}.$$

In view of the short discussion following the proof of Proposition 3, the metric $e^{-\kappa}$ satisfies the conclusions of Theorem 1. The proof of Theorem 1 is thus complete. □

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