THE NUMBER OF RATIONAL CURVES ON K3 SURFACES

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Abstract. Let \( X \) be a K3 surface with a primitive ample divisor \( H \), and let \( \beta = 2[H] \in H_2(X, \mathbb{Z}) \). We calculate the Gromov-Witten type invariants \( n_\beta \) by virtue of Euler numbers of some moduli spaces of stable sheaves. Eventually, it verifies Yau-Zaslow formula in the non primitive class \( \beta \).

Key words. Rational curve, K3 surface, stable sheaf, Euler number

AMS subject classifications. 14N35, 14D20

Introduction. Let \( X \) be a K3 surface with an ample divisor \( H \), and let \( C \in |H| \) be a reduced curve. By adjunction formula, the arithmetic genus of \( C \) is \( g = \frac{1}{2}H^2 + 1 \). Under the assumption that the homology class \( [H] \in H_2(X, \mathbb{Z}) \) is primitive, Yau and Zaslow [18] showed that the number of rational curves in the linear system \( |H| \) is equal to the coefficient of \( q^g \) in the series

\[
\Delta(q) = \frac{1}{\prod_{k>0} (1 - q^k)^{24}} = \sum_{d \geq 0} G_d q^d = 1 + 24q + 324q^2 + 3200q^3 + 25650q^4 + 176256q^5 + \cdots
\]

Here a multiplicity \( e(J) \) is assigned to each rational curve \( C \) in the counting[1]).

In [5], Fantechi, Götsche and van Straten gave an interpretation of the multiplicity \( e(JC) \). Let \( M_{0,0}(X,[H]) \) be the moduli space of genus zero stable maps \( f : \mathbb{P}^1 \to X \) with \( f_*[\mathbb{P}^1] = [H] \in H_2(X, \mathbb{Z}) \). \( M_{0,0}(X,[H]) \) is a zero dimensional scheme which is in general nonreduced. Let \( \iota : X \hookrightarrow X \) be a rational curve in the class \( [H] \), and \( n : \mathbb{P}^1 \to C \) its normalization. Then \( f = \iota \circ n : \mathbb{P}^1 \to X \) is a closed point of \( M_{0,0}(X,[H]) \) and \( e(JC) \) is equal to the multiplicity of \( M_{0,0}(X,[H]) \) at \( f \).

There is another formulation and generalization of Yau and Zaslow’s formula by virtue of Gromov-Witten invariants. For K3 surfaces, the usual genus 0 Gromov-Witten invariants vanish. To remedy this, one can use the notion of twistor family developed by Bryan and Leung in [2] provided that \( \beta \) is a primitive class. In general, there is an algebraic geometric approach proposed by Jun Li [11] using virtual moduli cycles. Roughly speaking, he defines Gromov-Witten type invariants \( N_\beta(\beta) \) on K3 surfaces by modifying the usual tangent-obstruction complex. When \( \beta \) is primitive, these invariants coincide with those defined by twistor family. Geometrically, \( N_\beta(\beta) \) can be thought as Gromov-Witten invariants of a one dimensional family of K3 surfaces, which actually count curves in the original surface. For the rigorous definitions, see [2],[11].

Bryan and Leung [2] proved a formula for \( N_\beta(\beta) \) when \( \beta \) is primitive. Let \( n_\beta = N_\beta(\beta) \). Then \( n_\beta = G_d \) with \( d = \frac{1}{2}\beta^2 + 1 \). It recovers the formula of Yau and Zaslow. For a non primitive class \( \beta \), the numbers \( N_\beta(\beta) \) are still unknown. However, there is a conjectural formula for \( N_0(\beta)([11]) \). Using the notation \( n_\beta \), it says

\[
n_\beta = \sum_k \frac{1}{k^3} G_d^{\frac{1}{2}(\frac{2}{d})^2 + 1}
\]
where the sum runs over all integers $k > 0$ such that $\frac{\beta}{k}$ is an integral homology class (see also [6]). The case $\beta = 2[H]$ with $[H]$ primitive and $H^2 = 2$ was proved by Gathmann in [6].

In this paper, we will prove the following result.

**Theorem 0.1.** Let $X$ be a K3 surface with an ample divisor $H$. Assume $[H] \in H_2(X, \mathbb{Z})$ is primitive. Let $\beta = 2[H]$ and $g = \frac{1}{4}H^2 + 1$. Then

$$n_\beta = G_{4g-3} + \frac{1}{8}G_g.$$

Now we sketch the proof of this theorem. It can be divided into two steps. First, we deform the pair $(X, H)$ to general position and then reduce the calculation of $n_\beta$ to $N_\beta$, which is the number of reduced and irreducible rational curves in $\beta$. The second step is the calculation of $N_\beta$. In Gathmann’s approach [6], the assumption $H^2 = 2$ is essential in this step. In this paper, we will generalize the approach of Yau and Zaslow [18] according to the suggestion in [11].

Next, we describe these two steps in details.

We begin with the first step. Let $(X, H)$ be a pair of a K3 surface $X$ and a primitively polarization $H$ on $X$. It is well known that two pairs $(X, H)$ and $(X', H')$ with $H^2 = H'^2$ are deformation equivalent. One can choose a general primitively polarized K3 surface $(X, H)$, such that $\text{Pic} X = \mathbb{Z} \cdot [H]$ and every rational curve in the linear system $|H|$ is nodal [3]. Moreover, using a generalization of the method in [3], one can also assume that any two rational curves in the system $|H|$ intersect transversely [4]. Now we fix such a pair $(X, H)$ once and for all. Since $n_\beta$ is a deformation invariant, we only need to calculate $n_\beta$ for such a surface.

By the enumerative interpretation of $n_\beta$, and follow up a similar argument as in [6], all stable maps $f : C \to X$ with $f_*([C]) = \beta$ can be decomposed into the following three types:

1) The domain $C$ is $\mathbb{P}^1$, and the image $f(C) \subset X$ is a reduced and irreducible curve in the linear system $|2H|$. We denote the number of such maps by $N_\beta$. The multiplicity of such $f$ is the Euler number of the compactified Jacobian of the image $f(C)$, as shown in [1].

2) The domain is a union of two $\mathbb{P}^1$ that intersect at one point $P$. In this case, the image is a union of two rational nodal curves that intersect at $H^2$ points. The image of $P$ has to be one of the intersections, hence there are $H^2$ such maps. Since the number of rational curves in the system $|H|$ is $G_g$, the total number of such maps is $\frac{1}{2}G_g(G_g - 1)H^2$.

3) $f : C \to X$ is a double cover onto the image $f(C)$. There are two different cases:

(a) Double covers that factor through the normalization of $f(C)$, this space has dimension 2.

(b) Double covers that do not factor through the normalization. In this case, the domain must be a union of two $\mathbb{P}^1$, which intersect at one point $P$. The image of $P$ is a node on the image curve $f(C)$, and there is only one map for each choice of node. Note that the number of nodes on $f(C)$ is equal to the arithmetic genus $g$, so there are totally $gG_g$ such maps.

By Lemma 4.1 in [6], the contribution of type (3a) is $\frac{1}{8}G_g$. Therefore,

$$n_\beta = N_\beta + \frac{1}{2}G_g(G_g - 1)H^2 + gG_g + \frac{1}{8}G_g.$$
Since the first step of the proof is already known, in this paper, we will focus on the second step, namely, the calculation of the number $N_\beta$ of reduced and irreducible rational curves in the linear system $|2H|$. To this end, we will work with the moduli space of sheaves on a $K3$ surface.

Let $(X,H)$ be the pair we fixed previously. Let $\mathcal{M}$ be the moduli scheme of stable sheaves $\mathcal{F}$ on $X$ such that $\dim \mathcal{F} = 1$, $c_1(\mathcal{F}) = 2H$ and $\chi(\mathcal{F}) = 1$. The Hilbert polynomial of $\mathcal{F}$ with respect to the polarization $H$ is $2H^2 \cdot n + 1$. Since there is no strictly semistable sheaf in $\mathcal{M}$, by [13], $\mathcal{M}$ is a smooth projective variety, and its Euler number $e(\mathcal{M})$ is $G_{2H^2+1}$([19]). In section 1, we will construct a morphism $\Phi : \mathcal{M} \to |2H|$ that sends $\mathcal{F} \in \mathcal{M}$ to its support in $|2H|$. For $D \in |2H|$, we denote by $\mathcal{M}_D$ the fiber of $\Phi$ over $D$ with the reduced subscheme structure. When $D$ is reduced and irreducible, $\mathcal{M}_D$ is the compactified Jacobian $\bar{J}D$ of $D$. In section 2, we will show that $e(\mathcal{M}_D) = 0$ if $D$ has an irreducible component whose geometric genus is positive.

Therefore only divisors with rational components contribute to the Euler number $e(\mathcal{M})$. Since $H$ is primitive, we have three types of these divisors in the linear system $|2H|$.

1) $D = C$, $C$ is a rational curve in homology class $\beta(= 2[H])$. In this case, $\mathcal{M}_D \cong \bar{J}D$. The number of such divisors $D$, counted with multiplicity $e(\bar{J}D)$, is equal to $N_\beta$.

2) $D = C_1 + C_2$, where $C_1$ and $C_2$ are different rational nodal curves. In this case, both $C_i$ are contained in the linear system $|H|$. There are totally $\frac{1}{2}G_g(G_g - 1)$ divisors of this type. We will show that $e(\mathcal{M}_D) = H^2$ in section 3.

3) $D = 2C_0$, where $C_0$ is a rational nodal curve and contained in $|H|$. The number of such divisors is $G_{g}$. In the last two sections we will prove $e(\mathcal{M}_D) = g$, which is equal to the number of nodes of $C_0$.

Since $e(\mathcal{M}) = \sum e(\mathcal{M}_D)$, where the sum runs over all divisors $D$ with rational components, we get

\begin{equation}
N_\beta = e(\mathcal{M}) - \frac{1}{2}G_g(G_g - 1)H^2 - gG_g
= G_{4g-3} - \frac{1}{2}G_g(G_g - 1)H^2 - gG_g.
\end{equation}

Together with (1), we prove

$$n_\beta = G_{4g-3} + \frac{1}{8}G_g.$$

Recently, J. Li and the author [12] proved the conjectured formula for non-primitive class $\beta = n[H]$ with $n < 6$, under the assumption that the transversality of rational curves still holds.

I am most grateful to Jun Li, from whom I learned moduli spaces of sheaves and Gromov-Witten invariants. During the preparation of this paper, his constant encouragement and discussions are invaluable. After finishing the manuscript, the author is informed that J. Lee and N. C. Leung [9] proved the same result using degeneration method and also counted genus 1 curves in $K3$ surfaces [10].

1. Decomposition of the moduli scheme $\mathcal{M}$. We start with some definitions and notations([15],[8]).

Let $X$ be a complex projective scheme with an ample line bundle $\mathcal{O}(1)$. For a coherent sheaf $\mathcal{E}$ of $\mathcal{O}_X$-module, the Hilbert polynomial $p(\mathcal{E}, n)$ of $\mathcal{E}$ is defined as

$$p(\mathcal{E}, n) = \dim H^0(X, \mathcal{E}(n)), \ n \gg 0.$$
The dimension of the support of $\mathcal{E}$ is equal to the degree of $p(\mathcal{E}, n)$. A coherent sheaf $\mathcal{E}$ is pure of dimension $d$ if for any nonzero coherent subsheaf $\mathcal{F} \subset \mathcal{E}$, $\dim \mathcal{F} = d$.

The Hilbert polynomial $p(\mathcal{E}, n)$ can be written as

$$p(\mathcal{E}, n) = \frac{a_0}{d!} n^d + \frac{a_1}{d(d-1)!} n^{d-1} + \cdots$$

with integral coefficients $a_i = a_i(\mathcal{E})$. We define the slope of $\mathcal{E}$ to be

$$\mu(\mathcal{E}) = \frac{a_0(\mathcal{O}_X)}{a_0(\mathcal{E})} - \frac{a_1(\mathcal{O}_X)}{a_0(\mathcal{E})}.$$ 

**Definition 1.1.** A coherent sheaf $\mathcal{E}$ is stable (resp. semistable) if it is pure, and if for any nonzero proper subsheaf $\mathcal{F} \subset \mathcal{E}$, there exists an $N$, such that for $n > N$,

$$\frac{p(\mathcal{F}, n)}{a_0(\mathcal{F})} < \frac{p(\mathcal{E}, n)}{a_0(\mathcal{E})} \quad (\text{resp.} \leq).$$ 

**Definition 1.2.** A coherent sheaf $\mathcal{E}$ is $\mu$-stable (resp. $\mu$-semistable) if it is pure, and if for any nonzero proper subsheaf $\mathcal{F} \subset \mathcal{E}$,

$$\mu(\mathcal{F}) < \mu(\mathcal{E}) \quad (\text{resp.} \leq).$$ 

**Theorem 1.3.** [15] Let $X$ be a complex projective scheme with an ample line bundle $\mathcal{O}(1)$. There is a projective coarse moduli scheme whose closed points represent the $S$-equivalence classes of semistable sheaves with Hilbert polynomial $P(n)$.

Let $X$ be a K3 surface with an ample line bundle $H$. By Riemann-Roch theorem, the Hilbert polynomial of a torsion free sheaf $\mathcal{E}$ is

$$p(\mathcal{E}, n) = \frac{r}{2} H^2 n^2 + (c_1 \cdot H)n + r \chi(\mathcal{O}_X) + \frac{1}{2} (c_1^2 - 2c_2),$$

where $r$ is the rank of $\mathcal{E}$ and $c_i = c_i(\mathcal{E})$. Let $\mathcal{F}$ be a pure sheaf of dimension 1 on $X$. By a locally free resolution, one can verify that the Hilbert polynomial of $\mathcal{F}$ is

$$p(\mathcal{F}, n) = (c_1(\mathcal{F}) \cdot H)n + \frac{1}{2} (c_1^2(\mathcal{F}) - 2c_2(\mathcal{F})).$$

It is clear that for such sheaves the notion of stability and $\mu$-stability coincide.

From now on, we fix a pair $(X, H)$ of a K3 surface $X$ and a polarization $H$ of $X$, such that

1) $\text{Pic } X = \mathbb{Z} \cdot [H]$;
2) every rational curve in $|H|$ is nodal; and
3) any two distinct rational curves in $|H|$ intersect transversely.

We let $\beta = 2[H] \in H_2(X, \mathbb{Z})$. Our immediate goal is to calculate $N_{\beta}$, the number of reduced and irreducible rational curves in $|2H|$ counted with multiplicity. To this end, we consider the moduli scheme $\mathfrak{M}$ of stable sheaves $\mathcal{F}$ of $\mathcal{O}_X$-modules that satisfy $\dim \mathcal{F} = 1$, $c_1(\mathcal{F}) = \beta$ and $\chi(\mathcal{F}) = 1$.

**Theorem 1.4.** [19] $\mathfrak{M}$ is a smooth projective variety. The Euler number $e(\mathfrak{M})$ is $G_{2H^2+1}$. 

Next we define the morphism $\Phi : \mathfrak{M} \rightarrow |2H|$ mentioned earlier.

Let $\mathcal{F}$ be a sheaf in $\mathfrak{M}$. Since $\mathcal{F}$ is pure of dimension 1, it admits a length 1 locally free resolution

$$0 \rightarrow \mathcal{E}_1 \xrightarrow{f} \mathcal{E}_0 \rightarrow \mathcal{F} \rightarrow 0,$$

with $r(\mathcal{E}_1) = r(\mathcal{E}_0)$. The homomorphism $f : \mathcal{E}_1 \rightarrow \mathcal{E}_0$ induces a homomorphism $\wedge^r f : \wedge^r \mathcal{E}_1 \rightarrow \wedge^r \mathcal{E}_0$, and a nonzero global section $s \in H^0(\wedge^r \mathcal{E}_1)^{-1} \otimes (\wedge^r \mathcal{E}_0)$, that defines an effective divisor $D = s^{-1}(0)$ on $X$. Since $(\wedge^r \mathcal{E}_1)^{-1} \otimes (\wedge^r \mathcal{E}_0) = c_1(\mathcal{F}) = 2H$, $D$ is contained in the linear system $|2H|$. The assignment $\mathcal{F} \rightarrow D$ defines a morphism $\Phi : \mathfrak{M} \rightarrow |2H|$.

We now give a specific decomposition of the projective space $|2H|$ according to the topological type of $D \in |2H|$.

We let $\mathcal{W}_1$ be the set of divisors $D$ which is reduced and irreducible. The arithmetic genus of $D$ is $p_a(D) = 2H^2 + 1$, which is an invariant for all $D \in \mathcal{W}_1$. We further stratify $\mathcal{W}_1$ according to the geometric genus of curves, $\mathcal{W}_1 = \bigsqcup_k \mathcal{W}_1^k$, where $\mathcal{W}_1^k$ consists of those $D$ that have geometric genus $k$. Clearly $\bigsqcup_{k \leq a} \mathcal{W}_1^k$ is closed in $\mathcal{W}_1$. Let $\mathcal{W}_2$ be the stratum of divisors $D = C_1 + C_2$ with $C_1 \neq C_2$ and $C_i \in |H|$. Without loss of generality, we can assume $p_g(C_1) \leq p_g(C_2)$. For $a \leq b$, we let $\mathcal{W}_2^{a,b} \subset \mathcal{W}_2$ be the subset of divisors $D$ with $p_g(C_1) = a$ and $p_g(C_2) = b$. Then $\mathcal{W}_2 = \bigsqcup \mathcal{W}_2^{a,b}$. Let $\mathcal{W}_3$ be the subset of divisors $D = 2C_0$ with $C_0 \in |H|$. Similarly, $\mathcal{W}_3 = \bigsqcup \mathcal{W}_3^k$, where $\mathcal{W}_3^k$ consists of $D = 2C_0$ with $p_g(C_0) = k$.

Put together,

$$|2H| = (\bigsqcup_k \mathcal{W}_1^k) \bigsqcup (\bigsqcup_{a \leq b} \mathcal{W}_2^{a,b}) \bigsqcup (\bigsqcup_k \mathcal{W}_3^k).$$

This induces a decomposition on $\mathfrak{M}$,

$$\mathfrak{M} = (\bigsqcup_k \Phi^{-1}(\mathcal{W}_1^k)) \bigsqcup (\bigsqcup_{a \leq b} \Phi^{-1}(\mathcal{W}_2^{a,b})) \bigsqcup (\bigsqcup_k \Phi^{-1}(\mathcal{W}_3^k)).$$

Now we state a general fact on the Euler number of varieties.

Let $Z$ be a complex variety. Let $Z = \bigsqcup Z_i$ be a decomposition into locally closed subset $Z_i$. Then the Euler number $e(Z) = \sum e(Z_i)$.

Apply this to the decomposition of $\mathfrak{M}$, we have

$$e(\mathfrak{M}) = e(\Phi^{-1}(\mathcal{W}_1)) + e(\Phi^{-1}(\mathcal{W}_2)) + e(\Phi^{-1}(\mathcal{W}_3))$$

$$= \sum_k e(\Phi^{-1}(\mathcal{W}_1^k)) + \sum_{a \leq b} e(\Phi^{-1}(\mathcal{W}_2^{a,b})) + \sum_k e(\Phi^{-1}(\mathcal{W}_3^k)).$$

**Proposition 1.5.** [1] Let $h : Y \rightarrow Z$ be a surjective morphism between complex algebraic varieties. Suppose that $e(h^{-1}(z)) = 0$ for every closed point $z \in Z$. Then $e(Y) = 0$.

The following proposition will be proved in the next section.

**Proposition 1.6.** Suppose $D$ is a divisor that has one irreducible component whose geometric genus is positive, then $e(\mathfrak{M}_D) = 0$.

Combine these results, we have

$$e(\mathfrak{M}) = e(\Phi^{-1}(\mathcal{W}_1^0)) + e(\Phi^{-1}(\mathcal{W}_2^{0,0})) + e(\Phi^{-1}(\mathcal{W}_3^0)).$$
Because \( e(\Phi^{-1}(\mathcal{W}_1)) \) is equal to \( N_\beta \), and \( e(\mathfrak{M}) = G_{2H^2+1} \). To calculate \( N_\beta \), it suffices to find the Euler numbers \( e(\Phi^{-1}(\mathcal{W}_{2}^{0,0})) \) and \( e(\Phi^{-1}(\mathcal{W}_{2}^{2})) \). The number \( e(\Phi^{-1}(\mathcal{W}_{2}^{0,0})) \) is essentially known, which is equal to \( \frac{1}{2}G_2(G_g - 1)H^2 \) as will be shown in section 3. The main body of the remainder of the paper is to show that \( e(\Phi^{-1}(\mathcal{W}_2^{2})) = gG_g \). Therefore,

\[
N_\beta = G_{2H^2+1} - \frac{1}{2}G_2(G_g - 1)H^2 - gG_g.
\]

Apply equality (1) in the introduction, we obtain the formula in the main theorem.

2. Proof of Proposition 1.6. We state a basic fact about the Euler number of a variety. Let \( X \) be a quasi-projective variety. If there exists a finite group action on \( X \) which is free of fixed point, then \( e(X) \) is divisible by the order of this group. Therefore, if for any positive integer \( N \), there is a finite group \( G_N \) whose order is greater than \( N \), and a free \( G_N \) action on \( X \), then \( e(X) \) is zero.

If \( D \) is a reduced and irreducible curve, then \( \mathfrak{M}_D \cong JD \). Since the geometric genus of \( D \) is positive, \( e(JD) = 0 \) (see [1]). Now if \( D = C_1 + C_2 \) with \( C_i \in \{H\} \) and by assumption the geometric genus \( p_g(C_2) > 0 \). From the restriction homomorphism \( \alpha : \text{Pic}D \rightarrow \text{Pic}C_2 \), we can choose a subgroup \( G \subset \text{Pic}D \), such that for \( L \in G \), \( L|_{C_1} \cong OC_1 \), and \( \alpha(L) = L|_{C_2} \) is trivial if and only if \( L \) is trivial. Next we show that the \( G \)-action on \( \mathfrak{M}_D \) defined by tensorization is free, i.e., for any sheaf \( F \in \mathfrak{M}_D \) and \( L \in G \), \( F \otimes L \cong F \) if and only if \( L \) is trivial. To this end, suppose \( F \otimes L \cong F \) for some \( F \) and \( L \). Let \( F_2 \) be the torsion free part of the restriction \( F|_{C_2} \). We obtain \( F_2 \otimes \alpha(L) \cong F_2 \) and therefore \( \alpha(L) \) is trivial by the same argument as in case 1. Finally, it implies \( L \) is trivial by the choice of the subgroup \( G \). Finally, \( D = 2C_0 \) is a divisor whose associated subscheme is a nonreduced curve \( C \), and a closed point in \( \mathfrak{M}_D \) is a sheaf of \( OC \)-modules. To prove this case, we first recall some facts on nonreduced curves.

Let \( F \) be a sheaf of \( OC \)-modules. An infinitesimal extension ([7], Exer II 8.7) of \( X \) by \( F \) is a scheme \( X' \), with an ideal sheaf \( I \), such that \( I^2 = 0 \) and \( (X', OC_{X'/I}) \cong (X, OC) \) and such that \( I \) with the induced structure of \( OC \)-module is isomorphic to the given sheaf \( F \). Let \( S \) be a smooth projective surface, and \( C_0 \subset S \) be a reduced and irreducible curve. There is an associated closed subscheme \( C \subset S \) to the divisor \( 2C_0 \). In fact, \( C \) is an infinitesimal extension of \( C_0 \) by \( I = OS(-(C_0)|_{C_0}) \).

Next we discuss the Picard group of \( C ([7], \text{Exer III 4.6}) \). From the exact sequence of sheaves of abelian groups

\[
0 \longrightarrow I \longrightarrow OC \longrightarrow OC_0 \longrightarrow 0,
\]

there is an induced exact sequence

\[
0 \longrightarrow H^1(C, I) \longrightarrow \text{Pic}C \longrightarrow \text{Pic}C_0 \longrightarrow 0.
\]

Notice that \( H^1(C, I) \) is a vector space and hence an injective \( Z \)-module, it implies that \( \text{Pic}C \cong \text{Pic}C_0 \oplus H^1(C, I) \) as groups. For \( L \in \text{Pic}C \), we let \( L_0 \in \text{Pic}C_0 \) be the restriction of \( L \) to \( C_0 \).

Now we continue the proof. Let \( \pi : \tilde{C}_0 \rightarrow C_0 \) be the normalization of \( C_0 \). Then \( \text{Pic}^0 \tilde{C}_0 \cong \text{Pic}^0 C_0 \oplus A \), where \( A \) is an affine commutative group. Since the genus of \( \tilde{C}_0 \) is positive, \( \text{Pic}^0 \tilde{C}_0 \) is nontrivial. For any odd prime \( p \), we can choose an order \( p \) subgroup \( G \subset \text{Pic}^0 C_0 \), such that for \( L \in G \), \( L \) is trivial if and only if \( \tilde{L} = \pi^* L_0 \) is...
trivial. There is a $G$-action on $\mathcal{M}_D$ defined by tensorization. Next we show that this group action is free and therefore $e(\mathcal{M}_D) = 0$.

Suppose $E \otimes E \cong E$ for some sheaf $E \in \mathcal{M}_D$ and $L \in G$. Restrict to $C_0$ and let $E_0$ be the torsion free part of $E \otimes \mathcal{O}_{C_0}$, we get $L_0 \otimes E_0 \cong E_0$.

1) If $IE \neq 0$, $E_0$ is a rank 1 torsion free sheaf on $C_0$. Using the same argument as in case 1, we obtain $E_0 \cong \mathcal{O}_{C_0}$, and hence $L \cong \mathcal{O}_C$, i.e., the group action is free.

2) If $IE = 0$, $E_0$ is a rank 2 torsion free sheaf on $C_0$. Let $E_0$ be the torsion free part of $\pi^*E_0$. Then we have $L_0 \otimes E_0 \cong E_0$. Take top wedge on both sides, we get $L_0^\otimes 2 \otimes \wedge^2 E_0 \cong \wedge^2 E_0$. Since $\wedge^2 E_0$ is invertible, $L_0^\otimes 2 \cong \mathcal{O}_{C_0}$. Note that $L \in G$ and $G$ has odd prime order $p$, it implies that $L \cong \mathcal{O}_C$. Hence the group action is free.

3. Calculation of $e(\Phi^{-1}(\mathcal{W}_2^{0,0}))$. Recall that $\mathcal{W}_2^{0,0}$ is a finite set of divisors $D = C_1 + C_2$ with $C_1, C_2 \in |H|$ being rational nodal curves and intersect transversally, $\Phi^{-1}(\mathcal{W}_2^{0,0}) = \bigcup \mathcal{M}_D$, with $D_i \in \mathcal{W}_2^{0,0}$. We will calculate $e(\mathcal{M}_D)$ for $D \in \mathcal{W}_2^{0,0}$ and then the Euler number $e(\Phi^{-1}(\mathcal{W}_2^{0,0}))$ follows.

A closed point in $\mathcal{M}_D$ is a stable sheaf $E$ of $\mathcal{O}_D$-modules, such that the restrictions $E|_{C_i}$ are rank 1 sheaves of $\mathcal{O}_{C_i}$-modules respectively. Let $x_1, x_2, \ldots, x_s$ be a list of intersections of $C_1$ and $C_2$. Then $s = H^2 > 0$. Since $E$ is stable, there is at least one point $x_i$, so that the stalk $E_{x_i}$ is isomorphic to $\mathcal{O}_{x_i}$. For otherwise, $E$ is the direct image of some sheaf on the disjoint union of $C_1$ and $C_2$, which violates the stability of $E$.

We let $S_{ij} \subset \mathcal{M}_D$ be the subset of stable sheaves $E$ such that $E_{x_i} \cong \mathcal{O}_{x_i}$, and $E_{x_j} \cong \mathcal{O}_{x_j}$, for two intersection points $x_i$ and $x_j$. We can find a subgroup $G \subset \text{Pic} D$ coming from the gluing of $\mathcal{O}_{C_1}$ and $\mathcal{O}_{C_2}$ at $x_i$ and $x_j$. $G \cong C^*$. Now follow a similar argument as in the previous section, the $G$-action on $S_{ij}$ is free. Therefore, the contribution to the Euler number $e(\mathcal{M}_D)$ come from stable sheaves $E$ whose stalks are not $\mathcal{O}$ at all nodes but one intersection point. Since both $C_1$ are rational curves, there is only one such stable sheaf corresponds to an intersection point. We have

**Proposition 3.1.** Let $D$ be a divisor in the set $\mathcal{W}_2^{0,0}$. Then $e(\mathcal{M}_D) = H^2$.

Since the number of rational curves in $|H|$ is $G$, $\mathcal{W}_2^{0,0}$ is a finite set with cardinality $\frac{1}{2} G_g(G_g - 1)$.

**Corollary 3.2.** $e(\Phi^{-1}(\mathcal{W}_2^{0,0})) = \frac{1}{2} G_g(G_g - 1) H^2$.

4. Calculation of $e(\Phi^{-1}(\mathcal{W}_3^{0,0}))$, Part I. In the remainder of this paper, we will calculate the Euler number $e(\Phi^{-1}(\mathcal{W}_3^{0,0}))$. Remember that $\mathcal{W}_3^{0,0}$ is a finite set of divisors $D = 2C_0$ with $C_0 \in |H|$ being rational nodal curves, there is a decomposition $\Phi^{-1}(\mathcal{W}_3^{0,0}) = \bigcup \mathcal{M}_D$. It suffices to calculate $e(\mathcal{M}_D)$ for $D \in \mathcal{W}_2^{0,0}$.

Recall that for every effective divisor, there is an associated subscheme. Let $C$ be the nonreduced curve associated to $D = 2C_0$. Then every closed point in $\mathcal{M}_D$ corresponds to a stable sheaf $E$ of $\mathcal{O}_C$-modules, such that the Hilbert polynomial $P_E(n)$ is $2H^2 + 1$.

There are two kinds of these sheaves. A sheaf $E$ in the first type satisfies $IE = 0$, where $I \subset \mathcal{O}_C$ is the nilpotent ideal sheaf. That is to say, $E$ is a rank 2 sheaf on $C_0$, the reduced part of $C$. Let $\mathcal{M}_D^1 \subset \mathcal{M}_D$ be the subset of sheaves of this type. The second type consists of sheaves $E$ satisfy $IE \neq 0$. It is direct to verify that for sheaves of this type, $E_{\eta} \cong \mathcal{O}_\eta$ with $\eta$ the generic point of $C$. Let $\mathcal{M}_D^2$ be the subset of sheaves of the second type. Then $e(\mathcal{M}_D) = e(\mathcal{M}_D^1) + e(\mathcal{M}_D^2)$.

In this section, we calculate $e(\mathcal{M}_D^1)$. The discussion of $\mathcal{M}_D^2$ is left to the next section. The result has been obtained by T. Teodorescu in his PhD thesis [16] which
We recall some standard facts about sheaves on a nodal curve. Since we will not talk about nonreduced curves in the next part of this section, we use $C$, instead of $C_0$, to denote a nodal curve. We always work on the complex topology.

Let $C$ be a projective curve with $n$ ordinary nodes $x_1, x_2, \ldots, x_n$ as singularities, and let $\pi : \tilde{C} \to C$ be the normalization of $C$. A torsion free sheaf $\mathcal{E}$ is locally free away from the nodes. It has the following nice local structure at each node $x_i \in C([14])$

$$\mathcal{E}_{x_i} \cong \mathcal{O}_{x_i}^{\oplus a_i} \oplus m_{x_i}^{\oplus \delta(r-a_i)},$$

where $m_{x_i} \subset \mathcal{O}_{x_i}$ is the maximal ideal, and $r$ is the rank of $\mathcal{E}$. Let $\hat{\pi} : \tilde{C} \to C$ be a partial normalization of $C$ at one node $x$. Then there exists a torsion free sheaf $\mathcal{F}$ on $\tilde{C}$ such that $\mathcal{E} \cong \hat{\pi}_* \mathcal{F}$ if and only if $\mathcal{E}_x \cong m_x^{\oplus \delta}.$

Let $r \geq 1$ be an integer and choose $n$ such that $(r, n) = 1$. There is a smooth projective variety $\mathcal{M}_{C}(r, n)$ whose closed points correspond to isomorphism classes of stable $\mathcal{O}_C$-modules $\mathcal{E}$, such that $r(\mathcal{E}) = r$ and $\chi(\mathcal{E}) = n$.

Next we introduce the notion of admissible quotients. It will be used to determine whether two torsion free sheaves $\mathcal{E}_1, \mathcal{E}_2$ are isomorphic. Let $\mathcal{E}$ be a torsion free sheaf and $(\pi^* E)^{\sharp}$ be the torsion free part of $\pi^* E$. There is a canonical exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \pi_*(\pi^* E)^{\sharp} \longrightarrow T \longrightarrow 0,$$

where $T \cong \oplus \mathcal{O}_{x_i}^{\oplus a_i}$ is a skyscraper sheaf supported at the nodes.

**Definition 4.1.** Let $\mathcal{V}$ be a rank $r$ locally free sheaf on $\tilde{C}$, and $\mathcal{Q} = \oplus \mathcal{O}_{x_i}^{\oplus a_i}$ be a skyscraper sheaf supported at the nodes of $C$. Let $\rho : \pi_* \mathcal{V} \to \mathcal{Q}$ be a surjective morphism and $\mathcal{E}$ be the kernel of $\rho$. $\rho$ is said to be an admissible quotient if there is a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{E} & \longrightarrow & \pi_* \mathcal{V} & \stackrel{\rho}{\longrightarrow} & \mathcal{Q} & \longrightarrow & 0 \\
& & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \mathcal{E} & \longrightarrow & \pi_*(\pi^* \mathcal{E})^\sharp & \longrightarrow & T & \longrightarrow & 0
\end{array}
$$

where the second row is the canonical exact sequence.

Let $p_i, q_i \in \tilde{C}$ be the inverse images of the node $x_i$. Then $(\pi_* \mathcal{V})_{x_i} = \mathcal{V}_{p_i} \oplus \mathcal{V}_{q_i}$. The homomorphism $\rho$ is given by $\rho_{x_i} : \mathcal{V}_{p_i} \oplus \mathcal{V}_{q_i} \to \mathcal{Q}_{x_i}$. Let $i^1_{x_i} : \mathcal{V}_{p_i} \to \mathcal{V}_{p_i} \oplus \mathcal{V}_{q_i}$ and $i^2_{q_i} : \mathcal{V}_{q_i} \to \mathcal{V}_{p_i} \oplus \mathcal{V}_{q_i}$ be the natural injections. By the definition of admissible quotients, $\rho^1_{x_i} = \rho_{x_i} \circ i^1_{x_i}$ are both surjective. Conversely, we have

**Proposition 4.2.** Let $\rho : \pi_* \mathcal{V} \to \mathcal{Q}$ be a quotient such that $\rho^1_{x_i}$, defined above are surjective for all $i = 1, 2, \ldots, n$ and $k = 1, 2$. Then $\rho$ is an admissible quotient.

**Proof.** Let $\mathcal{E}$ be the kernel of $\rho$. Apply the functor $\pi^*$ to the exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \pi_* \mathcal{V} \stackrel{\rho}{\longrightarrow} \mathcal{Q} \longrightarrow 0,$$

we get

$$\pi^* \mathcal{E} \stackrel{\psi_1}{\longrightarrow} \pi^*(\pi_* \mathcal{V}) \stackrel{\psi_2}{\longrightarrow} \pi^* \mathcal{Q} \longrightarrow 0.$$
Since \( \rho_{\psi} \) are surjective, the restriction of \( \psi_2 \) on the torsion part \( T' \subset \pi^*(\pi_* V) \) is surjective, i.e. \( \psi_2(T') = \pi^* Q \). It implies that the homomorphism \( \pi^* E \rightarrow (\pi^*(\pi_* V))^2 \) induced by \( \psi_1 \) is surjective. Because the kernel of \( \psi_1 \) is a skyscraper sheaf, \( \psi_1 \) induces an isomorphism \( (\pi^* E)^2 \rightarrow (\pi^*(\pi_* V))^2 \). Since every step is functorial, the result follows from the canonical isomorphism \( (\pi^*(\pi_* V))^2 \cong V \). \( \Box \)

**Proposition 4.3.** Let \( \rho_1, \rho_2 : \pi_* V \rightarrow Q \) be two admissible quotients and let \( E_1 = \ker \rho_1, E_2 = \ker \rho_2 \). Every isomorphism \( u : E_1 \cong E_2 \) can be extended to an isomorphism \( \psi : \pi_* V \cong \pi_* V \), i.e. we have a commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & E_1 & \longrightarrow & \pi_* V & \longrightarrow & \rho_1 & \longrightarrow & Q & \longrightarrow & 0 \\
& & \downarrow u & & \psi & & \downarrow \cong & & & & \\
0 & \longrightarrow & E_2 & \longrightarrow & \pi_* V & \longrightarrow & \rho_2 & \longrightarrow & Q & \longrightarrow & 0
\end{array}
\]

The next proposition deals with the automorphism group of \( \pi_* V \).

**Proposition 4.4.** Let \( V \) be a locally free sheaf on \( \mathcal{C} \). Every automorphism of \( \pi_* V \) as an \( \mathcal{O}_C \)-module can be induced from an automorphism of \( V \) as an \( \mathcal{O}_C \)-module. Hence there is a canonical isomorphism \( \text{Aut}_{\mathcal{O}_C}(\pi_* V) \cong \text{Aut}_{\mathcal{O}_C}(V) \).

**Proof.** Let \( u : \pi_* V \rightarrow \pi_* V \) be an automorphism of \( \pi_* V \) as an \( \mathcal{O}_C \)-module. It induces canonically an automorphism \( \tilde{u} : \pi^* \pi_* V \rightarrow \pi^* \pi_* V \) as an \( \mathcal{O}_C \)-module. Let \( T' \subset \pi^* \pi_* V \) be the torsion part. Then \( \tilde{u}(T') = T' \), and it induces an automorphism \( u^2 : (\pi^* \pi_* V)^2 \rightarrow (\pi^* \pi_* V)^2 \). Since \( V \) is locally free, there is a canonical isomorphism \( (\pi^* \pi_* V)^2 \cong V \). We obtain an automorphism \( \tilde{u} : V \rightarrow V \) as an \( \mathcal{O}_C \)-module. Since every step is functorial, it establishes an isomorphism \( \text{Aut}_{\mathcal{O}_C}(\pi_* V) \cong \text{Aut}_{\mathcal{O}_C}(V) \). \( \Box \)

Next we assume \( C \) is a rational nodal curve with \( n \) nodes. We describe a method to calculate \( e(M_C(r, n)) \).

Let \( E \) be a stable sheaf in \( M_C(r, n) \), and let \( V = (\pi^* E)^2 \) be the torsion free part of \( \pi^* E \). Then \( V \) is a locally free sheaf of rank \( r \) on \( \mathcal{C} \). Since \( \mathcal{C} \cong \mathbb{P}^1 \), by Grothendieck’s Lemma, \( V \cong \mathcal{O}(l_1) \oplus \mathcal{O}(l_2) \oplus \cdots \oplus \mathcal{O}(l_r) \) for some integers \( l_1 \leq l_2 \leq \cdots \leq l_r \). There is a decomposition of \( M_C(r, n) \),

\[
M_C(r, n) = \bigsqcup_{a_1, a_2, \ldots, a_n} M_{a_1, a_2, \ldots, a_n}^{l_1, \ldots, l_r}
\]

such that \([E] \in M_{a_1, a_2, \ldots, a_n}^{l_1, \ldots, l_r} \) if and only if

\[
(\pi^* E)^2 \cong \mathcal{O}(l_1) \oplus \mathcal{O}(l_2) \oplus \cdots \oplus \mathcal{O}(l_r)
\]

and

\[
E_{a_i} \cong \mathcal{O}_{a_i}^{\oplus r-a_i} \oplus m_{a_i}^{\oplus (r-a_i)}.
\]

Let \( E_1, E_2 \) be the kernels of two admissible quotients \( \rho_1, \rho_2 : \pi_* V \rightarrow Q \) respectively. The automorphism group of \( Q \) is a direct sum of automorphism groups of \( Q_{x_i} \). Let \( G_i = \text{Aut}(Q_i) \). Then \( G_i \cong GL(a_i, \mathbb{C}) \). There is an \( \text{Aut}(V) \times \prod G_i \) action on \( \text{Hom}(\pi_* V, Q) \), \( \rho \rightarrow g \circ \rho \circ u \), where \( \rho \in \text{Hom}(\pi_* V, Q) \), \( u \in \text{Aut}(V) \) and \( g \in \prod G_i \). Proposition 4.3 says that \( E_1 \cong E_2 \) if and only if \( \rho_1 \) and \( \rho_2 \) lie in the same orbit of \( \text{Hom}(\pi_* V, Q) \) under this group action.

Next we work out a matrix form of these results under suitable bases.
Let $V_i$ and $W_i$ be the fibres of $V$ at $p_i$ and $q_i$ respectively. Then $(\pi_*V) \otimes C_{x_i} \cong V_i \oplus W_i$. Since $Q_{x_i} = C^{\oplus a_i}$, every homomorphism $\rho : \pi_*V \to Q$ gives an element in the vector space

$$U = \bigoplus_{i=1}^n (\text{Hom}(V_i, C^{\oplus a_i}) \oplus \text{Hom}(W_i, C^{\oplus a_i})).$$

Fix an isomorphism $\mathcal{V} \cong \mathcal{O}(l_1) \oplus \mathcal{O}(l_2) \oplus \cdots \oplus \mathcal{O}(l_r)$ once and for all. For any summand $\mathcal{O}(l_i)$, there is an isomorphism of stalks $\mathcal{O}(l_i)_x \cong \mathcal{O}_x$ by the locally freeness of $\mathcal{O}(l_i)$. Those isomorphisms at $p_i$ and $q_i$ give rise to bases $e_i^k \in V_i$ and $f_i^k \in W_i$. Fix all these choices once and for all. Now an element $\rho \in U$ corresponds to a set of $a_i \times r$ matrixes

$${\rho}_i \in \text{Hom}(V_i, C^{\oplus a_i}), {\rho}_i'' \in \text{Hom}(W_i, C^{\oplus a_i})$$

Let $v_i = (v_i^1, v_i^2, \cdots, v_i^r) \in V_i$, $w_i = (w_i^1, w_i^2, \cdots, w_i^r) \in W_i$. Then

$${\rho}_i(v_i) = A_i \begin{pmatrix} v_i^1 \\ v_i^2 \\ \cdots \\ v_i^r \end{pmatrix}, {\rho}_i''(w_i) = B_i \begin{pmatrix} w_i^1 \\ w_i^2 \\ \cdots \\ w_i^r \end{pmatrix}.$$ 

**Corollary 4.5.** A quotient $\{A_i, B_i\}$ is admissible if and only if the ranks of $A_i$ and $B_i$ are both equal to $a_i$ for all $i$. In particular, for an admissible quotient $\{A_i, B_i\}$, one has $a_i \leq r$.

**Proof.** Follows from proposition 4.2. \[\square\]

Now we consider the $\text{Aut}(\mathcal{V}) \times \prod G_i$ action on $\text{Hom}(\pi_*\mathcal{V}, \mathbb{Q})$.

Evaluated at a closed point $x \in C$, every automorphism $u \in \text{Aut}(\mathcal{V})$ gives rise to an automorphism in $\text{Aut}(V_x)$, where $V_x$ is the fibre of $V$ at $x$. Therefore, every $u \in \text{Aut}(\mathcal{V})$ gives rise to an element

$$\prod_i u(p_i) \times \prod_i u(q_i) \in \prod_i \text{Aut}(V_i) \times \prod_i \text{Aut}(W_i).$$

Let $G' \subset \prod_i \text{Aut}(V_i) \times \prod_i \text{Aut}(W_i)$ be the subgroup of elements derived in this way. Since $G_i \cong GL(a_i, \mathbb{C})$, there is a $G' \times \prod GL(a_i, \mathbb{C})$ action on the vector space $U$ of all quotients $\{A_i, B_i\}_{i=1,2,\ldots,n}$, which is given by

$$\{A_i, B_i\} \mapsto \{g_i A_i u(p_i), g_i B_i u(q_i)\},$$

where $\prod u(p_i) \times \prod u(q_i) \in G'$ and $g_i \in GL(a_i, \mathbb{C})$. Two quotients $\{A_i, B_i\}$ and $\{A_i', B_i'\}$ are equivalent if they lie in one and the same orbit under this group action.

An application of this formulation is to determine whether $\mathcal{E} \otimes \mathcal{L} \cong \mathcal{F}$ for $\mathcal{L} \in \text{Pic}^0 C$.

Let $\rho : \pi_*\mathcal{V} \to \mathbb{Q}$ be an admissible quotient with corresponding matrixes $\{A_i, B_i\}$ and let $\mathcal{E} = \ker \rho$. Let $\mathcal{L} \in \text{Pic}^0 C$ be given by the matrixes $\{1, t_i\}$, where $t_i \in \mathbb{C}^*$. The exact sequence

$$0 \to \mathcal{E} \to \pi_*\mathcal{V} \xrightarrow{\rho} \mathbb{Q} \to 0$$

induces an exact sequence

$$0 \to \mathcal{E} \otimes \mathcal{L} \to \pi_*\mathcal{V} \otimes \mathcal{L} \xrightarrow{\rho \otimes 1} \mathbb{Q} \otimes \mathcal{L} \to 0.$$
Note that $\pi_* V \otimes \mathcal{L} \cong \pi_* \mathcal{V}$ and the quotient $\rho \otimes 1 : \pi_* V \otimes \mathcal{L} \to Q \otimes \mathcal{L}$ is also admissible. Fix an isomorphism $Q \otimes \mathcal{L} \cong Q$ and choose corresponding bases, $\rho \otimes 1$ is given by the matrixes $\{A_i, t_i B_i\}$.

We are now ready to calculate the Euler number $e(\mathcal{M}_C(r, n))$. For the purpose of this paper, we only consider the case $r = 2$ and $n = 1$.

**Proposition 4.6.** Let $\mathcal{M}^{l_1, l_2, \ldots, a_n}_{a_1, a_2, \ldots, a_n}$ be a stratum in $\mathcal{M}_C(2, 1)$ such that $\sum a_i \geq 2$. Then $e(\mathcal{M}^{l_1, l_2, \ldots, a_n}_{a_1, a_2, \ldots, a_n}) = 0$.

**Proof.** For simplicity, we consider only the stratum $\mathcal{M}^{l_1, l_2, \ldots, a_n}_{a_1, a_2, \ldots, a_n}$ as illustration. Let $[\mathcal{E}] \in \mathcal{M}^{l_1, l_2, \ldots, a_n}_{a_1, a_2, \ldots, a_n}$ be the kernel of an admissible quotient $\{A_i, B_i\}$. We can choose a suitable base such that $A_1 = A_2 = (1, 0), B_1 = B_2 = (0, 1)$. For an odd prime $p$, let $\mathcal{L}$ be given by $t_1 = 1, t_2 = \zeta$, where $\zeta$ is a $p$-th primitive root of unity. Then $\mathcal{L} \otimes p = \mathcal{O}_C$ and $\mathcal{E} \otimes \mathcal{L}$ corresponds to the quotient $\{A_i, t_i B_i\}$. It is direct to verify that $\{A_i, t_i B_i\}$ and $\{A_i, B_i\}$ are not equivalent, hence $\mathcal{E} \otimes \mathcal{L}$ and $\mathcal{E}$ are not isomorphic. So we get a free $\mathbb{Z}/(p)$ action on $\mathcal{M}^{l_1, l_2, \ldots, a_n}_{a_1, a_2, \ldots, a_n}$. Because $p$ can be chose arbitrarily large, $e(\mathcal{M}^{l_1, l_2, \ldots, a_n}_{a_1, a_2, \ldots, a_n}) = 0$. \(\square\)

Since $\mathcal{M}^{l_1, l_2, \ldots, a_n}_{a_1, a_2, \ldots, a_n}$ is empty, by this proposition, the contribution to the Euler number $e(\mathcal{M}_C(2, 1))$ comes from strata $\mathcal{M}^{l_1, l_2, \ldots, a_n}_{a_1, a_2, \ldots, a_n}$ with $\sum a_i = 1$. Because $\chi(\mathcal{E}) = 1$ and $\mathcal{E}$ fits into an exact sequence

$$0 \to \mathcal{E} \to \pi_* (\mathcal{O}(l_1) \oplus \mathcal{O}(l_2)) \to C_{x_i} \to 0,$$

the stability of $\mathcal{E}$ forces $l_1 = l_2 = 0$. Every $\mathcal{M}^{l_1, l_2, \ldots, a_n}_{a_1, a_2, \ldots, a_n}$ is a set of a single point. Therefore,

**Proposition 4.7.** The Euler number $e(\mathcal{M}_C(2, 1))$ is equal to $n$, which is the number of nodes on $C$.

Let $D = 2C_0$ be a divisor in the set $\mathcal{W}_3$. Then $\mathcal{M}^1_D$ is isomorphic to $\mathcal{M}_C^0(2, 1)$. The number of nodes on $C_0$ is equal to the arithmetic genus $g = \frac{1}{2}H^2 + 1$ of $C_0$. Therefore,

**Proposition 4.8.** $e(\mathcal{M}^1_D) = g$.

5. **Calculation of $e(\Phi^{-1}(\mathcal{W}_3^0))$, Part II.** This is the second part of the calculation of $e(\Phi^{-1}(\mathcal{W}_3^0))$. As we mentioned in the previous section, $\mathcal{M}_D$ is a disjoint union of $\mathcal{M}^1_D$ and $\mathcal{M}^2_D$ for $D \in \mathcal{W}_3$. We have calculated $e(\mathcal{M}^1_D)$. In this section, we will show that $e(\mathcal{M}^2_D) = 0$.

Let $C_0 \subset S$ be a nodal curve, and let $C$ be the associated nonreduced curve to the divisor $2C_0$. Let $p$ be a node on $C_0$, and $\pi_0 : \hat{C}_0 \to C_0$ be the partial normalization of $C_0$ at $p$. Now we construct a curve $\hat{C}$, which is an infinitesimal extension of $\hat{C}_0$, and a finite morphism $\pi : \hat{C} \to C$ called a partial normalization of $C$.

We pick a small neighborhood $U$ around $p$ on the surface $S$, such that $C$ is defined by $x^2 y^2 = 0$ in $U$. Let $\mathcal{C}\{x, y\}$ be the ring of holomorphic functions on $U$. Then $\mathcal{O}_C(U \cap C) = \mathcal{C}\{x, y\} / (x^2 y^2)$. The injective homomorphism

$$\psi : \mathcal{C}\{x, y\} / (x^2 y^2) \to \mathcal{C}\{x, u\} / (u^2) \oplus \mathcal{C}\{v, y\} / (v^2)$$

is a local isomorphism except at $p$. Remove the point $p$ on $C$ and glue the pieces defined by the ringed space $\mathcal{C}\{x, u\} / (u^2) \oplus \mathcal{C}\{v, y\} / (v^2)$ along $\psi$, we get a curve $\hat{C}$, and a finite map $\pi : \hat{C} \to C$. There is a canonical exact sequence

$$0 \to \mathcal{O}_C \to \pi_* \mathcal{O}_{\hat{C}} \to A \to 0,$$
where $A \cong \mathbb{C}[x,y]/(x^2,y^2)$ is a skyscraper sheaf supported at $p$. Moreover, there is a commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & \pi_*\mathcal{O}_C & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_{C_0} & \longrightarrow & \pi_*\mathcal{O}_{C_0} & \longrightarrow & C_p & \longrightarrow & 0.
\end{array}
$$

Let $I$ and $\tilde{I}$ be the nilpotent ideal sheaves of $\mathcal{O}_C$ and $\mathcal{O}_{C_0}$ respectively. Then $\chi(\tilde{I}) = \chi(I) + 3$. It implies that $\deg \tilde{I} = \deg I + 2$. Let $C$ be a rational nodal curve on a $K3$ surface and let $\tilde{C} \to C$ be the normalization of $C$. Then $\deg I = 2 - 2g = H^2$. Because the number of nodes on $C$ is equal to $g$, $\deg \tilde{I} = \deg I + 2g = 2$.

**Proposition 5.1.** Let $C$ be a nonreduced curve with nilpotent ideal sheaf $I$. Suppose $I$ is invertible as a sheaf of $\mathcal{O}_{C_0}$-modules and $\deg I > 0$. Let $\mathcal{E}$ be a pure sheaf of $\mathcal{O}_C$-modules such that $\mathcal{E}_\eta \cong \mathcal{O}_\eta$ at the generic point $\eta$ of $C$. Then $\mathcal{E}$ is not stable.

**Proof.** Let $\mathcal{E}$ be such a sheaf and let $\mathcal{E}_0^\sharp$ be the torsion free part of $\mathcal{E}_0 = \mathcal{E} \otimes \mathcal{O}_{C_0}$, considered as a sheaf of $\mathcal{O}_{C_0}$-modules. There is a canonical homomorphism $\mathcal{E} \to \mathcal{E}_0^\sharp$. Every quotient $\mathcal{E} \to \mathcal{F}$ with $\mathcal{F}$ a torsion free $\mathcal{O}_{C_0}$-module is equivalent to $\mathcal{E} \to \mathcal{E}_0^\sharp$. Therefore, for the stability of $\mathcal{E}$, it is enough to check the quotient $\mathcal{E} \to \mathcal{E}_0^\sharp$.

We start with the exact sequence

$$0 \longrightarrow I \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{C_0} \longrightarrow 0.$$

Tensoring with $\mathcal{E}$, we obtain

$$\mathcal{E}_0 \otimes I \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_0 \longrightarrow 0.$$

Let $T'$ be the torsion part of $\mathcal{E}_0 \otimes I$, and let $(\mathcal{E}_0 \otimes I)^\sharp = (\mathcal{E}_0 \otimes I)/T'$. There is an exact sequence

$$0 \longrightarrow (\mathcal{E}_0 \otimes I)^\sharp \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_0 \longrightarrow 0.$$

On the other hand, we have

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_0^\sharp \longrightarrow 0.$$

Because $I$ is an invertible sheaf of $\mathcal{O}_{C_0}$-modules, the torsion part of $\mathcal{E}_0$ is isomorphic to $T'$. $\chi(\mathcal{K}) = \chi(\mathcal{E}) - \chi(\mathcal{E}_0^\sharp) = \chi(\mathcal{E}_0 \otimes I)$. Because $\chi(\mathcal{E}_0 \otimes I) = \chi(\mathcal{E}_0) + \deg I > \chi(\mathcal{E}_0^\sharp) = \chi(\mathcal{E}_0^\sharp)$, $\mathcal{E}$ is not stable. 

Let $\pi : \tilde{C} \to C$ be the partial normalization of $C$ at $p$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_C$-modules which is pure of dimension 1. Then there is a canonical homomorphism $\mathcal{F} \to \pi_*(\pi^*\mathcal{F})$. Let $T_0 \subset \pi^*\mathcal{F}$ be the maximal subsheaf of dimension 0, we get a sheaf $(\pi^*\mathcal{F})^\sharp = \pi^*\mathcal{F}/T_0$ which is pure of dimension 1, and there is a canonical injective homomorphism $\mathcal{F} \to \pi_*(\pi^*\mathcal{F})^\sharp$. The cokernel $\mathcal{I}$ is a skyscraper sheaf supported at $p$. Note that if $\mathcal{F}$ satisfies $I\mathcal{F} \neq 0$, so does $(\pi^*\mathcal{F})^\sharp$ as a sheaf of $\mathcal{O}_C$-modules.

The notion of admissible quotients can be defined in the same way as in section 4, and propositions 4.2-4.4 are also true in this case.

Let $\rho : \pi_*\mathcal{E} \to \mathcal{Q}$ be an admissible quotient, and let $p$ be a node on $C$ with $\pi^{-1}(p) = \{q_1, q_2\}$. Then $(\pi_*\mathcal{E})_p = \mathcal{E}_{q_1} \oplus \mathcal{E}_{q_2}$. We let $i_1 : \mathcal{E}_{q_1} \to (\pi_*\mathcal{E})_p$ and $i_2 : \mathcal{E}_{q_2} \to (\pi_*\mathcal{E})_p$. 

Let \( \rho : \pi_*\mathcal{E} \rightarrow \mathcal{Q} \), be the natural injections and projections respectively. Define \( \rho^i \) as compositions

\[
\rho^i : \pi_*\mathcal{E}_p \xrightarrow{\sim} (\pi_*\mathcal{E})_p \xrightarrow{\rho} \mathcal{Q}.
\]

Because \( \rho \) is admissible, \( \rho^i \) are both surjective homomorphisms. Clearly \( \rho = \rho^1p_1 + \rho^2p_2 \). For \( t \in \mathbb{C}^* \), we define \( \rho_t = \rho^1p_1 + t\rho^2p_2 \). It gives rise to a surjective homomorphism \( \rho_1 : \pi_*\mathcal{E} \rightarrow \mathcal{Q} \) which is also admissible.

Let \( \pi : \tilde{C} \rightarrow C \) be the partial normalization of \( C \) at \( p \). Apply the above construction to the canonical exact sequence

\[
0 \rightarrow \mathcal{O}_C \rightarrow \pi_\#\mathcal{O}_{\tilde{C}} \xrightarrow{\rho} \mathcal{A} \rightarrow 0,
\]

and let \( \mathcal{L}_t = \ker \rho_t \). Then \( \mathcal{L}_t \) is invertible for every \( t \in \mathbb{C}^* \). The set of these invertible sheaves form a subgroup \( G_\rho \subset \text{Pic}^0 C \). Clearly \( G_\rho \cong \mathbb{C}^* \).

For any admissible quotient \( \rho : \pi_*\mathcal{E} \rightarrow \mathcal{Q} \), let \( \mathcal{K}_t \) be the kernel of \( \rho_t \).

**Lemma 5.2.** There is a canonical isomorphism between \( \mathcal{L}_t \otimes \mathcal{K}_s \) and \( \mathcal{K}_{st} \).

Now we give a decomposition on \( \mathcal{M}_D^2 \) for \( D \in \mathcal{W}_3^0 \). Let \( \pi : \tilde{C} \rightarrow C \) be the normalization of \( C \). Let \( \mathcal{M}_{,T} \) be the subset consists of stable sheaves \( F \) such that \( (\pi^*\mathcal{F})^\sharp \cong \mathcal{F} \) and \( \pi_*\mathcal{F}/\mathcal{F} \cong T \). We get a decomposition \( \mathcal{M}_D^2 = \sqcup \mathcal{M}_{,T} \). In fact, for every nonempty stratum \( \mathcal{M}_{,T} \), \( T \) is nonempty. Because if \( \mathcal{M}_{,0} \) is nonempty, a sheaf \( F \) in \( \mathcal{M}_{,0} \) is the direct image of a sheaf on \( \tilde{C} \), i.e. \( F \cong \pi_*\mathcal{E} \) for a sheaf \( \mathcal{E} \) of \( \mathcal{O}_{\tilde{C}} \)-modules. By proposition 5.1, \( \mathcal{E} \) is not stable, which violates the stability of \( \mathcal{F} \).

Let \( \mathcal{M}_{,T} \) be a stratum, and let \( p \in C \) be a node such that \( T_p \neq 0 \). There is a subgroup \( G_p \subset \text{Pic}^0 C \) defined as above, and a \( G \)-action on \( \mathcal{M}_{,T} \) defined by tensorization. Next we will show that this group action is free on \( \mathcal{M}_{,T} \). The following lemma is useful in the proof.

Let \( \mathcal{E} \) be a pure sheaf of \( \mathcal{O}_C \)-modules such that \( \mathcal{E}_\eta \cong \mathcal{O}_\eta \) at the generic point \( \eta \) of \( C \). Let \( \mathcal{E}'' \) be the torsion free part of \( \mathcal{E}_0 = \mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{O}_{C_0} \), and let \( \mathcal{E}' \) be the kernel of the restriction homomorphism \( f : \mathcal{E} \rightarrow \mathcal{E}'' \). Since \( f \) is not an isomorphism, \( \mathcal{E}' \) and \( \mathcal{E}'' \) are both rank 1 torsion free sheaves of \( \mathcal{O}_{C_0} \)-modules whose automorphism groups are \( \mathbb{C}^* \).

**Lemma 5.3.** Let \( \psi : \mathcal{E} \rightarrow \mathcal{E} \) be an automorphism and let \( c : \mathcal{E}' \rightarrow \mathcal{E}' \) and \( d : \mathcal{E}'' \rightarrow \mathcal{E}'' \) be the induced automorphisms. Then they fit into the commutative diagram

\[
0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0
\]

\[
\downarrow c \quad \quad \quad \downarrow \psi \quad \quad \quad \downarrow d
\]

\[
0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0
\]

and \( c = d \).

**Proof.** Consider \( \psi' = \psi - c \cdot id : \mathcal{E} \rightarrow \mathcal{E} \). Clearly \( \psi' (\mathcal{E}') = 0 \). It induces a homomorphism \( u : \mathcal{E}'' \rightarrow \mathcal{E} \). Composed with \( \mathcal{E} \rightarrow \mathcal{E}'' \), we get \( h : \mathcal{E}'' \rightarrow \mathcal{E}'' \). Since \( \mathcal{E}'' \) is torsion free and has rank 1 as an \( \mathcal{O}_{C_0} \)-module, \( h \) is a multiplication by \((d - c)\). If \( h \neq 0 \), then after scaling \( u \) by \( \frac{1}{h} \), \( u \) splits the exact sequence and hence \( \mathcal{E} \cong \mathcal{E}' \oplus \mathcal{E}'' \), which contradicts to \( \mathcal{E}_\eta \cong \mathcal{O}_\eta \). Therefore \( h = 0 \), i.e. \( c = d \).
Proposition 5.4. Let $\mathcal{M}_{\mathcal{F},T}$ be a stratum such that $T_p \neq 0$ for a node $p \in \mathcal{C}$. Then the associated $G_p$-action on $\mathcal{M}_{\mathcal{F},T}$ is free. Therefore, by the decomposition of $\mathcal{M}_T^2$, $c(\mathcal{M}_T^2) = 0$.

Proof. Let $\pi : \hat{\mathcal{C}} \to \mathcal{C}$ be the partial normalization of $\mathcal{C}$ at $p$. Let $\mathcal{F}$ be a stable sheaf in $\mathcal{M}_{\mathcal{F},T}$. Then $\mathcal{F}$ fits into the exact sequence

$$0 \to \mathcal{F} \to \pi_* (\pi^* \mathcal{F})^\sharp \to T \to 0.$$  

Let $\mathcal{E} = (\pi^* \mathcal{F})^\sharp$. Then $\rho : \pi_* \mathcal{E} \to T$ is clearly an admissible quotient. We let $\mathcal{K}_i$ be the kernel of $\rho_i$. Then $\mathcal{F} = \mathcal{K}_1$ and $\mathcal{F} \otimes \mathcal{L}_i = \mathcal{K}_i$. Suppose $\mathcal{F} \otimes \mathcal{L}_i \cong \mathcal{F}$ for some $\mathcal{L}_i \in G_p$, there is a commutative diagram

\[
\begin{array}{ccc}
0 & \to & K_1 \\
\downarrow & & \downarrow_{\cong} \\
0 & \to & K_i
\end{array}
\]

It induces the following diagram on the stalks at the node $p$,

\[
\begin{array}{cccc}
\mathcal{E}_{q_1} \oplus \mathcal{E}_{q_2} & \overset{\rho_1}{\to} & T_p & \to 0 \\
\downarrow_{\psi} & & \downarrow_h \\
\mathcal{E}_{q_1} \oplus \mathcal{E}_{q_2} & \overset{\rho_i}{\to} & T_p & \to 0
\end{array}
\]

where $\{q_1, q_2\} = \pi^{-1}(p)$.

Recall that there is a canonical exact sequence for $\mathcal{E}$,

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0,$$

where $\mathcal{E}'$ and $\mathcal{E}''$ are nonzero torsion free sheaves of $O_{\mathcal{C}_0}$-modules. Let $T_i \subset T_p$ be the images of $\mathcal{E}_{q_i}$ under the surjective homomorphisms $\rho_i : \mathcal{E}_{q_i} \to T_p$. Then we have the following three cases.

Case 1. $T^1 \neq 0$ and $T^2 \neq 0$.

Since $\psi : \pi_* \mathcal{E} \to \pi_* \mathcal{E}$ is induced from an automorphism of $\mathcal{E}$, $\psi(\mathcal{E}_{q_i}) = \mathcal{E}_{q_i}$. Consider the restriction of the diagram to $\mathcal{E}_{q_i}$ respectively, by Lemma 5.3, there are commutative diagrams

\[
\begin{array}{cccc}
\mathcal{E}_{q_1}^' \overset{\rho_1}{\to} & T^1 & \mathcal{E}_{q_2}^' \overset{\rho_2}{\to} & T^2 \\
\downarrow c & & \downarrow h_1 & & \downarrow c & & \downarrow h_2 \\
\mathcal{E}_{q_1}^' \overset{\rho_1}{\to} & T^1 & \mathcal{E}_{q_2}^' \overset{\rho_2}{\to} & T^2
\end{array}
\]

If $T^1 \cap T^2 \neq \{0\}$, let $0 \neq x \in T^1 \cap T^2$. Then from the left diagram, $h(x) = h_1(x) = cx$, and from the right diagram, $h(x) = h_2(x) = ct_2x$. It implies that $t = 1$ and therefore the group action is free. Next we assume $T^1 \cap T^2 = \{0\}$. Let $x \in T^1$ be a nonzero element. Then the image $\bar{x}$ of $x$ in $T_p/T^2$ is nonzero. From the commutative diagram

\[
\begin{array}{cccc}
\mathcal{E}_{q_2}^'' \overset{\rho_2}{\to} & T_p/T^2 & \mathcal{E}_{q_2}^'' \overset{t \rho_2}{\to} & T_p/T^2 \\
\downarrow c & & \downarrow h_3 & & \downarrow h_3 \\
\mathcal{E}_{q_2}^'' & \overset{t \rho_2}{\to} & T_p/T^2
\end{array}
\]
we have $h_3(\bar{x}) = ct \bar{x}$. Because $h(x) = h_1(x) = cx$, $h_3(\bar{x}) = c\bar{x}$. It implies that $t = 1$.

Case 2. $T^1 \neq 0$ and $T^2 = 0$ (Or equivalently $T^1 = 0$ and $T^2 \neq 0$).

Since $T^2 = 0$ and $\mathcal{E}_{q_2} \rightarrow T_p$ is surjective, there is a surjective morphism $\rho^2 : \mathcal{E}''_{q_2} \rightarrow T_p$ and a commutative diagram

$$
\begin{array}{ccc}
\mathcal{E}''_{q_2} & \xrightarrow{\rho^2} & T_p \\
\downarrow c & & \downarrow h \\
\mathcal{E}''_{q_2} & \xrightarrow{t\rho^2} & T_p
\end{array}
$$

Let $0 \neq x \in T^1$. We already have $h(x) = h_1(x) = cx$. However, from the above diagram, $h(x) = ct \bar{x}$. Hence $t = 1$.

Case 3. $T^1 = 0$ and $T^2 = 0$.

Since $T^1 = 0$ and $T^2 = 0$, we get commutative diagrams

$$
\begin{array}{ccc}
\mathcal{E}''_{q_1} & \xrightarrow{\rho^1} & T_p \\
\downarrow c & & \downarrow h_1 \\
\mathcal{E}''_{q_1} & \xrightarrow{\rho^1} & T_p
\end{array}
\begin{array}{ccc}
\mathcal{E}''_{q_2} & \xrightarrow{\rho^2} & T_p \\
\downarrow c & & \downarrow h_2 \\
\mathcal{E}''_{q_2} & \xrightarrow{\rho^2} & T_p
\end{array}
$$

Apply the same argument as in case 1, we get $t = 1$. □

Because the cardinality of the finite set $\mathcal{W}_0^3$ is $G_g$, combine proposition 4.8 and 5.4, we conclude

**Proposition 5.5.** $e(\Phi^{-1}(\mathcal{W}_0^3)) = gG_g$.

**REFERENCES**


