CANONICAL MAPS OF SURFACES DEFINED BY ABELIAN COVERS

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Abstract. In this paper, we classified the surfaces whose canonical maps are abelian covers over $\mathbb{P}^2$. Moreover, we give defining equations for Perssson’s surface and Tan’s surfaces with odd canonical degrees explicitly.

Key words. Canonical map, abelian cover.

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1. Introduction. Let $X$ be a minimal surface of general type over $\mathbb{C}$. As usual, let $p_g$, $q$, $\chi(\mathcal{O}_X)$, $K_X^2$ be the numerical invariants of $X$,

$$\varphi_{K_X} : X \to \Sigma \subset \mathbb{P}^{p_g-1}$$

be the canonical map of $X$ and $d = \deg \varphi_{K_X}$ be the canonical degree of $X$. It is well known that the behavior of $\varphi_{K_X}$ is quite complicated, however the degree of it can not be too large. In 1979, Beauville ([Bea]) proved that when $\varphi_{K_X}$ is a generically finite map, the degree is at most 36. Furthermore, $\deg \varphi_{K_X} = 36$ if and only if $K_X^2 = 36$, $p_g = 3$, $q = 0$, $\Sigma = \mathbb{P}^2$ and $|K_X|$ is base point free. Later, Xiao also found some restrictions on surfaces with high canonical degrees ([Xiao]). For surfaces of general type, since the canonical degree is bounded above, it is interesting to know which positive integers $d$’s occur as the degree of the canonical map. Among all known surfaces with highest canonical degree less than 36 is the surface with canonical degree 16 which was constructed by Persson([Per]) as follows. let $\pi : X \to Y$ be a double cover of a Campedelli surface $Y$ branched along $2K_Y$, where the Campedelli surface is the surface of general type with $p_g = q = 0$, $K_Y^2 = 2$, and $|2K_Y|$ is base point free. He found that $\varphi_{K_X} = \pi \circ \varphi_{K_Y}$, hence $\varphi_{K_X}$ is degree 16 over $\mathbb{P}^2$. We see that his construction is based on the existence of Campedelli surface, however the construction of a Campedelli surface is not an easy work ([Pet]). The remaining problem that whether there exists a surface of general type over $\mathbb{P}^2$ with canonical degree of the gap between 17 and 35 is still open.

If $\Sigma$ is a canonical surface, there are plenty of examples (see [Bea], [Cat1], [V-Z]) with canonical degrees being 2. For $d = 3$ and $d = 5$, Tan ([Tan]) and Pardini ([Par1]) constructed several surfaces independently. When $p_g(\Sigma) = 0$, Beauville ([Bea]) constructed surfaces with $\chi(\mathcal{O}_X)$ arbitrarily large and the canonical degrees 2, 4, 6 and 8. For $d = 9$, Tan also constructed a surface in [Tan]. However, as far as we know that it is still not clear which positive integers $d$’s occur as the degree of the canonical map even when $\Sigma = \mathbb{P}^2$. 

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We list our main theorems as follows.

**Theorem 1.1.** Assume \( \varphi : X \rightarrow \mathbb{P}^2 \) is an abelian cover.
(1) Let \( \deg \varphi = 16 \), then \( \varphi = \varphi_{K_X} \) if and only if \( X \) is defined by
\[
 z_1^2 = \ell_1 \ell_3 \ell_4 \ell_7, \quad z_2^2 = \ell_1 \ell_2 \ell_4 \ell_5, \quad z_3^2 = \ell_1 \ell_2 \ell_3 \ell_6, \quad z_4^2 = \ell_2 \ell_5 \ell_6 \ell_8.
\]
In particular, the surface defined by the first three equations is a Campedelli surface.
(2) Let \( \deg \varphi = 9 \), then \( \varphi = \varphi_{K_X} \) if and only if \( X \) is defined by \( z_1^3 = a_1 a_2, z_2^3 = a_1 a_2 a_3 \).
(3) Let \( \deg \varphi = 8 \), then \( \varphi = \varphi_{K_X} \) if and only if \( X \) is defined by either \( z_1^2 = \ell_1 \ell_2 \ell_3 \ell_8, z_2^2 = \ell_3 \ell_4 \ell_5 \ell_8, z_3^2 = \ell_5 \ell_6 \ell_7 \ell_8 \), or \( z_1^2 = a_1 a_4, z_2^2 = a_2 a_4, z_3^2 = a_3 a_4 \), or \( z_1^2 = a_1 a_2, z_2^2 = a_3 a_3 \).
(4) Let \( \deg \varphi = 6 \), then \( \varphi = \varphi_{K_X} \) if and only if \( X \) is defined by \( z_1^2 = a_1 a_2, z_2^3 = a_2 a_3 \).
(5) Let \( \deg \varphi = 4 \), then \( \varphi = \varphi_{K_X} \) if and only if \( X \) is defined by either \( z^4 = a_1^2 b_2 \), or \( z_1^2 = b_1, z_2^2 = b_2 \).
(6) Let \( \deg \varphi = 3 \), then \( \varphi = \varphi_{K_X} \) if and only if \( X \) is defined by either \( z^3 = c_1 \), or \( z^3 = c_2 \).
(7) Let \( \deg \varphi = 2 \), then \( \varphi = \varphi_{K_X} \) if and only if \( X \) is \( z^2 = h \).
Here \( \ell_i \)'s define different lines and there are at most three lines among them passing through one point, for \( i = 1 \cdots 8 \); \( a_i \)'s are reduced of degree 2; \( b_i \)'s are reduced of degree 4; \( c_1, c_2 \) are reduced of degree 6; \( h \) is reduced of degree 8 and all the irreducible components of \( a_i \)'s , \( b_i \)'s, \( c_1, c_2 \) and \( h \) are simply normal crossing.

**Theorem 1.2.** If
\[
 \varphi = \varphi_{K_X} : X \rightarrow \mathbb{P}^2,
\]
is an abelian cover, then \( d \) is equal to 2, 3, 4, 6, 8, 9, or 16. In particular, if the canonical degree of \( X \) is 36, then \( \varphi_{K_X} \) can not be an abelian cover.

In section 2, we present the main facts on abelian covers which are the key points for solving our problem. In section 3, we prove our main theorems. The defining equations of Tan’s examples will be given in section 4.

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**2. Abelian covers.** The theory of cyclic covers of algebraic surfaces was studied first by Comessatti in [Com]. Then F. Catanese ([Cat2]) studied smooth abelian covers in the case \((\mathbb{Z}_2)^{\oplus 2}\) and R. Pardini ([Par2]) analyzed the general case. In this section, we shall recall some basic definitions and results for abelian covers which will facilitate our subsequent discussion. Since our point of view is to find the defining equations for covering surfaces by explicit calculation, we use the expressions and notations appearing in [Gao].
Let \( \varphi : X \to Y \) is an abelian cover associated to abelian group \( G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k} \), i.e., function field \( \mathbb{C}(X) \) of \( X \) is an abelian extension of the rational function field \( \mathbb{C}(Y) \) with Galois group \( G \). Without lose of generality, we can assume \( n_1 | n_2 | \cdots | n_k \).

**Definition 2.1.** The dates of abelian cover over \( Y \) with group \( G \) are \( k \) effective divisors \( D_1, \cdots, D_k \) and \( k \) linear equivalent relations

\[
D_i \sim n_i L_i, \quad i = 1, \cdots, k.
\]

Let \( L_i = \mathcal{O}_Y(L_i) \) and \( f_i \) be the defining equation of \( D_i \), i.e., \( D_i = \text{div}(f_i) \), where \( f_i \in H^0(Y, \mathcal{L}_i^{n_i}) \). Denote \( V(\mathcal{L}_i) = \text{Spec} S(\mathcal{L}_i) \) to be the line bundle corresponding to \( \mathcal{L}_i \), where \( S(\mathcal{L}_i) \) is the sheaf of symmetric \( \mathcal{O}_Y \) algebra. Let \( z_i \) be the fiber coordinate of \( V(\mathcal{L}_i) \). Then the abelian cover can be realized by the normalizing of surface \( V \) defined by the system of equations

\[
z_1^{n_1} = f_1, \cdots, z_k^{n_k} = f_k.
\]

We summerize our main results as follows.

**Theorem 2.2.** (See [Gao]) Denote by \([Z]\) the integral part of a \( \mathbb{Q} \)-divisor \( Z \),

\[
L_g = - \sum_{i=1}^k g_i L_i + \left[ \sum_{i=1}^k \frac{g_i}{n_i} D_i \right].
\]

Then

\[
\varphi_* \mathcal{O}_X = \bigoplus_{g \in G} \mathcal{O}_Y(-L_g).
\]

The following result will be used to calculate the ramification index.

**Theorem 2.4.** Let \( P \) be an irreducible and reduced hypersurface in \( Y \), let \( \bar{P} = \pi^{-1}(P) \) be the reduced preimage of \( P \) in \( X \), and let \( a_i \) be the multiplicity of \( P \) in \( D_i = \text{div}(f_i) \). Then

\[
\pi^* P = \left| \frac{G}{d_P} \right| \bar{P},
\]
where

\[ d_P = \gcd \left( |G|, |G\frac{a_1}{n_1}|, \ldots, |G\frac{a_k}{n_k}| \right) \]

is the number of points in the preimage of a generic point on \( P \).

3. Canonical map. Let \( X \) be a surface of general type whose canonical map \( \varphi_{K_X} \) is a generically finite cover of a surface in \( \mathbb{P}^2 \).

**Lemma 3.1.** Assume \( \varphi : X \to \mathbb{P}^2 \) is a finite cover of degree 36, \( \varphi_{K_X} = \varphi \) if and only if

\[ \varphi_*O_X = O_{P^2} \oplus E \oplus O_{P^2}(-4), \]

where \( E \) is a rank 34 bundle satisfying \( E^\vee \cong E(4) \) and \( H^0(E(1)) = 0 \).

**Proof.** If \( \varphi \) is a finite cover, we take \( E_0 \) as the trace free part of \( \varphi_*O_X \), so \( \varphi_*O_X = O_{P^2} \oplus E_0 \). By relative duality,

\[ \varphi_*O_X \cong (\varphi_*O_X)^\vee \otimes \omega_{P^2} = O_{P^2}(-3) \oplus E_0^\vee(-3). \]

Since \( \omega_X = \varphi^*(O_{P^2}(1)) \), by projection formula, \( \varphi_*\omega_X = \varphi_*\varphi^*(O_{P^2}(1)) = O_{P^2}(1) \otimes \varphi_*O_X \), so

\[ O_{P^2}(1) \oplus E_0(1) \cong O_{P^2}(-3) \oplus E_0^\vee(-3). \]

We see that \( E_0 = E \oplus O_{P^2}(-4) \), and \( E^\vee = E(4) \).

On the other hand, if \( \varphi_*O_X = O_{P^2} \oplus E \oplus O_{P^2}(-4) \), then

\[ \varphi_* (\omega_X \otimes \varphi^*O_{P^2}(-1)) = \varphi_* \omega_X \otimes \varphi_*O_{P^2}(-1) \]

\[ = (\varphi_*O_X)^\vee \otimes \omega_{P^2} \otimes \varphi_*O_{P^2}(-1) \]

\[ = (O_{P^2} \oplus E \oplus O_{P^2}(-4))^\vee \otimes O_{P^2}(-4) \]

\[ = (O_{P^2} \oplus E \oplus O_{P^2}(4)) \otimes O_{P^2}(-4) \]

\[ = O_{P^2}(-4) \oplus E \oplus O_{P^2}, \]

\( O_{P^2}(-4) \oplus E \oplus O_{P^2} \) admits a non zero global section, so \( \omega_X \otimes \varphi^*O_{P^2}(-1) \) admits also non zero global section. Namely there is an effective divisor \( Z \) such that \( K_X = \varphi^*H + Z \), where \( H \) is the divisor of the hyperplane. Because

\[ p_g(X) = h^2(\varphi_*O_X) = h^2(O_{P^2} \oplus E \oplus O_{P^2}(-4)) = h^0(E(1)) + 3 = 3. \]

\( Z \) is the fixed part of \( |K_X| \). It implies that

\[ K_X^2 \geq (\varphi^*H)^2 = 36 = 9(p_g(X) + 1) \geq 9\chi(O_X), \]

by Miyaoka-Yau inequality, \( K_X^2 = 36 \), \( Z = 0 \) and \( q(X) = 0 \). So \( \varphi_{K_X} = \varphi \). \( \Box \)

**Lemma 3.2.** If \( \varphi = \varphi_{K_X} \) is a finite abelian cover of degree \( d \) over \( \mathbb{P}^2 \), then

\[ \varphi_*O_X = O_{P^2} \oplus O_{P^2}(-2)^{\oplus d - 2} \oplus O_{P^2}(-4). \]

**Proof.** Because \( \varphi \) is an abelian cover, \( \varphi_*O_X \) is a direct sum of the line bundle.

\[ \varphi_*O_X = O_{P^2} \oplus \bigoplus_{i=1}^{d-1} O_{P^2}(-l_i). \]
Assume $0 < l_1 \leq l_2 \leq \cdots \leq l_{d-1}$. Since $K_X = \varphi^*(\mathcal{O}_{\mathbb{P}^2}(1))$, for any $m \geq 1$,

$$p_m(X) = h^0(mK_X) = h^0(\varphi^*(\mathcal{O}_{\mathbb{P}^2}(m))) = h^0(\mathcal{O}_{\mathbb{P}^2}(m)) + \sum_{i=1}^{d-1} h^0(\mathcal{O}_{\mathbb{P}^2}(m - l_i)).$$

Because $p_g(X) = p_1(X) = h^0(\varphi^*(\mathcal{O}_{\mathbb{P}^2}(1))) = h^0(\mathcal{O}_{\mathbb{P}^2}(1)) = 3$, we see that

$$h^0(\mathcal{O}_{\mathbb{P}^2}(1 - l_i)) = 0, \forall i$$

So $l_i \geq 2$.

And $h^2(\varphi_*\mathcal{O}_X) = h^2(\mathcal{O}_{\mathbb{P}^2}) + \sum_{i=1}^{d-1} h^2(\mathcal{O}_{\mathbb{P}^2}(-l_i))$, so $3 = \sum_{i=1}^{d-1} h^0(\mathcal{O}_{\mathbb{P}^2}(l_i - 3))$, then $l_i \leq 4$.

Then there are two cases:

1. $l_1 = \cdots = l_{d-2} = 2, l_{d-1} = 4$, and
2. $l_1 = \cdots = l_{d-4} = 2, l_{d-3} = l_{d-2} = l_{d-1} = 3$.

On the other hand, if $m = 2$, we have the second plurigenus $p_2(X) = K_X^2 + \chi(\mathcal{O}_X) = d + 4$. So

$$d + 4 = h^0(\mathcal{O}_{\mathbb{P}^2}(2)) + \sum_{i=1}^{d-1} h^0(\mathcal{O}_{\mathbb{P}^2}(2 - l_i)).$$

The second case does not satisfy the equation. So the lemma is proved.

Now suppose $\varphi : X \to \mathbb{P}^2$ be an abelian cover associated to an abelian group $G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$. Then $X$ is the normalization of the surface defined by

$$\frac{1}{n_1} \alpha_1, \cdots, \frac{1}{n_k} \alpha_k = \prod_{\alpha} \alpha_i,$$

where $p_\alpha$’s are prime factors and $\alpha = (\alpha_1, \cdots, \alpha_k) \in G$. Denote $x_\alpha$ to be the degree of $p_\alpha$, $e_i = (0, \cdots, 0, 1, 0, \cdots, 0) \in G$, $1 \leq i \leq k$, and $l_g$ be the degree of $L_g$. So $x_\alpha$ and $l_g$ are all integers. Then

(3.1) \hspace{1cm} n_i l_{e_i} = \sum_{\alpha} \alpha_i x_\alpha \hspace{1cm} i = 1, \cdots k,

(3.2) \hspace{1cm} l_g = \sum_{i=1}^{k} g_i l_{e_i} - \sum_{\alpha} \left[ \sum_{i=1}^{k} \frac{g_i \alpha_i}{n_i} \right] x_\alpha.

**Lemma 3.3.** Using the notation as above, if $\varphi = \varphi_{K_X}$, then there exists $g' = (g'_1, \cdots, g'_k) \in G \cong \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$, such that $x_\alpha$ satisfies the following equations

\[
\begin{align*}
    n_i l_{e_i} &= \sum_{\alpha} \alpha_i x_\alpha \\
    l_g' &= \sum_{i=1}^{k} g'_i l_{e_i} - \sum_{\alpha} \left[ \sum_{i=1}^{k} \frac{g'_i \alpha_i}{n_i} \right] x_\alpha = 4 \\
    l_g &= \sum_{i=1}^{k} g_i l_{e_i} - \sum_{\alpha} \left[ \sum_{i=1}^{k} \frac{g_i \alpha_i}{n_i} \right] x_\alpha = 2,
\end{align*}
\]

$g \neq g', g \in G$.

**Proof.** It comes from Lemma 3.2 directly.

By the above lemma, finding surfaces whose canonical map are abelian covers over $\mathbb{P}^2$ is equivalent to finding the integral roots $\{x_\alpha\}$ of the above equations.
THEOREM 3.4. If the canonical degree of $X$ is 36, then $\varphi_{K_X}$ can not be an abelian cover.

Proof. From the famous theorem of Beauville([Bea]), we know the image of $\varphi_{K_X}(X)$ is $\mathbb{P}^2$. We assume that $\varphi_{K_X} : X \to \mathbb{P}^2$ is an abelian cover associated to abelian group $G$. Then $|G| = 36$.

If $G = \mathbb{Z}_{36}$, then there exists some $g' \in \mathbb{Z}_{36}$ and corresponding $l$ such that $\{x_\alpha\}$ must be satisfied the following equations:

$$\begin{align*}
    l_\alpha &= g' l_1 - \sum_{\alpha=1}^{35} [g_\alpha |_{\mathbb{Z}_{36}}] x_\alpha = 4, \\
    l_\beta &= g l_1 - \sum_{\alpha=1}^{35} [g_\alpha |_{\mathbb{Z}_{36}}] x_\alpha = 2, \\
    36l_1 &= \sum_{\alpha=1}^{35} \alpha x_\alpha.
\end{align*}$$

After computation, we find that there is no non-negative integral solution for the above equations for any $1 \leq g' \leq 35$.

Using the same method, we find there is no non-negative integral solution for (*) as $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2}$ and $n_1 | n_2$.

So $\varphi_{K_X}$ can not be abelian. □

THEOREM 3.5. Assume $\varphi : X \to \mathbb{P}^2$ is an abelian cover.

(1) Let $\deg \varphi = 16$, then $\varphi = \varphi_{K_X}$ if and only if $X$ is defined by

$$z_1^2 = \ell_1 \ell_2 \ell_3 \ell_4, \quad z_2^2 = \ell_1 \ell_2 \ell_4 \ell_5, \quad z_3^2 = \ell_1 \ell_2 \ell_3 \ell_5 \ell_6, \quad z_4^2 = \ell_2 \ell_5 \ell_6 \ell_8.$$

In particular, the surface defined by the first three equations is a Campedelli surface.

(2) Let $\deg \varphi = 9$, then $\varphi = \varphi_{K_X}$ if and only if $X$ is defined by $z_1^3 = a_1 a_2^2, z_2^3 = a_1 a_2 a_3$.

(3) Let $\deg \varphi = 8$, then $\varphi = \varphi_{K_X}$ if and only if $X$ is defined by either $z_1^2 = a_1 a_4, z_2^2 = a_2 a_4, z_3^2 = a_3 a_4$, or $z_1^2 = a_1 a_2, z_3^2 = a_3 a_3$.

(4) Let $\deg \varphi = 6$, then $\varphi = \varphi_{K_X}$ if and only if $X$ is defined by $z_1^2 = a_1 a_2, z_2^3 = a_2 a_3$.

(5) Let $\deg \varphi = 4$, then $\varphi = \varphi_{K_X}$ if and only if $X$ is defined by either $z_1^2 = b_1 b_2$, or $z_1^2 = b_1, z_2^2 = b_2$.

(6) Let $\deg \varphi = 3$, then $\varphi = \varphi_{K_X}$ if and only if $X$ is defined by either $z_1^2 = c_1$, or $z_1^3 = c_2$.

(7) Let $\deg \varphi = 2$, then $\varphi = \varphi_{K_X}$ if and only if $X$ is $z^2 = h$.

Here $\ell_i$'s define different lines and there are at most three lines among them passing through one point, for $i = 1 \cdots 8$; $a_i$'s are reduced of degree 2; $b_i$'s are reduced of degree 4; $c_1, c_2$ are reduced of degree 6; $h$ is reduced of degree 8 and all the irreducible components of $a_i$'s, $b_i$'s, $c_1, c_2$ and $h$ are simply normal crossing.

Proof. If $\varphi = \varphi_{K_X}$, then $K_X = \varphi^*(\mathcal{O}_{\mathbb{P}^2}(1))$, $|K_X|$ has no fixed part and is base point free.

Let $G = \mathbb{Z}_{n_1} \oplus \cdots \oplus \mathbb{Z}_{n_k}$. Since the arguments are similar, we only prove the most complicated case when $|G| = 16$. By Lemma 3.3, we only need to find the integral solution of the equations (*). After computation, we find if and only if $G = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, there are integral solutions.

$$\begin{align*}
(1) \quad l_{(1,0,0,0)} &= 4, \quad x_{(1,1,0,1)} = x_{(1,1,1,0)} = x_{(1,0,1,1)} = x_{(1,0,0,1)} = x_{(1,0,0,0)} = x_{(1,1,1,1)} = 1, \quad x_{(0,1,0,0)} = x_{(0,0,0,1)} = x_{(0,0,0,0)} = x_{(0,0,1,1)} = 0.
\end{align*}$$
(2) \( l_{(1,1,0,0)} = 4 \), \( x_{(0,1,1,1)} = x_{(1,0,1,1)} = x_{(0,1,0,0)} = x_{(1,0,0,1)} = x_{(1,1,1,1)} = x_{(0,1,1,0)} = x_{(1,1,0,1)} = 1 \), \( x_{(0,0,1,1)} = x_{(1,1,0,0)} = \), \( x_{(0,0,1,0)} = 0 \),

(3) \( l_{(1,1,1,0)} = 4 \), \( x_{(1,1,1,0)} = x_{(1,0,1,0)} = x_{(0,1,0,1)} = x_{(1,0,0,0)} = x_{(0,1,1,0)} = x_{(0,1,1,1)} = 1 \), \( x_{(1,0,0,1)} = x_{(1,0,1,0)} = x_{(0,1,1,1)} = x_{(0,0,1,0)} = x_{(1,0,1,1)} = x_{(0,1,0,0)} = x_{(1,0,0,0)} = x_{(1,0,1,0)} = 0 \),

(4) \( l_{(1,1,1,1)} = 4 \), \( x_{(0,1,1,1)} = x_{(1,1,1,0)} = x_{(1,1,0,1)} = x_{(1,0,1,1)} = x_{(1,0,0,0)} = x_{(1,0,1,0)} = x_{(0,0,1,0)} = x_{(0,0,1,1)} = x_{(0,1,0,1)} = x_{(0,1,1,0)} = x_{(0,1,1,1)} = 0 \).

Although there are 4 sets of solution, these defining equations are equivalent after quotient the group \( sl(4, \mathbb{Z}_2) \) which means that they are same surfaces. So the surface \( X \) is defined by (3.3). Next we are going to discuss the configurations of \( \ell_i \)'s as following three cases.

- If \( \ell_i \)'s are in general position, i.e. they are simply normal crossing. Now we prove \( X \) is smooth.
  Let \( p_{ij} \) be the intersection point of \( \ell_i \) and \( \ell_j \). The cover is locally defined by

\[
\begin{align*}
  z_1^2 &= x^{a_{11}}y^{a_{12}}, \\
  z_2^2 &= x^{a_{21}}y^{a_{22}}, \\
  z_3^2 &= x^{a_{31}}y^{a_{32}}, \\
  z_4^2 &= x^{a_{41}}y^{a_{42}},
\end{align*}
\]

where \( a_{ij} = 0 \) or \( 1 \) for all \( i, j \).

It is easy to check that \( \{(a_{11}, a_{12}) \} \neq \{(1, 1), (0, 0)\} \) i.e., at least one pair \( \{(a_{11}, a_{12}) \} = \{(1, 0)\} \) or \( \{(0, 1)\} \). Without lose of generality, we can assume \( (a_{11}, a_{21}) = (1, 0) \), i.e. \( z_1^2 = x \). The cover is branched along the smooth line.

So the surface is smooth.

From Theorem 2.4, the ramification index of \( H_i = \text{Div}(\ell_i) \) is 2. Thus \( K_X = \varphi^*(O_{\mathbb{P}^2}(-3) + \frac{1}{2} \sum_{i=1}^8 H_i) = \varphi^*(O_{\mathbb{P}^2}(1)) \), which means \( \varphi = \varphi_K \).

- If there are three of \( \ell_i \)'s intersecting at a point, we blow up these triple points at first. Let \( \sigma : P \to \mathbb{P}^2 \) be the blowing-ups of these triple points with \( \{E_s\} \) the except curves and \( \pi : \Sigma \to P \) be the corresponding abelian cover, i.e., the pull back of \( \varphi \) by \( \sigma \). Then it is easy to check that \( E_s \)'s are in the branch locus of \( \pi \) and the ramification indices are also 2. Similarly, we can show that \( \Sigma \) is smooth. Then the canonical divisor

\[
K_\Sigma = \pi^*(\sigma^*(-3H)) + \sum_s E_s + \frac{1}{2}(\sigma^*(8H) - 3 \sum_s E_s) + \frac{1}{2} \sum_s E_s
\]

where \( H \) is a hyperplane on \( \mathbb{P}^2 \). So the surface is minimal and the degree of the canonical map is 16. In this case, \( X \) has only \( A_1 \) type singularities.

- If there are \( d \) lines among \( \ell_i \)'s passing through a point, \( d \geq 4 \). We blow up these points such that the branch locus are normal crossing. Let \( \sigma : P \to \mathbb{P}^2 \) be the blowing-ups of these points with \( \{E_s\} \) the except curves and \( \pi : \Sigma \to P \) be the corresponding abelian cover. Then \( \Sigma \) is smooth. \( \{E_s\} \) decomposes into 2 parts: \( \{E_s\} = \{E_s\} \) are in the branch locus of
\[ K_\Sigma = \pi^*(\sigma^*(-3H) + \sum_{s \in S} E_s + \frac{1}{2}(\sigma^*(8H) - d \sum_{s \in S_1} E_s) + \frac{1}{2} \sum_{s \in S_2} E_s) \]

\[ = \pi^*(\sigma^*(H) + \frac{3 - d}{2} \sum_{s \in S_1} E_s + \frac{2 - d}{2} \sum_{s \in S_2} E_s), \]

where \( H \) is a hyperplane on \( \mathbb{P}^2 \). Obviously, \( K_\Sigma^2 = 16 \). There is no \(-1\) curve on \( \Sigma \) by resolution and the singularities on \( X \) are not rational double points.

Therefore the configuration of \( \ell_i \) has at most triple points.

Let \( X_1 \) be the surface defined by the first three equations of (3.3) and suppose \( \pi' : X_1 \to \mathbb{P}^2 \). When \( \ell_i \)'s are in general position, we can show that \( X_1 \) is smooth and \( K_{X_1}^2 = 2 \). By Theorem 2.2, \( L_g \sim 2H \) for \( g \neq 0 \), where \( H \) is a hyperplane on \( \mathbb{P}^2 \). So \( h^0(-L_g) = h^0(K_{X_1} + L_g) = 0 \) for all nonzero \( g \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). Due to Corollary 2.3, \( p_g(X_1) = q(X_1) = 0 \). Therefore the surface \( X_1 \) is a Campedelli surface.

Now we compute the fundamental group of the surface \( X_1 \).

According to the result of the canonical resolution of double cover, the preimage of each \( L_i = \text{Div}(\ell_i) \) by \( \pi' \) is irreducible. Then assume \( \pi'^*(L_i) = 2A_i \), we can get \( A_i^2 = 2 \).

Due to \( 2A_i \sim \pi'^*(H) \), \( 2(A_i - A_j) \sim 0 \). Assume \( A_i \sim A_j \), \( i \neq j \), then \( h^0(A_i) \leq 2 \) since \( h^0(A_i + A_j) \geq h^0(A_i) + h^0(A_j) - 1 \). And the equation holds if and only if \( A_i \) and \( A_j \) do not intersect, which is contradict to \( A_i^2 = 2 \). So \( h^0(A_i) = 1 \), which means that \( A_i \) is not linear equivalent to \( A_j \).

Therefore each \( A_i - A_j \) is a 2-torsion element in \( X_1 \). So \( X_1 \) has at least 6 2-torsion elements.

Denote \( B = \sum_{i=1}^7 L_i \) to be the branch locus in \( \mathbb{P}^2 \). \( \pi_1(\mathbb{P}^2 - B) = \mathbb{Z}^6 \) is an abelian group. We know that \( X_1 - \pi'^*(B) \) is an étale cover over \( \mathbb{P}^2 - B \). So \( \pi_1(X_1 - \pi'^*(B)) \) is a subgroup of \( \pi_1(\mathbb{P}^2 - B) \) which is abelian and \( \pi_1(X_1) \) is the quotient group of \( \pi_1(X_1 - \pi'^*(B)) \), then \( \pi_1(X_1) \) is also an abelian group, which means \( \pi_1(X_1) = H_1(X_1, \mathbb{Z}) \).

As \( q = 0 \), \( H^1(X_1, \mathbb{C}) = 0 \). Using the universal coefficient theorem for cohomology, we know \( H^1(X_1, \mathbb{C}) = H_1(X_1, \mathbb{C}) = 0 \). Since \( H_1(X_1, \mathbb{C}) = H^1(X_1, \mathbb{Z}) \otimes \mathbb{C} = 0 \), \( H^1(X_1, \mathbb{Z}) \) only has torsion parts.

Due to the following exact sequence

\[ H^1(X_1, \mathcal{O}_{X_1}) \longrightarrow H^1(X_1, \mathcal{O}_{X_1}^*) \longrightarrow H^2(X_1, \mathbb{Z}) \longrightarrow H^2(X_1, \mathcal{O}_{X_1}) \]

and \( p_g(X_1) = q(X_1) = 0 \), we know \( \text{Pic}X_1 = H^1(X_1, \mathcal{O}_{X_1}^*) = H^2(X_1, \mathbb{Z}) \).

From the following sequence

\[ 0 \longrightarrow \text{Ext}(H_2(X_1, \mathbb{Z}), \mathbb{Z}) \longrightarrow H^2(X_1, \mathbb{Z}) \longrightarrow \text{Hom}(H_2(X_1, \mathbb{Z}), \mathbb{Z}) \longrightarrow 0, \]

we have \( H_1(X_1, \mathbb{Z}) = \text{Tor}(H^2(X_1, \mathbb{Z})) \).

For any Campedelli surface \( X \), it is well known that \( |\pi_1(X)| \leq 9(|\text{Rei}|) \). So the torsion group of the Picard group of \( X_1 \) is \( G = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \), i.e., \( \pi_1(X_1) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \).

Remark 3.6.

(1) From the above theorem, we know the defining equations of Persson’s surface.

(2) The Campedelli surface constructed in the proof is as same as that given by Kulikov ([Kul]). In fact, Campedelli surfaces with fundamental group of order 8 are completely classified in [ML-P-R].
Corollary 3.7. If
\[ \varphi = \varphi_{K_X} : X \rightarrow \mathbb{P}^2, \]
is an abelian cover, then \( d \) is equal to 2, 3, 4, 6, 8, 9 or 16.

4. Defining equations of Tan’s examples. In [Tan], Tan constructed a series of nice surfaces whose canonical maps are of odd degrees by using a so-called conception of blowing-up the \( n \)-fold cyclic cover determined by \( A \) and \( q \).

We get the 3 fold cyclic cover determined by \( X \) if \( X \) is of degree 3, Tan calculated the invariants of such surface. But the computation of the invariants of the surface is pretty complicated. The main part of [Tan] is the computation of \( p_g(X) \).

We can reconstruct the Tan’s example by explicitly equations. Let \( \varphi : X \rightarrow \mathbb{P}^2 \) be an abelian cover branched over the line configuration described by Tan. Assume \( \ell_1, \ell_2, \ell_3 \) are the 3 lines through \( q_1 \), \( \ell_4, \ell_5, \ell_6 \) are the 3 lines through \( q_2 \) and \( \ell_7, \ell_8, \ell_9 \) are the 3 lines through \( q_3 \). \( \varphi \) is defined by the following equations.

\[ \pi : z_1^3 = \ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6 \ell_7 \ell_8 \ell_9, \quad z_2^3 = \ell_1 \ell_2 \ell_3 \ell_4^2 \ell_5^2 \ell_6. \]

Let \( \sigma : P \rightarrow \mathbb{P}^2 \) be blowing-ups of \( q_1, q_2, q_3 \). \( \pi : X \rightarrow P \) is the pull back of \( \varphi \) by \( \sigma \). Let \( \Sigma \) be the minimal resolution of \( \pi' : Y \rightarrow \mathbb{P}^2 \) which is a cyclic cover defined by \( z^3 = \ell_1 \ell_2 \ell_3 \ell_4 \ell_5 \ell_6 \ell_7 \ell_8 \ell_9 \). It is clear that \( X \) and \( Y \) have at most \( A_1 \)-type singularities. We calculate invariants \( q(\Sigma) = q(X) = 0 \) and \( p_g(X) = p_g(\Sigma) = 8 - n \) by Theorem 2.2. The canonical map of \( X \) factorizes through that of \( \Sigma \), so it is of degree 3.

In [Tan], the author also constructed a surface whose canonical map is of degree 5 by using \( \mathbb{Z}_5 \) set. We can also reconstruct it as follows.

Take five lines \( \ell_1, \cdots, \ell_5 \) in general position on \( \mathbb{P}^2 \). \( \pi : X \rightarrow \mathbb{P}^2 \) is an abelian cover defined by

\[ z_1^5 = \ell_1 \ell_2 \ell_3 \ell_4 \ell_5, \quad z_2^5 = \ell_1 \ell_2 \ell_3 \ell_4^3 \ell_5. \]

The 5 fold covering surface \( \Sigma \) over \( \mathbb{P}^2 \) is defined by the first equation. Similarly, we get \( K_X = 25, K_\Sigma = 5, \chi(\mathcal{O}_X) = \chi(\mathcal{O}_\Sigma) = 5, p_g(\mathcal{O}_X) = p_g(\mathcal{O}_\Sigma) \). Therefore, the canonical map of \( X \) factorizes through that of \( \Sigma \). This means that the degree of the canonical map of \( X \) is 5.

REFERENCES

References


[Rei] M. Reid, Surfaces with \( p_g = 0, K^2 = 2 \), preprint available at http://www.warwick.ac.uk/~masda/surf/.

