A RESULT ON RICCI CURVATURE AND THE SECOND BETTI NUMBER

JIANMING WAN

Abstract. We prove that the second Betti number of a compact Riemannian manifold vanishes under certain Ricci curved restriction. As consequences we obtain an interesting curved restriction for compact Kähler-Einstein manifolds and a homology sphere theorem in dim = 4, 5.

Key words. Ricci curvature, Betti number.

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1. Introduction. The study of relation between curvature and topology is the central topic in Riemannian geometry. One of the strong tool is Bochner technique. It plays a very important role in understanding relation between curvature and Betti numbers. The first result in this field is Bochner’s classical result (c.f. [6])

Theorem 1.1. (Bochner 1946) Let $M$ be a compact Riemannian manifold with $\text{Ric}_M > 0$. Then the first Betti number $b_1(M) = 0$.

Berger investigated that in what case the second Betti number vanishes. He proved the following (c.f. [1], also see [2] theorem 2.8)

Theorem 1.2. (Berger) Let $M$ be a compact Riemannian manifold of dimension $n \geq 5$. Suppose that $n$ is odd and the sectional curvature satisfies that $\frac{n-3}{4n-9} \leq K_M < 1$. Then the second Betti number $b_2(M) = 0$.

Consider a different curvature condition, Micallef and Wang proved (c.f. [4], also see [2] theorem 2.7)

Theorem 1.3. (Micallef-Wang) Let $M$ be a compact Riemannian manifold of dimension $n \geq 4$. Suppose that $n$ is even and $M$ has positive isotropic curvature. Then the second Betti number $b_2(M) = 0$.

Here positive isotropic curvature means, for any four othonormal vectors $e_1, e_2, e_3, e_4 \in T_pM$, the curvature tensor satisfies

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} > 2|R_{1234}|.$$

Recall that the Rauch-Berger-Klingenberg’s sphere theorem (c.f. [1]) states that a simple connected compact Riemannian manifold is homeomorphic to a sphere if the sectional curvatures lie in $\left(\frac{1}{4}, 1\right]$. A generalization of sphere theorem (dues to Micallef-Moore c.f. [5]) says that a compact simply connected Riemannian manifold with positive isotropic curvature is a homotopy sphere. Hence with the help of Poincare conjecture it is homeomorphic to a sphere. From the two theorems we know that theorems 1.2 and 1.3 can not cover too many examples.

In this note we shall use Ricci curvature to give a relaxedly sufficient condition for the second Betti number vanishing. Our main result is
Theorem 1.4. Let $M$ be a compact Riemannian manifold. The dimension $\dim(M) = 2m$ or $2m + 1$. Let $\bar{k}$ (resp. $\underline{k}$) be the maximal (resp. minimal) sectional curvature of $M$. If the Ricci curvature of $M$ satisfies that

\begin{equation}
\text{Ric}_M > \bar{k} + \frac{2m-2}{3}(\bar{k} - \underline{k}),
\end{equation}

then the second Betti number $b_2(M) = 0$.

Particularly, if $M$ is a compact Riemannian manifold with nonnegative sectional curvature, then the second Betti number vanishes provided

\begin{equation}
\text{Ric}_M > \frac{2m+1}{3}\underline{k}.
\end{equation}

Note that there is no dimensional restriction in theorem 1.4.

Any compact Kähler manifold does not satisfy (1.1) since it has $b_2 \geq 1$.

The condition 1.1 is a Ricci pinching condition. We mention that several other Ricci pinching type theorems obtained by Gu and Xu (c.f. [3] [7],).

As an immediate consequence, we obtain a curvature restriction for special Einstein manifolds.

Corollary 1.5. Let $M$ be a compact Einstein manifold with nonzero second Betti number. Then the Ricci curvature satisfies

\begin{equation}
\text{Ric} \leq \bar{k} + \frac{2m-2}{3}(\bar{k} - \underline{k}).
\end{equation}

In addition, if the sectional curvature is nonnegative, one must have

\begin{equation}
\text{Ric} \leq \frac{2m+1}{3}\underline{k}.
\end{equation}

Particularly (1.3) holds for any compact Kähler-Einstein manifold.

Remark 1.6. 1) The condition (1.1) implies that the maximal sectional curvature $\bar{k} > 0$: If $\bar{k} \leq 0$, then

$$\bar{k} \geq \text{Ric}_M > \bar{k} + \frac{2m-2}{3}(\bar{k} - \underline{k}).$$

We get $\bar{k} < \underline{k}$. This is a contradiction.

2) Since $\bar{k} > 0$, of course (1.1) implies $\text{Ric}_M > 0$.

3) If the minimal sectional curvature $\underline{k} < 0$. Since $\bar{k} > 0$. If $\dim(M) = 2m + 1$, from

$$2m\bar{k} \geq \text{Ric}_M > \bar{k} + \frac{2m-2}{3}(\bar{k} - \underline{k}),$$

one has

$$\bar{k} > \frac{2m-2}{4m-1}|\underline{k}|.$$

Similarly

$$\bar{k} > \frac{1}{2}|\underline{k}|.$$
provided \( \dim(M) = 2m \).

We use theorem 1.4 to test some simple examples.

**Example 1.7.** 1) The space form \( S^n, \bar{k} = k = 1, Ric = n - 1 = \bar{k} \) for \( n = 2 \) and \( Ric = n - 1 > \bar{k} \) for \( n \neq 2 \), \( b_2(S^2) = 1 \) and \( b_2(S^n) = 0 \) for \( n \neq 2 \).

2) \( S^2 \times S^2 \) with product metric, \( \bar{k} = 1, k = 0, Ric = 1 < \bar{k} + \frac{2n-2}{3}(\bar{k} - k) \), \( b_2(S^2 \times S^2) = 2 \).

3) \( S^m \times S^m, m > 4 \) with product metric, \( \bar{k} = 1, k = 0, Ric = m - 1 > \frac{2m+1}{3}\bar{k} \), \( b_2 = 0 \).

4) \( \mathbb{CP}^n \) with Fubini-Study metric, \( \bar{k} = 4, k = 1, Ric = 2n + 2 = \bar{k} + \frac{2n-2}{3}(\bar{k} - k) \), \( b_2(\mathbb{CP}^n) = 1 \).

From the examples we know that the inequality (1.1) is sharp.

The proof of theorem 1.4 is also based on Bochner technique. But comparing with Berger and Micallef-Wang’s results, we consider a different side. This allows us to get a uniform result (without dimensional restriction).

**2. Proof of the theorem.**

**2.1. Bochner formula.** Let \( M \) be a compact Riemannian manifold.

\[
\Delta = d\delta + \delta d
\]

be the Hodge-Laplacian, where \( d \) is the exterior differentiation and \( \delta \) is the adjoint to \( d \).

Let \( \varphi \in \Omega^k(M) \) be a smooth \( k \)-form. Then we have the well-known Weitzenböck formula (c.f. [6])

\[
(2.1) \quad \Delta \varphi = \sum_i \nabla^2_{v_i, v_i} \varphi - \sum_{i,j} \omega^i \wedge i(v_j)R_{v_i, v_j} \varphi,
\]

here \( \nabla^2_{X,Y} = \nabla_X \nabla_Y - \nabla_{\nabla_X Y} \) and \( R_{X,Y} = -\nabla_X \nabla_Y + \nabla_Y \nabla_X + \nabla_{[X,Y]} \). The \( \{v_i, 1 \leq i \leq n\} \) are the local orthonormal vector fields and \( \{\omega_i, 1 \leq i \leq n\} \) are the duality.

A \( k \)-form \( \varphi \) is called harmonic if \( \Delta \varphi = 0 \).

The famous Hodge theorem states that the de Rham cohomology \( H^k_{\text{dR}}(M) \) is isomorphic to the space spanned by \( k \)-harmonic forms.

Let \( \varphi = \sum_{i,j} \varphi_{ij} \omega^i \wedge \omega^j \) be a harmonic 2-form. By (2.1), under the normal frame we can get (c.f. [2] or [1])

\[
(2.2) \quad \Delta \varphi_{ij} = \sum_k (Ric_{ik} \varphi_{kj} + Ric_{jk} \varphi_{ik}) - 2 \sum_{k,l} R_{ikjl} \varphi_{kl},
\]

where \( R_{ijkl} = \langle R(v_i, v_j)v_k, v_l \rangle \) is the curvature tensor and \( Ric_{ij} = \sum_k \langle R(v_k, v_i)v_k, v_j \rangle \) is the Ricci tensor.

So we have

\[
\Delta |\varphi|^2 = 2 \sum_{i,j} \varphi_{ij} \Delta \varphi_{ij} + 2 \sum_{i,j} \sum_k (v_k \varphi_{ij})^2 \\
\geq 2 \sum_{i,j} \varphi_{ij} \Delta \varphi_{ij} \\
\triangleq 2F(\varphi).
\]
Note that by (2.1) one has the global form of above formula

$$0 = - \langle \Delta \varphi, \varphi \rangle = \sum_i |\nabla v_i \varphi|^2 + \sum_{i, j} \omega^i \land i(v_j) R_{v_i v_j} \varphi, \varphi \rangle - \frac{1}{2} \Delta |\varphi|^2.$$ 

The $F(\varphi)$ is just the curvature term $\langle \sum_{i, j} \omega^i \land i(v_j) R_{v_i v_j} \varphi, \varphi \rangle$.

2.2. Proof of Theorem 1.4. By Hodge theorem, we only need to show that every harmonic 2-form vanishes.

Case 1: Assume $\dim(M) = 2m$. For any $p \in M$, we can choose an orthonormal basis $\{v_1, w_1, ..., v_m, w_m\}$ of $T_p M$ such that $\varphi(p) = \sum \lambda_{\alpha} v_{\alpha} \land w_{\alpha}^*$ (for instance c.f. [1] or [2]). Here $\{v_{\alpha}^*, w_{\alpha}^*\}$ is the dual basis. Then

$$F(\varphi) = \sum_{\alpha=1}^{m} \lambda_{\alpha}^2 [Ric(v_{\alpha}, v_{\alpha}) + Ric(w_{\alpha}, w_{\alpha})] - 2 \sum_{\alpha, \beta=1}^{m} \lambda_{\alpha} \lambda_{\beta} R(v_{\alpha}, w_{\alpha}, v_{\beta}, w_{\beta})$$

The term

$$- 2 \sum_{\alpha, \beta=1}^{m} \lambda_{\alpha} \lambda_{\beta} R(v_{\alpha}, w_{\alpha}, v_{\beta}, w_{\beta})$$

$$= - 2 \sum_{\alpha \neq \beta} \lambda_{\alpha} \cdot \lambda_{\beta} \cdot R(v_{\alpha}, w_{\alpha}, v_{\beta}, w_{\beta}) - 2 \sum_{\alpha=1}^{m} \lambda_{\alpha}^2 R(v_{\alpha}, w_{\alpha}, v_{\alpha}, w_{\alpha})$$

$$\geq - \frac{4}{3} (\bar{k} - \bar{k}) \sum_{\alpha \neq \beta} |\lambda_{\alpha}| \cdot |\lambda_{\beta}| - 2 \bar{k} \sum_{\alpha=1}^{m} \lambda_{\alpha}^2$$

$$\geq - \frac{2}{3} (\bar{k} - \bar{k}) \sum_{\alpha \neq \beta} (\lambda_{\alpha}^2 + \lambda_{\beta}^2) - 2 \bar{k} |\varphi|^2$$

$$= - \frac{2}{3} (\bar{k} - \bar{k})(2m - 2)|\varphi|^2 - 2 \bar{k} |\varphi|^2$$

$$= - 2[\bar{k} + \frac{2m - 2}{3}(\bar{k} - \bar{k})] |\varphi|^2.$$ 

The first ”$\geq$” follows from Berger’s inequality (c.f. [1]): For any orthonormal 4-frames $\{e_1, e_2, e_3, e_4\}$, one has

$$|R(e_1, e_2, e_3, e_4)| \leq \frac{2}{3} (\bar{k} - \bar{k}).$$

On the other hand, by the condition (1.1) we have

$$\sum_{\alpha=1}^{m} \lambda_{\alpha}^2 [Ric(v_{\alpha}, v_{\alpha}) + Ric(w_{\alpha}, w_{\alpha})] \geq 2[\bar{k} + \frac{2m - 2}{3}(\bar{k} - \bar{k})] |\varphi|^2,$$

the equality holds if and only if $\varphi(p) = 0$.

This leads to

$$F(\varphi) \geq 0$$

with equality if and only if $\varphi(p) = 0$. Since

$$\int_M F(\varphi) \leq \frac{1}{4} \int_M \Delta |\varphi|^2 = 0,$$
we get

\[ F(\varphi) \equiv 0. \]

Thus the harmonic 2-form \( \varphi \equiv 0 \).

**Case 2:** If \( \dim(M) = 2m+1 \). For any \( p \in M \), we also can choose an orthonormal basis \( \{ u, v_1, w_1, ..., v_m, w_m \} \) of \( T_pM \) such that \( \varphi(p) = \sum_\alpha \lambda_\alpha v_\alpha^* \wedge w_\alpha^* \) (c.f. [1] or [2]).

We also have

\[ F(\varphi) = \sum_{\alpha=1}^{m} \lambda_\alpha^2 [Ric(v_\alpha, v_\alpha) + Ric(w_\alpha, w_\alpha)] - 2 \sum_{\alpha,\beta=1}^{m} \lambda_\alpha \lambda_\beta R(v_\alpha, w_\alpha, v_\beta, w_\beta). \]

Thus the argument is same to the even dimensional case.

This completes the proof of the theorem.

**3. Sphere theorem in \( \dim 4 \text{ and } 5 \).**

**Theorem 3.1.** Let \( M \) be a compact Riemannian manifold. \( \dim M = 4 \text{ or } 5 \). If

\[ \text{Ric}_M > \frac{5\bar{k} - 2k}{3}, \]

then \( M \) is a real homology sphere, i.e. \( b_i(M) = 0 \) for \( 1 \leq i \leq \dim M - 1 \).

**Proof.** Since \( \text{Ric}_M > 0 \), from theorem 1.1 we know that \( b_1(M) = 0 \). Theorem 1.4 implies that \( b_2(M) = 0 \). With the help of Poincare duality, we obtain the theorem. \( \Box \)

Finally we mention a differential sphere theorem for Ricci curvature obtained by Gu and Xu (c.f. [3] theorem D).

**Theorem 3.2.** Let \( M \) be a simple connected compact Riemannian \( n \)-manifold. If

\[ \text{Ric}_M > (n - \frac{11}{5})\bar{k}, \]

then \( M \) is diffeomorphic to \( S^n \).

**REFERENCES**
