HYPOEILLIPTICITY OF THE $\bar{\partial}$-NEUMANN PROBLEM AT A POINT OF INFINITE TYPE

LUCA BARACCO$^\dagger$, TRAN VU KHANH$^\ddagger$, AND GIUSEPPE ZAMPIERI$^\dagger$

Abstract. We prove local hypoellipticity of the complex Laplacian $\Box$ in a domain which has superlogarithmic estimates outside a curve transversal to the CR directions and for which the holomorphic tangential derivatives of a defining function are superlogarithmic multipliers in the sense of [10].

Key words. Hypoellipticity, $\bar{\partial}$-Neumann problem, superlogarithmic estimate, infinite type.

AMS subject classifications. 32F10, 32F20, 32N15, 32T25.

1. Introduction. For the pseudoconvex domain $\Omega \subset \mathbb{C}^n$ whose boundary is defined in local coordinates $z = x + iy$ in a neighborhood $U$ of $z_o = 0$, by

\begin{equation}
2x_n = \exp \left( -\frac{1}{\left( \sum_{j=1}^{n-1} |z_j|^2 \right)^{\frac{s}{2}}} \right), \quad s > 0,
\end{equation}

the tangential Kohn Laplacian $\Box_b = \bar{\partial}_b \partial_b^* + \partial_b^* \bar{\partial}_b$ as well as the full Laplacian $\Box = \bar{\partial} \partial^* + \partial^* \bar{\partial}$ show very interesting features especially in comparison with the “tube domain” whose boundary is defined by

\begin{equation}
2x_n = \exp \left( -\frac{1}{\left( \sum_{j=1}^{n-1} |x_j|^2 \right)^{\frac{s}{2}}} \right), \quad s > 0.
\end{equation}

(Here $z_j$ have been replaced by $x_j$ at exponent.) Energy estimates are the same for the two domains. For the problem on the boundary $b\Omega$, they come as

\begin{equation}
\| (\log \Lambda)^{\frac{1}{2}} u \|_{b\Omega} \lesssim \| \bar{\partial}_b u \|_{b\Omega}^2 + \| \bar{\partial}_b^* u \|_{b\Omega}^2 + \| u \|_{b\Omega}^2
\end{equation}

for any form $u \in C_c^\infty (b\Omega \cap U)^k$ of degree $k \in [1, n-2]$.

Here $\log \Lambda$ is the tangential pseudodifferential operator with symbol $\log(1+|\xi|^2)^{\frac{s}{2}}$, $\xi \in \mathbb{R}^{2n-1}$, the dual real tangent space. As for the problem on the domain $\Omega$, one has simply to replace $\bar{\partial}_b, \partial_b^*$ by $\partial, \partial^*$ and take norms over $\Omega$ for forms $u$ in $D_{\partial^*}$, the domain of $\partial^*$, of degree $1 \leq k \leq n - 1$; this can be seen, for instance, in [13]. In particular, these are superlogarithmic estimates if and only if $s < 1$; otherwise, for any $s > 0$ they are compactness estimates. A related problem is that of the local hypoellipticity of the Kohn Laplacian $\Box_b$ or, with equivalent terminology, the local regularity of the inverse (modulo harmonics) Green operator $N_b = \Box_b^{-1}$. Similar is the notion of hypoellipticity of the Laplacian $\Box$ or the regularity of the inverse Neumann operator $N = \Box^{-1}$. It has been proved by Kohn in [17] and by the two last authors in [14] that superlogarithmic estimates suffice for local hypoellipticity of
the problem in the boundary and in the domain. (Note that hypoellipticity for the domain, [17] Theorem 8.3, is deduced from microlocal hypoellipticity for the boundary, [17] Theorem 7.1, but a direct proof is also available, [10] Theorem 5.4.) In particular, for (1.1) and (1.3), there is local hypoellipticity when \( s < 1 \).

As for the more delicate hypoellipticity, in the critical range of indices \( s \geq 1 \), only the tangential problem has been studied and the striking conclusion is that the behavior of (1.1) and (1.2) split. The first stays always hypoelliptic for any \( s \) (Kohn [16]) whereas the second is not for \( s \geq 1 \) (Christ [6]). When one tries to relate \((\bar{\partial}_b, \bar{\partial}^*_b)\) on \( b\Omega \) to \((\bar{\partial}, \bar{\partial}^*_b)\) on \( \Omega \), estimates go well through (Kohn [17] Section 8 and Khanh [10] Chapter 4) but not regularity. In particular, the two conclusions about tangential hypoellipticity of \( \Box_b \) for (1.1) and non-hypoellipticity for (1.2) when \( s \geq 1 \), cannot be automatically transferred from \( b\Omega \) to \( \Omega \). Now, for the non-hypoellipticity in \( \Omega \) in case of the tube (1.2) the authors have obtained in [3] a result of propagation which is not equivalent but intimately related. The real planes of the variables \( x_1, ..., x_{n-1} \) are propagators of holomorphic extendibility from \( \Omega \) across \( b\Omega \). What we prove in the present paper is the converse, that is, hypoellipticity in \( \Omega \) for (1.1) when \( s \geq 1 \).

Indeed, we prove local regularity not only for (1.1) but also for the case of higher exponential type at 0. The model of our domain is the one with boundary

\[
2x_n = \exp\left(-\frac{1}{\sum_{j=1}^{n-1} \exp(-\frac{1}{|z_j|^m})^2}\right), \quad m_j < 1, \ s > 0.
\]

Here, the best possible estimate at \( z_1 = 0, ..., z_{n-1} = 0 \) is worse than for the domain with boundary (1.1), that is,

\[
||\log^{\frac{1}{m}}(1 + \log^{\frac{1}{m}} A) u||_2^2 \lesssim Q(u, u), \ \text{for} \ m := \max m_j.
\]

When \( z_j \neq 0 \) for any \( j \) we have of course a \( \frac{1}{2} \)-subelliptic estimate but, if \( z_j = 0 \) for some \( j \), then we do not have a subelliptic estimate as it was for (1.1) but just a \( \log^{\frac{1}{m}} \)-estimate; however it is strong enough for our need because it is superlogarithmic on account of \( m_j < 1 \) for any \( j \). Also, at \( z_1 = 0, ..., z_{n-1} = 0 \), the functions \( \partial_{z_j} r, \ j = 1, ..., n-1 \) are no more subelliptic multipliers (as in (1.1)); however, they are superlogarithmic multipliers (again, for \( m_j < 1, \ j = 1, ..., n-1 \)). Thus, (1.4) serves as a model of our main result

**Theorem 1.1.** Let \( \Omega \) be a pseudoconvex, rigid, domain of \( \mathbb{C}^n \) in a neighborhood of \( z_o = 0 \) and assume that the \( \bar{\partial} \)-Neumann problem satisfies the following properties for forms of degree \( \geq 1 \)

(i) there is a superlogarithmic estimate for \((z_1, ..., z_{n-1}) \neq 0 \),

(ii) \( \partial_{z_j} r, \ j = 1, ..., n-1 \), are superlogarithmic multipliers \((\text{cf. [15] and [10]})\).

Then \( \Box \) is locally hypoelliptic at \( z_o \) for forms of any degree \( k \geq 0 \).

The proof follows in Section 2. It consists in relating the system on \( \Omega \) to the tangential system on \( b\Omega \) along the guidelines of [17] Section 8, and then in using the argument of [16] to control the commutators of the energy \( Q \) with the derivatives \( D^s \) and the cut-off functions \( \zeta \).

**Remark 1.2.** What we prove is in fact, for a pair \( \zeta_o \prec \zeta \) of “nested” cut-off in tangential directions having support in a neighborhood \( U \) of \( z_o \),

\[
||\zeta_o u||_s \lesssim ||\zeta \bar{\partial} u||_s + ||\zeta \bar{\partial}^* u||_s + ||u||_0, \ \text{for any} \ u \in \mathcal{H}^k, \ k \geq 0,
\]
where $\mathcal{H}^k_\perp$ is the orthogonal to the space of harmonic $k$-forms and $s$ is the index of the norm in the Sobolev spaces $H^s$. Note here that we have in fact $\mathcal{H}_k = \{0\}$ for any $k \geq 1$.

We now observe that $b\Omega$ is given only locally in a neighborhood of $z_o$. We can continue $b\Omega$ leaving it unchanged in a neighborhood of $z_o$, making it strongly pseudoconvex elsewhere, in such a way that it bounds a relatively compact pseudoconvex domain $\Omega \subset\subset \mathbb{C}^n$ (cf. [19]). Thus, by the $L^2$-theory of $\bar{\partial}$, there is well defined the Neumann operator $N = \Box^{-1}$. As an immediate consequence of (1.5) we have that

$$\bar{\partial}^*N \text{ and } \bar{\partial}N \text{ are exactly locally } H^s\text{-regular at } z_o$$

over $\ker \bar{\partial} \text{ and } \ker \bar{\partial}^*$ respectively.

We specify the action of $N$ on $q$-forms by the notation $N_q$ and denote by $B_q := \text{Id} - \bar{\partial}^* N_{q+1} \bar{\partial}$ the Bergman projection and by $K(z, w)$ the Bergman kernel respectively. From the regularity of $\bar{\partial}^*N$ it follows that the Bergman projection $B$ is also regular. (Notice that exact regularity is perhaps lost by the presence of $\bar{\partial}$ in $B$.) To prove the local regularity at $z_o$ of $N$ itself, we follow now the method of Boas-Straube and exploit formula (5.36) in [20] in unweighted norms, that is, for $t = 0$:

$$N_q = B_q(N_q\bar{\partial})(\text{Id} - B_{q-1})(\bar{\partial}^*N_q)B_q + (\text{Id} - B_q)(\bar{\partial}^*N_{q+1})B_{q+1}(N_{q+1}\bar{\partial})(\text{Id} - B_q).$$

Now, in the right side, the $\bar{\partial}N$'s and $\bar{\partial}^*N$'s are evaluated over $\ker \bar{\partial}^*$ and $\ker \bar{\partial}$ respectively; thus they are exactly locally regular at $z_o$. The $B$'s are also locally regular at $z_o$ and therefore such is $N$. We put in a separate statement our conclusions for $B$ and also give a complement about $K$.

**Theorem 1.3.** We have, for a neighborhood $U$ of $z_o$ and for any pair of cut-off $\zeta_o < \zeta$ with support in $U$

$$\left\{ \begin{array}{l}
||\zeta_o B_q\alpha||_s \lesssim ||\zeta\alpha||_{s+1} + ||\alpha||_0, \\
K(z, w)|_{U \times U} \in C^\infty((\bar{\Omega} \times \bar{\Omega}) \setminus \text{Diagonal}).
\end{array} \right.$$  

**Proof.** The first of (1.7) has already been discussed. The second follows from the first by the method of Kerzman [9]. Note that it is explicit, in particular according to the note added in the proof at p.158, that only the local regularity of $B_q$ in the form of the first of (1.7) is needed to get the second. $\square$

2. **Proofs.** We need several preliminary results

**Proposition 2.1.** If (i) and (ii) hold for forms $u$ of degree $\geq 1$, they also holds for functions $u \in \ker \bar{\partial}^\perp$.

**Proof.** Since $\bar{\partial}$ has closed range, then, given $u \in \ker \bar{\partial}^\perp$, we can find a solution $v$ of degree 1 of

$$\left\{ \begin{array}{l}
\bar{\partial}^*v = u, \quad \bar{\partial}v = 0, \\
||v|| \lesssim ||u||.
\end{array} \right.$$
Let $U$ be the neighborhood of $z_0$ in which (i) and (ii) hold. We have for $\zeta_o \in C^\infty_c(U)$,

\begin{equation}
|| \log(\Lambda)\zeta_o u ||^2 = \left( \log(\Lambda)\zeta_o u, \log(\Lambda)\zeta_o \partial^* v \right)
\end{equation}

\begin{align*}
&= \left( \partial\zeta_o u, \log^2(\Lambda)\zeta_o v \right) + \left( \log(\Lambda)\zeta_o u, \log^{-1}(\Lambda) || \log^2(\Lambda)\zeta_o, \partial^* || v \right) \\
&\leq \epsilon || \partial\zeta_o u ||^2 + c_\epsilon || \log^2(\Lambda)\zeta_o v ||^2 + || \log(\Lambda)\zeta_o u ||^2 + c_\epsilon || \log^{-1}(\Lambda) || \log^2(\Lambda)\zeta_o, \partial^* || v ||^2. \\
&\quad \text{good} \quad \text{absorbable} \quad \text{(a)} \quad \text{absorbable} \quad \text{(b)}
\end{align*}

Observe that, for $\zeta > \zeta_o$

\begin{equation}
\epsilon Q(\zeta \log(\Lambda)\zeta_o v, v) + c_\epsilon || \zeta \log(\Lambda)\zeta_o v ||_0^2 + || v ||_{-\infty}^2 \\
\leq \epsilon \left( || \zeta \log(\Lambda)\zeta_o \partial^* v ||^2 + || (\partial^* \zeta \log(\Lambda)\zeta_o v ) ||^2 \right) + c_\epsilon || \zeta \log(\Lambda)\zeta_o v ||_0^2 + || v ||_{-\infty}^2,
\end{equation}

where $\partial^*(\cdot)$ denotes either $\partial^*$ or $\partial$. Now,

$$[\partial^*(\cdot), \zeta \log(\Lambda)\zeta_o] \sim \zeta \log(\Lambda)\zeta_o + \zeta \log(\Lambda)\zeta_o + \zeta \log(\Lambda)\dot{\zeta}_o.$$ 

Hence (*) and (**) are of type $|| \zeta'' \log(\Lambda)\zeta' v ||^2$ and can therefore be estimated by

\begin{equation}
\epsilon Q(\zeta' v, \zeta' v) + c_\epsilon || \zeta'' v ||^2 \\
\leq \epsilon \left( || \zeta' u ||^2 + || (\partial^* \zeta) \zeta' ||^2 \right) + c_\epsilon || \zeta'' v ||^2 \\
\leq c_\epsilon || u ||_0^2.
\end{equation}

(2.3)

As for (b), we notice that

$$\log^{-1}(\Lambda) || \log^2(\Lambda)\zeta_o, \partial^* || \sim \log(\Lambda)\dot{\zeta}_o + \zeta_o,$$

and hence, for $\zeta > \zeta_o$,

\begin{equation}
(b) \leq || \log(\Lambda)\zeta v ||^2 + || \zeta v ||_{\Lambda^{-1}}^2 \\
\leq \epsilon Q(\zeta v, \zeta v) + c_\epsilon || \zeta v ||_0^2 \\
\leq \epsilon || \zeta u ||_0^2 + || (\partial^* \zeta) v ||_0^2 + c_\epsilon || \zeta v ||_0^2 \\
\leq c_\epsilon || u ||_0^2.
\end{equation}

Hence (2.1) can be continued by

$$\leq \epsilon || \partial\zeta_o u ||^2 + c_\epsilon || u ||_0^2.$$ 

Thus (i) also holds for $u$. The proof that we have the same conclusion for (ii) is the same as above.
We note now that (i) implies a compactness estimate, that is, for any $\epsilon$ and for suitable $c_\epsilon$
\begin{equation}
\|u\|_0^2 \leq \epsilon Q(u, u) + c_\epsilon \|u\|_{-1}^2
\end{equation}
for any $u \in C^\infty_c(\Omega) \cap D^k_{\partial\Omega}, k \geq 1$ or $u \in C^\infty_c(\Omega) \cap \ker \partial^+_{\Omega}, k = 0$.

This follows from a more general fact: a totally real submanifold of $b\Omega$, such as the $y_n$-line, is a removable set of non-compactness (and we are in this situation since a superlogarithmic estimate is stronger than a compactness estimate).

**Lemma 2.2.** Assume that there are compactness estimates on $b\Omega \cap U$ except from a totally real subset $S$. Then we have in fact compactness estimates in the whole $b\Omega \cap U$.

**Proof.** We first prove (2.4) for $k \geq 1$. For this we introduce the family of weights
\begin{equation}
\{\phi_\epsilon\}_\epsilon = \{\frac{d S}{\epsilon^2}\}_\epsilon
\end{equation}
where $d S$ is the distance to $S$. These weights are bounded and their Levi form grows by the rate $1/\epsilon$ when $d S < \epsilon$. With these weights in hand and by the compactness outside $S$, we get (2.4) from the basic estimate for $k \geq 1$ by the same argument as in [12]. To prove the estimate for $k = 0$, we make repeated use of (2.4) in degree 1. This first implies that $\partial^* v$ has closed range on 1-forms. In particular, $(\ker \partial)^\perp = \text{range } \partial^*$. Thus, if $u \in (\ker \partial)^\perp$, then there exists a solution $v \in (L^2)^1$ to the equation $\partial^* v = u$. Moreover, we can choose $v$ belonging to $\ker \partial$. By the basic estimate for $v$ we have
\begin{equation}
\|v\|_0^2 \lesssim \|\partial^* v\|_0^2.
\end{equation}
We also have
\begin{equation}
\|v\|_1^2 \leq \epsilon \|\partial^* v\|_0^2 + c_\epsilon \|\partial^* v\|_{-1}^2.
\end{equation}
This can be proved by contradiction. If (2.6) is violated, then there is a sequence $v_\nu \in (\ker \partial^*)^\perp$ such that $\|v_\nu\|_{-1} = 1, \|\partial^* v_\nu\|_{-1} \to 0$ and $\|\partial^* v_\nu\|_0 \leq c$. But we also have from (2.5), $\|\partial^* v_\nu\|_0 \sim \|v_\nu\|_0 \geq \|v_\nu\|_{-1} = 1$. Thus any subsequential $L^2$-weak limit of $\partial^* v_\nu$ must be 0 and $\neq 0$. We use the notation $lc$ and $sc$ for a large and small constant respectively. We have for any function $u$
\begin{equation}
\|u\|_0^2 = (u, \partial^* v)
\end{equation}
\begin{equation}
= (\partial^* u, v)
\end{equation}
\begin{equation}
\leq \|\partial^* u\| \|v\|
\end{equation}
\begin{equation}
\leq \|\partial^* u\| (\epsilon \|\partial^* v\| + c_\epsilon \|v\|_{-1})
\end{equation}
(2.4) for $v$
\begin{equation}
\lesssim \|\partial^* u\| (\epsilon \|u\| + c_\epsilon \|u\|_{-1})
\end{equation}
(2.6)
\begin{equation}
\leq lc_1 \epsilon^2 \|\partial^* u\|^2 + sc_1 \|u\|^2 + lc_2 \epsilon_c^2 \|u\|_{-1}^2 + sc_2 \|\partial^* u\|^2
\end{equation}
\begin{equation}
\leq \epsilon' \|\partial^* u\|^2 + c_\epsilon \|u\|^2 + sc_1 \|u\|_{-1}^2,
\end{equation}
for $\epsilon' = lc_1 \epsilon^2 + sc_2$ and $c_\epsilon = lc_2 \epsilon_c^2$. By choosing $sc_1$ so that $sc_1 \|u\|^2$ is absorbed in the left, (2.7) yields (2.4) for $u$ in degree 0.

We decompose a $k$-form into the tangential and normal components $u = u^\tau + u^\nu$ and further decompose microlocally $u^\tau = u^{\tau^+} + u^{\tau^0} + u^{\tau^-}$ (cf. [17]). By elliptic
estimate for $Q$ over terms which vanish at $b\Omega$, we have, in particular, that (1.5) is fulfilled by $u^\nu$. The same is true for $u^{\tau 0}$ and $u^{\tau -}$ (cf. [17] Lemma 8.5). So we only need to prove (1.5) for $u^{\tau +}$ that we write as $u$ from now on. We further decompose $u = u^{(H)} + u^{(0)}$ where $u^{(H)}$ is the "holomorphic" component in the sense of [11] and $u^{(0)}$ is the complement; note that $u^{(0)}|_{b\Omega} = 0$. Along with $\zeta < \zeta'$ with support in $U$, we consider an additional tangential cut-off $\sigma$ with $\sigma < \zeta$ and denote by $R^\tau$ the pseudodifferential tangential operator with symbol $(1 + |\xi|^2)^{-\alpha/2}$. Here $(a, r) \in \mathbb{R}^{2n-1} \times \mathbb{R}$ is a local system of coordinates and $\xi$ are dual to the $a$'s. We choose all our cut-off as functions of product type $\zeta = \zeta_1(z')\zeta_2(y_n)$, $\zeta' = \zeta'_1(z')\zeta'_2(y_n)$ and $\sigma = \sigma_1(z')\sigma_2(y_n)$. We denote by $Q^\tau$ the tangential component of $Q$; thus $Q(u, u) = Q^\tau(u, u) + \|\tilde{L}_n u\|^2$ if $L_1, \ldots, L_n$ is a system of $(1, 0)$ vector fields dual to an orthonormal system of forms $\omega_1, \ldots, \omega_n$ in which $\omega_n = \partial r$. We point out the crucial property of the component $u^{(H)}$, that is, $\tilde{L}_n u^{(H)} \sim r\Lambda u^{(H)}$.

**Proposition 2.3.** In the hypotheses (i) and (ii) of Theorem 1.1, we have for any $\varepsilon$ for suitable $c_\varepsilon$ and for $\zeta' > \zeta$

\[
||((\zeta R^\tau \zeta)u^{(H)})||^2 \leq \varepsilon Q^\tau_{\zeta R^\tau \zeta}(u^{(H)}, u^{(H)}) + c_\varepsilon ||\zeta' u^{(H)}||^2_0,
\]

for $u \in D^b_\delta$, $k \geq 1$ and $u \in \ker \tilde{\partial}^\perp$, $k = 0$,

where $Q^\tau_{(\zeta R^\tau \zeta)}(u^{(H)}, u^{(H)}) = ||((\zeta R^\tau \zeta)\tilde{\partial}^{\tau} u^{(H)})||^2 + ||((\zeta R^\tau \zeta)\tilde{\partial}^{\tau^*} u^{(H)})||^2$.

**Remark.** In our discussion all estimates are obtained from basic estimates and thus they only hold, in principle, for smooth forms $u$. However, $b\Omega$ being rigid, they are readily converted into genuine estimates. For this, we use an approximation $\chi_\nu(y_n)$ of the identity in the variable $y_n$, and define $u^+_\nu := u^* + \chi_\nu; \chi_\nu \in C^\infty$. By the rigidity of the boundary we have $\tilde{\partial}_b^{(\ast)}(u^+_\nu) = (\tilde{\partial}_b^{(\ast)} u^+_\nu) + \tilde{\omega}$ and $\tilde{\partial}^{(\ast)}(u^+_\nu) = (\tilde{\partial}^{(\ast)} u^+_\nu) + \tilde{\omega}$ where $\tilde{\omega}$ denotes a microlocal component supported by the elliptic region. Then the a-priori estimate applied to $u^+_\nu$, in addition to the elliptic estimates for $\tilde{\omega}$ imply the following. If $\tilde{\partial} u, \tilde{\partial}^\ast u \in H^s$ in a neighborhood of $\text{supp} \langle \zeta \rangle$, then $||((\zeta R^\tau \zeta)u)|| < +\infty$ (in particular $u \in H^s(\{z : \zeta_0(z) = 1\})$ for $\zeta_0 < \sigma$).

**Proof.** Proposition 2.1 shows how to transfer (2.8) from forms to functions $u \in \ker \tilde{\partial}^\perp$; so we only prove the result for forms. We start by applying the compactness estimate (2.4) for $u$ replaced by $((\zeta R^\tau \zeta)u^{(H)})$

\[
||((\zeta R^\tau \zeta)u^{(H)})||^2 \leq \varepsilon \left( Q^\tau((\zeta R^\tau \zeta)u^{(H)}, (\zeta R^\tau \zeta)u^{(H)}) + ||\tilde{L}_n (\zeta R^\tau \zeta)u^{(H)}||^2 \right) + c_\varepsilon ||\zeta' u^{(H)}||^2_0.
\]

We wish to estimate the terms with a factor of $\varepsilon$ on the right. First,

\[
||\tilde{L}_n (\zeta R^\tau \zeta)u^{(H)}|| = ||\tilde{L}_n ((\zeta R^\tau \zeta)u^{(H)})||
\]

\[
= ||\tau \Lambda ((\zeta R^\tau \zeta)u^{(H)})||
\]

\[
\sim lc ||((\zeta R^\tau \zeta)u^{(H)}|| + sc ||\partial_r \Lambda^{-1}((\zeta R^\tau \zeta)u^{(H)}||
\]

\[
\sim lc ||((\zeta R^\tau \zeta)u^{(H)}|| + sc ||\tilde{L}_n (\zeta R^\tau \zeta)u^{(H)}||_{\Lambda^{-1}}
\]

where lc and sc denote a large and small constant respectively and where in the last inequality we have used that $\zeta R^\tau \zeta$ commutes with the operation of taking holomorphic
extension \((H)\). Next, we claim that
\[
Q^\tau((\zeta R^s \zeta)u^{(H)}, (\zeta R^s \zeta)u^{(H)}) \leq Q^{\tau}_{\zeta R^s \zeta}(u^{(H)}, u^{(H)}) + c||((\zeta R^s \zeta)u^{(H)})|^2 + c||((\zeta^s R^s \zeta)u^{(H)})||^2_{A^{-1}}.
\]
To see it, we observe that
\[
\left\{(\bar{\partial}^{(s)} \tau, \zeta R^s \zeta) = \zeta R^s \zeta + \zeta[\bar{\partial}^{(s)} \tau, R^s] \zeta + \zeta R^s \zeta, \right.
\]
\[
\left\{[\bar{\partial}^{(s)} \tau, R^s] \leq \sum_{j=1}^{n-1} s|\sigma_1 z_j(z)\sigma_2(y_n) + \sigma_1(z^r)z_j, \sigma_2(y_n)| \log(\Lambda) R^s.
\]
It follows
\[
Q^\tau((\zeta R^s \zeta)u^{(H)}, (\zeta R^s \zeta)u^{(H)}) \leq Q^\tau_{\zeta R^s \zeta}(u^{(H)}, u^{(H)}) + ||s \zeta \alpha \log(\Lambda) R^s \zeta u^{(H)}||^2 + ||\zeta' u^{(H)}||^2_0,
\]
where
\[
\alpha = \sum_{j=1}^{n-1} |\sigma_1 z_j(z^r)\sigma_2(y_n)| + |\sigma_1(z^r)z_j, \sigma_2(y_n)|.
\]
We recall the hypotheses (i) and (ii) of Theorem 1.1: there is a superlogarithmic estimate for \(z^r \neq 0\), in particular on supp(\(\sigma_j(z^r)\)) for any \(j\) and \(r_j\), are superlogarithmic multipliers. It follows
\[
||s \zeta \alpha \log(\Lambda) R^s \zeta u^{(H)}||^2 \leq scQ^\tau((\zeta R^s \zeta)u^{(H)}, (\zeta R^s \zeta)u^{(H)}) + lc||((\zeta R^s \zeta)u^{(H)})||^2_0 + c||((\zeta R^s \zeta)u^{(H)})||^2_{A^{-1}},
\]
where \(sc\) and \(lc\) denote again a small and large constant respectively. Combination of (2.13) with (2.12) yields the claim (2.11). If one plugs (2.11) and (2.10) into (2.9) and uses induction to reduce \(||((\zeta' R^s \zeta)u^{(H)})||^2_{A^{-1}}\) to \(||\zeta' u^{(H)}||^2_0\) (for a new \(\zeta'\)), one gets
\[
||((\zeta R^s \zeta)u^{(H)})||^2 \leq \epsilon \left(Q^\tau_{\zeta R^s \zeta}(u^{(H)}, u^{(H)}) + \underbrace{||((\zeta R^s \zeta)u^{(H)})||^2}_{\text{absorbable}}\right) + c_\epsilon ||\zeta' u^{(H)}||^2_0,
\]
which concludes the proof of the proposition. \(\Box\)

To carry on our proof we introduce our main technical result

**Proposition 2.4.** In the hypotheses (i) and (ii), we have
\[
||((\zeta R^s \zeta)u||^2 \leq Q_{\zeta R^s \zeta}(u, u) + Q_{\partial \Lambda^{-1} \zeta R^s \zeta}(u, u) + ||u||^2 + Q_{\Lambda^{-1} \zeta'}(u, u)
\]
for \(u \in D^{b}_{\overline{\partial}} \cap C^\infty(\overline{\Omega}), k \geq 1\) or \(u \in \ker \overline{\partial}^\perp\), \(k = 0\).

**Proof.** Again, \(u\) can be a form or a function in \(\ker \overline{\partial}^\perp\). We first focus our attention to (2.8) and wish to remove \((H)\) from the right. We notice that
\[
Q^\tau_{\zeta R^s \zeta}(u^{(H)}, u^{(H)}) \leq Q^\tau_{\zeta R^s \zeta}(u_b, u_b) + ||((\zeta R^s \zeta)u_b||^2_{-\frac{1}{2}}
\]
\[
\leq Q^\tau_{\zeta R^s \zeta}(u, u) + Q^\tau_{\partial \Lambda^{-1} \zeta R^s \zeta}(u, u)
\]
\[
+ ||((\zeta R^s \zeta)u||^2_0 + ||\partial \Lambda^{-1} (\zeta R^s \zeta)u||^2_0.
\]
Owing to \( \partial r = \bar{L}_n + \text{Tan} \), we have the estimate for the last term above

\[
\begin{align*}
||\partial_t \Lambda^{-1}(\zeta R^s \zeta) u||^2 \\
\leq Q_{\Lambda^{-1}(\zeta R^s \zeta)}(u, u) + ||(\zeta R^s \zeta) u||^2 + ||(\overline{\partial}^{(s)} \Lambda^{-1}(\zeta R^s \zeta)) u||^2. \\
\end{align*}
\]

(2.16)

It follows,

\[
\begin{align*}
||((\zeta R^s \zeta) u^{(H)})||^2 &\leq \varepsilon Q_{\zeta R^s \zeta}^*(u^{(H)}, u^{(H)}) + c_e ||\zeta' u^{(H)}||^2_0 \\
&\leq Q_{\zeta R^s \zeta}^*(u, u) + Q_{\overline{\partial}_t \Lambda^{-1}(\zeta R^s \zeta)}^*(u, u) \\
&\leq \varepsilon \left( ||(\zeta R^s \zeta) u||^2_0 + ||\partial_t \Lambda^{-1}(\zeta R^s \zeta) u||^2_0 \right) + c_e ||\zeta' u||^2_0 \\
&\leq Q_{\zeta R^s \zeta}^*(u, u) + Q_{\overline{\partial}_t \Lambda^{-1}(\zeta R^s \zeta)}^*(u, u) \\
&\quad + \varepsilon \left( ||(\zeta R^s \zeta) u||^2_0 + ||\zeta' u||^2_0 + ||\zeta' \partial_t \Lambda^{-1} u||^2_0 \right) + c_e ||\zeta' u^{(H)}||^2_0,
\end{align*}
\]

(2.17)

where in the last inequality the lower order term which occurs in (2.16) has been reduced to \((||\zeta' u||^2_0 + ||\zeta' \partial_t \Lambda^{-1} u||^2_0)\) by iteration. Next we turn our attention to the term \((0)\) and remark that

\[
\begin{align*}
||((\zeta R^s \zeta) u^{(0)})||^2 &\leq \text{Garding} Q_{\Lambda^{-1}(\zeta R^s \zeta)}(u^{(0)}, u^{(0)}) + ||(\zeta R^s \zeta) u^{(0)}||^2_0 + ||\zeta' u^{(0)}||^2_0 \\
&\leq Q_{\Lambda^{-1}(\zeta R^s \zeta)}(u, u) + Q_{\Lambda^{-1}(\zeta R^s \zeta)}(u^{(H)}, u^{(H)}) \\
&\quad + ||(\zeta R^s \zeta) u^{(0)}||^2_0 + ||\zeta' u^{(0)}||^2_0 \\
&\leq Q_{\Lambda^{-1}(\zeta R^s \zeta)}(u, u) + Q_{\Lambda^{-1}(\zeta R^s \zeta)}(u^{(H)}, u^{(H)}) \\
&\quad + ||(\zeta R^s \zeta) u^{(0)}||^2_0 + ||(\zeta R^s \zeta) u^{(0)}||^2_0 + ||\zeta' u^{(0)}||^2_0 \\
&\quad \leq Q_{\lambda \leq \Lambda}^* \left( ||(\zeta R^s \zeta) u^{(0)}||^2_0 + ||(\zeta R^s \zeta) u^{(0)}||^2_0 + ||\zeta' u^{(0)}||^2_0 \right)
\end{align*}
\]

(2.18)

where in the third inequality above we have decomposed \( Q(u, u) = Q^*(u, u) + ||\bar{L}_n u||^2 \) (over a tangential form \( u \)) and used (2.10) to estimate the term with \( \bar{L}_n \). The second term in the right of the last inequality is estimated by (2.17). Finally, combination of
(2.17) and (2.18) yields

$$
\| (\zeta R^s \zeta) u \|^2 \leq \| (\zeta R^s \zeta) u^{(H)} \|^2 + \| (\zeta R^s \zeta) u^{(0)} \|^2 
\lesssim Q_{\zeta R^s \zeta} (u, u) + Q_{\partial^s \Lambda^{-1} (\zeta R^s \zeta)} (u, u) + \epsilon \| (\zeta R^s \zeta) u^{(H)} \|^2 
+ \| (\zeta R^s \zeta) u^{(H)} \|_{A^{-1} \log (\Lambda)}^2 
+ \left( \| \zeta' u \|_0^2 + \| \Lambda^{-1} \partial_s \zeta' u \|_0^2 \right) 
+ \| \zeta' u \|_0^2 + c_s \| \zeta' u \|_0^2 
\leq (I) + (II),
$$

(2.19)

where in the last inequality we have estimated \( \| (\zeta R^s \zeta) u^{(0)} \|^2 \|_{A^{-1} \log (\Lambda)} \lesssim (I) + (II) \) because of

$$
\Lambda^{-1} (\zeta R^s \zeta) \leq R^{-1} (\zeta R^s \zeta) 
= \zeta R^{-1} \zeta + \text{Order 0},
$$

and from induction. Finally (II) is estimated as follows. As for \( u^{(H)} \):

$$
\| \zeta' u^{(H)} \|_0^2 \lesssim \| \zeta' u \|_0^2 \frac{1}{2} 
\lesssim \| \zeta' u \|_0^2 + \| \Lambda^{-1} \partial_s \zeta' u \|^2 
\lesssim \| \zeta' u \|_0^2 + \| \Lambda^{-1} \bar{L} \zeta' u \|^2 + \| \Lambda^{-1} \tan \zeta' u \|^2 
\lesssim \| u \|_0^2 + Q_{\Lambda^{-1} \zeta'} (u, u).
$$

The same inequality holds for \( u^{(H)} \) replaced by \( u^{(0)} \) on account of the identity \( u^{(0)} = u + u^{(H)} \). Thus \( (II) \lesssim c_s \| u \|_0^2 + Q_{\Lambda^{-1} \zeta'} (u, u) \) and if we plug this into (2.19), we get (2.14). \( \square \)

We are ready for

**Proof of Theorem 1.1.** We recall that we are writing \( u \) for \( u^\tau + \) (or \( u^+ \) in case of a function). We begin by noticing that, for \( \zeta_o < \sigma < \zeta \)

$$
\| \Lambda^s \zeta_o u \| \lesssim \| R^s \zeta_o u \| + \| u \| 
= \| R^s \zeta_o \zeta^2 u \| + \| u \| 
\lesssim \| R^s \zeta^2 u \| + \| [R^s, \zeta_o] \zeta^2 u \| + \| u \| 
\lesssim \| R^s \zeta^2 u \| + \| u \| 
\lesssim \| \zeta R^s \zeta u \| + \| [R^s, \zeta] \zeta u \| + \| u \| 
\lesssim \| \zeta R^s \zeta u \| + \| u \|.
$$

Using (2.14) of Proposition 2.4 we get (1.5) in tangential version, that is,

$$
\| \Lambda^s \zeta_o u \|^2 \lesssim Q_{\Lambda^s \zeta} (u, u) + Q_{\partial^s \Lambda^{-1} \zeta} (u, u) + Q_{\Lambda^{-1} \zeta'} (u, u) + \| u \|_0^2 
\lesssim \| \zeta' \bar{\partial} u \|_0^2 + \| \zeta' \partial^* u \|_0^2 + \| u \|_0^2.
$$

(2.20)
Finally, by non-characteristicity (cf. eg the end of Section 8 of [17]), we can replace $\|\Lambda^s \zeta u\|_s^2$ by $\|\zeta u\|_s^2$ in the left of (2.20); we also replace the notation $\zeta'$ by $\zeta$ on the right and get (1.5). From (1.5) the local regularity of $\partial^{(1)} N$ readily follows which implies the regularity of $B$ and $N$ by the argument before the statement of Theorem 1.3. This concludes the proof of Theorem 1.1. □

REFERENCES