FUNCTORIAL RELATIONSHIPS BETWEEN $QH^*(G/B)$ AND $QH^*(G/P)$, (II)*

CHANGZHENG LI†

Abstract. We show a canonical injective morphism from the quantum cohomology ring $QH^*(G/P)$ to the associated graded algebra of $QH^*(G/B)$, which is with respect to a nice filtration on $QH^*(G/B)$ introduced by Leung and the author. This tells us the vanishing of a lot of genus zero, three-pointed Gromov-Witten invariants of flag varieties $G/P$.

Key words. Quantum cohomology, flag varieties, filtered algebras, Gromov-Witten invariants.

AMS subject classifications. 14N35, 14M15.

1. Introduction. The (small) quantum cohomology ring $QH^*(G/P)$ of a flag variety $G/P$ is a deformation of the ring structure on the classical cohomology $H^*(G/P)$ by incorporating three-pointed, genus zero Gromov-Witten invariants of $G/P$. Here $G$ denotes a simply-connected complex simple Lie group, and $P$ denotes a parabolic subgroup of $G$. There has been a lot of intense studies on $QH^*(G/P)$ (see e.g. the survey [8] and references therein). In particular, there was an insight on $QH^*(G/P)$ in the unpublished work [20] of D. Peterson, which, for instance, describes a surprising connection between $QH^*(G/P)$ and the so-called Peterson subvariety. When $P = B$ is the Borel subgroup of $G$, Lam and Shimozono [15] proved that $QH^*(G/B)$ is isomorphic to the homology of the group of the based loops in a maximal compact Lie subgroup of $G$ with the ring structure given by the Pontryagin product, after equivariant extension and localization (see also [20], [18]). Woodward proved a comparison formula [21] of Peterson that all genus zero, three-pointed Gromov-Witten invariants of $G/P$ are contained in those of $G/B$. As a consequence, we can define a canonical (injective) map $QH^*(G/P) \hookrightarrow QH^*(G/B)$ as vector spaces. In [16], Leung and the author constructed a natural filtration $\mathcal{F}$ on $QH^*(G/B)$ which comes from a quantum analog of the Leray-Serre spectral sequence for the natural fibration $P/B \to G/B \to G/P$. The next theorem is our main result in the present paper, precise descriptions of which will be given in Theorem 2.4.

MAIN THEOREM. There is a canonical injective morphism of algebras from the quantum cohomology ring $QH^*(G/P)$ to the associated graded algebra of $QH^*(G/B)$ with respect to the filtration $\mathcal{F}$.

The above statement was proved by Leung and the author under an additional assumption on $P/B$. Here we do not require any constraint on $P/B$. That is, we prove Conjecture 5.3 of [16]. Combining the main results therein with the above theorem, we can tell a complete story as follows.

THEOREM 1.1. Let $r$ denote the semisimple rank of the Levi subgroup of $P$ containing a maximal torus $T \subset B$.

1. There exists a $\mathbb{Z}^{r+1}$-filtration $\mathcal{F}$ on $QH^*(G/B)$, respecting the quantum product structure.

*Received July 4, 2013; accepted for publication October 16, 2013.
†Kavli Institute for the Physics and Mathematics of the Universe (WPI), Todai Institutes for Advanced Study, The University of Tokyo, 5-1-5 Kashiwa-no-Ha,Kashiwa City, Chiba 277-8583, Japan (changzheng.li@ipmu.jp).
2. There exist an ideal $\mathcal{I}$ of $QH^*(G/B)$ and a canonical algebra isomorphism

$$QH^*(G/B)/\mathcal{I} \xrightarrow{\sim} QH^*(P/B).$$

3. There exists a subalgebra $A$ of $QH^*(G/B)$ together with an ideal $J$ of $A$, such that $QH^*(G/P)$ is canonically isomorphic to $A/J$ as algebras.

4. There exists a canonical injective morphism of graded algebras

$$\Psi_{r+1}: QH^*(G/P) \hookrightarrow Gr_{(r+1)} F(QH^*(G/B))$$

together with an isomorphism of graded algebras after localization

$$Gr_{(r)} F(QH^*(G/B)) \cong \bigotimes_{j=1}^r Gr_{(j)} F(QH^*(P_j/P_{j-1})), $$

where $P_j$'s are parabolic subgroups constructed in a canonical way, forming a chain $B := P_0 \subseteq P_1 \subseteq \cdots \subseteq P_{r-1} \subseteq P = P \subseteq G$. Furthermore, $\Psi_{r+1}$ is an isomorphism if and only if the next hypothesis (Hypo1) holds: $P_j/P_{j-1}$ is a projective space for any $1 \leq j \leq r$.

All the relevant ideals, subalgebras and morphisms above will be described precisely in Theorem 4.6. To get a clearer idea of them here, we use the same toy example of the variety of complete flags in $\mathbb{C}^3$ as in [16].

**Example 1.2.** Let $G = SL(3, \mathbb{C})$ and $B \subseteq P \subseteq G$. Then we have $G/B = \{V_1 \leq V_2 \leq \mathbb{C}^3 \mid \dim C V_1 = 1, \dim C V_2 = 2\}$, and the natural projection $\pi: G/B \to G/P$ is given by forgetting the vector subspace $V_1$ in the complete flag $V_1 \subseteq V_2 \subseteq \mathbb{C}^3$. In particular, $P/B \cong \mathbb{P}^1$, $G/P \cong \mathbb{P}^2$, and the semisimple rank $r$ of the Levi subgroup of $P$ containing a maximal torus $T \subseteq B$ equals 1. In this case, the quantum cohomology ring $QH^*(G/B) = (H^*(G/B) \otimes \mathbb{Q}[q_1, q_2], \ast)$ has a $\mathbb{Q}$-basis $\sigma^w q_1^a q_2^b$, indexed by $(w, (a, b)) \in W \times \mathbb{Z}^2_{\geq 0}$, and we define a grading map $\text{gr}(\sigma^w q_1^a q_2^b) := (2a - b, 3b) + \text{gr}(\sigma^w) \in \mathbb{Z}^2$. Here $W := \{1, s_1, s_2, s_1 s_2, s_2 s_1, s_1 s_2 s_1\}$ is the Weyl group (isomorphic to the permutation group $S_3$). The grading $\text{gr}(\sigma^w)$ is the usual one from the Leray-Serre spectral sequence, respectively given by $(0, 0)$, $(1, 0)$, $(0, 1)$, $(0, 2)$, $(1, 1)$, $(1, 2)$. Using the above gradings together with the lexicographical order on $\mathbb{Z}^2$ (i.e., $(x_1, x_2) < (y_1, y_2)$ if and only if either $x_1 < y_1$ or $(x_1 = y_1$ and $x_2 < y_2$)), we have the following conclusions.

1. There is a $\mathbb{Z}^2$-filtration $F = \{F_x\}_{x \in \mathbb{Z}^2}$ on $QH^*(G/B)$, defined by $F_x := \bigoplus_{\text{gr}(\sigma^w q_1^a q_2^b) \leq x} \mathbb{Q}\sigma^w q_1^a q_2^b \subseteq QH^*(G/B)$. Furthermore, $F$ respects the quantum product structure. That is, $F_x \ast F_y \subseteq F_{x+y}$.

2. $\mathcal{I} := \bigoplus_{\text{gr}(\sigma^w q_1^a q_2^b) \in \mathbb{Z} \times \mathbb{Z}^+} \mathbb{Q}\sigma^w q_1^a q_2^b$ is an ideal of $QH^*(G/B)$. We take the standard ring presentation $QH^*(\mathbb{P}^1) = \mathbb{Q}[x, q]/(x^2 - q)$. Note $P/B \cong \mathbb{P}^1$. Then $\sigma^{a_1} + \mathcal{I} \mapsto x$ and $q_1 + \mathcal{I} \mapsto q$ define an isomorphism of algebras from $QH^*(G/B)/\mathcal{I}$ to $QH^*(P/B)$.

3. $A := \sum_{k \in \mathbb{Z}} F_{(0, k)}$ is a subalgebra of $QH^*(G/B)$, and $J := F_{(0, -1)}$ is an ideal of $A$. Write $QH^*(\mathbb{P}^2) = \mathbb{Q}[z, t]/(z^3 - t)$. Note $G/P \cong \mathbb{P}^2$. Then $z \mapsto \sigma^{s_2} + J$, $z^2 \mapsto \sigma^{s_1 s_2} + J$ and $t \mapsto \sigma^{s_1 q_2} + J$ define an isomorphism of algebras from $QH^*(G/P)$ to $A/J$.

4. $G_{(2)} := \bigoplus_{k \in \mathbb{Z}} F_{(0, k)}/\sum_{x < (0, k)} F_x$ is a graded subalgebra of $Gr_{(2)} F(QH^*(G/B))$, and it is canonically isomorphic to $A/J$ as algebras. Combining it with (3), we have an isomorphism of (graded) algebras
In addition, by taking the classical limit, $F|_{q=0}$ gives the usual $\mathbb{Z}^2$-filtration on $H^*(G/B)$ from the Leray-Serre spectral sequence. The classical limit of $\pi_q^*$ also coincides with the induced morphism $\pi^*: H^*(G/P) \hookrightarrow H^*(G/B)$ of algebras.

In the present paper, we will prove Theorem 1.1 in a combinatorial way. It will be very interesting to explore a conceptual explanation of the theorem. Such an explanation may involve the notion of vertical quantum cohomology in [1]. As an evidence, part (2) of Theorem 1.1 turns out to coincide with equation (2.17) of [1] in the special case when $G = \text{SL}(n+1, \mathbb{C})$. In a future project, we plan to investigate the relation between our results and those from [1]. We would like to remind that a sufficient condition (Hypo2) for $\Psi_{r+1}$ to be an isomorphism was provided in [16], which says that $P/B$ is isomorphic to a product of complete flag varieties of type $A$. It is not a strong constraint, satisfied for all flag varieties $G/P$ of type $A, G_2$ as well as for most of flag varieties $G/P$ of each remaining Lie type. The necessary and sufficient condition in the above theorem is slightly more general. For instance for $G$ of type $F_4$, there are 16 flag varieties $G/P$ in total (up to isomorphism together with the two extremal cases $G/B, \{\text{pt}\}$ being counted). Among them, there are 13 flag varieties satisfying both hypotheses (Hypo1) and (Hypo2), while one more flag variety satisfies (Hypo1). Precisely, for $G$ of type $F_4$, (Hypo1) holds for all $G/P$ except for the two (co)adjoint Grassmannians that respectively correspond to (the complement of) the two ending nodes of the Dynkin diagram of type $F_4$.

The notion of quantum cohomology was introduced by the physicists in 1990s, and it can be defined over a smooth projective variety $X$. It is a quite challenging problem to study the quantum cohomology ring $QH^*(X)$, partially because of the lack of functorial property. Namely, in general, a reasonable morphism between two smooth projective varieties does not induce a morphism on the level of quantum cohomogy. However, Theorem 1.1 tells us a beautiful story on the “functoriality” among the special case of the quantum cohomology of flag varieties. We may even expect nice applications of it in future research. Despite lots of interesting studies of $QH^*(G/P)$, they are mostly for the varieties of partial flags of subspaces of $\mathbb{C}^{n+1}$, i.e., when $G = \text{SL}(n+1, \mathbb{C})$. For $G$ of general Lie type, ring presentations of the quantum cohomology are better understood for either complete flag varieties $G/B$ [14] or most of Grassmannians, i.e., when $P$ is maximal (cf. [5], [6] and references therein). The special case of the functorial property [16] when $P/B \cong \mathbb{P}^1$ has led to nice applications on the “quantum to classical” principle [17], as further applications of which Leung and the author obtained certain quantum Pieri rules [19] as well as alternative proofs of the main results of [4]. On the other hand, our main result could also be treated as a kind of application of the “quantum to classical” principle. As we can see later, the proof requires knowledge on the vanishing of a lot of Gromov-Witten invariants as well as explicit calculations of certain non-vanishing Gromov-Witten invariants that all turn out to be equal to 1. Although Leung and the author have showed an explicit combinatorial formula for those Gromov-Witten invariants (with sign cancelation involved) in [18], it would exceed the capacity of a computer in some cases if we use the formula directly. Instead, we will apply the “quantum to classical” principle developed in [17].

The paper is organized as follows. In section 2, we introduce a (non-recursively defined) grading map $gr$ and state the main results of the present paper. The whole of section 3 is devoted to a proof of Main Theorem when the Dynkin diagram of the
Levi subgroup of $P$ containing a maximal torus $T \subset B$ is connected, the outline of which is given at the beginning the section. The proofs of some propositions in section 3 require arguments case by case. Details for all those cases not covered in the section are given in section 5. In section 4, we describe Theorem 1.1 in details and provide a sketch proof of it therein, in which there is no constraint on $P/B$. We also greatly clarify the grading map defined recursively in [16], by showing the coincidence between it and the map $gr$ defined in section 2. Both the definition of $gr$ and the conjecture of the coincidence between the two grading maps were due to the anonymous referee of [16]. It is quite worth to prove the coincidence, because the grading map was used to establish a nice filtration on $QH^*(G/B)$, which is the heart of the whole story of the functoriality.

2. Main results.

2.1. Notations. We will follow most of the notations used in [16], which are repeated here for the sake of completeness. Our readers can refer to [12] and [9] for more details.

Let $G$ be a simply-connected complex simple Lie group of rank $n$, $B$ be a Borel subgroup, $T \subset B$ be a maximal complex torus with Lie algebra $\mathfrak{h} = \text{Lie}(T)$, and $P \supset B$ be a proper parabolic subgroup of $G$. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subset \mathfrak{h}^*$ be a basis of simple roots and $\{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \subset \mathfrak{h}$ be the simple coroots. Each parabolic subgroup $\bar{P} \supset B$ is in one-to-one correspondence with a subset $\Delta_{\bar{P}} \subset \Delta$. Conversely, by $P_{\Delta}$ we mean the parabolic subgroup containing $B$ that corresponds to a given subset $\Delta \subset \Delta$. Here $B$ contains the one-parameter unipotent subgroups $U_{\alpha_i}$, $\alpha_i \in \Delta$. Clearly, $P_{\Delta} = G$, $\Delta_B = \emptyset$ and $\Delta_P \subset \Delta$. Let $\{\omega_1, \ldots, \omega_n\}$ (resp. $\{\omega_1^\vee, \ldots, \omega_n^\vee\}$) denote the fundamental (co)weights, and $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ denote the natural pairing. Let $\rho := \sum_{i=1}^n \omega_i$.

The Weyl group $W$ is generated by $\{s_{\alpha_i} \mid \alpha_i \in \Delta\}$, where each simple reflection $s_i := s_{\alpha_i}$, maps $\lambda \in \mathfrak{h}$ and $\beta \in \mathfrak{h}^*$ to $s_i(\lambda) = \lambda - \langle \alpha_i, \lambda \rangle \alpha_i^\vee$ and $s_i(\beta) = \beta - \langle \beta, \alpha_i^\vee \rangle \alpha_i$, respectively. Let $\ell : W \to \mathbb{Z}_{\geq 0}$ denote the standard length function. Given a parabolic subgroup $\bar{P} \supset B$, we denote by $W_{\bar{P}}$ the subgroup of $W$ generated by $\{s_{\alpha} \mid \alpha \in \Delta_{\bar{P}}\}$, in which there is a unique element of maximum length, say $w_{\bar{P}}$. Given another parabolic subgroup $\bar{P}$ with $B \subset \bar{P} \subset \bar{P}$, we have $\Delta_{\bar{P}} \subset \Delta_{\bar{P}}$. Each coset in $W_{\bar{P}}/W_P$ has a unique minimal length representative. The set of all these minimal length representatives is denoted by $W_{\bar{P}}(\subset W_{\bar{P}} \subset W)$. Note that $W_B = \{1\}$, $W_{\bar{P}} = W_{\bar{P}}$ and $W_G = W$. We simply denote $w_0 := w_G$ and $W_P := W_{\bar{P}}$.

The root system is given by $R = W \cdot \Delta = R^+ \sqcup (-R^+)$, where $R^+ = R \cap \bigoplus_{i=1}^n \mathbb{Z}_{\geq 0} \alpha_i$ is the set of positive roots. It is a well-known fact that $\ell(w) = |\text{Inv}(w)|$ where $\text{Inv}(w)$ is the inversion set of $w \in W$ given by

$$\text{Inv}(w) := \{\beta \in R^+ \mid w(\beta) \in -R^+\}.$$ 

Given $\gamma = w(\alpha_i) \in R$, we have the coroot $\gamma^\vee := w(\alpha_i^\vee)$ in the coroot lattice $Q^\vee := \bigoplus_{i=1}^n \mathbb{Z} \alpha_i^\vee$ and the reflection $s_{\gamma} := ws_\alpha w^{-1} \in W$, which are independent of the expressions of $\gamma$. For the given $P$, we denote by $R_P = R_P^+ \sqcup (-R_P^-)$ the root subsystem, where $R_P^+ = R^+ \cap \bigoplus_{\alpha_i \in \Delta_P} \mathbb{Z} \alpha_i$ and $Q_P^\vee := \bigoplus_{\alpha_i \in \Delta_P} \mathbb{Z} \alpha_i^\vee$.

The (co)homology of the flag variety $G/P$ has an additive basis of Schubert (co)homology classes $\sigma_u$ (resp. $\sigma^u$) indexed by $W_P$. In particular, we can identify $H_2(G/P, \mathbb{Z}) = \bigoplus_{\alpha_i \in \Delta \setminus \Delta_P} \mathbb{Z} \sigma_{s_{\alpha_i}}$ with $Q_P^\vee / Q_P^\vee$ canonically, by mapping $\sum_{\alpha_i \in \Delta \setminus \Delta_P} a_{\alpha_i} \sigma_{s_{\alpha_i}}$ to $\lambda_P = \sum_{\alpha_i \in \Delta \setminus \Delta_P} a_{\alpha_i} \alpha_i^\vee + Q_P^\vee$. For each $\alpha_j \in \Delta \setminus \Delta_P$, we introduce a formal variable $g_{\alpha_j}^\vee + Q_P^\vee$. For such $\lambda_P$, we denote $q_{\lambda_P} = \prod_{\alpha_j \in \Delta \setminus \Delta_P} g_{\alpha_j}^\vee + Q_P^\vee$. 


The (small) quantum cohomology ring $QH^*(G/P) = (H^*(G/P) \otimes \mathbb{Q}[q], \star_P)$ of $G/P$ also has a natural $\mathbb{Q}[q]$-basis of Schubert classes $\sigma^u = \sigma^u \otimes 1$, for which

$$\sigma^u \star_P \sigma^v = \sum_{w \in W^P, \lambda_P \in Q^\vee/Q^\vee_2} N_{u,v}^{w,\lambda_P} q_{\lambda_P} \sigma^w.$$  

The quantum product $\star_P$ is associative and commutative. The quantum Schubert structure constants $N_{u,v}^{w,\lambda_P}$ are all non-negative, given by genus zero, 3-pointed Gromov-Witten invariants of $G/P$. When $P = B$, we have $Q^\vee_2 = 0$, $W^P = \{1\}$ and $W^F = W$. In this case, we simply denote $\star = \star_P$, $\lambda = \lambda_P$ and $q_j = q_{a_j}$. 

2.2. Main results. We will assume the Dynkin diagram $Dyn(\Delta_P)$ to be connected throughout the paper except in section 4. Denote $r := |\Delta_P|$. Note $1 \leq r < n$. Recall that a natural $\mathbb{Q}$-basis of $QH^*(G/B)[q_1^{-1}, \cdots, q_n^{-1}]$ is given by $q_{\lambda} \sigma^w$ labelled by $(w, \lambda) \in W \times Q^\vee$. Note that $q_{\lambda} \sigma^w \in QH^*(G/B)$ if and only if $q_{\lambda} \in \mathbb{Q}[q]$ is a polynomial. In [16], Leung and the author introduced a grading map 

$$gr : W \times Q^\vee \longrightarrow \mathbb{Z}^{r+1} = \bigoplus_{i=1}^{r+1} \mathbb{Z} e_i.$$  

Due to Lemma 2.12 of [16], the following subset 

$$S := \{gr(w, \lambda) \mid q_{\lambda} \sigma^w \in QH^*(G/B)\}^1$$  

is a totally-ordered sub-semigroup of $\mathbb{Z}^{r+1}$. Here we are using the lexicographical order on elements $a = (a_1, \cdots, a_{r+1}) = \sum_{i=1}^{r+1} a_i e_i$ in $\mathbb{Z}^{r+1}$. Namely $a < b$ if and only if there exists $1 \leq j \leq r+1$ such that $a_j < b_j$ and $a_i = b_i$ for all $1 \leq i < j$. We can define a family $F = \{F_a\}_{a \in S}$ of subspaces of $QH^*(G/B)$, in which 

$$F_a := \bigoplus_{gr(w, \lambda) \leq a} \mathbb{Q} q_{\lambda} \sigma^w \subset QH^*(G/B).$$  

It is one of the main theorems in [16] that 

**Proposition 2.1** (Theorem 1.2 of [16]). $QH^*(G/B)$ is an $S$-filtered algebra with filtration $F$. Furthermore, this $S$-filtered algebra structure is naturally extended to a $\mathbb{Z}^{r+1}$-filtered algebra structure on $QH^*(G/B)$. 

As a consequence, we obtain the associated $\mathbb{Z}^{r+1}$-graded algebra 

$$Gr^F(QH^*(G/B)) := \bigoplus_{a \in \mathbb{Z}^{r+1}} Gr^a_F,$$  

where $Gr^a_F := F_a/ \bigoplus_{b < a} F_b$. In particular, we have a graded subalgebra 

$$Gr^F_{(r+1)} := \bigoplus_{i \in \mathbb{Z}} Gr^F_{ie_{r+1}}.$$  

Recall the next Peterson-Woodward comparison formula [21] (see also [15]). 

**Proposition 2.2.** For any $\lambda_P \in Q^\vee/Q^\vee_2$, there exists a unique $\lambda_B \in Q^\vee$ such that $\lambda_P = \lambda_B + Q^\vee_2$ and $\langle \alpha, \lambda_B \rangle \in \{0, -1\}$ for all $\alpha \in R^+_B$. Furthermore for every $u, v, w \in W^P$, we have 

$$N_{u,v}^{w,\lambda_P} = N_{u,v}^{w,wPwP^P,\lambda_B},$$  

$(w, \lambda)$ is simply denoted as $wq_{\lambda}$ in [16].
where $\Delta_{P'} = \{ \alpha \in \Delta_P \mid \langle \alpha, \lambda_B \rangle = 0 \}$.

The above formula, comparing Gromov-Witten invariants for $G/P$ and for $G/B$, induces an injective map

$$
\psi_{\Delta, \Delta_P} : QH^*(G/P) \rightarrow QH^*(G/B);
$$

$$
\sum a_{w, \lambda_P} q_{\lambda_P} \sigma^w \mapsto \sum a_{w, \lambda_P} q_{\lambda_B} \sigma^{w_{P'}} \sigma^w,
$$

and we call $\lambda_B$ the Peterson-Woodward lifting of $\lambda_P$. The next proposition is another one of the main theorems in [16] (see Proposition 3.24 and Theorem 1.4 therein).

**Proposition 2.3.** Suppose that $\Delta_P$ is of $A$-type. Then the following map

$$
\Psi_{r+1} : QH^*(G/P) \hookrightarrow Gr_F^{\mathbb{Z}}(r+1);
$$

$$
q_{\lambda_P} \sigma^w \mapsto \psi_{\Delta, \Delta_P}(q_{\lambda_P} \sigma^w)
$$

is well-defined, and it is an isomorphism of (graded) algebras.

Conjecture 5.3 of [16] tells us the counterpart of the above proposition, and it is the main result of the present paper that such a conjecture does hold. Namely

**Theorem 2.4.** Suppose that $\Delta_P$ is not of $A$-type. Then the map $\Psi_{r+1}$ given in Proposition 2.3 is well-defined, and it is an injective morphism of (graded) algebras. Furthermore, $\Psi_{r+1}$ becomes an isomorphism if and only if $r = 2$ together with either case (C1B) or case (C9) of Table 2.1 occurring.

**Remark 2.5.** The algebra $QH^*(G/P)$ is equipped with a natural $\mathbb{Z}$-grading: a Schubert class $\sigma^w$ is of grading $\ell(w)$, and a quantum variable $q_{\alpha^V + \mathbb{Q}^P}$ is of grading $\langle \sigma_{s_\alpha}, c_1(G/P) \rangle$. Once we show that $\Psi_{r+1}$ is a morphism of algebras, the way of defining $\Psi_{r+1}$ automatically tells us that it preserves the $\mathbb{Z}$-grading as well.

We will provide the proof in the next section, one point of which is to compute certain Gromov-Witten invariants explicitly.

In order to define the grading map $gr$ in [16], Leung and the author introduced an ordering on the subset $\Delta_P$ first. In our case when $\Delta_P$ is not of type $A$, such an ordering is equivalent to the assumption that $\Delta_P = \{ \alpha_1, \ldots, \alpha_r \}$ with all the possible Dynkin diagrams $Dyn(\Delta)$ being listed in Table 2.1. These are precisely the cases for which Theorem 2.4 is not covered in [16]. In addition, Table 2.1 has exhausted all the possible cases of fiberations $G/B \rightarrow G/P$ such that $Dyn(\Delta_P)$ is connected but not of type $A$. Therein the cases are basically numbered according to those for $Dyn(\{ \alpha_1, \ldots, \alpha_{r-1} \})$ in Table 2 of [16].

**Remark 2.6.** In Table 2.1, we have treated bases of type $E_6$ and $E_7$ as subsets of a base of type $E_8$ canonically. $Dyn(\Delta_P)$ is always given by a unique case in Table 2.1 except when $\Delta$ is of $E_6$-type together with $r = 5$. In this exceptional case, both C5) and C7) occur and we can choose either of them. Note $2 \leq r < n$. The case of $G_2$-type does not occur there.

In [16], the grading map $gr$ was defined recursively by using the Peterson-Woodward comparison formula together with the given ordering on $\Delta_P$. Here we will define $gr$ as below, following the suggestion of the referee of [16] (see also Remark 2.10 therein).
Definition 2.7. Let us choose the ordering of $\Delta_P$ as given in Table 2.1. For each $1 \leq j \leq r$, we denote $\Delta_j := \{\alpha_1, \ldots, \alpha_j\}$. Set $\Delta_0 := \emptyset$ and $\Delta_{r+1} := \Delta$. Denote by $P_i := P_{\Delta_i}$ the parabolic subgroup corresponding to $\Delta_i$ for all $0 \leq i \leq r + 1$. Recall that we have denoted by $\{e_1, \ldots, e_{r+1}\}$ the standard basis of $\mathbb{Z}^{r+1}$. Define a grading map $gr$ by

$$gr: W \times Q^\vee \longrightarrow \mathbb{Z}^{r+1};$$

$$(w, \lambda) \mapsto gr(w, \lambda) = \sum_{i=1}^{r+1} \left| \text{Inv}(w) \cap (R_{P_i}^+ \setminus R_{P_{i-1}}^+) \right| + \sum_{\beta \in R_{P_i}^+ \setminus R_{P_{i-1}}^+} \langle \beta, \lambda \rangle e_i.$$  

Say $gr(u, \eta) = \sum_{i=1}^{r+1} a_i e_i$. Let $1 \leq j \leq k \leq r + 1$. As usual, we define

$$|gr(u, \eta)| := \sum_{i=1}^{r+1} a_i, \quad gr_{[j,k]}(u, \eta) := \sum_{i=j}^{k} a_i e_i.$$  

As a known fact, we have (see also the proof of Proposition 4.3 for detailed explanations)

$$gr(w, 0) = \sum_{j=1}^{r+1} \ell(w_j) e_j,$$

where $w_j \in W_{P_{j-1}}^{P_j}$ are the unique elements such that $w = w_{r+1} w_r \cdots w_1$.

We will show the next conjecture of the referee of [16].

<table>
<thead>
<tr>
<th>C1B)</th>
<th>Dynkin diagram of $\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cdots \circ \cdots \circ \circ \alpha_{r+1} \alpha_1 \alpha_r$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>C1C)</th>
<th>Dynkin diagram of $\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cdots \circ \cdots \circ \alpha \alpha_{r+1} \alpha_1 \alpha_{r-1}$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>C2)</th>
<th>Dynkin diagram of $\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha \alpha_7 \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>C4)</th>
<th>Dynkin diagram of $\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_7 \alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>C5)</th>
<th>Dynkin diagram of $\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_0 \alpha_5 \alpha_3 \alpha_2 \alpha_1$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>C7)</th>
<th>Dynkin diagram of $\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_6 \alpha_1 \alpha_2 \alpha_4 \alpha_5$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>C9)</th>
<th>Dynkin diagram of $\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_4 \alpha_1 \alpha_2 \alpha_3 \alpha_5 \alpha_6 \alpha_7 \alpha_8$</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>C10)</th>
<th>Dynkin diagram of $\Delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha_1 \alpha_2 \alpha_3 \alpha_4 \alpha_5 \alpha_6 \alpha_7 \alpha_8$</td>
<td></td>
</tr>
</tbody>
</table>
Theorem 2.8. The two grading maps by Definition 2.7 above and by Definition 2.8 of [16] coincide with each other.

Because of the coincidence, Proposition 2.1 holds with respect to the grading map \( gr \). Namely for any Schubert classes \( \sigma^u, \sigma^v \) of \( QH^*(G/B) \), if \( q_\lambda \sigma^w \) occurs in the quantum multiplication \( \sigma^u \star \sigma^v \), then

\[
gr(w, \lambda) \leq gr(u, 0) + gr(v, 0).
\]

The proof of Theorem 2.8 will be given in section 4, which is completely independent of section 3. Due to the coincidence, the proofs of several main results in [16] may be simplified substantially. We can describe the explicit gradings of all the simple coroots as follows, which were obtained by direct calculations using Definition 4.2 of the grading map \( gr' \). (See section 3.5 of [16] for more details on the calculations.)

Proposition 2.9. Let \( \alpha \in \Delta \). We simply denote \( gr(\alpha^\vee) := gr(id, \alpha^\vee) \).

1. \( gr(\alpha^\vee) = 2e_{r+1}, \text{ if } Dyn(\{\alpha\} \cup \Delta_P) \text{ is disconnected.} \)
2. \( gr(\alpha^\vee) = (1 + j)e_j + (1 - j)e_{j-1}, \text{ if } \alpha = \alpha_j \text{ with } j \leq r - 1 \text{ where } 0 \cdot e_0 := 0. \)
3. \( gr(\alpha^\vee) \) is given in Table 2.2, if \( \alpha = \alpha_r \) or \( \alpha_{r+1} \).

**Table 2.2**

<table>
<thead>
<tr>
<th>( \alpha^\vee )</th>
<th>( gr(\alpha^\vee_{r+1}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1B)</td>
<td>( 2re_r - (2r - 2)e_{r-1} )</td>
</tr>
<tr>
<td>C1C)</td>
<td>( (r + 1)e_r - (r - 1)e_{r-1} )</td>
</tr>
<tr>
<td>C2)</td>
<td>( 2(r - 1)e_r + (2 - r)(e_{r-1} + e_{r-2}) )</td>
</tr>
<tr>
<td>C4)</td>
<td>( (3r - 7)e_r + (3 - r) \sum_{j=r-3}^{r-1} e_j )</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>C5)</td>
<td>( 2(r - 1)e_r + (2 - r)(e_{r-1} + e_{r-2}) )</td>
</tr>
<tr>
<td>C7)</td>
<td>( 2re_r - (2r - 2)e_{r-1} )</td>
</tr>
<tr>
<td>C9)</td>
<td></td>
</tr>
<tr>
<td>C10)</td>
<td></td>
</tr>
</tbody>
</table>

4. The remaining cases happen when there are two nodes adjacent to \( Dyn(\Delta_P) \), namely the node \( \alpha_{r+1} \) and the other node, say \( \alpha_{r+2} \). Then we have either of the followings.

(a) \( gr(\alpha^\vee_{r+2}) = 2re_{r+1} + (1 - r)e_r - \sum_{j=1}^{r-1} e_j \), which holds if C7) occurs and \( r \leq 6; \)

(b) \( gr(\alpha^\vee_{r+2}) = 5e_3 - 2e_2 - e_1 \), which holds if C9) occurs and \( r = 2 \).

In particular, we have \( |gr(\alpha^\vee)| = 2 \) for any \( \alpha \in \Delta \).

3. Proof of Theorem 2.4. Recall that we have defined a grading map \( gr : W \times Q^\vee \rightarrow \mathbb{Z}^{r+1} \). For convenience, for any \( q_\lambda \sigma^w \in QH^*(G/B)[q_1^{-1}, \ldots, q_n^{-1}] \), we will also use the following notation

\[
gr(q_\lambda \sigma^w) := gr(w, \lambda).
\]
The injective map \( \psi_{\Delta, \Delta_P} : QH^*(G/P) \to QH^*(G/B) \) induces a natural map \( QH^*(G/P) \to Gr^F(QH^*(G/B)) \). That is, \( q_{\lambda, \mu} \sigma_w \mapsto \psi_{\Delta, \Delta_P}(q_{\lambda, \mu} \sigma_w) \in Gr^F \subset Gr^F(QH^*(G/B)) \), where \( a = gr(\psi_{\Delta, \Delta_P}(q_{\lambda, \mu} \sigma_w)) \). We state the next proposition, which extends Proposition 3.24 of [16] in the case of parabolic subgroups \( P \) such that \( \Delta_P \) is not of type \( A \).

**Proposition 3.1.** For any \( q_{\lambda, \mu} \sigma_w \in QH^*(G/P) \), we have

\[
gr_{[1, r]}(\psi_{\Delta, \Delta_P}(q_{\lambda, \mu} \sigma_w)) = 0.
\]

Hence, \( a \in \mathbb{Z} e_{r+1} \). That is, the map \( \Psi_{r+1} \) as in Proposition 2.3 is well-defined. We can further show

**Proposition 3.2.** \( \Psi_{r+1} \) is an injective map of vector spaces. Furthermore, \( \Psi_{r+1} \) is surjective if and only if \( r = 2 \) and either case C1(B) or case C9) occurs.

We shall also show

**Proposition 3.3.** \( \Psi_{r+1} \) is a morphism of algebras. That is, for any \( q_{\lambda, \mu} \sigma_w, \sigma_v \) in \( QH^*(G/P) \), we have

1. \( \Psi_{r+1}(\sigma'' \star_P \sigma''') = \Psi_{r+1}(\sigma'') \star \Psi_{r+1}(\sigma''') \);
2. \( \Psi_{r+1}(q_{\lambda, \mu} \star_P \sigma) = \Psi_{r+1}(q_{\lambda, \mu}) \star \Psi_{r+1}(\sigma) \);
3. \( \Psi_{r+1}(q_{\lambda, \mu} \star_P q_{\mu}) = \Psi_{r+1}(q_{\lambda, \mu}) \star \Psi_{r+1}(q_{\mu}) \).

To achieve the above proposition, we will need to show the vanishing of a lot of Gromov-Witten invariants occurring in certain quantum products in \( QH^*(G/B) \), and will need to calculate certain Gromov-Witten invariants, which turn out to be equal to 1.

Clearly, Theorem 2.4 follows immediately from the combination of the above propositions. The rest of this section is devoted to the proofs of these propositions. Here we would like to remind our readers of the following notation conventions:

(a) Whenever referring to an element \( \lambda_P \in Q^\vee/Q_P^\vee \), by \( \lambda_B \) we always mean the unique Peterson-Woodward lifting in \( Q^\vee \) defined in Proposition 2.2. Namely, \( \lambda_B \in Q^\vee \) is the unique element that satisfies \( \lambda_P = \lambda_B + Q_P^\vee \) and \( \langle \alpha, \lambda_B \rangle \in \{0, -1\} \) for all \( \alpha \in R_P^\vee \).

(b) We simply denote \( \hat{P} := P_{r-1} \). Namely, we have \( \Delta_{\hat{P}} := \{\alpha_1, \ldots, \alpha_{r-1}\} \).

(c) Whenever an element in \( \lambda \in Q^\vee \) is given first, we always denote \( \lambda_P := \lambda + Q_P^\vee \in Q^\vee/Q_P^\vee \) and \( \hat{\lambda}_P := \lambda + Q_P^\vee \in Q^\vee/Q_P^\vee \). Note that the three elements \( \lambda, \lambda_B \) and \( \hat{\lambda}_B \) (which is the Peterson-Woodward lifting of \( \hat{\lambda}_P \)) are all in \( Q^\vee \), and they may be distinct with each other in general.

### 3.1. Proofs of Proposition 3.1 and Proposition 3.2

In analogy with [16], we introduce the following notion with respect to the given pair \( (\Delta, \Delta_P) \).

**Definition 3.4.** An element \( \lambda \in Q^\vee \) is called a virtual null coroot, if \( \langle \alpha, \lambda \rangle = 0 \) for all \( \alpha \in \Delta_P \). An element \( \mu_P \in Q^\vee/Q_P^\vee \) is called a virtual null coroot, if its Peterson-Woodward lifting \( \mu_B \in Q^\vee \) is a virtual null coroot.

By the definition of \( gr \), every virtual null coroot \( \lambda \) satisfies

\[
gr_{[1, r]}(q_{\lambda}) = 0.
\]

**Example 3.5.** Suppose \( \alpha \in \Delta \) satisfies that \( Dyn(\{\alpha\} \cup \Delta_P) \) is disconnected. Then \( \alpha \in \Delta \setminus \Delta_P \), and \( \alpha^\vee \) is a virtual null coroot. Furthermore, for \( \lambda_P := \alpha^\vee + Q_P^\vee \in Q^\vee/Q_P^\vee \), we have \( \lambda_B = \alpha^\vee \). Therefore \( \lambda_P \) is also a virtual null coroot.
Lemma 3.6. Given \( \lambda_P, \mu_P \in Q^\vee / Q_P^\vee \), we denote \( \kappa_P := \lambda_P + \mu_P \). If \( \mu_P \) is a virtual null coroot, then we have \( \kappa_B = \lambda_B + \mu_B \). Consequently,

\[
\mathcal{L} := \{ \eta_P \in Q^\vee / Q_P^\vee \mid \eta_P \text{ is a virtual null coroot} \}
\]

is a sublattice of \( Q^\vee / Q_P^\vee \).

Proof. Clearly, \( \kappa_P = \kappa_B + Q_P^\vee \), and we have \( \langle \alpha, \kappa_B \rangle = \langle \alpha, \lambda_B \rangle \in \{0, -1\} \) for all \( \alpha \in R_P^+ \). Thus the statement follows from the uniqueness of the lifting. \( \square \)

We will let \( \mathcal{L}_B \) denote the set of virtual null coroots in \( Q^\vee \):

\[
\mathcal{L}_B := \{ \lambda \in Q^\vee \mid \langle \alpha, \lambda \rangle = 0, \forall \alpha \in \Delta_P \}.
\]

Denote by \( \Lambda^\vee \) the set of coweights of \( G \) and by \( \Lambda_P^\vee \) the set of coweights of the derived subgroup \( (L, L) \) of the Levi factor \( L \) of \( P \). Denote by \( \{ \omega^\vee_1, \ldots, \omega^\vee_r \} \) the fundamental coweights in \( \Lambda_P^\vee \) dual to \( \{ \alpha_1, \ldots, \alpha_r \} \). Denote by \( \partial \Delta_P \) the simple roots in \( \Delta \setminus \Delta_P \) which are adjacent to \( \Delta_P \). The next uniform description of the quotient \( (Q^\vee / Q_P^\vee ) / \mathcal{L} \) is provided by the referee.

Proposition 3.7. The quotient \( (Q^\vee / Q_P^\vee ) / \mathcal{L} \) is isomorphic to the subgroup of \( \Lambda_P^\vee / Q_P^\vee \) generated by

\[
\{ \omega^\vee_{i,P} \mid \alpha_i \text{ is adjacent to } \partial \Delta_P \}.
\]

Proof. Recall that \( \Lambda_P^\vee \) is the set of integral valued linear form on \( Q_P^\vee \). In particular, there is a natural morphism \( R : Q^\vee \to \Lambda_P^\vee \) obtained by restriction: \( R(\lambda) = \lambda|_{Q_P^\vee} \). We have \( R(Q_P^\vee) = Q_P^\vee \).

Furthermore, this map factors through the quotient \( Q^\vee \to \mathcal{L}_B \) and the induced map is injective. In particular, the quotient \( (Q^\vee / Q_P^\vee ) / \mathcal{L} \) is isomorphic to the image \( R(Q^\vee / Q_P^\vee ) \). Since for \( \alpha \in \Delta \) with \( \Delta_P \cup \{ \alpha \} \) disconnected we have \( \alpha^\vee \in \mathcal{L}_B \), it follows that \( R(Q^\vee / Q_P^\vee ) \) is the subgroup of \( \Lambda_P^\vee / Q_P^\vee \) generated by \( R(\alpha^\vee) \) for \( \alpha \in \partial \Delta_P \). But \( R(\alpha^\vee) = -\omega^\vee_{i,P} \) for \( \alpha_i \) adjacent to \( \alpha \) and the result follows. \( \square \)

Remark 3.8. The group \( \Lambda_P^\vee / Q_P^\vee \) is a finite abelian group. It is the center of simply-connected cover of \( (L, L) \), and is generated by the cominuscule coweights. One recovers this way the groups \( (Q^\vee / Q_P^\vee ) / \mathcal{L} \) in Table 3.1.

Recall that each monomial \( q_\lambda = q_{i_1}^{a_1} \cdots q_{i_n}^{a_n} \) corresponds to a coroot \( \lambda = \sum_{j=1}^n a_j \alpha^\vee_j \). Given a sequence \( I = [i_1, i_2, \ldots, i_m] \), we simply denote by \( s_{i_1 i_2 \cdots i_m} \) or \( s_I \) the product \( s_{i_1} s_{i_2} \cdots s_{i_m} \), and define \( |I| := m \).

Proposition 3.9. The virtual coroot lattice \( \mathcal{L}_B \) is generated by the virtual null roots \( \mu_B \in Q^\vee \) given in Table 3.1.

For each case in Table 3.2, the corresponding coroot \( \lambda \) satisfies \( \langle \alpha_k, \lambda \rangle = -1 \) for the given number \( k \) in the table, and \( \langle \alpha_j, \lambda \rangle = 0 \) for all \( j \in \{1, \ldots, r\} \setminus \{k\} \).

Furthermore, we have \( \psi_{\Delta, \Delta_P}(q_{\lambda B}) = q_{\lambda u}^{\sigma u} \) with \( q_\lambda \) and \( u \) being shown in Table 3.2 as well (which implies \( \lambda = \lambda_B \)). In particular, each \( u \) is of the form \( s_I s_{r-1} s_{r-2} \cdots s_1 \), \( s_I s_J \) or \( s_I \) where \( I \) (resp. \( J \)) is a sequence of integers ending with \( r \) (resp. \( r-1 \)) in the table. The grading \( gr(\sigma u) \) is then given by \( |I| e_r + \sum_{i=1}^{r-1} e_i, |I| e_r + |J| e_{r-1} \) and \( |J| e_r \), respectively.

Proof. Assume that case \( C1B \) occurs, then we have a unique \( \mu_B = 2\alpha_{r+1} + (\alpha^\vee_r + 2 \sum_{j=1}^{r-1} \alpha^\vee_j) \) and a unique \( \lambda = \alpha^\vee_{r+1} \) in the tables. Clearly, \( \langle \alpha, \mu_B \rangle = 0 \) for all
\[\begin{array}{|c|c|c|}
\hline
\mu_B & (Q^\vee/Q^\vee_P)/\mathcal{L} \\
\hline
C1B) & 2\alpha_{r+1}^{\vee} + (\alpha_r + 2 \sum_{i=1}^{r-1} \alpha_i^{\vee}) & \mathbb{Z}/2\mathbb{Z} \\
C1C) & \alpha_{r+1} + (\sum_{i=1}^{r-1} \alpha_i^{\vee}) & \{\text{id}\} \\
C2) & 2\alpha_{r+1}^{\vee} + (\alpha_r + \alpha_{r+1} + 2 \sum_{j=1}^{r-2} \alpha_j^{\vee}) & \mathbb{Z}/2\mathbb{Z} \\
r = 6 & 3\alpha_7^{\vee} + (4\alpha_6^{\vee} + 5\alpha_5^{\vee} + 6\alpha_3^{\vee} + 4\alpha_4^{\vee} + 2\alpha_5^{\vee} + 3\alpha_7^{\vee}) & \mathbb{Z}/3\mathbb{Z} \\
r = 7 & 2\alpha_8^{\vee} + (3\alpha_7^{\vee} + 4\alpha_6^{\vee} + 5\alpha_5^{\vee} + 6\alpha_3^{\vee} + 4\alpha_4^{\vee} + 2\alpha_5^{\vee} + 3\alpha_7^{\vee}) & \mathbb{Z}/2\mathbb{Z} \\
C4) & 4\alpha_6^{\vee} + (5\alpha_5^{\vee} + 6\alpha_3^{\vee} + 4\alpha_4^{\vee} + 2\alpha_5^{\vee} + 3\alpha_7^{\vee}) & \mathbb{Z}/4\mathbb{Z} \\
r = 4 & 2\alpha_8^{\vee} + (\alpha_1 + 2\alpha_2^{\vee} + \alpha_3^{\vee} + 2\alpha_4^{\vee}) & \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\
r = 5 & 2\alpha_8^{\vee} + \alpha_2^{\vee} + (2\alpha_2^{\vee} + 3\alpha_3^{\vee} + 4\alpha_4^{\vee} + 2\alpha_5^{\vee} + 3\alpha_7^{\vee}) & \mathbb{Z}/4\mathbb{Z} \\
r = 6 & 2\alpha_7^{\vee} + (\alpha_1 + 2\alpha_2^{\vee} + 3\alpha_3^{\vee} + 4\alpha_4^{\vee} + 2\alpha_5^{\vee} + 3\alpha_7^{\vee}) & \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \\
r = 7 & 4\alpha_7^{\vee} + (2\alpha_2^{\vee} + 4\alpha_4^{\vee} + 6\alpha_3^{\vee} + 8\alpha_4^{\vee} + 10\alpha_3^{\vee} + 5\alpha_4^{\vee} + 7\alpha_7^{\vee}) & \mathbb{Z}/4\mathbb{Z} \\
C7) & 2\alpha_7^{\vee} + (2\alpha_2^{\vee} + \alpha_3^{\vee} + 2\alpha_4^{\vee}) & \mathbb{Z}/2\mathbb{Z} \\
r = 2 & 2\alpha_7^{\vee} + (\alpha_1 + 2\alpha_2^{\vee} + \alpha_3^{\vee} + 2\alpha_4^{\vee}) & \mathbb{Z}/2\mathbb{Z} \\
C9) & 2\alpha_7^{\vee} + (2\alpha_2^{\vee} + 4\alpha_4^{\vee} + 3\alpha_3^{\vee}) & \mathbb{Z}/2\mathbb{Z} \\
C10) & 2\alpha_7^{\vee} + (\alpha_1 + 2\alpha_2^{\vee} + 3\alpha_3^{\vee}) & \mathbb{Z}/2\mathbb{Z} \\
\hline
\end{array}\]

\(\alpha \in \Delta_P\). Thus \(\mu_B\) is a virtual null coroot, and it is the expected Peterson-Woodward lifting of \(\mu_P := \mu_B + Q_P^\vee = 2\alpha_{r+1}^{\vee} + Q_P^\vee\). Hence, \(\mu_P\) is a virtual null coroot in \(Q^\vee/Q^\vee_P\) by definition. It follows from Example 3.5 that all the elements in the sublattice \(\mathcal{L}'\) generated by \(\{\mu_P\} \cup \{\alpha \in \Delta \mid D_{\text{Dyn}}(\{\alpha\} \cup \Delta_P)\) is disconnected) are virtual null coroots. Clearly, \((Q^\vee/Q_P^\vee)/\mathcal{L}' \cong \mathbb{Z}/2\mathbb{Z}\). Since \(\mathcal{L}' \subset \mathcal{L} \subset Q^\vee/Q^\vee_P\), we have a surjective morphism \((Q^\vee/Q_P^\vee)/\mathcal{L}' \to (Q^\vee/Q^\vee_P)/\mathcal{L} \cong \mathbb{Z}/2\mathbb{Z}\). Hence, this is an isomorphism, and \(\mathcal{L} = \mathcal{L}'\).

It is clear that for \(k = 1\), we have \(\langle \alpha_k, \lambda \rangle = -1\) and \(\langle \alpha_j, \lambda \rangle = 0\) for all \(j \in \{1, \cdots, r\} \setminus \{k\}\). Note that \(\Delta_P\) is of \(B_r\)-type, and that any positive root \(\gamma \in R_P^+\) is of the form \(\varepsilon \alpha_1 + \sum_{j=2} \varepsilon_j \alpha_j\) where \(\varepsilon \in \{0, 1\}\) (see e.g. [3]). Thus \((\gamma, \lambda) \in \{0, -1\}\). Hence, \(\lambda = \lambda_B\) is the Peterson-Woodward lifting of \(\lambda_P\). Consequently, \(\lambda_P\) is not a virtual null coroot, as \(\lambda_B\) is not. That is, our claim holds.

By definition, \(\psi_{\Delta_P}(\sigma_{\lambda_P}) = q_\lambda \sigma_{w_P w_P'}\) where \(\Delta_P = \Delta_P \setminus \{\alpha_k\}\) in this case. Note that \(w_P w_P'\) is the unique element of maximal length in \(W_P^F\), whose length is equal to \(|R_P^+| - |R_P^+|\). In order to show \(u = s_1 s_2 \cdots s_r s_{r-1} s_{r-2} \cdots s_1\) coincides with \(w_P w_P'\), it suffices to show: (1) the above expression of \(u\) is reduced of expected length; (2) \(u \in W_P^F\), i.e., \(u(\alpha) \in R^+\) for all \(\alpha \in \Delta_P\). Indeed, in the case of \(C1B\), \(\Delta_P = \{\alpha_2, \cdots, \alpha_r\}\) is of \(B_{r-1}\)-type. For \(1 \leq j \leq r, s_1 \cdots s_{j-1}(\alpha_j) = \alpha_1 + \cdots + \alpha_j \in R^+\). For \(r-1 \geq i \geq 1, s_1 \cdots s_{r-1} s_i (\alpha_i) = \alpha_1 + \cdots + \alpha_i + 2\alpha_{i+1} + \cdots + 2\alpha_r \in R^+\). Thus the expression of \(u\) is reduced, and \(\ell(u) = r + r - 1 = r^2 - (r-1)^2 = |R_P^+| - |R_P^+|\). For all \(2 \leq j \leq r\), we note \(u(\alpha_j) = \alpha_j \in R^+\). Therefore both (1) and (2) hold.

The expression \(u = s_1 s_{r-1} s_{r-2} \cdots s_1\), where \(I = [1, 2, \cdots, r]\), is reduced. Thus the subexpression \(s_1 = s_1 \cdots s_r\) is also reduced. Clearly, \(s_1 \in W_{P'}^{F-1}\), and \(s_1(\alpha_j) \in R^+\) for all \(\alpha_j \in \Delta_{P_{r-1}} = \{\alpha_1, \cdots, \alpha_{r-1}\}\), which implies \(s_1 \in W_{P'}^{F-1}\). Hence, \(gr(\sigma^u) = \)}
\[
\ell(s_i)e_r + \sum_{i=1}^{r-1} \ell(s_i)e_i = |I|e_r + \sum_{i=1}^{r-1} e_i.
\]

The arguments for the remaining cases are all the same. □

**Remark 3.10.** We obtain both tables using case by case analysis, which gives an alternative proof of Proposition 3.7 by studying the quotient \((Q^\vee / Q'_P) / \mathcal{L}'\) first.

**Lemma 3.11.** Let \(\lambda_P, \mu_P \in Q^\vee / Q'_P\). Write \(\psi_{\Delta, \Delta_P}(q_{\lambda_P}) = q_{\lambda_B} \sigma^u\). If \(\mu_P\) is a virtual null coroot, then we have

\[
\psi_{\Delta, \Delta_P}(q_{\mu_P}) = q_{\mu_B} \quad \text{and} \quad \psi_{\Delta, \Delta_P}(q_{\lambda_P + \mu_P}) = q_{\lambda_B + \mu_B} \sigma^u.
\]

Consequently, we have

\[
gr_{[1,r]}(\psi_{\Delta, \Delta_P}(q_{\mu_P})) = 0 \quad \text{and} \quad gr_{[1,r]}(\psi_{\Delta, \Delta_P}(q_{\lambda_P + \mu_P})) = gr_{[1,r]}(\psi_{\Delta, \Delta_P}(q_{\lambda_P})).
\]

**Proof.** For \(\kappa_P \in Q^\vee / Q'_P\), by definition we have \(\psi_{\Delta, \Delta_P}(q_{\kappa_P}) = q_{\kappa_B} \sigma^{w_P w_P'}\) with \(\Delta_P' = \{\alpha \in \Delta_P \mid (\alpha, \kappa_B) = 0\}\). If \(\kappa_P = \mu_P\), then \(\Delta_P' = \Delta_P\) since \(\mu_P\) is a virtual null coroot. Thus \(w_P w_P' = \text{id}\) and consequently \(\psi_{\Delta, \Delta_P}(q_{\mu_P}) = q_{\mu_B}\). If \(\kappa_P = \lambda_P + \mu_P\),

\[
|P| = 1
\]

\[
|P'| = 1
\]

\[
|P''| = 1
\]

\[
|P'''| = 1
\]

\[
|P''''| = 1
\]
then $\kappa_B = \lambda_B + \mu_B$. Write $\Delta_{P'} = \{ \alpha \in \Delta_P \mid \langle \alpha, \kappa_B \rangle = 0 \} = \{ \alpha \in \Delta_P \mid \langle \alpha, \lambda_B \rangle = 0 \}$. That is, we have $u = w_P w_{P'}$ and $\psi_{\Delta, \Delta_{P'}}(q_{\kappa_P}) = q_{\kappa_B} \sigma^u$. The two identities on the gradings are then a direct consequence. \(\Box\)

**Proof of Proposition 3.1.** The initial proof used case by case analysis with Table 3.2. Here we provide a uniform proof from the referee.

To prove the statement using Lemma 3.11, we only need to prove that

$$gr_{[1,r]}(\psi_{\Delta, \Delta_{P'}}(q_{\alpha^\vee + Q_{P'}})) = 0$$

for $\alpha \in \partial \Delta_{P'}$.

Let $\alpha \in \partial \Delta_{P'}$ and let $\alpha_i$ be the unique element in $\Delta_{P'}$ adjacent to $\alpha$. We have $\langle \alpha_i, \alpha^\vee \rangle \neq 0$. By definition we have $gr_{[1,r]}(\psi_{\Delta, \Delta_{P'}}(q_{\alpha^\vee + Q_{P'}})) = gr_{[1,r]}(w_{P'}, \alpha^\vee)$ and

$$gr_{[1,r]}(w_{P'}, \alpha^\vee) = \sum_{j=1}^r \left( |\text{Inv}(w_{P'}) \cap (R_{j}^+ \setminus R_{j-1}^+)\right) + \sum_{\beta \in R_{j}^+ \setminus R_{j-1}^+} \langle \beta, \alpha^\vee \rangle \right) e_j.$$

We first remark that the above grading does only depend on the restriction of $\alpha^\vee$ to $\Delta_{P'}$ so on $R(\alpha^\vee) = -\omega_i^{\vee}_{P'}$ as defined in the proof of Proposition 3.7. For $w \in W_P$ and $\lambda \in \Lambda_{P'}$, we define

$$gr_{[1,r]}(w, \lambda) = \sum_{j=1}^r \left( |\text{Inv}(w_{P'}) \cap (R_{j}^+ \setminus R_{j-1}^+)\right) + \sum_{\beta \in R_{j}^+ \setminus R_{j-1}^+} \langle \beta, \lambda \rangle \right) e_j.$$

For $a = \sum_{j=1}^r a_j e_j$, define

$$||a|| := \sum_{j=1}^r |a_j|.$$

Next remark (see Corollary 3.13 of [7]) that for $w \in W_P$ and $\lambda \in \Lambda_{P'}$, we have

$$\ell(wt\lambda) = ||gr_{[1,r]}(w, \lambda)||$$

where $\ell$ denotes the length function on $\tilde{W}_{aff}$ the extended affine Weyl group and where we consider the element $wt\lambda$ as an element of the extended affine Weyl group $\tilde{W}_{aff}$ (see Definition 3.9 of [7]).

Now for $P'$ defined by $\Delta_{P'} = \{ \beta \in \Delta_{P} \mid \langle \beta, \omega_i^{\vee}_{P'} \rangle = 0 \}$, the element

$$\tau_i := w_{P'} t_{-\omega_i^{\vee}_{P'}}$$

is the element $\tau_i$ defined on page 9 of [7]. In particular this element satisfies $\ell(\tau_i) = 0$ (since this element is in the stabiliser of the fundamental alcove, see also page 5 of [15]). As a consequence we get

$$||gr_{[1,r]}(\psi_{\Delta, \Delta_{P'}}(q_{\alpha^\vee + Q_{P'}}))|| = 0 \quad \text{and} \quad gr_{[1,r]}(\psi_{\Delta, \Delta_{P'}}(q_{\alpha^\vee + Q_{P'}})) = 0.$$

\(\Box\)

To prove Proposition 3.2, we need the next lemma.

**Lemma 3.12** (Lemma 4.1 (1) of [16]). For any $d = \sum_{i=1}^{r-1} d_i e_i \in \mathbb{Z}^{r-1} \times \{(0,0)\} \subset \mathbb{Z}^{r+1}$, there exists a unique $(w, \eta) \in W \times Q^\vee$ such that $gr(q_{\eta} \sigma^w) = d$.\(\Box\)
Proof of Proposition 3.2. Since \( \psi_{\Delta, \Delta_P} \) is injective, so is \( \Psi_{r+1} \).

For a nonzero element \( \bar{q}_{\mu} \sigma^w \in G_P^{(r+1)} \), we write \( w = vu \) where \( v \in W^P \) and \( u \in W_P \), and write \( \mu = \mu' + \mu'' \), where \( \mu' \in \bigoplus_{i=1}^{r-1} \mathbb{Z}_{\geq 0} \alpha_i^{\vee} \) and \( \mu'' \in \bigoplus_{i=1}^r \mathbb{Z}_{\geq 0} \alpha_i^{\vee} \). Note \( gr_{[1,r]}(q_{\mu} \sigma^v) \in \bigoplus_{i=1}^r \mathbb{Z}_{\leq 0} e_i \), and \( 0 = gr_{[1,r]}(q_{\mu} \sigma^w) = gr_{[1,r]}(q_{\mu'} \sigma^u) + gr_{[1,r]}(q_{\mu''} \sigma^v) \). Thus we have \( gr_{[1,r]}(q_{\mu'} \sigma^u) \in \bigoplus_{i=1}^r \mathbb{Z}_{\geq 0} e_i \). Setting \( \lambda_P := \mu + Q_P \), we have \( \psi_{\Delta, \Delta_P}(q_{\lambda_P} \sigma^v) = q_{\lambda_B} \sigma^{w_P \cdot w_P} \) with \( \lambda_B = \mu' + \lambda'' \) for some \( \lambda'' \in \bigoplus_{i=1}^r \mathbb{Z}_{\geq 0} \alpha_i^{\vee} \). Note \( gr_{[1,r]}(q_{\lambda_B} \sigma^{w_P \cdot w_P}) = gr_{[1,r]}(q_{\lambda'} \sigma^{w_P \cdot w_P} + gr_{[1,r]}(q_{\mu''} \sigma^v) \), and it is equal to \( 0 \) by Proposition 3.1. Hence, \( d := gr_{[1,r]}(q_{\lambda'} \sigma^{w_P \cdot w_P}) = gr_{[1,r]}(q_{\mu''} \sigma^v) \in \bigoplus_{i=1}^r \mathbb{Z}_{\geq 0} e_i \). Thus the map \( \Psi_{r+1} \) is surjective as soon as there is a unique element \( q_{\mu''} \sigma^u \) of grading \( d \).

Suppose \( r = 2 \) and either case C1B or case C9 occurs. Note \( gr(q_1) = 2e_1 \), \( gr(q_2) = 4e_2 - 2e_1 \), and \( u = u_2u_1 \) for a unique \( u_1 \in W_{P_1} = \{1, s_1\} \) and \( u_2 \in W_{P_2} = \{1, s_2, s_1s_2, s_2s_1s_2\} \). Note for given \( d_1, d_2 \geq 0 \), the next equalities

\[
d_1e_1 + d_2e_2 = gr_{[1,r]}(q_{s_1s_2^{-1}} \sigma^u) = a_1 \cdot gr(q_1) + a_2 \cdot gr(q_2) + \ell(u_1)e_1 + \ell(u_2)e_2
\]
determine a unique \( (a_1, a_2, \ell(u_1), \ell(u_2)) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \{0, 1\} \times \{0, 1, 2, 3\} \). The pair \( (\ell(u_1), \ell(u_2)) \) further determines a unique \( (u_1, u_2) \in W_{P_1} \times W_{P_2} \). Hence, \( q_{\mu''} \sigma^u = q_{\mu''} \sigma^{w_P \cdot w_P} \) follows from the uniqueness.

In order to show \( \Psi_{r+1} \) is not surjective for the remaining cases, it suffices to consider the virtual null roots \( \mu_P \) in Proposition 3.7, for which we note \( \Psi_{r+1}(\mu_P) = \bar{q}_{\mu_P} \). The point is to show \( gr_{[r,r]}(q_r) \leq \ell(w_P w_P^{-1}) e_r \). Once this is done, we show the existence of \( q_{r_{r-1}}^{r-r-1} \sigma^{w_P} \) satisfying \( gr_{[r,r]}(q_r^{r-r-1} \sigma^{w_P}) = gr_{[r,r]}(q_{\lambda_B} \sigma^{w_P \cdot w_P}) \), where \( u_r \in W_{P_{r-1}} \), and \( a_r \) denotes the power of \( q_r \) in the monomial \( q_{\lambda_B} \). Then we apply Lemma 3.12 to construct an element in \( W_{P_{r-1}} \times \bigoplus_{i=1}^{r-1} \mathbb{Z}_{\geq 0} \alpha_i^{\vee} \) with grading \( gr_{[1, r-1]}(q_{\lambda_B} \sigma^{w_P \cdot w_P}) = gr_{[1, r-1]}(q_{r_{r-1}}^{r-r-1} \sigma^{w_P}) \). In this way, we obtain an element of the same grading as \( gr(q_{\lambda_B} \sigma^{w_P \cdot w_P}) \) that is not in the image of \( \Psi_{r+1} \). Precise arguments are given as follows.

For case C1C, we have \( \mu_P = \alpha_{r+1}^\vee + Q_P^r \) and \( \mu_B = \alpha_1^\vee + \cdots + \alpha_r^\vee \). Note \( gr_{[r,r]}(q_r) = (r+1)e_r \), and \( w_P w_P^{-1} \) is the longest element in \( W_{P_{r-1}} \), which is of length \( \ell(w_P w_P^{-1}) = |R_P^r| - |R_P^{r-1}| = r^2 - \frac{(r-1)r}{2} = \frac{r^2 + r}{2} \geq r + 1 \). Hence, there exists \( u_r \in W_{P_{r-1}} \) of length \( r + 1 \). Note for each \( 1 \leq j \leq r-1 \), \( u_j := s_j \in W_{P_{j-1}} \). Thus \( gr(q_{r+1} \sigma_{r-r-1}^{u_r \cdots u_2 u_1}) = (2r + 2) e_{r+1} = gr(q_1 \cdots q_{r+1}) \). However, \( q_{r+1} \sigma_{r-r-1}^{u_r \cdots u_2 u_1} \notin \psi_{\Delta, \Delta_P}(QH^*(G/B)) \).

For case C1B with \( r \geq 3 \), we have \( \mu_P = 2\alpha_{r+1}^\vee + Q_P^r \) and \( \mu_B = 2\alpha_{r+1}^\vee + \alpha_r^\vee + 2 \sum_{j=1}^{r-1} \alpha_j^\vee \). Note \( gr_{[r,r]}(q_r) = 2r e_r \), and \( \ell(w_P w_P^{-1}) = \frac{r(r+1)}{2} \geq 2r \). Hence, there exists \( u_r \in W_{P_{r-1}} \) of length \( 2r \). Set \( u_j = s_j \cdots s_1 \) for \( 2 \leq j \leq r-1 \). Then \( gr_{[r,r]}(q_{r+1} q_1 \sigma_{u_r \cdots u_2}) = 0 \) so that \( q_{r+1} q_1 \sigma_{u_r \cdots u_2} \in G_P^{(r+1)} \). However, \( q_{r+1} q_1 \sigma_{u_r \cdots u_2} \notin \psi_{\Delta, \Delta_P}(QH^*(G/B)) \).

The arguments for the remaining cases are also easy and similar. \( \square \)

3.2. Proof of Proposition 3.3 (1). For \( v', v'' \in W^P \), we note

\[
\Psi_{r+1}(v') = \Psi_{r+1}(v') \ast \Psi_{r+1}(v'') = \Psi_{r+1}(v' \ast v'') = \sum N_{\nu', \nu''}^{w', \lambda} q_{\lambda} \sigma^w,
\]
the summation over those \( q_{\lambda} \sigma^w \in QH^*(G/B) \) satisfying \( gr_{[1,r]}(q_{\lambda} \sigma^w) = 0 \) and \( q_{\lambda} \sigma^w \notin \psi_{\Delta, \Delta_P}(QH^*(G/B)) \). It suffices to show the vanishing of all the coefficients \( N_{\nu', \nu''}^{w', \lambda} \) (if any). In particular, it is already done, if \( \Psi_{r+1} \) is an isomorphism of vector spaces. Therefore, if \( r = 2 \), then both C1B and C9 could be excluded in the rest of this subsection.
To do this, we will use the same idea occurring in section 3.5 of [16]. Namely, we consider the fibration $G/B \to G/\tilde{P}$ where $\Delta_{\tilde{P}} = \{\alpha_1, \cdots, \alpha_{r-1}\}$. Set $\varepsilon := r - 1$, and note that $\Delta_{\tilde{P}}$ is of $A$-type satisfying the assumption on the ordering as in [16]. Using Definition 2.7 with respect to $(\Delta, \Delta_{\tilde{P}})$, we have a grading map

$$
g r : W \times Q^\vee \longrightarrow \mathbb{Z}^{r+1} = \bigoplus_{i=1}^r \mathbb{Z} e_i \hookrightarrow \mathbb{Z}^{r+1},$$

which satisfies the next obvious property

$$
g r_{r-1} = g r_{r-1}.$$

Consequently, we obtain a filtration $\tilde{F}$ on $Q H^* (G/B)$ and a (well-defined) induced map $\tilde{\Psi}_{r-1} : Q H^* (G/\tilde{P}) \to Gr_{r-1} \subset Gr_{r-1} (Q H^* (G/B))$ as well. Furthermore, all the results of [16] hold with respect to the fibration $G/B \to G/\tilde{P}$. In particular, we have the next proposition (which follows immediately from Theorem 1.6 of [16]).

**Proposition 3.13.** Let $u, v \in W \tilde{P}$ and $w \in W \tilde{P}$. In $Gr_{r-1} (Q H^* (G/B))$, we have

1. $\sigma^u \star \sigma^v = \psi_{\Delta, \Delta_{\tilde{P}}} (\sigma^u \star \sigma^v)$;
2. $\sigma^u \star \sigma^w = \sigma^{uw}.$

**Lemma 3.14.** For any $u, v \in W \tilde{P}$, we have in $Q H^* (G/B)$ that

$$
\sigma^u \star \sigma^v = \sum N_{u, v}^{w, \lambda} q_{\lambda} \sigma^w + \sum N_{u, v}^{w', \lambda'} q_{\lambda'} \sigma^{w'} + \sum N_{u, v}^{w'', \lambda''} q_{\lambda''} \sigma^{w''},
$$

where the first summation is over those $q_{\lambda} \sigma^w \in \psi_{\Delta, \Delta_{\tilde{P}}} (Q H^* (G/P))$, the second sumation is over those $q_{\lambda'} \sigma^{w'} \in \psi_{\Delta, \Delta_{\tilde{P}}} (Q H^* (G/P)) \setminus \psi_{\Delta, \Delta_{\tilde{P}}} (Q H^* (G/P))$, and the third summation is over those $q_{\lambda''} \sigma^{w''}$ satisfying $g r_{r-1} (\lambda, \sigma^{w''}) < 0$.

**Proof.** Since $\Delta_{\tilde{P}} \subset \Delta_P$, we have $u, v \in W \tilde{P}$. By Proposition 3.13(1), we have

$$
\sigma^u \star \sigma^v = \sum N_{u, v}^{w, \lambda} q_{\lambda} \sigma^w + \sum N_{u, v}^{w', \lambda'} q_{\lambda'} \sigma^{w'} + \sum N_{u, v}^{w'', \lambda''} q_{\lambda''} \sigma^{w''},
$$

where

- If $q_{\lambda} \sigma^w \in \psi_{\Delta, \Delta_{\tilde{P}}} (Q H^* (G/P))$, then $\lambda = \lambda_B = \lambda + Q \vee$ and $w = w_1 w_2 w_{p-2}$ with $w_1$ being the minimal length representative of the coset $w W P$. Since $R^+_p \subset R^+_p$, $\lambda_B = \lambda_B = \lambda + Q \vee$. Note that $\Delta_{\tilde{P}} = \{\alpha \in \Delta_{\tilde{P}} | \langle \alpha, \lambda_B \rangle = 0\} \subset \{\alpha \in \Delta | \langle \alpha, \lambda_B \rangle = 0\} = \Delta_{\tilde{P}}$. Thus $\text{Inv} (w_1 w_2 w_{p-2}) = R^+_p \setminus R^+_p \cap (R^+_p \setminus R^+_p) \cup (R^+_p \setminus R^+_p)$. Hence, we have $w_1 w_2 w_{p-2} = w_2 w_1 w_{p-2}$, where $w_2$ is the minimal length representative of the coset $w W P = w_1 w_2 w_{p-2}$, for which we have $\text{Inv} (w_2) = (R^+_p \setminus R^+_p) \cup (R^+_p \setminus R^+_p)$. Note $w_1 w_2 \in W \tilde{P}$. Thus we have $q_{\lambda} \sigma^w = \psi_{\Delta, \Delta_{\tilde{P}}} (q_{\lambda} \sigma^{w_1 w_2}) \in \psi_{\Delta, \Delta_{\tilde{P}}} (Q H^* (G/P))$. Therefore the statement follows by noting $gr_{r-1} = gr_{r-1}$. 

Due to the above lemma, it remains to show that for any element $q_{\lambda} \sigma^{w'}$ in $\psi_{\Delta, \Delta_{\tilde{P}}} (Q H^* (G/P))$, either $N_{u, v}^{w', \lambda'} = 0$ or $gr (q_{\lambda} \sigma^{w'}) < 0$ holds. The latter claim could be further simplified as $gr (q_{\lambda} \sigma^{w'}) < 0$, by noting $gr_{r-1} (q_{\lambda} \sigma^{w'}) = 0$. For this purpose, we need the next main result of [17], which
is in fact an application of [16] in the special case of $P/B \cong \mathbb{P}^1$. For each $\alpha \in \Delta$, we define a map $\text{sgn}_\alpha : W \to \{0, 1\}$ by $\text{sgn}_\alpha(w) := 1$ if $\ell(w) - \ell(ws_\alpha) > 0$, and 0 otherwise.

**Proposition 3.15** (Theorem 1.1 of [17]). Given $u, v, w \in W$ and $\lambda \in Q^\vee$, we have

1. $N_{w, \lambda}^w = 0$ unless $\text{sgn}_\alpha(w) + \langle \alpha, \lambda \rangle \leq \text{sgn}_\alpha(u) + \text{sgn}_\alpha(v)$ for all $\alpha \in \Delta$.
2. Suppose $\text{sgn}_\alpha(w) + \langle \alpha, \lambda \rangle = \text{sgn}_\alpha(u) + \text{sgn}_\alpha(v) = 2$ for some $\alpha \in \Delta$, then

$$N_{w, \lambda}^w = N_{u, \lambda}^w - \alpha^\vee x,$$

where $x = 0$ if $\text{sgn}_\alpha(w) = 0$ and $x = 1$ if $\text{sgn}_\alpha(w) = 1$.

**Corollary 3.16.** Let $u, v \in W^P$. Suppose $N_{u, v}^w, \lambda \neq 0$ for some $w \in W$ and $\lambda \in Q^\vee$. Then we have

1. $\langle \alpha, \lambda \rangle \leq 0$ for all $\alpha \in \Delta_P$;
2. Set $\lambda_p := \lambda + Q^+_P$, and denote by $w_1$ the minimal length representative of the coset $\Delta \alpha P/B$. If $\lambda = \lambda_B$ and $gr_{[1, r]}(q_\lambda^w) = 0$, then $q_\lambda^w = \psi_{\Delta, P}(q_\lambda^w)$.

**Proof.** Assume $\langle \alpha, \lambda \rangle > 0$ for some $\alpha \in \Delta_p$, then we have $\text{sgn}_\alpha(u) + \text{sgn}_\alpha(v) = 0 < \langle \alpha, \lambda \rangle \leq \text{sgn}_\alpha(w) + \langle \alpha, \lambda \rangle$. Thus $N_{u, v}^w, \lambda = 0$ by Proposition 3.15 (1), contradicting with the hypothesis.

Since $N_{u, v}^w, \lambda \neq 0$, we have $\text{sgn}_\alpha(w) = 0$ for any $\alpha \in \Delta_p = \{\beta \in \Delta_p | \langle \beta, \lambda_B \rangle = 0\}$, following from Proposition 3.15 (1) again; that is, $w(\alpha) \in R^+$. Thus $w \in W^P$ and consequently $w_1 = w_2 \in W^P_r$. Since $gr_{[1, r]}(q_\lambda^w) = gr_{[1, r]}(q_\lambda^w) = gr_{[1, r]}(q_\lambda^w) = gr_{[1, r]}(q_\lambda^w) = w_2$. Since $w_2, w_3, w_4 \in W^P$, $gr_{[1, r]}(q_\lambda^w) = 0 = gr_{[1, r]}(q_\lambda^w)$. Therefore $\langle \lambda, \lambda \rangle = |gr(w_2)| = |gr(w_3, w_4)| = |gr(w_2)|$. Hence, $w_2 = w_3, w_4$ by the uniqueness of elements of maximal length in $W^P$. Thus the statement follows.

**Lemma 3.17.** Let $u, v \in W^P$. Suppose $N_{u, v}^w, \lambda \neq 0$ for some $w \in W$ and $\lambda \in Q^\vee$. Assume $gr_{[1, r]}(q_\lambda^w) = 0$ and $\lambda \neq \lambda_B$ where $\lambda_B := \lambda + Q^+_P$. Then we have

$$gr_{[r, r]}(q_\lambda) = (R^+_P \cup R^+_P) e_r$$

where $\Delta_p := \{\alpha \in \Delta_p | \langle \alpha, \lambda \rangle = 0\}$.

**Proof.** Write $\lambda = \sum_{j=1}^n a_j \alpha_j^\vee, gr_{[r, r]}(q_{\alpha}) = x e_r$ and $gr_{[r, r]}(q_{r+1}) = y e_r$. Whenever $r + 2 \leq n$, we denote $gr_{[r, r]}(q_{r+2}) = z e_r$. Note $gr_{[r, r]}(q_{\alpha}) = (x_a + y a_{r+1} + z a_{r+2}) e_r$ (where $z = 0$ unless case C7) occurs with $r \leq 6$). Let $\varepsilon_j = -\langle \alpha_j, \lambda \rangle, j = 1, \ldots, r$.

Note that Proposition 3.3 holds with respect to $QH^*(G/P)$ and $gr_{[1, r]}(q_\lambda^w) = gr_{[1, r]}(q_\lambda^w) = 0$. Hence, $\lambda$ is the unique Peterson-Woodward lifting of $\lambda + Q^+_P \in Q^\vee/Q^+_P$ to $Q^\vee$. Thus $\langle \gamma, \lambda \rangle \in \{0, -1\}$ for all $\gamma \in R^+_P$. Consequently, $\varepsilon_j = 0$ for all $j$ in $\{1, \ldots, r - 1\}$ with at most one exception, and if there exists such an exception, say $k$, then $\varepsilon_k = 1$. Furthermore, we have $\varepsilon_r \geq 0$, by noting $N_{u, v}^w, \lambda \neq 0$ and using Corollary 3.16.

Assume case C1B (resp. case C1C) occurs, then we have $-2y = x = 2r$ (resp. $y = x = r + 1$) and $z = 0$. In this case, we note $a_{r+1} + \frac{x}{y} a_r = \sum_{j=1}^r j \varepsilon_j$ (resp. $\frac{x}{y} a_r + \sum_{j=1}^{r-1} j \varepsilon_j$ where $\varepsilon_r = 2a_{r-1} - 2a_r$ is even). If $\varepsilon_r > 0$, then we have $(y a_{r+1} + x a_r) \geq y r > \frac{r(r+1)}{2} = r^2 - \frac{(r-1)r}{2} = |R^+_P| - |R^+_P| \geq R^+_P - R^+_P$. If
$\varepsilon_r = 0$, then there exists such an exception $k$ with $2 \leq k \leq r-1$ (resp. $1 \leq k \leq r-1$).
(For case C1B), each positive root in $R^+_P$ is of the form $\gamma = \varepsilon + \sum_{j=2}^r b_j \alpha_j$ where $\varepsilon = 0$ or 1. If $k = 1$, it would imply that $\lambda = \lambda_B$ is the Peterson-Woodward lifting of $\lambda_P$, contradicting with the hypothesis.) Consequently, we have $|R^+_P| - |R^+_P \cup R^+_P| = |R^+_P| - |R_P^+| + |R^+_P \setminus (\alpha_k)| = r^2 - \frac{r(r-1)}{2} - (k-1)^2 + (k-1)^2 + \frac{(r-k+1)(r-k)}{2} = kr - \frac{k(k-1)}{2} - yk = -(ya_{r+1} + xa_r).

Assume case C2) occurs, then we have $-2y = x = 2(r-1)$ and $z = 0$. Note $a_{r+1} + \frac{x}{2}a_r = \frac{z}{2} \varepsilon_r + \frac{r-2}{2} \varepsilon_{r-1} + \sum_{j=1}^{r-2} j \varepsilon_j$, and $\varepsilon_r - \varepsilon_{r-1} = 2a_{r-1} - 2a_r \equiv 0$ (mod 2).
If $\varepsilon_r > 0$, then $-ya_{r+1} + xa_r \leq (r-1)(\frac{x}{2} \varepsilon_r + \frac{r-2}{2} \varepsilon_{r-1} + 0) \leq (r-1)^2 > \frac{r(r-1)}{2} = |R^+_P| - |R^+_P| + |R^+_P \cup R^+_P|$. If $\varepsilon_r = 0$, then there exists such an exception $k$ with $2 \leq k \leq r-2$ (since $\lambda \neq \lambda_B$). Consequently, we have $|R^+_P| - |R^+_P \cup R^+_P| = kr - \frac{k(k+1)}{2} < r(k-1) = -(ya_{r+1} + xa_r).

For the remaining cases, the arguments are all similar, and the details will be given in section 5.1.

Proof of Proposition 3.3 (1). Since $QH^*(G/B)$ is an $S$-filtered algebra, we have $gr_{[1,r]}(q^w_\sigma w) \leq gr_{[1,r]}(\sigma w^\lambda) + gr_{[1,r]}(\sigma w^\lambda) = 0$ if $N^\lambda_\nu \neq 0$. Due to Lemma 3.14, it is sufficient to show $gr_{[1,r]}(q_\lambda w_\sigma) < 0$ whenever both $N_\nu \mu,\nu \neq 0$ and $q_\lambda w_\sigma \in \psi_{\Delta_\alpha}(QH^*(G/P)) \setminus \psi_{\Delta_\alpha}(QH^*(G/P))$ hold. For the latter hypothesis, we only need to check that either of the following holds: (a) $gr_{[1,r]}(q_\lambda w_\sigma) = 0$, $\lambda \neq \lambda_B$; (b) $gr_{[1,r]}(q_\lambda w_\sigma) = 0$, $\lambda = \lambda_B$, $w \neq w_1 w_{P} w_{P}$, where $w_1$ is the minimal length representative of the coset $w_1 w_{P}$. If (b) holds, then it is done by Corollary 3.16 (2). Write $w = w_1 w_2$ where $w_1 \in W_P$ and $w_2 \in W_P$. By Proposition 3.15 (1), we conclude $w_2(\alpha) \in R^+$ whenever $\alpha \in \Delta_\alpha = \{ \beta \in \Delta_\alpha | \langle \beta, \lambda \rangle = 0 \}$. Thus $w_2 \in W_P$. Hence, $gr_{[1,r]}(w_2) = |Inv(w_2) \cap (R^+_P \setminus R^+_P)|_{e_\alpha} \leq |(R^+_P \setminus R^+_P) \cap (R^+_P \setminus R^+_P)|_{e_\alpha} = |(R^+_P \setminus R^+_P) \cup R^+_P|_{e_\alpha}$. Thus if (a) holds, then the statement follows as well, by noting $gr_{[1,r]}(q_\lambda w_\sigma) = gr_{[1,r]}(q_\lambda) + gr_{[1,r]}(\sigma w_\sigma)$ and using Lemma 3.17.

3.3. Proof of Proposition 3.3 (2). The statement to prove is a direct consequence of the next proposition.

PROPOSITION 3.18. Let $u \in W_{P}$ and $v \in W_{P}$. In $QH^*(G/B)$, we have
\[
\sigma^u \star \sigma^u = \sigma^{uvu} + \sum_{w, \lambda} b_{w, \lambda} q_{\lambda} \sigma^u
\]
with $gr(q_{\lambda} \sigma^u) < 0$ whenever $w_{b, \lambda} \neq 0$.

REMARK 3.19. Proposition 3.18 here extends Proposition 3.23 of [16] in the case of parabolic subgroups $P$ such that $\Delta_\alpha$ is not of type A. In Proposition 3.23 of [16], the same property for $\sigma^u \star \sigma^u$ was discussed, under the assumptions that $\Delta_\alpha$ is of type A, $s_j \in W_{P}$ and $v \in W_{P}$. By modifying the proof therein slightly, the assumption “$v \in W_{P}^+$” could be generalized to “$v \in W$ with $gr_{[1,j]}(v) \leq je_j$”.

Proof of Proposition 3.3 (2). This follows immediately from Proposition 3.18:
\[
\Psi_{r+1}(q_{\lambda_P} \star_P \sigma^u) = q_{\lambda_P} \sigma^{w_{P}w_{P}w_{P}^\prime} = q_{\lambda_P} \star \sigma^u \star \sigma^{w_{P}w_{P}w_{P}^\prime} = \Psi_{r+1}(q_{\lambda_P}) \star \Psi_{r+1}(\sigma^u).
\]
To show Proposition 3.18, we prove some lemmas first.

**Lemma 3.20.** Let $v \in W^P$ and $u \in W_P$. Take any $w \in W$ and $\lambda \in Q^\vee$ satisfying $\text{gr}(q_\lambda \sigma^u) = \text{gr}(\sigma^u) + \text{gr}(\sigma^v)$. If $\lambda$ is a virtual null coroot, then we have

$$N_{v,u}^{w,\lambda} = \begin{cases} 1, & \text{if } (w, \lambda) = (vu, 0) \\ 0, & \text{otherwise} \end{cases}.$$  

**Proof.** Write $w = w_1w_2$ where $w_1 \in W^P$ and $w_2 \in W_P$. Take a reduced expression $w_2 = s_{i_1} \cdots s_{i_m}$ (i.e., $\ell(w_2) = m$). Since $v \in W^P$ and $\lambda$ is a virtual null coroot, we have $\text{sgn}_\alpha(v) = 0 = \langle \alpha, \lambda \rangle$ for all $\alpha \in \Delta_P$. Note $\alpha_{i_j} \in \Delta_P$ and $\text{sgn}_{\alpha_{i_j}}(w_1s_{i_1} \cdots s_{i_j}) = 1$ for all $1 \leq j \leq m$. Applying the tuple $(u,v,w,\lambda,\alpha)$ of Proposition 3.15 (2) to the case $(us_{i_m} \cdots s_{i_{j+1}}, vs_{i_j}, w_1s_{i_1} \cdots s_{i_j}, \lambda + \alpha_{i_j}, \alpha_{i_j})$, we have $N_{v,u}^{w_1s_{i_1} \cdots s_{i_j}, \lambda} = N_{v,u}^{w_1s_{i_1} \cdots s_{i_{j-1},i_j}, \lambda}$ if $\ell(us_{i_m} \cdots s_{i_{j+1}}s_{i_j}) = \ell(us_{i_m} \cdots s_{i_{j+1}}) - 1$, or $0$ otherwise. Hence, we have $N_{v,u}^{w,\lambda} = N_{v,u}^{w_1,\lambda}$ if $\ell(u) = \ell(w) - \ell(w_2)$, or $0$ otherwise.

Note that $\ell(u) = \ell(u) = |gr(u)| = |gr[1,r](u) + gr[1,r](v)| = |gr[1,r](w)| = |gr[1,r](w)| = \ell(w_2)$. When $\ell(u \cdot \omega_2^{-1}) = 0$, we have $u \cdot \omega_2^{-1} = \text{id}$, and consequently $N_{v,u}^{w,\lambda} = 1$ if $(w, \lambda) = (v, 0)$, or $0$ otherwise. Thus the statement follows. □

Recall $\Delta_P = \{\alpha_1, \ldots, \alpha_{r-1}\}$, whose Dynkin diagram is of type $A_{r-1}$. It is easy to see the next combinatorial fact (see e.g. Lemma 2.8 and Remark 2.9 of [17]).

**Lemma 3.21.** Let $\lambda \in Q^\vee$ be a nonzero effective coroot, i.e., $\lambda = \sum_{i=1}^n a_i \alpha_i^\vee \neq 0$ satisfies $a_i \geq 0$ for all $i$. Then there exists $\alpha \in \Delta$ such that $\langle \alpha, \lambda \rangle > 0$. Furthermore if $a_i = 0$ for $i = r, r+1, \ldots, n$, and if there exists only one such $\alpha$, then $\langle \alpha, \lambda \rangle > 1$.

**Lemma 3.22.** Let $\lambda \in Q^\vee$, and $\lambda_B$ be the Peterson-Woodward lifting of $\lambda + Q^\vee_P$. If $\lambda$ is the Peterson-Woodward lifting of $\lambda + Q^\vee_P$, then either $\lambda - \lambda_B$ or $\lambda_B - \lambda$ is effective. Furthermore if $\lambda - \lambda_B = \sum_{j=1}^r c_j \alpha_j^\vee \neq 0$, then the coefficient $c_r \neq 0$.

**Proof.** If follows from the definition of a Peterson-Woodward lifting that $\langle \alpha, \lambda_B \rangle = 0$ (resp. $\langle \alpha, \lambda \rangle = 0$) for all $\alpha \in \Delta_P$ (resp. $\Delta$) with at most one exception, and if such exception $\alpha_k$ (resp. $\alpha_{k'}$) exists, then $\langle \alpha, \lambda_B \rangle = -1$ (resp. $\langle \alpha_{k'}, \lambda \rangle = 1$). When there does not exist such exception, we denote $k = k' = n + 1$ for notation conventions.

We may assume $\langle \alpha_r, \lambda - \lambda_B \rangle \geq 0$ (otherwise we consider $\lambda_B - \lambda$). Then $\lambda - \lambda_B \in Q^\vee_P$ is given by the difference between a dominate coweights and a fundamental coweights in $Q^\vee_P$. Therefore it is well known that $\lambda - \lambda_B$ is either a nonpositive combination or a nonnegative combination of $\alpha_1^\vee, \ldots, \alpha_r^\vee$. (For instance, we can prove this by direct calculations using Table 1 of [11]).

Now we assume $c_j \geq 0$ for all $j$ (otherwise we consider $\lambda_B - \lambda$). Since $\langle \alpha_i, \lambda - \lambda_B \rangle$, $i = 1, \ldots, r-1$, are all nonpositive with at most an exception of value 1, we conclude $c_r > 0$. Otherwise, it would make a contradiction with the second half of the statement of Lemma 3.21. □

Recall that $\partial \Delta_P$ denotes the set of simple roots in $\Delta \setminus \Delta_P$ which are adjacent to $\Delta_P$.

**Lemma 3.23.** Let $v \in W^P$, $u \in W_P$ and $w \in W$. Let $\lambda \in Q^\vee$ be effective, and $\lambda_B$ be the Peterson-Woodward lifting of $\lambda + Q^\vee_P$. If $\lambda - \lambda_B = \sum_{j=1}^r c_j \alpha_j^\vee$ satisfies $c_r < 0$ and $c_j \leq 0$ for all $j$, then $N_{v,u}^{w,\lambda} = 0$.  

Proof. Let \( \alpha \in \Delta \setminus (\Delta_P \cup \partial \Delta_P) \). Then \( \text{sgn}_{\alpha}(u) = 0 \) and \( v_{s\alpha} \in W^P \). If \( \langle \alpha, \lambda \rangle > 0 \), then the coefficient of \( \alpha^\vee \) in \( \lambda \) must be positive. By Proposition (1), we have \( N_{v,u}^{w,\lambda} = 0 \) unless \( \text{sgn}_{\alpha}(v) = \langle \alpha, \lambda \rangle = 1 - \text{sgn}_{\alpha}(w) = 1 \). Furthermore when this holds, we have \( N_{v,u}^{w,\lambda} = N_{w,u}^{s\alpha,\lambda - \alpha^\vee} \) by Proposition 3.15 (2). Clearly, \( \lambda - \lambda_B = (\lambda - \alpha^\vee) - (\lambda - \alpha^\vee) \). Therefore by induction, we can assume \( \langle \alpha, \lambda \rangle \leq 0 \) for all \( \alpha \in \Delta \setminus (\Delta_P \cup \partial \Delta_P) \).

The boundary \( \partial \Delta_P \) consists one or two nodes. We assume \( \partial \Delta_P = \{\alpha_{r+1}\} \) first. Then by Lemma 3.21, we have \( \langle \alpha_{r+1}, \lambda_B \rangle \geq 1 \).

Assume that \( \alpha_1 \) is adjacent to \( \alpha_{r+1} \). This happens in cases C5), C7) with \( r = 3 \), C9) with \( r = 3 \), and C10). Then \( \langle \alpha_{r+1}, \lambda_B \rangle = \langle \alpha_{r+1}, \lambda_B \rangle + c_r \langle \alpha_{r+1}, \alpha^\vee \rangle \geq 2 \). Since \( \text{sgn}_{\alpha_{r+1}}(u) = 0 \), we have \( N_{v,u}^{w,\lambda} = 0 \) by Proposition 3.15 (1).

Assume that \( \alpha_1 \) is adjacent to \( \alpha_{r+1} \). This happens in cases C1B), C1C), C2) and C4). If \( \lambda_B \) is of the form \( a\mu_B + \alpha^\vee_{r+1} + \sum_{j=r+2}^{n} a_j \alpha_j^\vee \), then by the hypotheses \( \langle \alpha_j, \lambda_B \rangle \leq 0 \) for all \( j \geq r + 2 \), and the precise description of \( \mu_B \) in Table 3.1, we can easily conclude that \( \langle \alpha_{r+1}, \lambda_B - \alpha^\vee_{r+1} \rangle \geq 0 \). Hence, we obtain \( N_{v,u}^{w,\lambda} = 0 \) again by the same arguments above. If \( \lambda_B \) is not of the aforementioned form, then \( \lambda_B \) is the combination of a virtual null coroot and a non-simple coroot in Table 3.1 (or zero coroot). Such \( \lambda_B \) satisfies \( \langle \alpha_i, \lambda_B \rangle = 0 \) for all \( 1 \leq i \leq r_a \), where \( \alpha_{r_a} \) is the simple root adjacent to \( \alpha_r \). Let \( m \) be the minimum of the set \( \{i | 1 \leq i \leq r_a, c_i < 0 \} \) if nonempty, or \( m := r_a \) otherwise. Then \( \langle \alpha_m, \lambda \rangle = \langle \alpha_m, \sum_{i=1}^{r} c_i \alpha_i^\vee \rangle > 0 \). Since \( \text{sgn}_{\alpha_m}(v) = 0, \ N_{v,u}^{w,\lambda} = 0 \) unless \( \text{sgn}_{\alpha_m}(u) = \langle \alpha_m, \lambda \rangle = 1 - \text{sgn}_{\alpha_m}(w) = 1 \). When this holds, we have \( N_{v,u}^{w,\lambda} = N_{v,u}^{w,\lambda - \alpha_m^\vee} \) with \( w \in W^P \) and \( \lambda - \alpha^\vee_m = \lambda_B + (-1) \cdot \alpha^\vee_m + \sum_{j=1}^{r} c_j \alpha_j^\vee \), by Proposition 3.15. Hence, by reduction, we can assume \( c_1 = 0 \). Consequently, we have \( \langle \alpha_{r+1}, \lambda \rangle \geq 2 \), and then obtain \( N_{v,u}^{w,\lambda} = 0 \).

Now we assume \( \partial \Delta_P = \{\alpha_{r+1}, \alpha_{r+2}\} \). That is, case C7) with \( 4 \leq r \leq 6 \), or case C9) with \( r = 2 \) occurs. If \( \langle \alpha_{r+1}, \lambda_B \rangle > 0 \), then we are done by the same arguments as above, since \( \alpha_r \) is adjacent to \( \alpha_{r+1} \) and \( c_r < 0 \). If \( \langle \alpha_{r+1}, \lambda_B \rangle > 0 \), then \( \langle \alpha_{r+2}, \lambda_B \rangle > 0 \). If \( \lambda_B = \tau + \alpha^\vee_{r+2} \) with \( \tau \) a virtual null coroot, then we conclude \( \langle \alpha_{r+2}, \tau \rangle \geq 0 \). (For instance when case 7) with \( r = 6 \) occurs, we have \( \tau = a\mu_B + b\mu_B \), where \( \mu_B^{(1)}, \mu_B^{(2)} \) denote the corresponding two coroots in Table 3.1, and \( a, b \geq 0 \). We have \( \langle \alpha_7, \lambda_B \rangle = a - b \leq 0 \) and \( \langle \alpha_8, \lambda_B \rangle = 2b - a + 2 > 0 \). This implies \( 2b - a > 0 \). The arguments for the remaining cases are similar.) If \( \lambda_B \) is not of the aforementioned form, then by Table 3.2 we conclude that \( \langle \alpha_i, \lambda_B \rangle = 0 \) for all \( 1 \leq i \leq r_a \), where \( \alpha_{r_a} \) is the simple root of \( \Delta_P \) adjacent to \( \alpha_r \). Therefore, we are done by the same arguments as above. \( \square \)

Proof of Proposition 3.18. Since \( QH^*(G/B) \) is an \( S \)-filtered algebra, by Lemma 3.20, we have

\[
\sigma^v \ast \sigma^u = \sigma^{vu} + \sum N_{v,u}^{w,\lambda} q_{\lambda} \sigma^w + \sum b_{w',\lambda} q_{\lambda} \sigma^{w'}.
\]

Here \( gr(q_{\lambda} \sigma^w) < gr(\sigma^v) + gr(\sigma^u) \). The first summation is over those \( q_{\lambda} \sigma^w \) satisfying both

(i) \( \lambda = \sum_{j=1}^{n} a_j \alpha_j^\vee \) is not a virtual null coroot, where \( a_j \geq 0 \) for all \( j \),

and (ii) \( gr(q_{\lambda} \sigma^w) = gr(\sigma^v) + gr(\sigma^u) \). The hypothesis (ii) is equivalent to

(ii)' \( gr_{[1,r]}(q_{\lambda} \sigma^w) = gr_{[1,r]}(\sigma^u) \),

following from the dimension constraint of Gromov-Witten invariants \( N_{v,u}^{w,\lambda} \) (see also Lemma 3.11 of [16]) and the assumption that \( v \in W^P \).
By Proposition 3.13 (1), we conclude that elements in the first summation also satisfy

\[(iii) \sigma^v q_\lambda = \psi_{\Delta, \Delta_P}(q_{\lambda, P}) \text{ where } \tilde{\lambda}_P := \lambda + Q_{P'}^\vee.\]

Therefore, it suffices to show \(N_{w, u}^{\alpha, \lambda} = 0\) whenever all (i), (ii)', and (iii) hold.

Let \(\lambda_B\) denote the Peterson-Woodward lifting of \(\lambda + Q_{P'}^\vee\). By Lemma 3.22, the coefficients \(c_j\) of \(\lambda - \lambda_B = \sum_{j=1}^t c_j \alpha_j\) are all nonpositive or all nonnegative, and \(c_r \neq 0\) due to (i). If \(c_r < 0\), then we are done by Lemma 3.23. Therefore we assume \(c_r > 0\) in the following.

Since all \(c_j \geq 0\), we write \(\lambda = \lambda_B + \sum_{i=1}^t \beta_i^\vee\). The set \(\Delta_{\lambda'} = \{\alpha \in \Delta_P \mid \langle \alpha, \lambda_B \rangle = 0\}\) coincides with either \(\Delta_{\lambda'} \cap \Delta_P\) or \(\Delta_P \setminus \{\alpha_k\}\) for a unique \(\alpha_k \in \Delta_P\) with \(\langle \alpha_k, \lambda_B \rangle = -1\). Therefore we can further assume that \(\beta_i \in \Delta_{\lambda'}, i = 1, 2, \ldots, t\), satisfy \(\langle \beta_i, \lambda_B \rangle + \langle \beta_j, \sum_{i=j+1}^t \beta_i^\vee \rangle > 0\) for all \(1 \leq j \leq t\). (This can be done: if \(\langle \alpha, \lambda - \lambda_B \rangle > 0\) holds for some \(\alpha\) in \(\Delta_P\) distinct from \(\alpha_k\), then we simply choose \(\beta_1 = \alpha\). If not, then \(\alpha = \alpha_k\), and the coefficient of \(\alpha_k\) in the highest root of the root subsystem \(R_P\) is equal to 1.) Hence we conclude \(\langle \alpha, \lambda - \lambda_B \rangle \geq 2\).

Since \(v \in W_P\), \(\text{sgn}_\alpha(v) = 0\) for all \(\alpha \in \Delta_{\lambda'}\). For each \(1 \leq j \leq t\), we have \(N_{v, us_{\beta_1} \cdots s_{\beta_{j-1}} \beta_j}^{w_{s_{\beta_1} \cdots s_{\beta_{j-1}}}, \lambda_B + \sum_{i=j}^t \beta_i^\vee} = 0\) unless \(\ell(us_{\beta_1} \cdots s_{\beta_{j-1}}) = \ell(us_{\beta_1} \cdots s_{\beta_{j-1}}) - 1\), \(\ell(us_{\beta_1} \cdots s_{\beta_{j-1}}) = (\ell(us_{\beta_1} \cdots s_{\beta_{j-1}}) + 1)\) and \(\langle \beta_j, \lambda_B \rangle + \langle \beta_j, \sum_{i=j+1}^t \beta_i^\vee \rangle = 1\) all hold, by Proposition 3.15 (1). Furthermore when all these hypotheses hold, we have

\[N_{v, us_{\beta_1} \cdots s_{\beta_{j-1}} \beta_j}^{w_{s_{\beta_1} \cdots s_{\beta_{j-1}}}, \lambda_B + \sum_{i=j}^t \beta_i^\vee} = N_{v, us_{\beta_1} \cdots s_{\beta_{j}}}^{w_{s_{\beta_1} \cdots s_{\beta_{j-1}}}, \lambda_B + \sum_{i=j+1}^t \beta_i^\vee}\]

by applying the tuple \((u, v, w, \lambda, \alpha)\) of Proposition 3.15 (2) to the tuple \((\tilde{u}s_{\beta_1} \cdots s_{\beta_{j-1}}, ws_{\beta_1} \cdots s_{\beta_{j-1}} \beta_j, \lambda_B + \sum_{i=j+1}^t \beta_i^\vee)\). Denote \(u' := us_{\beta_1} \cdots s_{\beta_{j}}\). Combining all these, we have

\[N_{v, u'} = N_{v, u}^{w_{s_{\beta_1} \cdots s_{\beta_{j}}} \cdot \beta_j, \lambda_B}\]

if all the hypotheses \(\dagger\) hold:

\[\ell(ws_{\beta_1} \cdots s_{\beta_t}) = \ell(w) + t, \ell(u') = \ell(u) - t, \langle \beta_j, \lambda_B \rangle + \langle \beta_j, \sum_{i=j}^t \beta_i^\vee \rangle = 1, j = 1, \ldots, t,\]

or 0 otherwise. In particular if \(\lambda_B = 0\), then we are done since the hypotheses on the step \(j = t\) cannot hold.

It suffices to show \(N_{v, u}^{w_{s_{\beta_1} \cdots s_{\beta_{t}}} \cdot \beta_j, \lambda_B} = 0\) under the hypotheses \(\dagger\) and \(\lambda_B \neq 0\). If \(\ell(u') = 0\), then \(u' = \text{id}\), and we are done. Assume \(\ell(u') > 0\) now. For any \(\eta \in Q_{P'}^\vee\), we have \(|gr_{[1, r]}(q_\eta)| = |gr_{[1, r]}(q_\eta)| = |gr_{[1, r]}(q_\eta)| = |gr_{[1, r]}(q_\eta)| = \ell(u)\).

By Proposition 3.1, \(-|gr_{[1, r]}(q_\lambda)| = |gr_{[1, r]}(w_P w_P^\vee)| = \ell(w_P w_P^\vee) = |R_P^+| - |R_P^-|\).

Combining both, we have

\[|gr_{[1, r]}(w_{s_{\beta_1} \cdots s_{\beta_t}})| = |gr_{[1, r]}(w)| + t = \ell(u') - |gr_{[1, r]}(q_\lambda)| = \ell(u') + |R_P^+| - |R_P^-|\cdot\]

Thus there is \(\alpha \in \Delta_{\lambda'}\) such that \(\text{sgn}_\alpha(w_{s_{\beta_1} \cdots s_{\beta_t}}) = 1\) (otherwise, \(w_{s_{\beta_1} \cdots s_{\beta_t}}(\alpha) \in R^+\) for all \(\alpha \in \Delta_{\lambda'}\)).
v ∈ W^P, sgn_α(v) = 0. By Proposition 3.15 (1), we have N_{v,w}^{w_1} · · · s_{β_1} · · · s_{β_t} · · · s_{α} = 0 unless sgn_α(u') = 1, i.e., ℓ(u') = ℓ(u) − 1. Furthermore when this holds, we have N_{v,u'}^{w_1} · · · s_{β_1} · · · s_{α} = N_{v,u'}^{w_1} · · · s_{β_1} · · · s_{α} · · · s_{β_t} · · · s_{α} · · · s_{α} (by applying the tuple (u,v,w,λ,α) of Proposition 3.15 (2) to (u,v,w,α,β)). By induction, we conclude N_{v,u'}^{w_1} · · · s_{β_1} · · · s_{λ} = 0 unless both u' ∈ W^P and ℓ(ws_{β_1} · · · s_{β_t} · · · s_{α} · · · s_{α} · · · s_{α} · · · s_{α}) − ℓ(u') hold. Furthermore when both hypotheses hold, we have N_{v,u'}^{w_1} · · · s_{β_1} · · · s_{λ} = N_{v,u'}^{w_1} · · · s_{β_1} · · · s_{λ} · · · s_{α} · · · s_{α} · · · s_{α} = 0 since λ_B ≠ 0.

3.4. Proof of Proposition 3.3 (3). The statement tells us that the elements Ψ_{r+1}(q_{β}) in Gr^{F}_{(r+1)} do behave like monomials. Due to Lemma 3.11, it suffices to show those q_{λ}σ^u behave like the non-identity elements of the finite abelian group (Q^u/Q^p)/L as in Table 3.2. For any one of the cases C1B, C1C, C2 and C9, we use the first virtual null coroot μ_B in Table 3.1 and the unique element q_{λ}σ^u in Table 3.2. Namely for the exceptional case when C9 with r = 2 occurs, there are two virtual null coroots, and we will use the one μ_B = 2α′_1 + 2α′_2 + α′_2. For any one of these cases, we only need to use check one quantum multiplication as in the next proposition, which we assume first. The remaining cases require verifications of more quantum multiplications, which will be discussed in section 5.3.

Proposition 3.24. Assume C1B, C2 or C9 occurs. In QH^*(G/B), we have

q_{λ}σ^u · q_{λ}σ^u = q_{μ_B} + \sum b_{w',λ}q_{λ}σ^{w'}

with gr(q_{λ}σ^{w'}) < gr(q_{μ_B}) whenever b_{w',λ} ≠ 0.

Proof of Proposition 3.3 (3). Let q_{κ_p}, q_{κ_p}' ∈ QH^*(G/P). If case C1C occurs, then we note ψ_{Δ}Δ_p(q_{κ_p}) * ψ_{Δ}Δ_p(q_{κ_p}') = q_{κ_p} * q_{κ_p}' = q_{κ_p + κ_p}' = ψ_{Δ}Δ_p(q_{κ_p + κ_p}'). Therefore, Ψ_{r+1}(q_{κ_p}) * Ψ_{r+1}(q_{κ_p}') = Ψ_{r+1}(q_{κ_p + κ_p}'). Assume that case C1B, C9 or C2 occurs now. If either κ_p or κ_p' is a virtual null coroot, then we are done, by using Lemma 3.11. Otherwise, by Proposition 3.9 we have κ_p = τ_p + λ_p and κ_p' = τ_p + λ_p for some virtual null coroots τ_p, τ_p', and consequently κ_p + κ_p' = τ_p + τ_p' + (μ_B + Q^p). Here μ_B and ψ_{Δ}Δ_p(q_{κ_p}) = q_{λ}σ^u are given in Table 3.1 and Table 3.2 respectively. Hence, we have Ψ_{r+1}(q_{κ_p}) = Ψ_{r+1}(q_{τ_p}) * Ψ_{r+1}(q_{λ_p}) and Ψ_{r+1}(q_{κ_p}') = Ψ_{r+1}(q_{τ_p'}) * Ψ_{r+1}(q_{λ_p}), by Lemma 3.11. Using Proposition 3.24, we have Ψ_{r+1}(q_{κ_p}) * Ψ_{r+1}(q_{λ_p}) = q_{λ}σ^u · q_{λ}σ^u = q_{μ_B}. Hence, we have Ψ_{r+1}(q_{κ_p}) * Ψ_{r+1}(q_{κ_p}') = q_{μ_B} · q_{μ_B} = q_{μ_B + τ_p' + μ_B} = Ψ_{r+1}(q_{κ_p + κ_p}'). For the remaining cases in Table 2.1, the statements follows from the arguments given in section 5.2. Thus we are done.

Now we prepare some lemmas in order to prove Proposition 3.24. The reduced expressions of the longest element w_P in W^P are not unique. There is a conceptual approach to construct w_P of the form w^± whenever h is even (see e.g. Chapter 3 of [13]). Here h denotes the Coxeter number of W_P, and it is equal to 2r (resp. 2r − 2) for Δ_p of type B_r (resp. D_r). The next lemma provides a special choice of the above w ∈ W^P.

Lemma 3.25. For Δ_p of type B_r or D_r, (s_1 · · · s_r)^{1/2} is a reduced expression of the longest element w_P.

Proof. It is easy to check that the given element maps all simple roots in Δ_p to negative roots, and note ℓ(w_P) = r^2 (resp. r(r − 1)). Thus the statement follows.

Recall that for u in Table 3.2, ū denotes the minimal length representative of uW_P.
Lemma 3.26. Let \( v = s_{\beta_1} \cdots s_{\beta_p} \in W_P \) be a reduced expression. Assume C1B), C2) or C9) occurs, then \( \bar{u}^{-1} \leq v \) if and only if there exists a subsequence \([i_1, \ldots, i_{\frac{p}{2}}]\) of \([1, \ldots, p]\) such that \([\beta_{i_1}, \ldots, \beta_{i_{\frac{p}{2}}}]=[\alpha_r, \alpha_{\frac{r-1}{2}}, \ldots, \alpha_2, \alpha_1]\).

Proof. Note \( \ell(\bar{u}^{-1})=\frac{p}{2} \). It is a general fact that \( \bar{u}^{-1} \leq v \) if and only if there exists a subsequence \([i_1, \ldots, i_{\frac{p}{2}}]\) of \([1, \ldots, p]\) such that \( \bar{u}^{-1}=s_{\beta_{i_1}} \cdots s_{\beta_{i_{\frac{p}{2}}}} \). Since the simple reflections in \( \bar{u}^{-1}=s_{r}s_{\frac{r-1}{2}} \cdots s_2s_1 \) are distinct, we conclude that the two sets \([\alpha_r, \alpha_{\frac{r-1}{2}}, \ldots, \alpha_2, \alpha_1]\) and \([\beta_{i_1}, \ldots, \beta_{i_{\frac{p}{2}}}]=[\alpha_r, \alpha_{\frac{r-1}{2}}, \ldots, \alpha_2, \alpha_1]\) coincide with each other. Then the coincidence of the corresponding two ordered sequences follows immediately from the obvious observation that \( s_{r}s_{\frac{r-1}{2}} \cdots s_{j+1}s_j(\alpha_j) \in -R^+ \) for all \( j \).

The next well-known fact works for arbitrary \( \Delta_P \) (see e.g. Theorem 3.17 (iv) of [2]).

Lemma 3.27. Let \( w, v \in W_P \). If \( w^{-1} \not\leq v^{-1}w_P \), then \( \sigma^w \cup \sigma^v = 0 \) in \( H^*(P/B) \).

Corollary 3.28. For case C1B), C2) or C9), we have \( \sigma^\bar{u} \cup \sigma^\bar{u} = 0 \) in \( H^*(P/B) \).

Proof. By Lemma 3.25, \( \bar{u}^{-1}w_P \) is equal to \((s_1s_2 \cdots s_r)^{-1} \) if case C1B) or C9) occurs, or equal to \( s_{r-1}(s_1s_2 \cdots s_r)^{-2} \) if case C2) occurs (since \( s_1s_{r-1}=s_{r-1}s_1 \)). Clearly, there does not exist a subsequence \([i_1, \ldots, i_{\frac{p}{2}}]\) satisfying \([\alpha_i, \alpha_{\frac{i-1}{2}}, \ldots, \alpha_2, \alpha_1]\) such that \( \bar{u}^{-1} \not\leq \bar{u}^{-1}w_P \) by Lemma 3.26. Hence, the statement follows from Lemma 3.27.

Proof of Proposition 3.24. Due to the filtered algebra structure of \( QH^*(G/B) \), we have \( q_\lambda \sigma^u \ast q_\lambda \sigma^u = \sum_{\eta, \eta'} N^u,\eta N^\eta,\eta' q_\eta+2\lambda \sigma^w + \sum b_{\eta', \lambda} q_\lambda \sigma^w \), where \( gr(q_{\eta+2\lambda} \sigma^w)=2gr(q_{\eta} \sigma^w) \) and \( gr(q_{\eta} \sigma^w) \). Since \( gr_{[r+1, r]}(\sigma^u)=0 \), we conclude \( w \in W_P \) and \( \eta = \sum_{i=1}^{r} b_i \alpha_i \), where \( b_i \geq 0 \). Note \( gr_{[r+1, r]}(q_{\eta})=he_r \) by Table 2.2. Write \( w = w_1w_2 \) where \( w_1 \in W_P \) and \( w_2 \in W_P \). Using \( gr_{[r, r]} \), we conclude \( \ell(w_1)+b_r h = h \). If \( b_r = 0 \), then \( gr(q_{\eta} \sigma^{w_2}) = gr_{[r+1, r]}(q_{\eta} \sigma^{w_1w_2}) = 2gr_{[r, r-1]}(u) = 2\sum_{i=1}^{r-1} e_i = q_{\eta_1} q_{\eta_1-1} \cdots q_{\eta_1-1} \) where \( v_j := s_{j-1}s_j \) for each \( 2 \leq j \leq r+1 \). Thus we have \( q_{\eta_1} q_{\eta_1} = q_\sigma q_{\eta_1-1} \cdots q_{\eta_1-1} \) by Lemma 3.12. In \( Gr^f(QH^*(G/B)) \), we have \( \sigma^u \ast \sigma^v = N_{\eta'_{\lambda}, \eta} q_{\eta} q_{\eta_{\lambda}} q_{\eta_{\lambda-1}} \cdots v_{\lambda-1} + \text{other terms} \). On the other hand, by Proposition 3.13, we have \( \sigma^u \ast \sigma^v = (\sigma^u \ast \sigma_{\eta_{\lambda-1}})^2 = (N_{\eta'_{\lambda}, \eta} q_{\eta} q_{\eta_{\lambda-1}} + \text{term 1}) \ast (\text{term 2}) \).

Here term 1 is a nonnegative combination of the form \( q_\Delta q_{\lambda} q_{\sigma v_{\lambda}} \) with either \( \eta'_{\lambda} \neq 0 \) or \( w_1 \neq w_1' \in W_P \), and term 2 is a combination of elements \( q_{\lambda} q_{\sigma v_{\lambda}} \) with \( (\eta', \lambda') \in W_P \times Q^V_P \). Hence, \( N_{\eta', \lambda} \neq 0 \) only if \( N_{\eta', \lambda} = 0 \), the latter of which is the coefficient of \( \sigma w_{\lambda-1} \) in \( \sigma \sigma_{\lambda-1} \in H^*(G/B) \). It is a general fact (following from the surjection \( H^*(G/B) \rightarrow H^*(P/B) \)) that \( N_{\eta', \lambda} \) coincides with the coefficient of \( \sigma w_{\lambda-1} \) in \( \sigma \sigma_{\lambda-1} \in H^*(P/B) \), and therefore it is equal to 0 by Corollary 3.28. Thus \( N_{\eta', \lambda} = 0 \) whenever \( b_r = 0 \). It remains to deal with the case \( b_r = 1 \). By Lemma 3.12, there is exactly one such term, which turns out to be \( q_{\eta} q_{\sigma v_{\lambda}} = q_{\mu} \sigma v_{\lambda} \). Thus it suffices to show \( N_{\mu, \eta}^{\eta_{\lambda-1}} = 1 \). Using Proposition 3.15 (2) repeatedly, we conclude the followings.

---

2The Schubert cohomology classes \( \sigma^v \) are denoted as \( P_{v-1} \) in [2].
1. If case C1(B) or C9) occurs, then \( \eta = \mu_B - 2\lambda = \alpha_r^\vee + 2\sum_{i=1}^{r-1} \alpha_i^\vee \), and we have

\[
N_{u,u}^{id \cdot \eta} = N_{s_{l-1}s_{l-2}\cdots s_{r-1}q_{r-1}}^{s_1 \cdots s_{r-1}q_{r-1}} = N_{s_{l-1}s_{l-2}\cdots s_{r-1}q_{r-1}}^{s_1 \cdots s_{r-1}q_{r-1}} = N_{s_{l-1}s_{l-2}\cdots s_{r-1}q_{r-1}}^{s_1 \cdots s_{r-1}q_{r-1}}.
\]

2. If C2) occurs, then \( \eta = \mu_B - 2\lambda = \alpha_r^\vee + \alpha_r^\vee + 2\sum_{i=1}^{r-2} \alpha_i^\vee \), and we have

\[
N_{u,u}^{id \cdot \eta} = N_{s_{l-1}s_{l-2}\cdots s_{r-1}q_{r-1}}^{s_1 \cdots s_{r-1}q_{r-1}} = N_{s_{l-1}s_{l-2}\cdots s_{r-1}q_{r-1}}^{s_1 \cdots s_{r-1}q_{r-1}} = N_{s_{l-1}s_{l-2}\cdots s_{r-1}q_{r-1}}^{s_1 \cdots s_{r-1}q_{r-1}}.
\]

Namely, we always have \( N_{u,u}^{id \cdot \mu_B - 2\lambda} = N_{u''u''}^{id \cdot 0} \) with \( u'' = s_1 \cdots s_{\frac{r}{2} - 2} \in W_{P''} \) and \( v'' = s_1 \cdots s_r \in W_{P''} \), where \( \Delta_{P''} = \{ \alpha_1, \ldots, \alpha_{\frac{r}{2} - 2} \} \). Thus it is equal to 1 by Lemma 3.20. \( \square \)

4. Conclusions for general \( \Delta_P \). In this section, we allow \( P/B \) to be reducible, namely the Dynkin diagram \( \text{Dyn}(\Delta_P) \) could be disconnected. We will first show the coincidence between the grading map \( gr \) defined in section 2.2 and the one introduced in [16]. Then we will refine the statement of Theorem 5.2 of [16], and will sketch the proof of it.

Whenever referring to the subset \( \Delta_P = \{ \alpha_1, \ldots, \alpha_r \} \), in fact, we have already given an ordering on the \( r \) simple roots in \( \Delta_P \), in terms of \( \alpha_i \)'s. As we can see in Definition 2.7, the grading map \( gr : W \times Q^\vee \to \mathbb{Z}^{r+1} \) depends only on such an ordering of \( \alpha_i \)'s in \( \Delta_P \), which has nothing to do with the connectedness of \( \text{Dyn}(\Delta_P) \). Therefore we can use the same definition even if \( \text{Dyn}(\Delta_P) \) is disconnected. We want to show \( gr \) coincides with the grading map given by Definition 2.8 (resp. 5.1) of [16] when \( \text{Dyn}(\Delta_P) \) is connected (resp. disconnected).

Recall \( \Delta_0 := \emptyset, \Delta_{P+1} := \Delta, \Delta_i := \{ \alpha_1, \ldots, \alpha_i \} \) for \( 1 \leq i \leq r \), and \( P_j := P_{\Delta_j} \) for all \( j \). Denote \( \rho_j := \frac{1}{2} \sum_{\beta \in R^+_j} \beta \) where \( \rho_0 := 0 \). Then for any \( \lambda \in Q^\vee \), we have

\[
gr(id, \lambda) = \sum_{j=1}^{r+1} (2\rho_j - 2\rho_{j-1}, \lambda)e_j \text{ by Definition 2.7.}
\]

**Lemma 4.1.** For any \( \alpha \in \Delta_j \), we have \( gr_{[j+1,r+1]}(id, \alpha^\vee) = 0 \) and \( |gr(id, \alpha^\vee)| = 2 \).

**Proof.** It is well-known that \( \rho_k \) equals the sum of fundamental weights in the root subsystem \( R_{P_k} \). That is, we have \( \langle \rho_k, \alpha^\vee \rangle = 1 \) for any \( \alpha \in \Delta_k \). Hence, for \( j \leq k \leq r + 1 \), we have \( |gr_{[1,k]}(id, \alpha^\vee)| = \sum_{i=1}^{k} (2\rho_i - 2\rho_{i-1}, \alpha^\vee) = (2\rho_k, \alpha^\vee) = 2 \). Thus if \( i > j \), then \( |gr_{[i,j]}(id, \alpha^\vee)| = |gr_{[1,i]}(id, \alpha^\vee)| - |gr_{[1,i-1]}(id, \alpha^\vee)| = 2 - 2 = 0 \). \( \square \)

By abuse of notation, we still denote by \( \psi_{\Delta_{j+1}, \Delta_j} : W_{P_{j+1}}^{P_j} \times Q_{P_{j+1}}^\vee / Q_{P_j}^\vee \to W \times Q^\vee \) induced from the Peterson-Woodward comparison formula. We recall Definition 2.8 of [16] as follows.

**Definition 4.2.** Define a grading map \( gr' : W \times Q^\vee \to \mathbb{Z}^{r+1} \) associated to \( \Delta_P = (\alpha_1, \ldots, \alpha_r) \) as follows.

1. For \( w \in W \), we take its (unique) decomposition \( w = v_{r+1} \cdots v_1 \) where \( v_j \in W_{P_j}^{P_{j-1}} \). Then we define \( gr'(w, 0) = \sum_{i=1}^{r+1} \ell(v_i) e_i \).
2. For $\alpha \in \Delta$, we can define all $gr'(id, \alpha^\vee)$ recursively in the following way. Define $gr'(id, \alpha_1^\vee) = 2e_1$; for any $\alpha \in \Delta_{j+1} \setminus \Delta_j$, we define

$$gr'(id, \alpha^\vee) = (\ell(w_j, w_{j+1}) + 2 + \sum_{i=1}^j 2a_i)e_{j+1} - gr'(w_j, 0) - \sum_{i=1}^j a_igr'(id, \alpha_i^\vee),$$

where $w_j, w_{j+1}$ and $a_i$'s are defined by the image $\psi_{j+1}(id, \alpha^\vee + Q_j^\vee) = (w_j, w_{j+1}, \alpha^\vee + \sum_{i=1}^j a_\alpha_i^\vee)$.

3. In general, we define $gr'(w, \sum_{k=1}^n b_k\alpha_k^\vee) = gr'(w, 0) + \sum_{k=1}^n b_kgr'(id, \alpha_k^\vee)$.

One of the main results of [16], i.e., Proposition 2.1, tells us that the grading $gr'$ respects the quantum multiplication. Precisely for any Schubert classes $\sigma^u, \sigma^v$ of $QH^*(G/B)$, if $q_\lambda \sigma^w$ occurs in the quantum multiplication $\sigma^u \star \sigma^v$, then

$$gr'(w, \lambda) \leq gr'(u, 0) + gr'(v, 0).$$

**Proposition 4.3.** If $Dyn(\Delta_P)$ is connected, then $gr = gr'$.

**Proof.** While it is a general fact that $gr|_{W \times \{0\}} = gr'|_{W \times \{0\}}$, we illustrate a little bit details here. For each $j$, $v_j \cdots v_1 \in W_P$ preserves $R^+$, and $v_{j+1} v_j \cdots v_1 \in W^P$ maps $R^+_P$ (resp. $-R^+_P$) to $R^+$ (resp. $-R^+$). Thus for any $\beta \in R^+_P$, $w(\beta) \in -R^+$ and only if $v_j \cdots v_1(\beta) \in -R^+ \subset -R^+$. That is, we have $\ell(v_j \cdots v_1) = |Inv(v_j \cdots v_1)| = |Inv(w) \cap R^+_P|$. Hence, $\ell(v_j) = \ell(v_j \cdots v_1) - \ell(v_{j-1} \cdots v_1) = |Inv(w) \cap R^+_P| - |Inv(w) \cap (R^+_P \setminus R^+_{P-1})|$. Note $\{\gamma, \lambda\} = \{0, -1\}$ for any $\gamma \in R^+_P$, and $\Delta^+_{P'} = \{ \beta \in \Delta_k \mid (\beta, \lambda) = 0 \}$. Thus we have $\{\gamma, \lambda\} = -1$ if $\gamma \in R^+_P \setminus R^+_{P-1}$, or $0$ if $\gamma \in R^+_{P-1}$. Hence, for any $1 \leq i \leq k$, we have $-2\rho_i - 2\rho_{i-1}, \ldots, -2\rho_1, -2\rho_0$.

When $Dyn(\Delta_P)$ is not connected, we use the same ordering on $\Delta_P$ as in section 5 of [16]. Namely, we write $\Delta_P = \bigsqcup_{k=1}^m \Delta^{(k)}$ such that each $Dyn(\Delta^{(k)})$ is a connected component of $Dyn(\Delta_P)$. Clearly, $\Delta^{(k)}$'s are all of $A$-type with at most one exception, say $\Delta^{(m)}$ if it exists. We fix a canonical order on $\Delta_P$. Namely, we say $\Delta_P = (\Delta^{(1)}, \ldots, \Delta^{(m)}) = (\alpha_1, \ldots, \alpha_r)$ such that for each $k$, $\Delta^{(k)} = \{\alpha_{k,1}, \ldots, \alpha_{k,r_k}\}$ satisfying (1) if $\Delta^{(k)}$ is of $A$-type, then $Dyn(\Delta^{(k)})$ is given by $\alpha_{k,1} \alpha_{k,2} \cdots \alpha_{k,r_k}$ together with

$$\alpha_{k,1} \alpha_{k,2} \cdots \alpha_{k,r_k}.$$
the same way of denoting an ending point (by $\alpha_{k,1}$ or $\alpha_{k,r_k}$) as in section 2.4 of [16]; (2) if $\Delta^{(k)}$ is not of $A$-type, then $Dyn(\Delta^{(k)})$ is given in the way of Table 2.1. We also denote the standard basis of $Z^{r+1}$ as $\{e_1, \ldots, e_{r_1}, \ldots, e_{m,1}, \ldots, e_{m,r_m}, e_{m+1,1}\}$. In order words, we have $e_i^P = e_j + \sum_{i=1}^{k-1} r_i$ and $\alpha_{k,i} = \alpha_{i+r-1} + \sum_{i=1}^{k-1} r_i$ in terms of our previous notations of $e_j$’s and $\alpha_j$’s respectively.

Using Definition 4.2 (resp. 2.7) with respect to $\Delta^{(k)}$, we obtain a grading map

$$gr^{(k)}: W \times Q^\vee \rightarrow Z^{r+1} = \bigoplus_{i=1}^{r+1} Z e_{k,i}$$

(resp. $gr^{(k)}: W \times Q^\vee \rightarrow Z^{r+1} = \bigoplus_{i=1}^{r+1} Z e_{k,i}$). Note $W_P = W_1 \times \cdots \times W_m$ where each $W_k$ is the Weyl subgroup generated by simple reflections from $\Delta^{(k)}$. In particular for any $(w, \lambda) \in W_k \times (\bigoplus_{\alpha \in \Delta^{(k)}} Z \alpha^\vee) \subset W \times Q^\vee$, we have $gr^{(k)}(w, \lambda) \in \bigoplus_{i=1}^{r_k} Z e_{k,i} \hookrightarrow Z^{r+1}$ which we treat as an element of $Z^{r+1}$ via the natural inclusion. Now we recall Definition 5.1 of [16] for general $\Delta_F$ as follows.

**Definition 4.4.** We define a grading map as follows, say again $gr^i: W \times Q^\vee \rightarrow Z^{r+1}$ by abuse of notation.

1. Write $w = e_{m+1} e_m \cdots v_1$ (uniquely), in which $(v_1, \ldots, v_m, v_{m+1}) \in W_1 \times \cdots \times W_m \times W^F$. Then $gr^i(w, 0) := \ell(v_{m+1}) e_{m+1} + \sum_{k=1}^{r_k} gr^{(k)}(w_k, 0)$.
2. For each $\alpha_{k,i} \in \Delta^{(k)}$, $gr^i(id, q_{\alpha_{k,i}}) := gr^{(k)}(id, q_{\alpha_{k,i}})$. For $\alpha \in \Delta \setminus \Delta_P$, we write $\psi_{\Delta, \Delta_P}(q_{\alpha^\vee + q_F^\vee}) = w_P w_P q_{\alpha^\vee} \prod_{k=1}^{m} \prod_{i=1}^{r_k} q_{\alpha_{k,i}}$ and then define

$$gr^i(id, \alpha^\vee) = (\ell(w_P w_P) + 2 + \sum_{k=1}^{m} \sum_{j=1}^{r_k} 2a_{k,i}) e_{m+1} - gr^{(k)}(w_P w_P, 0)$$

$$- \sum_{k=1}^{m} \sum_{i=1}^{r_k} a_{k,i} gr^{(k)}(id, \alpha_{k,i}).$$

3. In general, $gr^i(w, \sum_{\alpha \in \Delta} b_{\alpha} \alpha^\vee) := gr^i(w, 0) + \sum_{\alpha \in \Delta} b_{\alpha} gr^{(k)}(id, \alpha^\vee)$.

By abuse of notation, we denote $\pi_k$ for both of the natural projections

$$Z^{r+1} = \bigoplus_{i=1}^{r_{k+1}} Z e_{k,i} \hookrightarrow \bigoplus_{i=1}^{r_k} Z e_{k,i}$$
and

$$Z^{r+1} = \bigoplus_{i=1}^{r_{j+1}} Z e_{j,i} \hookrightarrow \bigoplus_{i=1}^{r_j} Z e_{j,i}.$$

**Lemma 4.5.** For $1 \leq k \leq m$, we have $\pi_k \circ gr = \pi_k \circ gr^{(k)}$ and $\pi_k \circ gr^i = \pi_k \circ gr^i(k)$.

**Proof.** It follows immediately from the definition that $\pi_k \circ gr^i(w, \alpha^\vee) = \pi_k \circ gr^{(k)}(w, \alpha^\vee)$ for $(w, \alpha^\vee) \in W \times \Delta^{(k)}$. For $\beta \in \Delta^{(k)}$ where $k \neq k$, we note $\langle \alpha, \beta \rangle = 0$. Thus $\psi_{\Delta, \Delta_P}(q_{\alpha^\vee + q_F^\vee}) = q_{\beta^\vee}$; furthermore if $\psi_{\Delta, \Delta_P}(q_{\alpha^\vee + q_F^\vee}) = w_P w_P q_{\alpha^\vee} \prod_{k=1}^{m} \prod_{i=1}^{r_k} q_{\alpha_{k,i}}$ where $\gamma \in \Delta \setminus \Delta_P$, then $\psi_{\Delta, \Delta_P}(q_{\alpha^\vee + q_F^\vee}) = w_P w_P q_{\alpha^\vee} \prod_{k=1}^{m} \prod_{i=1}^{r_k} q_{\alpha_{k,i}}$ with $w_P w_P$ given by the $W_k$-component of $w_P w_P$, implying $\pi_k \circ gr^i(w_P w_P) = \pi_k \circ gr^i(w_P w_P)$. Thus we have $gr^{(k)}(id, \beta^\vee) = 2e_{k,r_k+1}, \pi_k \circ gr^i(id, \beta^\vee) = \pi_k \circ gr^i(id, \beta^\vee)$, and consequently $\pi_k \circ gr^i(id, \gamma^\vee) = \pi_k \circ gr^i(id, \gamma^\vee)$. Hence, $\pi_k \circ gr^i = \pi_k \circ gr^i(k).$
Due to our notation conventions, we have \( e_j = e_{k,i} \) for \( j = i + \sum_{l=1}^{k-1} r_l \). Thus \( \pi_k \circ gr = \pi_k \circ gr(k) \) follows immediately, by noting \( R^+_P = R^+_{\Delta(k)} \bigcup \left( \bigcup_{l=1}^{k-1} R^+_P \right) \).

**Proof of Theorem 2.8.** For each \( 1 \leq k \leq m \), we have \( gr(k) = gr' \) by Proposition 4.3. Thus \( \pi_k \circ gr = \pi_k \circ gr' \) by Lemma 4.5. That is, we have \( gr^{[1,r]} = gr^{[1,r]} \). Note \( |gr(w,0)| = |gr'(w,0)| = \ell(w) \) and \( |gr(id,\alpha^\vee)| = |gr'(id,\alpha^\vee)| = 2 \) for any \( \alpha \in \Delta \). Thus we have \( |gr(w,\lambda)| = |gr'(w,\lambda)| \) for any \( (w,\lambda) \in W \times Q^v \). Hence, the statement follows. \( \square \)

For general \( \Delta_P \), the subset \( \{ gr(w,\lambda) \mid q_\lambda \sigma^w \in QH^*(G/B) \} \) of \( \mathbb{Z}^{r+1} \), denoted as \( S \) by abuse of notation, turns out again to be a totally-ordered sub-semigroup of \( \mathbb{Z}^{r+1} \). (The proof is similar to the one for Lemma 2.12 of [14] in the case when \( Dyn(\Delta_P) \) is connected.) In the same way as in section 2.2, we obtain an \( S \)-family of subspaces of \( QH^*(G/B) \); it naturally extends to a \( \mathbb{Z}^{r+1} \)-family, and induces graded vector subspaces. Namely, by abuse of notation, we have \( F = \{ F_a \} \) with \( F_a := \bigoplus_{gr(w,\lambda) \leq a} Q_\lambda \sigma^w \); \( Gr^F(\mathbb{Z}^{r+1}; QH^*(G/B)) := \bigoplus_{a \in \mathbb{Z}^{r+1}} Gr^F \), where \( Gr^F := F_a / \sum_{b \leq a} F_b \); for each \( 1 \leq j \leq r+1 \), \( Gr^F(\mathbb{J}) := \bigoplus_{i \in \mathbb{Z}} Gr^F \). In addition, we denote

\[
I := \bigoplus_{gr(w,\lambda) > 0} Q_\lambda \sigma^w \subset QH^*(G/B)
\]

and

\[
A := \psi_{\Delta,\Delta_P}(QH^*(G/P)) \oplus J \quad \text{where} \quad J := F_{-r+1}.
\]

For each \( 1 \leq j \leq r \), \( X_j := P_j/P_{j-1} \) is a Grassmannian (possibly of general type), and the quantum cohomology \( QH^*(X_j) \) is therefore isomorphic to \( H^*(X_j) \otimes \mathbb{Q}[t_j] \) as vector spaces. Note \( X_{r+1} := P_{r+1}/P_r = G/P \).

Now we can restate Theorem 1.1 in the introduction more precisely as follows.

**Theorem 4.6.**

1. \( QH^*(G/B) \) has an \( S \)-filtered algebra structure with filtration \( F \), which naturally extends to a \( \mathbb{Z}^{r+1} \)-filtered algebra structure on \( QH^*(G/B) \).
2. \( I \) is an ideal of \( QH^*(G/B) \), and there is a canonical algebra isomorphism

\[
QH^*(G/B)/I \cong QH^*(P/B).
\]

3. \( A \) is a subalgebra of \( QH^*(G/B) \) and \( J \) is an ideal of \( A \). Furthermore, there is a canonical algebra isomorphism (induced by \( \psi_{\Delta,\Delta_P} \))

\[
QH^*(G/P) \cong A/J.
\]

4. There is a canonical isomorphism of \( \mathbb{Z}^r \times \mathbb{Z}^{r+1} \)-graded algebras:

\[
Gr^F(\mathbb{Z}^{r+1}; QH^*(G/B) \otimes \mathbb{Q}[t^{-1}]) \cong \bigotimes_{j=1}^r QH^*(X_j) \otimes \mathbb{Q}[t^{-1}] \otimes Gr^F_{(r+1)}.
\]

There is also an injective morphism of graded algebras:

\[
\Psi_{r+1} : QH^*(G/P) \hookrightarrow Gr^F_{(r+1)},
\]

well defined by \( q_\lambda \sigma^w \mapsto \psi_{\Delta,\Delta_P}(q_\lambda \sigma^w) \). Furthermore, \( \Psi_{r+1} \) is an isomorphism if and only if either (a) \( \Delta(k) \)'s are all of \( A \)-type or (b) the only exception \( \Delta^{(m)} \) is of \( B_2 \)-type with \( \alpha_r \) being a short simple root.
Remark 4.7. Say \( \alpha_j \in \Delta^{(k)} \), i.e., \( j = i + \sum_{p=1}^{k-1} r_p \) for some \( 1 \leq i \leq r_k \). Whenever \( (k, i) \neq (m, r_m) \), we have \( X_j \cong \mathbb{P}^i \). If \( \Delta^{(m)} = \{ \alpha_{r-1}, \alpha_r \} \) is of type \( B_2 \) and \( \alpha_r \) is a short simple root, then we have \( X_{r-1} \cong \mathbb{P}^1 \) and \( X_r \cong \mathbb{P}^3 \). In other words, \( \Psi_{r+1} \) is an isomorphism if and only if all \( X_j \) (\( 1 \leq j \leq r \)) are projective spaces.

(Sketch) Proof of Theorem 4.6. The quantum cohomology ring \( QH^*(G/B) \) is generated by the divisor Schubert classes \( \{ \sigma^{a_1}, \ldots, \sigma^{a_n} \} \). The well-known quantum Chevalley formula (see [10]) tells us

\[
\sigma^u \ast \sigma^v = \sum_{\gamma} \langle \omega_i, \gamma \rangle \sigma^{u_{\gamma}} + \sum_{\gamma} \langle \omega_i, \gamma \rangle q_{\gamma} \sigma^{u_{\gamma}}
\]

where \( u \in W \) is arbitrary, and \( \{ \omega_1, \ldots, \omega_n \} \) denote the fundamental weights.

To show (1), it suffices to use induction on \( \ell(u) \) and the positivity of Gromov-Witten invariants \( N_{u,v}^{w,\lambda} \), together with the Key Lemma of [16] for general \( \Delta_p \). Namely, we need to show \( gr(us_{\gamma}, 0) \leq gr(u, 0) + gr(s_i, 0) \) (resp. \( gr(us_{\gamma}, \gamma^\vee) \leq gr(u, 0) + gr(s_i, 0) \)) whenever the corresponding coefficient \( \langle \omega_i, \gamma^\vee \rangle \neq 0 \). Under this hypothesis, the expected inequality will hold if we replace “\( gr^u \)” by “\( gr^{(k)} \)”, due to the Key Lemma of [16] which works for any \( \Delta^{(k)} \). Therein the proof of the Key Lemma is most complicated part of the paper. We used the notion of virtual null coroot to do some reductions, but still had to do a big case by case analysis. Hence, the expected inequality holds if we replace “\( gr^u \)” by “\( \pi_k \circ gr^u \)” (for any \( 1 \leq k \leq m \)), due to Lemma 4.5. That is, it holds when we replace “\( gr^u \)” by “\( gr^{(1)} \)”. Thus the expected inequality holds by noting that \( |gr(us_{\gamma}, 0)| \) (resp. \( |gr(us_{\gamma}, \gamma^\vee) \) is equal to \( |gr(u, 0)| + |gr(s_i, 0)| \).

The proof of (2) is exactly the same as the proof of Theorem 1.3 in [16]. The quotient \( P/B \) is again a complete flag variety, and therefore the quantum cohomology \( QH^*(P/B) \) is generated by the special Schubert classes \( \sigma^{a_i} \), \( i = 1, \ldots, r \). The proof is done by showing that \( QH^*(G/B)/\mathcal{I} \) is generated by \( \overline{\sigma^{a_i}} \), \( i = 1, \ldots, r \), respecting the same quantum Chevalley formula.

Statement (3) is in fact a consequence of (4).

The proof of (4) is similar to the above one for (1). Namely we reduce \( gr \) to \( \pi_k \circ gr = \pi_k \circ gr^{(k)} \). The expected statement will hold with respect to \( gr^{(k)} \) by using either the corresponding results of [16] for \( \Delta^{(k)} \) of type \( A \) or Theorem 2.4 when \( \Delta^{(k)} \) is not of type \( A \). The proof of the former case is much simpler than the latter one, although the ideas are similar. Here we also need to use the same observation that \( |gr(u, \lambda)| = |gr(u, 0)| + |gr(v, 0)| \) whenever the Gromov-Witten invariant \( N_{u,v}^{w,\lambda} \) in the quantum product \( \sigma^u \ast \sigma^v \) is nonzero. \( \square \)

5. Appendix.

5.1. Proof of Lemma 3.17 (Continued). Recall \( \epsilon_j = -\langle \alpha_j, \lambda \rangle \geq 0 \), \( j = 1, \ldots, r \). \( gr_{[r,r]}(q_r) = xe_r, gr_{[r,r]}(q_{r+1}) = ye_r, gr_{[r,r]}(q_{r+2}) = ze_r \). Define \( (c_1, \ldots, c_r) \) by

\[
\sum_{\beta \in R_{r_r} \setminus R_{r_{r-1}}} \beta = -y \sum_{i=1}^{r} c_i \alpha_i,
\]

which are described in Table 5.1 by direct calculations. Therein we recall that the case C9) with \( r = 2 \) has been excluded from the discussion. By definition, we have

\[
gr_{[r,r]}(q_\lambda) = (xa_r + ya_{r+1} + za_{r+2})e_r = (y \sum_{j=1}^{r} c_j \epsilon_j)e_r.
\]
Therefore, if $\varepsilon_r > 0$, then we have

$$-(xa_r + ya_{r+1} + za_{r+2}) = (-y)\sum_{j=1}^{r} c_j \varepsilon_j \geq (-y) \cdot c_r \cdot 1 \geq |R^+_P| - |R^-_P| \geq |R^+_P| - |R^+_P \cup R^-_P|,$$

If case C9) with $r = 3$ occurs, we are done. If $-gr_{[r,r]}(q_{\lambda}) = (|R^+_P| - |R^+_P \cup R^-_P|)\varepsilon_r$ held, then C5) or C7) occurs and all the above inequalities are equalities. This implies that $\varepsilon_r = 1$ and $\varepsilon_j = 0, j = 1, \ldots, r - 1$. Therefore we have $\langle \gamma, \lambda \rangle \in \{0, -1\}$ for any $\gamma \in R^+_P$, by noting that $\gamma = \varepsilon \alpha_r + \sum_{i=1}^{r-1} c_i \alpha_i$ (where $\varepsilon \in \{0, 1\}$) for all these three cases. That is, $\lambda = \lambda_B$ is the Peterson-Woodward lifting of $\lambda_P$, contradicting with the hypothesis. Hence, the statement follows if $\varepsilon_r > 0$.

Assume now $\varepsilon_r = 0$. Since $\lambda \neq \lambda_B$, we have $\varepsilon_j = 0$ for all $\alpha_j \in \Delta_P$ but exactly one exception, say $\alpha_k$. In addition, we have

$$-(xa_r + ya_{r+1} + za_{r+2}) = -yc_k \varepsilon_k = -yc_k,$$

together with the property that the coefficient $t_k$ in the highest root $\theta = \sum_{i=1}^{r} \theta_i \alpha_i$ of $R^+_P$ is not equal to 1. (Otherwise, $\lambda$ would be the Peterson-Woodward lifting of $\lambda_P$, contradicting with the hypothesis.) Thus if $c_k \geq c_r$, then we have

$$-(xa_r + ya_{r+1} + za_{r+2}) = -yc_k \geq -yc_r \geq |R^+_P| - |R^-_P| > |R^+_P| - |R^+_P \cup R^-_P|.$$ 

Here the last inequality holds since $\alpha_r \in R^+_P \setminus R^-_P$. If $c_k < c_r$, then all possible $k$, together with $-yc_k$ and the number $|R^+_P| - |R^+_P \cup R^-_P| = |R^+_P| - |R^-_P| - |R^+_P \Delta_P \setminus \alpha_k| + |R^+_P \Delta_P \setminus \alpha_k|$, are precisely given in Table 5.2 by direct calculations. In particular, we also have $-(xa_r + ya_{r+1} + za_{r+2}) = -yc_k > |R^+_P| - |R^+_P \cup R^-_P|$. 

5.2. Proof of Proposition 3.3 (3) (Continued). For each case, we uniformly denote those $(q_{\lambda}, u)$ in Table 3.2 in order as $(q_{\lambda_i}, u_i)$’s, and denote by $\tilde{u}_i$ the minimal length representative of $u_iW_P$ as before (i.e., $\tilde{u}_i$ is given by a subexpression $s_l$ of $u_i$ with the sequence ending with $r$). We also denote those virtual null coroot(s) $\mu_B$ in Table 3.1 in order as $\mu_1, \mu_2$. Namely if there is a unique $\mu_B$, then we denote $\mu_1 = \mu_2 = \mu_B$ for convenience.
Due to Lemma 3.11 again, it suffices to show all the equalities in Table 5.3 hold in $Gr^F(QH^*(G/B))$ for the corresponding cases. Note $\bar{\sigma}^{w^u} = \sum N_{u_i,u_j} q_n \sigma^w$ where $\bar{\sigma}^{w^u} = \sum N_{u_i,u_j} q_n \sigma^w$. Consequently, we have $w \in W_\rho$, $\eta = \sum \kappa_k b_k \alpha_k$, and $(\ell(u_i) + \ell(u_j) = b_i |gr_{r,r}(q_r)| + |gr_{r,r}(\sigma^w)|$. Since $0 \leq |gr_{r,r}(\sigma^w)| \leq |R^+_\rho| - |R^+_{\rho}|$, we have $\bar{b}^{\text{max}} \geq b_r \geq \bar{b}^{\text{min}} \geq 0$ for certain integers $\bar{b}^{\text{max}}, \bar{b}^{\text{min}}$. Write $w = \nu w_2$ where $v \in W_\rho$ and $w_2 \in W_\rho$. Once $b_r$ is given, both $\ell(v)$ and $(w_2, \eta)$ are fixed by the above equalities on gradings together with Lemma 3.12. There is a unique term, say $\nu_\sigma \sigma^w$, on the right hand side of each expected identity in Table 5.3 as well, which is of the form either $\nu_\sigma \sigma^w$ or $\nu_\sigma \sigma^w$. Consequently, we have $b^{\text{max}} \geq b_r \geq b^{\text{min}} \geq 0$ for certain integers $b^{\text{max}}, b^{\text{min}}$. Write $w = \nu w_2$ where $v \in W_\rho$ and $w_2 \in W_\rho$. Finally, we have $w = \nu w_2$ where $v \in W_\rho$ and $w_2 \in W_\rho$. Hence, in order to conclude the expected equality, it suffices to show

1. $N_{u_i,u_j} = 1$ if $v = \tilde{v}$, or 0 otherwise;
2. $N_{u_i,u_j} = 0$ whenever $b^{\text{max}} > b_r(n) \geq b^{\text{min}}$. Similar to the proof in section 3.4, this claim follows from the next two:
   (a) $\sigma^{u_i} \cup \sigma^{u_j} = 0$ in $H^*(P/B)$;
   (b) $N_{u_i,u_j} = 0$ whenever $l^{\text{max}} > b_r(n) \geq \max \{1, b^{\text{min}}\}$.

To show (1), we use Proposition 3.15 (2) repeatedly. As a consequence, we can conclude that $N_{\tilde{v}^{\nu_2}, \tilde{v}^{\nu_2}}$ coincides with a classical intersection number given in Table 5.3 as well, which is of the form either $N_{u_i,u_j}^{-1,0}$ or $N_{u_i,u_j}^{\nu_2,0}$. The formal one is equal to 1 by checking $u_i = u_j^{-1}$ easily. Denote $\Delta_\rho := \Delta_\rho \setminus \{\alpha_k\}$. For the latter one, it is easy to check that both $u', v'$ are in $W_\rho$, where $s_k$ denotes the last simple reflection in the reduced expression of $\tilde{v}_2$. Thus $N_{u',v'}^{\nu_2,0} = 0$ unless $\nu v_2$ is in $W_\rho$ as well. In addition, it is easy to check that $\ell(u') + \ell(v') = \ell(\nu v_2) = \dim P/\tilde{P}$, and that $u'$ is the minimal length representative of $W_\rho = w P/v' W_\rho$. Thus $u'$ is dual to $v'$ with respect to the canonical non-degenerated bilinear form on $H^*(P/\tilde{P})$. Hence, $N_{u',v'}^{\nu_2,0} = 1$. (See e.g. section 3 of [10] for these well-known facts.) That is, (1) follows. To illustrate the above reduction more clearly, we give a little bit more details for $N_{\tilde{v}^{\nu_2}, \tilde{v}^{\nu_2}}$ in case C4) with $r = 6$. In this case, we have $u_1 = s_{54362132436} s_{5438383281}$ and $q_\theta = q_{\lambda_2 - 2\lambda_1} = q_1^2 q_2^2 q_3 q_4 q_6$. Proposition 3.15 (2) $u_1 = s_{54362132436} s_{5438383281}$ and $q_\theta = q_{\lambda_2 - 2\lambda_1} = q_1^2 q_2^2 q_3 q_4 q_6$. Using Proposition 3.15 (2), we can first deduce

| Table 5.2 |
| --- | --- | --- |
| $r$ | $-yc_k$ | $|R^+_\rho| - |R^+_\rho \cup R^+_{\rho}|$ |
| C4) | $r = 7$ | 2 | 42 | 32 |
| | | 6 | 28 | 27 |
| C5) | $r = 5$ | 2 | 8 | 7 |
| | $r = 6$ | 2 | 10 | 9 |
| | $r = 7$ | 2 | 12 | 11 |
| | 3 | 18 | 15 |
| C9) | $r = 3$ | 2 | 6 | 5 |
| C10) | 1 | 4 | 3 |
We have a representative $\tilde{w}_1 \in W_{P/B}$ such that $\ell(\tilde{u}_1) = \ell(w_1) + \ell(\tilde{w}) = 1$. Then, we have $N_{u_1, u_1}^{w_1, 0} = N_{u_1, u_1}^{u_1, \tilde{u}_1} = N_{u_1, u_1}^{w_1, \tilde{w}} = N_{u_1, u_1}^{u_1, \tilde{w}} = 0$. Consequently, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Hence, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Therefore, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Consequently, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Hence, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Therefore, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Consequently, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Hence, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Therefore, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Consequently, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Hence, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Therefore, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Consequently, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Hence, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Therefore, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Consequently, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Hence, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Therefore, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Consequently, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Hence, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Therefore, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Consequently, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Hence, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Therefore, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Consequently, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Hence, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Therefore, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Consequently, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Hence, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$. Therefore, we have $N_{u_1, u_1}^{w_1, 0} = 0$ unless $w_1 = \tilde{u}_1$.
Similarly, for C4) with \( r = 6 \) we have \( N_{u,\eta_1}^{w,\eta} = 0 \) by considering \( \text{sgn}_6 \).

2) For case C4) with \( r = 7 \) and \( q_1 = q_2 = q_3 = q_4 = q_5 \), we can first conclude \( N_{u,\eta_1}^{w,\eta} = N_{u,\eta_1}^{w_1234574321,0} \) by using Proposition 3.15 (2) repeatedly. Here
\[
u' := u_1 s_{123474321} = s_{1234753654723456}.
\]
Denote \( \Delta_P := \Delta_P \setminus \{\alpha_6\} \), and note \( u' \in W_P^P \). Thus we can further conclude \( N_{u,\eta_1}^{w,\eta} = N_{u',\eta'}^{w,0} \). Since \( 2(\nu') = 34 > 63 - 30 = \text{dim} P/P' \), we have \( N_{u',\eta'}^{w,0} = 0 \). Similarly, (b) follows if case C7) occurs.

3) For case C10), we can conclude \( N_{u_1,u_1}^{w,\eta} = N_{s_{321},s_{321}}^{w,0} \). Therefore it is equal to
\[
0, \text{by noting} (s_{321})^{-1} w_P = s_{321} s_{321} \not\in (s_{321})^{-1} \text{ and using Lemma 3.27.}
\]
Hence, (b) follows.

Acknowledgements. The author is grateful to Naichung Conan Leung for valuable suggestions and constant encouragement. He also thanks Ivan Chi-Ho Ip and Leonardo Constantin Mihalecea for useful discussions and advice. Especially, he thanks the referee for his/her careful reading and very helpful comments. The author is partially supported by JSPS Grant-in-Aid for Young Scientists (B) No. 25870175.

REFERENCES


