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Metrics on D-brane Orbifolds

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Abstract

We calculate the metric on the D-brane vacuum moduli space for backgrounds of the form $\mathbb{C}^3/\Gamma$ for cyclic groups Γ. In the simplest procedure — starting with a flat "seed" metric on the covering space — we find that the resulting D-brane metric is not Ricci-flat. We argue that this is likely to be true of the true 0-brane metric at weak string coupling.

1 Introduction

During the last year, developments in nonperturbative string theory have made it possible to probe the structure of space-time on sub-stringy scales. In the context of weakly coupled IIa string theory, this emerges from the dynamics of gauge theory, the world-volume theories of Dirichlet zero-branes, and geometrical concepts appear to remain sensible all the way down to the eleven-dimensional Planck scale l_{p11} [1]. Short distance geometry is similar to the long distance geometry probed by fundamental strings, as one would expect if the two are simply limits of a single idea of 'geometry,' but with interesting differences.

In a recent paper [2] the first steps towards understanding short distance Calabi-Yau geometry were taken. Comparing with problems with higher supersymmetry, the most salient difference is that the metric can depend on an arbitrary function, the Kähler potential. Thus the questions of what metric is seen by D0-branes and whether it is determined by a local equation of motion can be studied directly.

In [2], a study was made of backgrounds of the form $\mathbb{C}^3/\Gamma$ for cyclic subgroups $\Gamma \in SU(3)$, focusing on comparing the topological properties probed by D0-branes with those probed by fundamental strings. As has been known for some time, fundamental strings probe a rich phase structure in Calabi-Yau backgrounds which typically includes regions which are most directly described in a non-geometrical abstract conformal field theory language. These non-geometrical phases emerge from the linear sigma model analysis of [3]. As pointed out in [4] and utilized in [5], if one allows for analytic continuation from geometrical regions and uses physically motivated parameters, even these non-geometrical phases appear to have geometrical content in terms of Calabi-Yau spaces in which part or even the whole manifold has been shrunk to string or sub-stringy scales.

The newfound power of D-branes allows us to go beyond the indirect method of analytically continued fundamental string conclusions. In [2], it was found that, in a rather novel manner, the topological properties probed by D0-branes in such backgrounds appear to match the analytically continued fundamental string results at short distances.

In this note we go beyond the topological questions considered in [2] and take a first look at metric properties. Our approach, although calculationally intensive beyond the simplest of examples, is a straightforward extension of the techniques used in [2, 6, 7]. Namely, we begin with D-branes on $\mathbb{C}^3$, arranged in the regular representation of the group Γ. We then truncate to the Γ-invariant sector of this D-brane Lagrangian. The result is a gauged sigma model whose low energy configuration space will be interpreted as sub-stringy space-time. This prescription can be justified by world-sheet computation at weak string coupling [6], while its primary justification at strong string coupling is simplicity and the need for the theory to remain non-singular in the orbifold limit [8].

In the case of D0-branes and quantum mechanics, quantum effects are controlled by the dimensionless parameters g_s/m_i^3, where the m_i are the masses of the degrees of freedom being integrated out to derive the low energy effective theory. Here the massive degrees of freedom are strings stretched between image branes, whose mass can be estimated as $m^2 \sim |\zeta|/\alpha'^2$ [9]. Restoring all the α''s, the condition $g_s/m_i^3 << 1$ translates into $|\zeta| >> l_{p11}^2$, so for blowup larger than the eleven-dimensional Planck scale the effective theory can be derived by classical considerations.

The classical moduli space is a Kähler quotient (the same proposed in [10]), which allows direct determination of the quotient space metric. As we shall see, the explicit calculations are facilitated by methods discussed in [11]. The result depends on the original (or "seed") metric of the gauged sigma model, but even starting with a flat seed metric produces quite non-trivial quotient metrics.

In the case of $\mathbb{C}^2/\Gamma$ for Γ lying in $SU(2)$ considered in [6, 7], the effective theory has eight real supercharges, and the quotient construction is of *hyperkähler* type [11]. This ensures that the resulting metric is Ricci-flat. In the Kähler quotients considered here, although the metric will be of the correct topological type (zero first Chern class), nothing ensures a similar Ricci-flatness condition. In fact, we shall find in the examples we study, that the resulting metric is not Ricci-flat. In the mathematical context, this was first pointed out by Sardo Infirri [10].

The result has different interpretations depending on whether we consider weak or strong string coupling. At weak coupling, we know that fundamental strings do not see a Ricci-flat metric, so it should not be surprising that D-branes do not see one either.

In principle the D0-brane metric could be computed from world-sheet computation, in two ways. One could start with the Calabi-Yau sigma model, introduce general boundary conditions along the lines of [12], and derive the conditions for an RG fixed point. One would presumably obtain the Nambu-Born-Infeld action with an effective metric related to the Ricci-flat metric by α' corrections, a priori different from those of bulk renormalization. Alternatively, one could compute the true seed metric of the gauged sigma model of [2] along the lines of [6], and then perform the quotient. In the orbifold limit, the seed metric is flat, and the analyticity of conformal perturbation theory implies that the true metric is an analytic function of the blow-up parameters. The expectation from both approaches is that, although the precise metric cannot be determined without additional world-sheet computation, there is no reason for it to be Ricci-flat.

D0-branes are also the fundamental degrees of strongly coupled IIa theory, and a precise form of this statement is the M(atrix)-theory conjecture [13]. In the regime we study, their dynamics is governed by eleven-dimensional supergravity, leading inescapably to the conclusion that they move in a background Ricci-flat metric.

Now the original proposal of [13] asserts that M-theory is produced only by taking the large N limit, and our result offers no contradiction to this statement. On the other hand, a more recent proposal of Susskind [14] asserts that the finite N theory also has an M-theory interpretation, as the theory with a compactified light-cone dimension. We see no reason why such compactification would modify the prediction of supergravity, in which case the model with a flat seed metric is in contradiction with this proposal.

Now there is no a priori reason to take a flat seed metric in this context and the simplest response to this result is to ask whether the model can be modified to produce a Ricci-flat metric. The simplest possibility would to be to define the model with a seed metric tuned to produce Ricci flatness after the quotient. Since the quotient metric has zero first Chern class and the seed metric has more parameters than the resulting Kähler potential, there is no obvious obstruction to doing this. However this leaves unanswered the question of what consistency condition forces this choice of seed metric.

Although we leave this question for future work, we would like to point out here that a promising context for answering it is to consider the theory with more than one D0-brane. Unlike the case of a single D0-brane, now it is possible for massive strings to become light (when component D0-branes approach to distances $d \sim l_{p11}$), and thus quantum effects can always become large. Thus we propose as an interesting goal for future work the construction of actions which realize the axioms of [15] while remaining non-singular in the orbifold limit, presumably by deforming the seed metric in the $N > 1$ version of the construction of [2].

2 D-branes on Orbifolds and Their Metrics

In this section we briefly review our procedure for discussing D-branes on orbifolds of the form $\mathbb{C}^3/\Gamma$. For more details the reader should consult [2,6,7].

Our starting point is an $\mathcal{N} = 4, d = 4$ $U(n)$ supersymmetric gauge theory where $n = \dim|\Gamma|$, arising from D-brane 'compactification' on $\mathbb{C}^3$. We then specify an action of Γ on the fields of the covering theory, and truncate the $\mathcal{N} = 4$ Lagrangian to the fields transforming trivially under this action. The result is a gauged supersymmetric linear sigma model with a nontrivial superpotential.

Specifically, we take Γ to act on the Chan-Paton degrees of freedom (which ultimately become the D-brane spatial coordinates) in the regular representation (other possibilities are discussed in [8] and [2]), and on the $\mathbb{C}^3$ coordinates Z^i via $Z^i \to \omega^{a_i} Z^i$ with $\omega = \exp(2\pi i/n)$ and $a_1 + a_2 + a_3 \cong 0$ (mod n). The latter condition ensures that Γ lies in $SU(3)$ and hence, from a four-dimensional perspective, we have $\mathcal{N} = 1$ in the open string sector, and $\mathcal{N} = 2$ in the closed string sector.

In [2] it was found to be convenient to think of the constraints arising from the superpotential as if they were D-term constraints in an auxiliary linear sigma model. The latter was shown to yield a vacuum phase structure which meshes well with that of the analytically continued fundamental string, at least as far as topological properties are concerned.

Our present purpose is to understand metric properties. We will do so by utilizing our supersymmetric gauged sigma models (or "linear sigma models") to produce classical moduli space metrics of this quotient form [11].

Specifically, we will determine the metric in two steps. First, we restrict the original flat metric to solutions of $\partial W = 0$ where W is the superpotential. Second, we restrict this metric to solutions of the D-flatness conditions and quotient by the gauge action. This second step is a symplectic reduction or "Kähler quotient" as it is guaranteed to produce a Kähler metric.

We note that if we were to take the superpotential in the first step to be the one determined by $\mathcal{N} = 2$, $d = 4$ supersymmetry and perform both steps, the resulting procedure is the hyperkähler quotient. On general grounds the hyperkähler quotient is guaranteed to produce a Ricci-flat metric. Physically, this would be the relevant construction if we were considering *two-dimensional* orbifolds $\mathbb{C}^2/\Gamma$. In the three-dimensional case of present interest, though, there is no guarantee that the Kähler quotient construction yields a Ricci-flat metric; indeed in the simplest example, a hypersurface with $c_1 = 0$ in a $\mathbb{P}^n$ realized as the usual quotient of $\mathbb{C}^{n+1}$ by a $U(1)$ action, it is not so. We emphasize that this is so in our case even though we take Γ to lie inside of $SU(3)$. This ensures that we preserve the $c_1 = 0$ condition, but it does not ensure that the specific metric produced is Ricci-flat. We shall explicitly see this in what follows.

3 Calculational Procedure

The two step procedure for determining the D-brane metric outlined above can be carried out as follows. For the time being, we start with the flat Kähler potential on the covering space

$$K_f = \sum_i |z_i|^2,$$

where the coordinates z_i run over the subset of the original $3|\Gamma|$ chiral fields that survive the orbifold projection.

The first step in the projection is simply to restrict to the submanifold by solving the conditions $\delta W = 0$. To carry out the second step, we use the complex reduction procedure of [11]. In physical terms, we write the action with the coupling to vector superfields V_i, and then integrate them out. The highest component of V becomes a Lagrange multiplier for the D-flatness condition. This action has a complexified gauge invariance which can then be fixed arbitrarily.

Explicitly, we write the gauged Kähler potential

$$K = \sum_i (|z_i|^2 \exp \sum_a t_i^a V_a) - \sum_a \zeta^a V_a \ ,$$

where t_i^a is the charge of z_i under the a'th $U(1)$, and ζ_a are the Fayet-Iliopoulos parameters. We will also let $q_a = \exp V_a$ in the following.

To derive an explicit metric, we choose a gauge slice X, with local coordinates x^i. The gauged Kähler potential on the slice is $K = K(\{x^\mu\}, \{q^j\}, \{\zeta_k\})$. We then determine the q^j by solving the equations

$$\partial_j K|_X = 0, \tag{3.1}$$

and find the metric

$$g_{\mu\nu} = \partial_\mu \partial_\nu K|_X.$$

In these and other expression, ∂_μ and ∂_j are with respect to x^μ and q^j. We can also write this as

$$g_{\mu\nu} = \partial_\mu\partial_\nu K + \partial_\mu\partial_j K \partial_\nu q^j + \partial_\nu \partial_j K \partial_\mu q^j + \partial_l \partial_j K \partial_\nu q^j \partial_\mu q^l.$$

The partials $\partial_\mu q^i$ are determined by considering

$$\frac{d}{dq^i}\partial_\mu K = 0,$$

which implies

$$\partial_\mu \partial_i K = -\partial_i \partial_j K \partial_\mu q^j$$

and

$$\partial_\mu q^j = -(\partial_\mu \partial_i K)(\partial_j \partial_i K)^{-1} ,$$

where the latter inverse is in the matrix sense. Using both of these equations we find

$$g_{\mu\nu} = \partial_\mu \partial_\nu K - A^t (B^{-1})^t A ,$$

where $A = (A_{i\mu}) = (\partial_i \partial_\mu K)$ and $B = (B_{ij}) = (\partial_i \partial_j B)$.

This determines the metric explicitly given the ability to solve the equations (3.1). In practice, except in the simplest of examples, this equation must be solved numerically.

We will describe the results in two examples below. As our interest will be to determine if the D-brane metrics are Ricci-flat, we first, as a point of comparison, briefly review relevant work of Calabi on the construction of Ricci-flat metrics.

4 Ricci Flat Metrics à la Calabi

In [16], explicit constant curvature Kähler metrics are constructed on non-compact spaces. One starts with a constant curvature Kähler metric on $\mathcal{M}$ of complex dimension $n-1$, and writes a metric ansatz on a line bundle $\mathcal{E}$ over $\mathcal{M}$ for which the differential equation $R_{i\bar{j}} = cg_{i\bar{j}}$ can be solved explicitly. If $\mathcal{E}$ is the canonical line bundle, the resulting metric will be Ricci-flat.

Let z^i be coordinates on $\mathcal{M}$ and $K_0(z,\bar{z})$ the Kähler potential, satisfying

$$\det g^{(0)}_{i\bar{j}} = \det \partial_i \bar{\partial}_{\bar{j}} K_0 = e^{-lK_0} |(holomorphic)|^2.$$

Let w be a coordinate on the fiber. The metric ansatz on $\mathcal{E}$ is then

$$K = K_0 + u(a(z,\bar{z})|w|^2). \tag{4.1}$$

Since $x \equiv a|w|^2$ is a Hermitian form on the fiber, it has an associated connection $L = \partial \log a$ and curvature $S = \bar{\partial} L$.

Metrics derived from the ansatz (4.1) can be expressed simply in terms of the basis of one-forms dz^i and $\nabla w = dw + wL$:

$$\partial\bar{\partial}K = (g^{(0)}_{i\bar{j}} + xu'(x)S_{i\bar{k}}J^{\bar{k}}_{\bar{j}})dz^i d\bar{z}^{\bar{j}} + (u'(x) + xu''(x))a|dw + wL|^2 \ ,$$

where J is the complex structure. In this basis, $\det g$ is not hard to compute.

Next, one takes for a the form induced on the canonical line bundle from the constant curvature metric. Coordinates can be chosen in which this is just $a = e^{lK_0}$, and $S_{i\bar{j}} = l\omega_{i\bar{j}}$. The equation $\det g = \text{const}$ then reduces to

$$(1 + lxu')^{n-1}(xu'' + u') = \text{const}.$$

This has the explicit real solution

$$u(x) = \frac{n}{l}\left(\sqrt[n]{1+cx} - 1\right) - \frac{1}{l}\sum_{j=1}^{n-1}(1-\omega^j)\log\left(\frac{\sqrt[n]{1+cx} - \omega^j}{1-\omega^j}\right)$$

with $\omega = e^{2\pi i/n}$. The holomorphic n-form is simply

$$\Omega^{(n)} = dz^1 \wedge \ldots \wedge dw.$$

We now take $\mathcal{M} = \mathbb{P}^{n-1}$, and apply this construction to obtain a Ricci-flat metric on $\mathcal{O}_{\mathbb{P}^{n-1}}(-n)$. For $n = 2$ this produces the Eguchi-Hanson metric. In general Calabi showed the metric is complete but the ALE nature of the space seems to have not been explicitly addressed. Fortunately, since the asymptotic behavior for large x will be governed by the first term in u, $K \sim x^{1/n} + O(\log x)$ with $x \sim |w|^2 e^{lK_0}$, it is fairly easy to show that a given example is ALE.

For $n = 3$, the resulting total space will be $\mathcal{O}_{\mathbb{P}^2}(-3)$, which is a blowup of $\mathbb{C}^3/\mathbb{Z}_3$. The constant curvature metric has $K_0 = \zeta \log \sum |z_i|^2$ with $l = 3$. The original $\mathbb{P}^2$ is $w = 0$ so ζ becomes the volume of the two-cycle.

5 Examples

In this section, we carry out the Kähler reduction to determine the orbifold and blown-up metric probed by D-branes.

5.1 Example: the Eguchi-Hanson Metric

As shown in [6,7], the D-brane model realizes Kronheimer's construction as a hyperkähler quotient of $\mathbb{C}^4$ by a $U(1)$ action. Usually this is done in a way which makes the $SU(2)$ acting on the complex structures manifest, and produces the metric in the form given by Gibbons and Hawking. Here we want a single complex structure manifest, so we do the complex reduction.

We start with four complex fields b_0, b_1, $\bar{b}_0$ and $\bar{b}_1$ with charges $+2$, -2, -2 and $+2$ respectively. The original holomorphic two-form is

$$\Omega_f = db_0 \wedge d\bar{b}_0 + db_1 \wedge d\bar{b}_1. \tag{5.1}$$

The gauged Kähler potential is

$$K = e^V(|b_0|^2 + |\bar{b}_1|^2) + e^{-V}(|b_1|^2 + |\bar{b}_0|^2) - \zeta_R V$$

while the constraints $\delta W = 0$ are

$$b_0\bar{b}_0 - b_1\bar{b}_1 = \zeta_C.$$

The resulting metric is invariant under rotating the parameter $\vec{\zeta}$, so we take $\zeta_C = 0$ and relabel $\zeta_R = \zeta$.

Write $q \equiv e^V$, $b_0 = z$, $\bar{b}_0 = w$, set the gauge $\bar{b}_1 = 1$ and solve $b_1 = zw$. We then have

$$K = q(1 + |z|^2) + \frac{1}{q}(1 + |z|^2)|w|^2 - \zeta \log q.$$

The condition $\partial K/\partial q = 0$ determines

$$\begin{aligned} 0 &= q^2 - |w|^2 - \tfrac{\zeta q}{1+|z|^2} \\ q &= \tfrac{1}{2(1+|z|^2)}\left(\zeta \pm \sqrt{\zeta^2 + 4(1+|z|^2)^2|w|^2}\right) \end{aligned}$$

and after substituting back,

$$K = \pm\sqrt{\zeta^2 + 4(1 + |z|^2)^2|w|^2} - \zeta \log q.$$

Notice that we have made a fortuitous choice of coordinates: the combination $(1+|z|^2)^2|w|^2$ is the x of Calabi's construction, and we get the metric exactly in his form. (The prefactor ζ is controlling the overall scale and could be put inside by change of coordinates.) The simple way to find this is to look for a choice which turns (5.1) into $dz \wedge dw$.

5.2 $\mathbb{C}^3/\mathbb{Z}_3$

Now, as in [2], the model starts with nine complex fields X^i, Y^i and Z^i; there are two $U(1)$ actions which we associate with vector superfields U and V. As before, let $U = \log p$ and $V = -\log q$, then the gauged Kähler form is

$$\begin{aligned} K = &\tfrac{1}{pq}(|X^0|^2 + |Y^0|^2 + |Z^0|^2) + q(|X^1|^2 + |Y^1|^2 + |Z^1|^2) \\ &+p(|X^2|^2 + |Y^2|^2 + |Z^2|^2) - \zeta_1 \log p - \zeta_2 \log q. \end{aligned} \tag{5.2}$$

There are nine constraints $X^iY^j = X^jY^i$, $X^iZ^j = X^jZ^i$ and $Y^iZ^j = Y^jZ^i$ of which four are independent, leaving us with a five-dimensional solution space. By using complexified gauge transformations we can go to the three-dimensional slice $Z_1 = Z_2 = 1$. The holomorphic three-form is

$$\begin{aligned} \Omega^{(3)} &= \textstyle\sum_i dX^i \wedge dY^{i+1} \wedge dZ^{i+2} \\ &\to dX^1 \wedge dY^2 \wedge dZ^0, \end{aligned} \tag{5.3}$$

which will be non-singular assuming these are single-valued coordinates. The constraints are solved by $Y^2 = Y^1$, $X^2 = X^1$, $Y^0 = Y^1Z^0$, $X^0 = X^1Z^0$.

Comparison of three-forms and the symmetry of the situation motivate the identification

$$z_1 = X^1 \qquad z_2 = Y^1 \qquad w = Z^0, \tag{5.4}$$

after which (5.2) becomes

$$K = \left(1 + |z_1|^2 + |z_2|^2\right)\left(p + q + \frac{|w|^2}{pq}\right) - \zeta_1 \log p - \zeta_2 \log q.$$

p and q can again be eliminated explicitly. Let $A = 1 + |z_1|^2 + |z_2|^2$, $B = A|w|^2$, then

$$K = A(p+q) + \frac{B}{pq} - \zeta_1 \log p - \zeta_2 \log q$$

and $\partial K = 0$ implies

$$\begin{aligned} 0 &= p^2qA - pq\zeta_1 - B \\ 0 &= pq^2A - pq\zeta_2 - B\ , \end{aligned}$$

which combine to

$$\begin{aligned} (p-q)A &= \zeta_1 - \zeta_2 \\ (p-q)B &= pq(q\zeta_1 - p\zeta_2). \end{aligned}$$

Let $u = p - q$ and $v = A(q\zeta_1 - p\zeta_2)/(\zeta_1 - \zeta_2)$, then

$$\begin{aligned}
u &= (\zeta_1 - \zeta_2)/A \\
p &= (v + \zeta_1)/A \\
q &= (v + \zeta_2)/A \\
x &= A^2 B = v(v+\zeta_1)(v+\zeta_2) \\
K &= \left(2 + \tfrac{1}{\zeta_1 - \zeta_2}\right) v - \zeta_1 \log(v+\zeta_1) - \zeta_2 \log(v+\zeta_2) + (\zeta_1+\zeta_2)\log A \ .
\end{aligned}$$

Notice that this expression shows a strong resemblance to Calabi's expression. In particular, if we take $x = v(v+\zeta_1)(v+\zeta_2)$ with $\zeta_1 = 1-\omega$ and $\zeta_2 = 1-\omega^2$ the result is exactly that of Calabi. However, the Fayet-Iliopoulos parameters are physically (and mathematically) dictated to be *real*. Therefore, the choice we have taken in order to make contact with Calabi's work is not physical; at best it is an analytic continuation from the physical domain. For real values of ζ_1, ζ_2 we find that the resulting metric is *not* Ricci-flat. Thus, although we have come tantalizingly close to a Ricci-flat metric, in reality the quotient construction will not produce one except in the trivial case $\zeta = 0$.

The non-Ricci flatness of this metric was first found in [10] using a less explicit argument. The construction presented here not only gives the explicit form of the quotient metric but indeed allows writing explicit one-parameter families of metrics interpolating between the Ricci-flat and quotient metrics, by allowing the parameters ζ to move off into the complex plane. The physical D0-brane metrics form such a family, interpolating between Ricci-flat for $\zeta >> \alpha'$ and a true quotient metric for $\zeta << \alpha'$, and one is led to wonder if they can be described in this way.

It is also interesting to note that while the Ricci-flat metrics form a family with one real parameter (the volume of the S^2), the non-Ricci flat metrics naturally depend on two real parameters, the ζ_i. The volume of the S^2 corresponds to one linear combination (which one depends on the signs of the ζ's as described in [2]), but the full metric depends on both. The other parameter becomes $\int B$ in the large blowup limit, and in this sense we see that the sub-stringy geometry explicitly depends on moduli which were non-metric moduli at large volume.

5.3 $\mathbb{C}^3/\mathbb{Z}_n, n > 3$

For $n > 3$, the analogous equations are easy to write down but do not appear to admit analytic solutions for general ζ. However, they are certainly amenable to numerical evaluation. To assess whether or not the metric is Ricci-flat, it is simplest to compute $\det g$ and see whether or not it is constant over the manifold. In the present circumstances, constant $\det g$ is equivalent to Ricci-flatness. We have done this for $n = 5$ and $n = 7$ for various choices of the Fayet-Iliopoulos parameters. We find that the metric, once again, is *not* Ricci-flat.

6 Conclusions

For weak string coupling and sub-stringy blowup parameters, D0-branes propagating on an ALE space of three complex dimensions see a metric which is calculable in principle by a combination of world-sheet techniques and the quotient construction described here. It depends on the full complexified Kähler class, not just the real part. There is no reason to expect the result to be Ricci-flat.

We gave explicit forms for the Ricci-flat metric on $\mathbb{C}^3/\mathbb{Z}_3$ and for the quotient metric obtained by starting with a flat seed metric. Although we found that the quotient construction could indeed produce a Ricci-flat metric in this case, this was obtained by a rather formal procedure of "complex reduction with complex Fayet-Iliopoulos parameters" for which we did not find a satisfactory physical (or mathematical) interpretation. (It would be interesting if there were one.) We also found no evidence that similar adaptation of the quotient construction could produce Ricci-flat metrics in general, and suspect that it arose from particular simplicities of the $\mathbb{Z}_3$ case. It does provide a nice family of metrics interpolating between Ricci-flat and quotient metrics.

In the M-theory limit, we believe it is possible to postulate a seed metric which reproduces the expected physical result, that after quotienting a D0-brane sees a Ricci-flat metric. We furthermore pointed out that the same construction applies to bound states of $N > 1$ D0-branes, and that quantum effects can become large in this case, possibly leading to new consistency conditions.

We close by listing further interesting questions to address in the present framework:

1. What is the geometric interpretation of the dependence of the metric on the complexified Kähler class? We know that in quantum geometry the effective volumes of two-cycles and four-cycles can be controlled independently [17]; this should translate into statements about the masses of wrapped branes outlined in [2], but can we relate these brane masses to the explicit metric? Can we postulate a B-field on the two-cycle which reproduces quantum volumes as proposed there?

2. We argued that on grounds of genericity there exists a seed metric for which the quotient metric will be Ricci-flat. Is it true that this seed metric will be the flat metric plus corrections analytic in ζ, as we might

expect to get either from conformal field theory considerations or from the background considerations in [9] ? This would allow corrections expressed as a double series in ζ/α' and ζ/z^2. An even stronger condition which might emerge from deeper study of the world-sheet computation is that the corrections to the seed metric should be real analytic in the coordinates as well. This would preclude ζ/z^2 corrections and imply that in the limit $|\zeta| << \alpha'$ the D0-brane metric reduces to the quotient metric computed here.

3. Is there any simple equation which the true weak coupling D0-brane metric should satisfy (e.g. a β-function equation with all α' corrections included). Because this metric is physically observable, this question is better motivated than for the analogous fundamental string metric. One possibility (suggested by G. Tian) is to try $0 = R_{ij} + \delta S/\delta g_{ij}$ for S some generating functional for an anomaly of a six-dimensional Dirac operator. S would be an eight-form constructed from curvatures, which fits with the known three-loop sigma model correction, but we have no real justification for this idea at present.

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