

Isolated Horizons: the Classical Phase Space

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Abstract

A Hamiltonian framework is introduced to encompass non-rotating (but possibly charged) black holes that are “isolated” near future time-like infinity or for a finite time interval. The underlying space-times need not admit a stationary Killing field even in a neighborhood of the horizon; rather, the physical assumption is that neither matter fields nor gravitational radiation fall across the portion of the horizon under consideration.

A precise notion of non-rotating isolated horizons is formulated to capture these ideas. With these boundary conditions, the gravitational action fails to be differentiable unless a boundary term is added at the horizon. The required term turns out to be precisely the Chern-Simons action for the self-dual connection. The resulting symplectic structure also acquires, in addition to the usual volume piece, a surface term which is the Chern-Simons symplectic structure. We show that these modifications affect in subtle but important ways the standard discussion of constraints, gauge and dynamics. In companion papers, this framework serves as the point of departure for quantization, a statistical mechanical calculation of black hole entropy and a derivation of laws of black hole mechanics, generalized to isolated horizons. It may also have applications in classical general relativity, particularly in the investigation of analytic issues that arise in the numerical studies of black hole collisions.

1 Introduction

In the seventies, there was a flurry of activity in black hole physics which brought out an unexpected interplay between general relativity, quantum field theory and statistical mechanics [18, 17, 22]. That analysis was carried out only in the semi-classical approximation, i.e., either in the framework of Lorentzian quantum field theories in curved space-times or by keeping just the leading order, zero-loop terms in Euclidean quantum gravity. Nonetheless, since it brought together the three pillars of fundamental physics, it is widely believed that these results capture an essential aspect of the more fundamental description of Nature. For over twenty years, a concrete challenge to all candidate quantum theories of gravity has been to derive these results from first principles, without having to invoke semi-classical approximations.

Specifically, the early work is based on a somewhat ad-hoc mixture of classical and semi-classical ideas—reminiscent of the Bohr model of the atom—and generally ignored the quantum nature of the gravitational field itself. For example, statistical mechanical parameters were associated with macroscopic black holes as follows. The laws of black hole mechanics were first derived in the framework of *classical* general

relativity, without any reference to the Planck's constant \hbar [17]. It was then noted that they have a remarkable similarity with the laws of thermodynamics if one identifies a multiple of the surface gravity κ of the black hole with temperature and a corresponding multiple of the area a_{hor} of its horizon with entropy. However, simple dimensional considerations and thought experiments showed that the multiples must involve \hbar , making quantum considerations indispensable for a fundamental understanding of the relation between black hole mechanics and thermodynamics [18]. Subsequently, Hawking's investigation of (test) quantum fields propagating on a black hole geometry showed that black holes emit thermal radiation at temperature $T_{\text{rad}} = \hbar\kappa/2\pi$ [22]. It therefore seemed natural to assume that black holes themselves are hot and their temperature T_{bh} is the same as T_{rad} . The similarity between the two sets of laws then naturally suggested that one associate entropy $S = a_{\text{hor}}/4\hbar$ with a black hole of area a_{hor} . While this procedure seems very reasonable, these considerations can not be regarded as providing a "fundamental derivation" of the thermodynamic parameters of black holes. The challenge is to derive these formulas from first principles, i.e., by regarding large black holes as statistical mechanical systems in a suitable quantum gravity framework.

Recall the situation in familiar statistical mechanical systems such as a gas, a magnet or a black body. To calculate their thermodynamic parameters such as entropy, one has to first identify the elementary building blocks that constitute the system. For a gas, these are molecules; for a magnet, elementary spins; for the radiation field in a black body, photons. What are the analogous building blocks for black holes? They can not be gravitons because the gravitational fields under consideration are stationary. Therefore, the elementary constituents must be non-perturbative in the field theoretic sense. Thus, to account for entropy from first principles within a candidate quantum gravity theory, one would have to: i) isolate these constituents; ii) show that, for large black holes, the number of quantum states of these constituents goes as the exponential of the area of the event horizon; iii) account for the Hawking radiation in terms of quantum processes involving these constituents and matter quanta; and, iv) derive the laws of black hole thermodynamics from quantum statistical mechanics.

These are difficult tasks, particularly because the very first step – isolating the relevant constituents– requires new conceptual as well as

mathematical inputs. Furthermore, in the semi-classical theory, thermodynamic properties have been associated not only with black holes but also with cosmological horizons. Therefore, the framework has to be sufficiently general to encompass these diverse situations. It is only recently, more than twenty years after the initial flurry of activity, that detailed proposals have emerged. The more well-known of these comes from string theory [35] where the relevant elementary constituents are associated with D-branes which lie outside the original perturbative sector of the theory. The purpose of this series of articles is to develop another scenario, which emphasizes the quantum nature of geometry, using non-perturbative techniques from the very beginning. Here, the elementary constituents are the quantum excitations of geometry itself and the Hawking process now corresponds to the conversion of the quanta of geometry to quanta of matter. Although the two approaches seem to be strikingly different from one another, we will see [4] that they are in certain sense complementary.

An outline of ideas behind our approach was given in [3, 9]. In this paper, we will develop in detail the classical theory that underlies our analysis. The next paper [4] will be devoted to the details of quantization and to the derivation of the entropy formula for large black holes from statistical mechanical considerations. A preliminary account of the black hole radiance in this approach was given in [25] and work is now in progress on completing that analysis. A derivation of the laws governing isolated horizons –which generalize the standard zeroth and first laws of black hole mechanics normally proved in the stationary context– is given in [5, 6].

The primary goal of our classical framework is to overcome three limitations that are faced by most of the existing treatments. First, isolated black holes are generally represented by *stationary* solutions of field equations, i.e., solutions which admit a translational Killing vector field *everywhere*, not just in a small neighborhood of the black hole. While this simple idealization was appropriate in the early development of the subject, it does seem overly restrictive. Physically, it should be sufficient to impose boundary conditions at the horizon to ensure *only that the black hole itself is isolated*. That is, it should suffice to demand only that the intrinsic geometry of the horizon be time independent although the geometry outside may be dynamical and admit gravitational and other radiation. Indeed, we adopt a similar viewpoint in

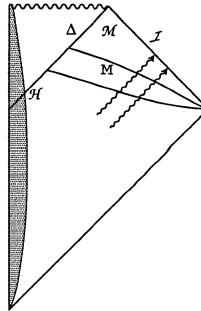


Figure 1: A typical gravitational collapse. The portion Δ of the horizon at late times is isolated. The space-time \mathcal{M} of interest is the triangular region bounded by Δ , \mathcal{I}^+ and a partial Cauchy slice M .

ordinary thermodynamics; in the standard description of equilibrium configurations of systems such as a classical gas, one usually assumes that only the system is in equilibrium and stationary, not the whole world. For black holes, in realistic situations, one is typically interested in the final stages of collapse where the black hole is formed and has “settled down” (Figure 1) or in situations in which an already formed black hole is isolated for the duration of the experiment. In such situations, there is likely to be gravitational radiation and non-stationary matter far away from the black hole, whence the space-time as a whole is not expected to be stationary. Surely, black hole thermodynamics should be applicable in such situations.

The second limitation comes from the fact that the classical framework is generally geared to *event* horizons which can only be constructed retroactively, after knowing the *complete* evolution of space-time. Consider for example, Figure 2 in which a spherical star of mass M undergoes a gravitational collapse. The singularity is hidden inside the null surface Δ_1 at $r = 2M$ which is foliated by a family of trapped surfaces and which would be a part of the event horizon if nothing further happens in the future. However, let us suppose that, after a very long time, a thin spherical shell of mass δM collapses. Then, Δ_1 would not be a part of the event horizon which would actually lie slightly outside Δ_1 and coincide with the surface $r = 2(M + \delta M)$ in distant future. However, on physical grounds, it seems unreasonable to exclude Δ_1 a priori from all thermodynamical considerations. Surely, one should be able to establish laws of black hole mechanics not only for the event

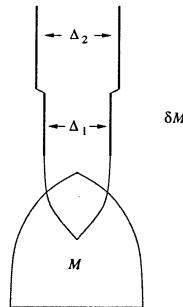


Figure 2: A spherical star of mass M undergoes collapse. Later, a spherical shell of mass δM falls into the resulting black hole. While Δ_1 and Δ_2 are both isolated horizons, only Δ_2 is part of the event horizon.

horizon but also for Δ_1 . Another example is provided by cosmological horizons in the de Sitter space-time. In this space-time, there are no singularities or event horizons. On the other hand, semi-classical considerations enable one to assign entropy and temperature to these horizons as well. This suggests that the notion of event horizons is too restrictive for thermodynamic considerations. We will see that this is indeed the case; as far as equilibrium properties are concerned, the notion of event horizons can be replaced by a more general, quasi-local notion of “isolated horizons” for which the familiar laws continue to hold. The surface Δ_1 in figure 2 as well as the cosmological horizons in de Sitter space-times are examples of isolated horizons.

The third limitation is that most of the existing classical treatments fail to provide a natural point of departure for quantization. In a systematic approach, one would first extract an appropriate sector of the theory in which space-time geometries satisfy suitable conditions at interior boundaries representing horizons, then introduce a well-defined action principle tailored to these boundary conditions, and finally construct the Hamiltonian framework by spelling out the symplectic structure, constraints, gauge and dynamics. By contrast, treatments of black hole mechanics are often based on differential geometric identities and field equations and not concerned with issues related to quantization. We will see that all these steps necessary for quantization can be carried out in the context of isolated horizons.

At first sight, it may appear that only a small extension of the stan-

dard framework based on stationary event horizons may be needed to overcome these three limitations. However, this is not the case. For example, if there is radiation outside the black hole, one can not identify the ADM mass with the mass of the black hole. Hence, to formulate the first law, a new expression of the black hole mass is needed. Similarly, in absence of a space-time Killing field, we need to generalize the notion of surface gravity in a non-trivial fashion. Indeed, even if the space-time happens to be static but only in a neighborhood of the horizon –already a stronger condition than what is contemplated above– the notion of surface gravity is ambiguous because the standard expression fails to be invariant under constant rescalings of the Killing field. When a *global* Killing field exists, the ambiguity is removed by requiring that the Killing field be unit at *infinity*. Thus, contrary to what one would intuitively expect, the standard notion of surface gravity of a stationary black hole refers not just to the structure at the horizon but also that at infinity. This “normalization problem” in the definition of surface gravity seems difficult especially in the cosmological context where Cauchy surfaces are compact. Apart from these conceptual problems, a host of technical issues need to be resolved because, in the Einstein-Maxwell theory, while the space of stationary solutions with event horizons is finite dimensional, the space of solutions admitting isolated horizons is *infinite* dimensional, since these solutions can admit radiation near infinity. As a result, the introduction of a well-defined action principle is subtle and the Hamiltonian framework acquires certain qualitatively new features.

The organization of this paper is as follows. Section 2 recalls the formulation of general relativity in terms of $SL(2, C)$ -spin soldering forms and self dual connections for asymptotically flat space-times *without* internal boundaries. Section 3 specifies the boundary conditions that define for us isolated, non-rotating horizons and discusses those consequences of these boundary conditions that are needed in the Hamiltonian formulation and quantization. It turns out that one has to add to the usual action (with its boundary term at infinity) a surface term at the horizon consisting of the Chern-Simons action. This unexpected interplay between general relativity and Chern-Simons theory is discussed in Section 4. It also contains a discussion of the resulting phase space, symplectic structure, constraints and gauge on which quantization [4] will be based. For simplicity, up to this point, the entire discussion refers to vacuum general relativity. The modifications that

are necessary for incorporating electro-magnetic and dilatonic hair are discussed in Section 5. Section 6 summarizes the main results.

Some of our constructions and results are similar to those that have appeared in the literature in different contexts. In particular, the ideas introduced in Section 3 are closely related to those introduced by Hayward in an interesting series of papers [23] which also aims at providing a more physical framework for discussing black holes. Our introduction in Section 4 of the boundary term in the action uses the same logic as in the work of Regge and Teitelboim [31], Hawking and Gibbons [21] and others. The specific form of the boundary term is closely related to that in the work of Momen [28], Balachandran et al [15], and Smolin [33]. However, the procedure used to arrive at the term and its physical and mathematical role are quite different. These similarities and differences are discussed at the appropriate places in the text.

2 Preliminaries: Review of connection dynamics

In this paper we use a formulation of general relativity [1] in which it is a dynamical theory of connections rather than metrics. This shift of the point of view does not change the theory classically¹ and is therefore not essential to the discussion of results that hold in classical general relativity, such as the (generalized) laws of black hole mechanics [5, 6]. However, this shift makes the kinematics of general relativity the same as that of SU(2) Yang-Mills theory, thereby suggesting new non-perturbative routes to quantization. The quantum theory, discussed in detail in the accompanying paper [4], uses this route in an essential way. Therefore, in this paper, we will discuss the boundary conditions, action and the Hamiltonian framework using connection variables. To fix notation and to acquaint the reader with the basic ideas, in this section we will recall some facts on the classical connection dynamics. (For further details see, e.g., [2].) For definiteness, we will tailor our main discussion to the case in which the cosmological constant Λ van-

¹The shift, does suggest natural extensions of general relativity to situations in which the metric may become degenerate. However, in this paper we will work with standard general relativity, i.e., assume that the metrics are non-degenerate.

ishes. Incorporation of a non-zero Λ only requires appropriate changes in the boundary conditions and surface terms at infinity. The structure at isolated horizons will remain unchanged.

2.1 Self-dual connections

Fix a 4-dimensional manifold \mathcal{M} with only one asymptotic region. Our basic fields will consist of a pair $(\sigma_a^{AA'}, A_a^{AB})$ of asymptotically flat, smooth fields where $\sigma_a^{AA'}$ is a soldering form (sometimes referred to as a ‘tetrad’) for primed and unprimed $SL(2, C)$ spinors and A_a^{AB} is an $SL(2, C)$ connection on unprimed spinors² [2]. Each pair $(\sigma_a^{AA'}, A_a^{AB})$ represents a possible history. The action is given by [30]

$$\begin{aligned} S'_{\text{Grav}}[\sigma, A] &= -\frac{i}{8\pi G} \left[\int_{\mathcal{M}} \text{Tr}(\Sigma \wedge F) - \int_{\mathcal{T}} \text{Tr}(\Sigma \wedge A) \right] \\ &\equiv \frac{i}{8\pi G} \frac{1}{4} \left[\int_{\mathcal{M}} d^4x \Sigma_{ab}^{AB} F_{cdAB} \eta^{abcd} - 2 \int_{\mathcal{T}} d^3x \Sigma_{ab}^{AB} A_{cAB} \eta^{abc} \right] \end{aligned} \quad (2.1)$$

Here \mathcal{T} is the time-like cylinder at infinity, the 2-forms Σ are given by $\Sigma^{AB} = \sigma^{AA'} \wedge \sigma_{A'}{}^B$, F is the curvature of the connection A , i.e., $F = dA + A \wedge A$, and η is the (metric-independent) Levi-Civita density. If we define a metric g of signature $-+++$ via $g_{ab} = \sigma_a^{AA'} \sigma_b^{BB'} \epsilon_{AB} \epsilon_{A'B'}$, then the 2-forms Σ are self-dual (see [2], Appendix A.):

$$\star \Sigma_{cd}^{AB} := \frac{1}{2} \varepsilon_{cd}^{ab} \Sigma_{ab}^{AB} = i \Sigma_{cd}^{AB}, \quad (2.2)$$

where ε is the natural volume 4-form defined by the metric g .

The equations of motion follow from variation of the action (2.1). Varying (2.1) with respect to A one gets

$$\mathcal{D} \wedge \Sigma = 0. \quad (2.3)$$

²Here, the term “spinors” is used in an abstract sense since we do not have a fixed metric on \mathcal{M} . Thus, our spinor fields α^A and $\beta^{A'}$ are just cross-sections of 2-dimensional complex vector bundles equipped with 2-forms ϵ_{AB} and $\epsilon_{A'B'}$. Spinor indices are raised and lowered using these two forms, e.g., $\alpha_A = \alpha^B \epsilon_{BA}$. For details, see, e.g., [2, 29] and appendix A.

This equation implies that the connection D defined by A coincides with the restriction to unprimed spinors of the torsion-free connection ∇ defined by the soldering form σ via $\nabla_a \sigma_b^{BB'} = 0$. The connection ∇ acts on tensor fields as well as primed and unprimed spinor fields. Thus, A is now the self-dual part of the spin-connection compatible with σ [29]. Hence, there is a relation between the curvature F and the Riemann curvature of the metric g determined by σ : F is the self-dual part of the Riemann tensor:

$$F_{ab}^{AB} = -\frac{1}{4} \Sigma_{cd}^{AB} R_{ab}^{cd}. \quad (2.4)$$

(see, e.g., [2], p. 292). Using this expression of the 2-form F in (2.1) one finds that the volume term in the action reduces precisely to the Einstein-Hilbert term:

$$\frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} R. \quad (2.5)$$

The numerical coefficients in (2.1) were chosen to ensure this agreement.

Varying the action (2.1) with respect to σ and taking into account the fact that A is compatible with σ , we obtain a second equation of motion

$$G_{ab} = 0 \quad (2.6)$$

where G is the Einstein tensor of g . Thus, the equations of motion that follow from the action (2.1) are the same as those that follow from the usual Einstein-Hilbert action; the two theories are classically equivalent.

We are now ready to perform a Legendre transform to pass to the Hamiltonian description. For this, we now assume that the space-time manifold \mathcal{M} is topologically $M \times \mathbb{R}$ for some 3-manifold M , with no internal boundaries and a single asymptotic region. The basic phase-space variables turn out to be simply the pull-backs to M of the connection A and the 2-forms Σ . To avoid proliferation of symbols, we will use the same notation for the four-dimensional fields and their pull-backs to M ; the context will make it clear which of the two sets we are referring to.

The geometrical meaning of these phase-space variables is as follows. Recall first that the four-dimensional $SL(2, C)$ -soldering form on \mathcal{M}

induces on M a 3-dimensional $SU(2)$ soldering form σ_{AB}^a for $SU(2)$ “space-spinors” on M via

$$\sigma_{AB}^a = i\sqrt{2}q_m^a \sigma_{AA'}^m \tau^{A'}_B,$$

where q_m^a is the projection operator of M and $\tau^{AA'} = \tau^a \sigma_a^{AA'}$ is the spinorial representation of the future directed, unit normal τ^a to M . The intrinsic metric q_{ab} on M can be expressed as $q_{ab} = \sigma_a^{AB} \sigma_{bAB}$. The 2-forms Σ , pulled-back to M , are closely related to the dual of these $SU(2)$ soldering forms. More precisely,

$$\frac{1}{2\sqrt{2}} \eta^{abc} \Sigma_{bc}^{AB} =: \tilde{\sigma}_a^{AB} \equiv \sqrt{q} \sigma_a^{AB}$$

where q is the determinant of the 3-metric q_{ab} on M . To see the geometric meaning of A , recall first that the $SU(2)$ soldering form σ determines a unique torsion-free derivative operator on (tensor and) spinor fields on M . Denote the corresponding spin-connection by Γ_a^{AB} . Then, assuming that the compatibility condition (2.3) holds, difference between the (pulled-back) connection A and Γ is given by the extrinsic curvature K_{ab} of M :

$$A_a^{AB} = \Gamma_a^{AB} - \frac{i}{\sqrt{2}} K_a^{AB},$$

where $K_a^{AB} = K_{ab} \sigma^{bAB}$. (The awkward factors of $\sqrt{2}$ here and related formulas in Sections 2.2 and 4.3 disappear if one works in the adjoint rather than the fundamental representation of $SU(2)$. See [2], chapter 5.) Thus, while Γ depends only on the spatial derivatives of the $SU(2)$ soldering form σ , A depends on both spatial and temporal derivatives.

The phase space consists of pairs $(A_a^{AB}, \Sigma_{ab}^{AB})$ of smooth fields on the 3-manifold M subject to asymptotic conditions which are induced by the asymptotic behavior of fields on \mathcal{M} . These require that the pair of fields be asymptotically flat on M in the following sense. To begin with, let us fix an $SU(2)$ soldering form $\overset{\circ}{\sigma}$ on M such that the 3-metric $\overset{\circ}{q}_{ab} = \overset{\circ}{\sigma}_a^{AB} \overset{\circ}{\sigma}_{bAB}$ it determines is flat outside of a compact set. Then the dynamical fields (Σ, A) are required to satisfy:

$$\begin{aligned} \Sigma_{ab} &= \left(1 + \frac{M(\theta, \phi)}{r}\right) \overset{\circ}{\Sigma}_{ab} + O\left(\frac{1}{r^2}\right) \\ \text{Tr}(\overset{\circ}{\sigma}^a A_a) &= O\left(\frac{1}{r^3}\right), \quad A_a + \frac{1}{3} \text{Tr}(\overset{\circ}{\sigma}^m A_m) \overset{\circ}{\sigma}_a = O\left(\frac{1}{r^2}\right), \end{aligned} \quad (2.7)$$

where r is the radial coordinate defined by the flat metric $\overset{\circ}{q}_{ab}$.

Using the “covariant phase space formalism”, one can use a standard procedure to obtain a symplectic one-form from the action and take its curl to arrive at the symplectic structure on the phase space. The result is:

$$\Omega|_{(A,\Sigma)}(\delta_1, \delta_2) = -\frac{i}{8\pi G} \int_M \text{Tr} [\delta_1 A \wedge \delta_2 \Sigma - \delta_2 A \wedge \delta_1 \Sigma], \quad (2.8)$$

where $\delta \equiv (\delta A, \delta \Sigma)$ denotes a tangent vector to the phase space at the point (A, Σ) . Note that, although the action has a boundary term at infinity, the symplectic structure does not.

This completes the specification of the phase space. On this space, the 3+1 form of Einstein’s equations is especially simple; all equations are low order polynomials in the basic phase space variables. Laws governing isolated horizons in classical general relativity are discussed within this framework in [6].

2.2 Real connections

For passage to quantum theory, however, this framework is not as suitable. To see this, note first that since we wish to borrow techniques from Yang-Mills theories, it is natural to use connections A as the configuration variables and let the quantum states be represented by suitable functions of (possibly generalized) connections. Then, the development of quantum theory would require tools from functional analysis on the space of connections. Moreover, to maintain diffeomorphism covariance, this analysis should be carried out without recourse to a background structure such as a metric. However, as noted above, the connections A are complex (they take values in the Lie algebra $\text{sl}(2, C)$ rather than $\text{su}(2)$) and, as matters stand, the required functional analysis has been developed fully only on the space of real connections [8, 10, 13, 27, 32, 14]; work is still in progress to extend the framework to encompass complex connections. Therefore, at this stage, the quantization strategy that has been most successful has been to perform a canonical transformation to manifestly real variables. Since the primary goal of this paper is to provide a Hamiltonian description which can serve as a platform for quantization in [4], we will now discuss these real variables.

The expression $A = \Gamma - (i/\sqrt{2})K$ of the $\text{SL}(2, C)$ connection in terms of real fields Γ and K suggests an appropriate strategy [16]. For each non-zero real number γ , let us set

$$\begin{aligned}\gamma A_a{}^{AB} &:= \Gamma_a{}^{AB} - \frac{\gamma}{\sqrt{2}} K_a{}^{AB} \\ \gamma \Sigma_{ab}{}^{AB} &:= \frac{1}{\gamma} \Sigma_{ab}{}^{AB}.\end{aligned}\quad (2.9)$$

It is not hard to check that variables $\gamma A, \gamma \Sigma$ are canonically conjugate in the sense that the symplectic structure is given by:

$$\Omega|_{(\gamma A, \gamma \Sigma)} (\delta_1, \delta_2) = \frac{1}{8\pi G} \int_M \text{Tr} [\delta_1 \gamma A \wedge \delta_2 \gamma \Sigma - \delta_2 \gamma A \wedge \delta_1 \gamma \Sigma] \quad (2.10)$$

where $\delta_1 \equiv (\delta_1 \gamma A, \delta_1 \gamma \Sigma)$ and $\delta_2 \equiv (\delta_2 \gamma A, \delta_2 \gamma \Sigma)$ are arbitrary tangent vectors to the phase space at the point $(\gamma A, \gamma \Sigma)$.

Thus, in the final picture, the real phase space is coordinatized by manifestly real fields $(\gamma A, \gamma \Sigma)$ which are smooth and are subject to the asymptotic conditions:

$$\begin{aligned}\gamma \Sigma_{ab} &= \frac{1}{\gamma} \left(1 + \frac{M(\theta, \phi)}{r} \right) \dot{\Sigma}_{ab} + O\left(\frac{1}{r^2}\right) \\ \text{Tr}(\dot{\sigma}^a A_a) &= O\left(\frac{1}{r^3}\right), \quad A_a + \frac{1}{3} \text{Tr}(\dot{\sigma}^m A_m) \dot{\sigma}_a = O\left(\frac{1}{r^2}\right),\end{aligned}\quad (2.11)$$

where, as before, r is the radial coordinate defined by the flat metric \dot{q}_{ab} . The symplectic structure is given by (2.10). Irrespective of the values of the real parameter γ , all these Hamiltonian theories are classically equivalent. They serve as the starting point for non-perturbative canonical quantization. However, it turns out that the corresponding quantum theories are unitarily *inequivalent* [24]. Thus, there is a quantization ambiguity and a one parameter family of inequivalent quantum theories³, parameterized by γ , which is referred to as the *Immirzi parameter*. This ambiguity is similar to the θ -ambiguity in QCD where, again, the classical theories are equivalent for all values of θ while the quantum theories are not [19]. The inequivalence will play an important role in the next paper [4].

³If one succeeds in developing the necessary steps in the functional calculus to carry out quantization directly using self-dual connections, this ambiguity will presumably arise from the presence of inequivalent measures in the construction of the quantum Hilbert space.

Remark. The original description in terms of pairs (A, Σ) is “hybrid” in the sense that Σ are $\text{su}(2)$ -valued 2-forms while A are $\text{sl}(2, C)$ valued one-forms. The phase space is, however, real. This is analogous to using $(z = q - ip, q)$ as “coordinates” on the real phase space of a simple harmonic oscillator. Consequently, there are “reality conditions” that have to be taken into account to eliminate the over-completeness of the basic variables. These are trivialized in the manifestly real description in terms of $(\gamma A, \gamma \Sigma)$ in the sense that now all variables are real. However, while A has a natural geometrical interpretation –it is the pull-back to M of the self-dual part of the connection compatible with the four-dimensional $\text{SL}(2, C)$ soldering form– the real connections γA do not. Indeed, their meaning in the four dimensional, space-time setting is quite unclear. Furthermore, the form of the Hamiltonian constraint in terms of $(\gamma A, \gamma \Sigma)$ is complicated which made real variables undesirable in the early literature. However, thanks to the more recent work of Thiemann [36], now this technical complication does not appear to be a major obstruction.

In the next two sections, we will return to the original, self-dual $\text{SL}(2, C)$ connections A and $\text{SL}(2, C)$ soldering form σ discussed in section 2.1 and extend the framework outlined in the beginning of this section to the context where there is an internal boundary representing an isolated horizon. After casting this extended framework in a Hamiltonian form, in section 4.3 we will again carry out the canonical transformation and pass to manifestly real variables.

3 Boundary conditions for isolated horizons

As explained in the introduction, in this series of papers, we wish to consider isolated horizons rather than stationary ones. Space-times of interest will now have an internal boundary, topologically $S^2 \times R$ and, as before, one asymptotic region. The internal boundary will represent an *isolated, non-rotating horizon*. (The restriction on rotation is only for technical simplicity and we hope to treat rotating horizons in subsequent papers.) A typical example is shown in Figure 1 which depicts a stellar gravitational collapse. The space-time of interest is the

wedge shaped region, bounded by the future piece Δ of the horizon, future null infinity \mathcal{I}^+ and a partial Cauchy surface extending from the past boundary of Δ to spatial infinity i° . In a realistic collapse one expects emission of gravitational waves to infinity, whence the underlying space-time can not be assumed to be stationary. There would be some back-scattering initially and a part of the emitted radiation will fall in to the black hole. But one expects, e.g., from numerical simulations, that the horizon will “settle down” rather quickly. In the asymptotic region near i^+ , we can assume that the part Δ of the horizon is non-dynamical and isolated to a very good approximation; here, the area of the horizon will be constant.

We now wish to impose on the internal boundary Δ precise conditions which will capture these intuitive ideas. While they will in particular incorporate isolated *event horizons*, as noted in the Introduction, the conditions are quasi-local and therefore also allow more general possibilities. All results obtained in this series of papers — the presence of the Chern-Simons boundary term in the action, the Hamiltonian formulation, the derivation of generalized laws of black hole mechanics and the calculation of entropy— will hold in this more general context. This strongly suggests that it is the notion of *isolated horizons*, rather than event horizons of stationary black holes that is directly relevant to the interplay between general relativity, quantum field theory and statistical mechanics, discovered in the seventies. For example, although there is no black hole in the de Sitter space-time, the cosmological horizons it admits are isolated horizons in our sense and our framework [4, 5, 6] leads to the Hawking temperature and entropy normally associated with these horizons [20].

3.1 Definition

We are now ready to give the general Definition and discuss the physical meaning and mathematical consequences of the conditions it contains. Although the primary applications of the framework will be to general relativity (possibly with a cosmological constant) coupled to Maxwell and dilatonic fields, in this and the next section we will allow general matter, subject only to the conditions stated explicitly in the Definition.

Definition. The internal boundary Δ of a history $(\mathcal{M}, \sigma_a^{AA'}, A_a^{AB})$

will be said to represent a *non-rotating isolated horizon* provided the following conditions hold⁴:

- (i) *Manifold conditions:* Δ is topologically $S^2 \times R$, foliated by a (preferred) family of 2-spheres S and equipped with a direction field l^a which is transversal to the foliation. We will introduce a coordinate v on Δ such that $n_a : \hat{\equiv} - D_a v$ is normal to the preferred foliation.
- (ii) *Conditions on the metric g determined by σ :* The surface Δ is null with l^a as its null normal.
- (iii) *Dynamical conditions:* All field equations hold at Δ .
- (iv) *Main conditions:* For any choice of the coordinate v on Δ , let l^a be so normalized that $l^a n_a \hat{\equiv} -1$. Then, if o^A and i^A is a spinor-dyad on Δ , satisfying $i^A o_A \hat{\equiv} 1$, such that $l^a = i\sigma_{AA'}^a o^A \bar{o}^{A'}$ and $n^a \hat{\equiv} i\sigma_{AA'}^a i^A \bar{i}^{A'}$, then the following conditions should hold:
 - (iv.a) $o^B \cancel{\mathcal{D}}_a o_B \hat{\equiv} 0$; and
 - (iv.b) $i^B \cancel{\mathcal{D}}_a i_B \hat{\equiv} i f(v) \overline{m}_a$,
 where \mathcal{D} is the derivative operator defined by A , f is a positive function on Δ (which is constant on each 2-sphere S in the foliation), and $\overline{m}_a : \hat{\equiv} -\sigma_a^{AA'} i_A \bar{o}_{A'}$ is a complex vector field tangential to the preferred family of 2-spheres.
- (v) *Conditions on matter:* On Δ the stress-energy tensor of matter satisfies the following requirements:
 - (v.a) Energy condition: $T_{ab} l^b$ is a past-directed, causal vector;
 - (v.b) The quantity $T_{ab} l^a n^b$ is constant on each 2-sphere S of the preferred foliation.

Note that these conditions are imposed *only* at Δ and, furthermore, the main condition involves only those geometrical fields which are defined intrinsically on Δ . Let us first discuss the geometrical and physical meaning of these conditions to see why they capture the intuitive notions discussed above. The first three conditions are rather weak

⁴Throughout this paper, the symbol $\hat{\equiv}$ will stand for “equal at points of Δ to”, a single under-arrow will denote pull-back to Δ and, a double under-arrow, pull-back to the preferred 2-sphere cross-sections S of Δ .

and are satisfied on a variety of null surfaces in, e.g., the Schwarzschild space-time (and indeed in any null cone in Minkowski space). In essence it is the fourth condition that pins down the surface as an isolated horizon.

The first condition is primarily topological. The second condition simply asks that Δ be a null surface with l^a as its null normal. The third is a dynamical condition, completely analogous to the one normally imposed at null infinity. The last condition restricts matter fields that may be present on the horizon. Condition (v.a) is mild; it follows from the (much more restrictive) dominant energy condition. It is satisfied by matter fields normally used in classical general relativity, and in particular by the Maxwell and dilatonic fields considered here. Condition (v.b) is a stronger restriction which is used in our framework to ensure that the black hole is non-rotating. Its specific form does seem somewhat mysterious from a physical viewpoint. However, we will see in section 5 that, in the Einstein-Maxwell-dilatonic system, this condition is well-motivated from detailed considerations of the matter sector.

Let us now turn to the key conditions (iv). These conditions restrict the pull-back to Δ of the connection \mathcal{D} defined by A , or, equivalently, of the connection ∇ compatible with σ since the dynamical condition (iii) implies that, on Δ , the action of these two operators agree on unprimed spinors. The pull-backs to Δ are essential because the spinor fields i^A and o^A are defined only on Δ . However, a subtlety arises because there is a rescaling freedom in i^A and o^A . To see this, let us change our labeling of the preferred foliation via $v \mapsto \tilde{v} = F(v)$ (with $F' > 0$). Then, n and l get rescaled and therefore also the spin-dyad: $i^A \mapsto \tilde{i}^A = G^{-1}i^A$ and $o^A \mapsto \tilde{o}^A = Go^A$ where $|G|^2 = F'(V)$. It is easy to verify that conditions (iv) continue to be satisfied in the tilde frame with $f(v) \mapsto \tilde{f}(v) = F'(v)f(v)$. Thus, conditions (iv) are unambiguous: If they are satisfied for a dyad (i^A, o^A) then they are satisfied by all dyads obtained from it by permissible rescalings.

The content of these conditions is as follows. (iv.a) is equivalent to asking that the null vector field l^a be geodesic, twist-free, expansion-free and shear-free. The first two of these properties follow already from (ii). Furthermore, (v.a) and the Raychaudhuri equation imply that if l is expansion-free, it is also shear-free. Thus, physically, the only new restriction imposed by (iv.a) is that l be expansion-free. It is equivalent

to the condition that the area 2-form of the 2-sphere cross-sections of Δ be constant in time which in turn captures the idea that the horizon is isolated. Condition (iv.b) is equivalent to asking that the vector field n^a is shear and twist-free, its expansion is spherically symmetric, given by $-2f(v)$ and its Newman Penrose coefficient $\pi := -l^a \bar{m}^b \nabla_a n_b$ vanish on Δ . These properties imply that the isolated horizon is non-rotating. Since, furthermore, f is required to be positive, we are asking that the expansion of the congruence n^a be *negative*. This captures the idea that we are interested in future horizons rather than past, i.e., black holes rather than white holes. Finally, note that rather than fixing a preferred foliation in the beginning, we could have required only that a foliation satisfying our conditions exists. The requirement (iv) implies that the foliation is unique. Furthermore, it has a natural geometrical meaning: since the expansion of n is constant in this foliation, it is the analog for null surfaces of the constant mean curvature slicing often used to foliate space-times.

Let us summarize. Non-rotating, isolated horizons Δ are null surfaces, foliated by a family of marginally trapped 2-spheres with the property that the expansion of the inward pointing null normal n^a to the foliation is constant on each leaf and negative. The presence of trapped surfaces motivates the label ‘horizon’ while the fact that they are *marginally* trapped —i.e., that the expansion of l^a vanishes— accounts for the adjective ‘isolated’. The condition that the expansion of n^a is negative says that Δ is a future horizon rather than past and the additional restrictions on the derivative of n^a imply that Δ is non-rotating. Boundary conditions refer only to the behavior of fields at Δ and the general spirit is very similar to the way one formulates boundary conditions at null infinity.

Remarks.

- a) All the boundary conditions are satisfied by static black holes in the Einstein-Maxwell-dilaton theory possibly with a cosmological constant. To incorporate rotating black holes, one would only have to weaken conditions (iv.b) and (v.b); the rest of the framework will remain unchanged. (Recent results of Lewandowski [26] show that we can continue to require that the expansion of n^a be spherically symmetric in the rotating case. However n^a would now have shear whence there would be an additional term proportional to m_a on the right side

of (iv.b).) Similarly, one may be primarily interested in solutions to Einstein's equations with matter without regard to whether the theory admits a well-defined action principle or a Hamiltonian formulation. Then, one may in particular want to allow matter rings and cages around the horizon. With such sources, the horizons can be distorted even in static situations. To incorporate such black holes, again, one would only have to weaken conditions (iv.b) and (v.b).

b) Note however that, in the non-static context, there may well exist physically interesting distorted black holes satisfying our conditions. Indeed, one can solve for all our conditions and show that the resulting 4-metrics need not be static or spherically symmetric near Δ [26]. (We will see explicitly in Sections 3.3 and 5.1 that the Weyl curvature and the Maxwell field need not be spherically symmetric near Δ .) Since the boundary conditions allow such histories and since we are primarily interested in histories —or, in the Hamiltonian formulation, the full phase space— rather than classical solutions in this paper and its sequel [4], we chose the adjective ‘non-rotating’ rather than ‘spherical’ while referring to these isolated horizons.

c) In the choice of boundary conditions, we have tried to strike the usual balance: On the one hand the conditions are strong enough to enable one to prove interesting results (e.g., a well-defined action principle, a Hamiltonian framework, and a generalization of black hole mechanics) and, on the other hand, they are weak enough to allow a large class of examples. As we already remarked, the standard non-rotating black holes in the Einstein-Maxwell-dilatonic systems satisfy these conditions. More importantly, starting with the standard static black holes and using known existence theorems one can specify procedures to construct new solutions to field equations which admit isolated horizons as well as radiation at null infinity [6]. These examples already show that, while the standard static solutions have only a finite parameter freedom, the space of solutions admitting isolated horizons is *infinite* dimensional. Thus, in the Hamiltonian picture, even the reduced phase-space is infinite dimensional; the conditions indeed admit a very large class of examples.

3.2 Symmetries and Gauge on Δ

In the bulk, the symmetry group is the group of automorphisms of the $SL(2, C)$ spin-bundle, i.e., the semi-direct product of local $SL(2, C)$ transformations on spinor fields with the diffeomorphism group of \mathcal{M} . The boundary conditions impose restrictions on dynamical fields and hence also on the permissible behavior of these transformations on boundaries. The restrictions at infinity are well-known: all transformations are required to preserve asymptotic flatness. Usually, these boundary conditions involve fixing (a trivialization of the spin-bundle and) a flat $SL(2, C)$ soldering form at infinity, imposing conditions on the fall-off of σ , A and matter fields and a requirement that the magnetic part of the Weyl curvature should fall faster than the electric part. Then, the asymptotic symmetry group reduces to the Poincaré group (see, e.g., [12] and references therein) and the asymptotic limits of the permissible automorphisms in the bulk have to belong to this group. In this sub-section, we will discuss the analogous restrictions at Δ .

Recall first that Δ is foliated by a family of 2-spheres ($v = \text{const}$) and a transversal direction field l^a . The permissible diffeomorphisms are those which preserve this structure. Hence, on Δ , these diffeomorphisms must be compositions of translations along the integral curves of l^a and general diffeomorphisms on a 2-sphere in the foliation. Thus, the boundary conditions reduce $\text{Diff}(\Delta)$ to a semi-direct product of the Abelian group of “translations” generated by vector fields $\alpha(v)l^a$ and $\text{Diff}(S^2)$. We will refer to this group as $\text{Symm}(\Delta)$.

The situation with the internal $SL(2, C)$ rotations is more subtle. Recall from section 2.1 that the 1-form n is tied to the preferred foliation: given a coordinate v whose level surfaces correspond to the preferred foliation, we set $n = dv$. Since the permissible changes in v are of the type $v \mapsto \tilde{v} = F(v)$, with $F'(v) > 0$, the co-vector field n_a is now unique up to rescalings $n_a \mapsto \tilde{n}_a = F'(v)n_a$. Since (l^a, n_a) are normalized via $l^a n_a \hat{\equiv} -1$, Δ is now equipped with a class of pairs (l^a, n_a) unique up to rescalings $(l^a, n_a) \mapsto (\tilde{l}^a, \tilde{n}_a) = (F'(v))^{-1}l^a, F'(v)n_a$. Hence, given any history (σ, A) satisfying the boundary conditions, we have a spin-dyad (i, o) unique up to rescalings

$$(\tilde{i}^A, \tilde{o}^A) = ((\exp \Theta) i^A, (\exp -\Theta) o^A), \quad (3.1)$$

where

$$\exp \operatorname{Re}(2\Theta) = F'(v) \quad \text{and} \quad \theta := \operatorname{Im} \Theta \quad \text{is an arbitrary function on } \Delta.$$

This suggests that we fix on Δ a spin-dyad (i, o) *up to these rescalings* and allow only those histories in which $\sigma_{AA'}^a$ maps $(i^A \bar{i}^{A'}, o^A \bar{o}^{A'})$ to one of the allowed pairs (n_a, l^a) on Δ . It is easy to check that this gauge-fixing can always be achieved. It reduces the group of local $\operatorname{SL}(2, C)$ gauge transformations to complexified $U(1)$ as in (3.1).

As is easy to check, under these restricted internal rotations, the fields $f(v)$ (which determines the expansion of n) and α, β (which determine \underline{A} via (3.4)) have the following gauge behavior:

$$f(v) \mapsto (\exp \operatorname{Re} 2\Theta) f(v), \quad \alpha_a \mapsto \alpha_a + \partial_a \Theta, \quad \beta_a \mapsto (\exp 2\Theta) \beta_a. \quad (3.2)$$

Thus, α_a transforms as a connection while f and β transform as “matter fields” on which the connection acts. Since f and $\operatorname{Re} \Theta$ are both positive functions of v alone, the transformation property of f suggests that we further reduce the gauge freedom to $U(1)$ by gauge-fixing f . This is not essential but it does clarify the structure of the degrees of freedom and frees us from keeping track of the awkward fact that $\operatorname{Re} \Theta$ depends only on v while $\operatorname{Im} \Theta$ is an arbitrary function on Δ as in local gauge theories.

Since $f(v)$ has the dimensions of expansion (i.e., $(\text{length})^{-1}$), and since the only (universally defined) quantity of the dimension of length is the horizon radius $r_\Delta = (a_\Delta/4\pi)^{1/2}$, it is natural to ask that f be proportional to $1/r_\Delta$. Furthermore, there is a remarkable fact: for all static black holes in the Einstein-Maxwell theory (with standard normalization of the Killing field), the expansion of n is given by $-2/r_\Delta$ *irrespective* of the values of charges or of the cosmological constant, so that, in all these solutions, $f(v) = 1/r_\Delta$. This fact can be exploited to extend the definition of surface gravity to non-static black holes [6]. Although we will not use surface gravity in this paper or its companion [4], for uniformity, we will use the same gauge and set $f(v) = 1/r_\Delta$. As is clear from (3.2), this choice can always be made and furthermore exhausts the gauge freedom in the real part of Θ . Thus, the internal $\operatorname{SL}(2, C)$ freedom now reduces to $U(1)$. We wish to emphasize however that none of the conclusions of this paper or [4] depend on this choice; indeed, we could have avoided gauge fixing altogether.

Let us summarize. With the gauge fixing we have chosen, only those automorphisms of the $\text{SL}(2, C)$ spin-bundle in the bulk are permissible which (reduce to identity at infinity and) belong, on Δ , to the semi-direct product of the local $U(1)$ gauge group and $\text{Sym } \Delta$. Under $U(1)$ gauge rotations, the basic fields transform as follows:

$$\alpha_a \mapsto \alpha_a + i\partial_a\theta, \quad \beta_a \mapsto \beta_a, \quad i^A \mapsto (\exp i\theta) i^A, \quad o^A \mapsto (\exp -i\theta) o^A, \quad (3.3)$$

where $\theta = \text{Im}\Theta$. We will see in the next section that these considerations match well with the action principle which will induce a $U(1)$ Chern-Simons theory on Δ . As usual, the structure of constraints in the Hamiltonian theory will tell us which of these automorphisms are to be regarded as gauge and which are to be regarded as symmetries.

3.3 Consequences of boundary conditions

In this sub-section, we will list those implications of our boundary conditions which will be needed in the subsequent sections of this paper and in the companion paper on quantization and entropy [4]. While some of these results are immediate, others require long calculations. Derivations are sketched in Appendix B. For alternate proofs, based on the Newman-Penrose formalism, see [6].

1. Condition (iv.a) implies that the Lie derivative of the intrinsic metric on Δ with respect to l^a must vanish; $\mathcal{L}_l \underline{g}_{ab} \hat{=} 0$. Thus, as one would intuitively expect, the intrinsic geometry of isolated horizons is time-independent. Note, however, that in general there is no Killing field even in a neighborhood of Δ . Indeed, since the main conditions (iv) restrict only on the pull-backs of various fields on Δ , we can not even show that the Lie derivative of the *full* curvature tensor with respect to l^a must vanish on Δ ; i.e., the 4-metric need not admit a Killing field even on Δ .

Nonetheless, since l^a is a Killing field for the *intrinsic* (degenerate) metric \underline{g} on Δ , it follows, as already noted, that the area of the 2-sphere cross-sections S of Δ is constant in time. We will denote it by a_Δ .

2. Conditions (iv) imply that the pull-back of the four-dimensional

self-dual connection \underline{A}_a^{AB} to Δ has the form⁵:

$$\underline{A}_a^{AB} \hat{=} -2\alpha_a i^{(A} o^{B)} - \beta_a o^{(A} i^{B)} \quad (3.4)$$

where, as before, $\hat{=}$ stands for “equals at points of Δ to”, α is a complex-valued 1-form on Δ , and the complex 1-form β is given by:

$$\beta_a \hat{=} i f(v) \overline{m}_a, \quad (3.5)$$

where $f(v)$ and \overline{m}_a are as in the boundary condition (iv.b).

Let us set

$$\alpha_a \hat{=} U_a + i V_a \quad (3.6)$$

where the one-forms U and V are real. It turns out that U is completely determined by the area a_Δ of the horizon and matter fields:

$$U_a \hat{=} r_\Delta \left[\frac{2\pi}{a_\Delta} - \frac{\Lambda}{2} - 4\pi G T_{ab} n^a l^b \right] \underline{n}_a \quad (3.7)$$

The one-form β_a is determined by the 1-form V and the value a_Δ of area via:

$$D_{[a} \beta_{b]} : \hat{=} \partial_{[a} \beta_{b]} - i V_{[a} \beta_{b]} \hat{=} 0 \quad \text{and} \quad \mathcal{F}_{ab} \hat{=} i \beta_{[a} \overline{\beta}_{b]}, \quad (3.8)$$

where D is the covariant derivative operator defined by the $U(1)$ connection V and \mathcal{F} is its curvature. (As discussed in section 3.2, β transforms as a $U(1)$ matter field, while V transforms as a $U(1)$ connection so that the first equation is gauge covariant.) Thus, the boundary conditions imply that the pull-back \underline{A}_a^{AB} to Δ of the four-dimensional $SL(2, C)$ connection A_a^{AB} is essentially determined by the real one-form V and area a_o of Δ . Finally, \underline{V}_a has a simple geometric interpretation: the group of tangent-space rotations of S is $SO(2)$ and \underline{V}_a is the natural spin-connection on the corresponding $U(1)$ -bundle. More precisely,

$$\underline{V}_a = -i \underline{\Gamma}_a^{AB} i_A o_B \quad (3.9)$$

⁵Note that it is redundant to pull-back forms such as α and β which are defined *only* on Δ . The derivative operator \underline{D}_a is given by: $\underline{D}_a \lambda_A = \partial_a \lambda_A + A_{aA}{}^B \lambda_B$ where ∂ is the unique derivative flat operator which annihilates i and o . Note that the fields i, o and A are all defined separately on the southern and the northern hemispheres of S and related on the overlap by a local $U(1)$ gauge transformation.

where, as before Γ is the spin-connection of the spatial soldering form σ on M .

3. Boundary condition (iii) enables us to express the curvature of \underline{A}_a^{AB} in terms of the pull-back to Δ of the Riemann curvature of the four-dimensional $\text{SL}(2, C)$ soldering form σ (see [2], Appendix A):

$$\underline{\underline{F}_{ab}}^{AB} \cong -\frac{1}{4} \underline{\underline{R}_{ab}}^{cd} \Sigma_{cd}^{AB}. \quad (3.10)$$

It turns out that conditions (iv) and (v) then severely restrict the Riemann curvature. To spell out these restrictions, it is convenient to use the Newman-Penrose notation (see Appendix B). First, the components $\Phi_{00}, \Phi_{01}, \Phi_{10}, \Phi_{02}$ and Φ_{20} of the Ricci tensor vanish. Second, the components Ψ_0, Ψ_1 and $\text{Im}\Psi_2$ also vanish. Third, Ψ_3 is not independent but equals Φ_{21} . Finally, the real part of Ψ_2 is constant on Δ . As a consequence, the following key equality relating the pull-backs of curvature F and self-dual 2-forms Σ holds on Δ :

$$\underline{\underline{F}_{ab}}^{AB} \cong -\frac{2\pi}{a_\Delta} \underline{\underline{\Sigma}_{ab}}^{AB} - 2i(3\Psi_2 - 2\Phi_{11}) \underline{\underline{\eta}}_{[a} \underline{\underline{m}}_{b]} o^A o^B. \quad (3.11)$$

Note that the first term on the right side is simple and its coefficient is universal, irrespective of the cosmological constant and values of electric, magnetic and dilatonic charges or details of other matter fields present at the horizon. This fact will play a key role in this paper as well as [4]. In particular, it will lead us to a universal action principle.

The relation (3.11) tells us that the curvature of the pulled-back connection \underline{A} is severely restricted. Note however that the above relation holds *only* at points of the isolated horizon Δ . In the interior of space-time \mathcal{M} , curvature can be quite arbitrary due to the presence of gravitational radiation and matter fields. Furthermore, even at points of Δ , the restriction is only on the pulled-back curvature since the main boundary conditions refer only to fields defined intrinsically on Δ . In particular, there is no restriction on the components Ψ_3, Ψ_4 of the Weyl curvature, or on the components Φ_{22}, Φ_{12} of Ricci curvature or the scalar curvature even at points of the boundary Δ . In particular, *they need not be spherically symmetric*.

4. Finally, one can further pull-back (3.11) to the preferred 2-sphere cross sections of Δ (i.e., transvect the equation with $m^{[a} \bar{m}^{b]}$). The result

can easily be obtained from (3.11) by noting that the second term has zero pull-back. Thus,

$$\underline{F_{ab}}^{AB} \hat{\equiv} -\frac{2\pi}{a_o} \underline{\Sigma}_{ab}^{AB} \quad (3.12)$$

This equation will play a key role in specification of the boundary condition on the phase space variables in section 4 and in the passage to quantum theory in [4]. Finally, the curvature on the left side of (3.12) is completely determined by the curvature \mathcal{F} of the $U(1)$ connection V :

$$\underline{\underline{F_{ab}}}^{AB} \hat{\equiv} -2i\mathcal{F}_{ab} i^{(A} o^{B)} \hat{\equiv} -4i\partial_{[a} V_{b]} i^{(A} o^{B)}. \quad (3.13)$$

Let us summarize. As one might expect, boundary conditions on Δ imply that the space-time fields (σ, A) that constitute our histories are restricted on Δ . While the restriction is not as severe as that at the boundary at infinity where σ must reduce to a fixed flat soldering form and A must vanish, they are nonetheless quite strong. Given the constant a_o —the value of the horizon area— the only unconstrained part of \underline{A} is the 1-form V and the pull-back $\underline{\Sigma}$ of Σ_{ab}^{AB} to the 2-sphere cross-sections is completely determined by the curvature of V by equations (3.12, 3.13).

4 Action and phase space

Recall from section 2 that in the absence of internal boundaries the action of general relativity is given by:

$$S'_{\text{Grav}}[\sigma, A] = -\frac{i}{8\pi G} \left[\int_{\mathcal{M}} \text{Tr}(\Sigma \wedge F) - \int_{\mathcal{T}} \text{Tr}(\Sigma \wedge A) \right] \quad (4.1)$$

Therefore, one might imagine that the presence of the internal boundary Δ could be accommodated simply by replacing the time-like cylinder \mathcal{T} at infinity in (4.1) by $\mathcal{T} \cup \Delta$. However, this simple strategy is unsuitable because the boundary conditions at Δ are quite different from those on \mathcal{T} . In section 4.1 we will show that the action can in fact be made differentiable by adding to it a Chern-Simons term at Δ . In section 4.2 we will perform a Legendre transform, obtain the phase space and analyze the notion of gauge in the Hamiltonian framework. Finally,

in section 4.3, we will perform a canonical transformation to obtain a Hamiltonian description (along the lines of section 2.2) in which all fields are real.

Since our primary motivation is to construct a Hamiltonian framework which will serve as a point of departure for the entropy calculation in the next paper [4], we will confine ourselves to histories with a *fixed* value of isolated horizon parameters⁶. In this section, the only parameter is the area a_Δ (or, the radius r_Δ , where $a_\Delta = 4\pi r_\Delta^2$). In the next section, we will also fix the electric, magnetic and dilatonic charges. In the classical theory we will thus be led to a phase space, each point of which admits an isolated horizon with given values of parameters. The idea is to quantize this sector in a way that allows for appropriate quantum fluctuations also at the boundary [4]. The surface states at the horizon in the resulting quantum theory will account for the entropy of a black hole (or cosmological) horizon with the specified area and charges.

4.1 Action Principle

Consider a 4-manifold \mathcal{M} , topologically $M \times R$. We will work with a fixed cosmological constant Λ , i.e., with a fixed theory. If $\Lambda \leq 0$, the ‘spatial’ 3-manifold M will be taken to be diffeomorphic to $S^2 \times R$ with an internal boundary S with a 2-sphere topology, while if $\Lambda > 0$, it will be taken to be the complement of an open ball in S^3 , again with an internal boundary S with a 2-sphere topology. Consider on \mathcal{M} smooth histories (A, σ) satisfying suitable boundary conditions. There are conditions at infinity which require the fields to be asymptotically flat in the standard sense [12] if $\Lambda = 0$, and asymptotically anti-de Sitter if $\Lambda < 0$ [11]. In all cases, the boundary conditions at the internal boundary $\Delta = S \times R$ will be the isolated horizon conditions spelled out in Section 3.1. While all main considerations go through irrespective of the value of the cosmological constant, as in Section 2, for definiteness we will set $\Lambda = 0$ in the main discussion.⁷

⁶A treatment which allows fields with arbitrary values of parameters is necessary in order to generalize the laws of black hole mechanics and is given in [6].

⁷In the final description, the case with $\Lambda \neq 0$ can be recovered by adding the obvious term $(\Lambda \text{Tr}\sigma \wedge \sigma \wedge \sigma)$ to the scalar (or Hamiltonian) constraint, and, ignoring the terms at infinity if $\Lambda > 0$ and replacing them appropriately [11] if $\Lambda < 0$.

Let us begin with the action $S'_{\text{Grav}}(A, \sigma)$. Fix a region of \mathcal{M} bounded by two (partial) Cauchy surfaces M_1 and M_2 which extend to spatial infinity in the asymptotic region and intersect the isolated horizon Δ in two 2-spheres S_1 and S_2 of our preferred foliation. Since σ appears undifferentiated in the action, and since we can replace Σ in (4.1) by its fixed boundary value on \mathcal{T} in the surface term, the variation with respect to σ is well-defined and gives rise only to the bulk equation of motion $\sigma_{AA'}^a F_{ab}^{AB} = 0$. The variation with respect to A , on the other hand, gives rise to a surface term

$$[\delta S'_{\text{Grav}}]_\Delta = -\frac{i}{8\pi G} \int_\Delta \text{Tr } \Sigma \wedge \delta A \quad (4.2)$$

Now, the boundary condition (3.4) implies that, for every A in our space of histories, $A_a^{AB} o_A o_B \hat{\equiv} 0$. Hence, $\delta A_a^{AB} o_A o_B \hat{\equiv} 0$. We can now use (3.11) to conclude that

$$\text{Tr } \underline{\Sigma} \wedge \underline{\delta A} = -\frac{a_\Delta}{2\pi} \text{Tr } \underline{F} \wedge \underline{\delta A}. \quad (4.3)$$

Since the 3-form in the integrand of (4.2) is pulled back to Δ , we have:

$$\begin{aligned} [\delta S'_{\text{Grav}}]_\Delta &= \frac{i}{8\pi G} \frac{a_\Delta}{2\pi} \int_\Delta \text{Tr } F \wedge \delta A \\ &= \frac{i}{8\pi G} \frac{a_\Delta}{4\pi} \delta \int_\Delta \text{Tr} (A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \end{aligned} \quad (4.4)$$

where in the second step we have used the fact that, since δA vanishes on M_1 and M_2 , it vanishes on the 2-spheres S_1 and S_2 on Δ . Note that the right side is precisely the variation of the Chern-Simons action for the connection \underline{A} on Δ . Hence, it immediately follows that the action

$$\begin{aligned} S_{\text{Grav}}(A, \sigma) &:= S'_{\text{Grav}}(A, \sigma) - \frac{i}{8\pi G} \frac{a_\Delta}{4\pi} S_\Delta^{\text{CS}} \\ &= -\frac{i}{8\pi G} \left[\int_{\mathcal{M}} \text{Tr } \Sigma \wedge F - \int_{\tau} \text{Tr } \Sigma \wedge A \right. \\ &\quad \left. + \frac{a_\Delta}{4\pi} \int_\Delta \text{Tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \right] \end{aligned} \quad (4.5)$$

has a well-defined variation with respect to A which gives rise only to the bulk equation of motion $\mathcal{D} \wedge \Sigma = 0$.

To summarize, with our boundary conditions at infinity and at the isolated horizon Δ , the action $S_{\text{Grav}}(\sigma, A)$ of (4.5) is differentiable and its variation yields *precisely* Einstein's equations on \mathcal{M} . In spite of the presence of boundary terms, there are no additional equations of motion either at the time-like cylinder τ at spatial infinity or on the isolated horizon Δ . In particular, although the boundary term at Δ is the Chern-Simons action for \underline{A} , we do *not* have an equation of motion which says that the curvature \underline{F} of \underline{A} vanishes. Indeed, \underline{F} is no-where vanishing and is given by (3.11). Nonetheless, we will see that the presence of this boundary term does give rise to an addition of the Chern-Simons term to the symplectic structure of the theory, which in turn plays a crucial role in the quantization procedure. Thus, the role of the surface term S_{Δ}^{CS} is subtle but important.

Remarks.

- a) The boundary conditions in section 3 were motivated by geometric considerations within general relativity and capture the idea that Δ is a non-rotating, isolated horizon. The fact that these then led to a consistent action principle is quite non-trivial by itself. The fact that the added boundary term has a simple interpretation as the Chern-Simons action for the self-dual connection \underline{A} on Δ is remarkable. We will see that this delicate interplay between classical general relativity and Chern-Simons theory also extends to quantum theory, where the matching extends even to precise numerical factors.
- b) Note that the Chern-Simons action arose because of equations (3.4) and (3.11). These equations have a “universality”: the inclusion of a cosmological constant or electric, magnetic and dilatonic charges have no effect on them. Consequently, in all these cases, the coefficient of the Chern-Simons term is always $a_{\Delta}/4\pi$. It is likely that this “universality” is directly related to the “universality” of the expression $S_{\text{bh}} = a_{\Delta}/4\ell_P^2$ of the Bekenstein-Hawking black hole entropy in general relativity.
- c) There are several contexts in which a Chern-Simons action has emerged as a boundary term. In some of these discussions, one begins with a theory, notices that the action is not fully gauge invariant and adds new boundary degrees of freedom to obtain a more satisfactory action for the extended system. The boundary degrees are typically connections and their dynamics is governed by the Chern-Simons ac-

tion (see, in particular, [28, 15]). By contrast, in the present work we did not add new degrees of freedom at all. The Chern-Simons piece did arise because the naive action $S'_{\text{Grav}}(A, \sigma)$ fails to admit a well-defined variational principle. However, the effect of the boundary conditions is to *reduce* the number of degrees of freedom: as usual, the boundary conditions impose relations between dynamical variables which are independent in the bulk. Furthermore, unlike in other discussions, these conditions arose from detailed geometrical properties of null vector fields l and n associated with isolated horizons in general relativity. This is also a major difference from the discussion in [33] where the Chern Simons action arose as a boundary term in *Euclidean* gravity subject to the condition that the spin-connection reduce to a right-handed SU(2) connection on the boundary. Also, while in other contexts the boundary connections are non-Abelian (typically SU(2)), as noted in Section 3, in our case, the independent degrees of freedom on Δ are coded in the Abelian connection V . This fact will play an important role in quantization.

We will conclude this section by expressing the action in terms of this Abelian connection. Using the expression (3.4) of the connection $\overset{\leftarrow}{A}$ on the boundary Δ and the expression (3.7) of U , it is easy to verify that the action can be re-written as

$$S_{\text{Grav}}(A, \sigma) = S'_{\text{Grav}}(A, \sigma) + \frac{i}{8\pi G} \frac{a_\Delta}{2\pi} \int_{\Delta} V \wedge dV - \frac{1}{8\pi G} \int_{\Delta} U \wedge {}^2\epsilon$$

where ${}^2\epsilon$ is the area 2-form on the preferred 2-spheres. It is easy to verify that, since a_Δ is fixed on the space of paths, the variation of the last integral vanishes in the Einstein-Maxwell-Dilaton theory. Hence,

$$\tilde{S}_{\text{Grav}}(A, \sigma) = S'_{\text{Grav}}(A, \sigma) + \frac{i}{8\pi G} \frac{a_\Delta}{2\pi} \int_{\Delta} V \wedge dV.$$

is also a permissible action which yields the same equations of motion as S_{Grav} . (In passing from S_{Grav} to \tilde{S}_{Grav} we have merely used the freedom to add to the action a function of dynamical variables which is constant on the space of paths.) In this form, we see explicitly that the surface term at Δ is the Chern-Simons action of the *Abelian* connection V . It is not surprising that the action depends only on the Abelian part V of the $\text{SL}(2, C)$ connection $\overset{\leftarrow}{A}$ since $\overset{\leftarrow}{A}$ is completely determined by V on our space of paths. However, it is pleasing that the functional

dependence of the action on V is simple, being just the Chern-Simons action for V .

Finally, the Abelian nature of V implies that the usual problem with large gauge transformations does not arise.

4.2 Legendre transform, constraints and gauge

We are now ready to pass to the Hamiltonian framework by performing the Legendre transform. Consider any history A, σ on \mathcal{M} and introduce a time function t on \mathcal{M} such that: i) each leaf $M(t)$ of the resulting foliation is space-like; ii) at infinity t reduces to a Minkowskian time coordinate of the flat fiducial metric near i° , each $M(t)$ passes through i° and the pull-backs of the dynamical fields A, Σ to $M(t)$ are asymptotically flat, i.e., satisfy (2.7); and, iii) on the isolated horizon Δ , the time-function t coincides with the function v labeling the preferred 2-spheres $S(t)$, and the unit normal τ^a to $M(t)$ is given by $\tau^a \equiv (l^a + n^a)/\sqrt{2}$ on each $S(t)$. Next, introduce a ‘time-evolution’ vector field t^a such that: i) $\mathcal{L}_t t = 1$; ii) at infinity, t^a is orthogonal to the leaves $M(t)$, i.e., $t^a = N\tau^a$ at i° for some ‘lapse function’ N ; and, iii) on Δ , $t^a \equiv l^a$. The conditions on t^a at the two boundaries imply that t and t^a define the same asymptotic rest frame at infinity and, at Δ , the frame coincides with the ‘rest frame’ of the isolated horizon. These two restrictions are not essential; they are introduced just to avoid some minor technical complications which are inessential to our main discussion.

Since the action is written in terms of forms, in contrast to the standard calculation in geometrodynamics, the Legendre transform is almost trivial to perform. One obtains:

$$\begin{aligned} & 8\pi iG S_{\text{Grav}}(A, \sigma) \\ &= - \int dt \int_{M(t)} \text{Tr} \left(\Sigma \wedge \mathcal{L}_t A + (A \cdot t) \mathcal{D}\Sigma - \Sigma \wedge (\vec{N} \cdot F) + iN\sqrt{2}\sigma \wedge F \right) \\ &\quad - \int dt \oint_{S_\infty(t)} \text{Tr} i\sqrt{2}N\sigma \wedge A + \frac{a_\Delta}{4\pi} \int dt \oint_{S_\Delta} \text{Tr} A \wedge \mathcal{L}_t A \end{aligned} \tag{4.6}$$

where, as usual, the lapse N and the shift \vec{N} are defined via $t^a = N\tau^a + \vec{N}^a$, the 1-form $\vec{N} \cdot F$ is defined via $(\vec{N} \cdot F)_b := \vec{N}^a F_{ab}$, and σ is the spatial, $SU(2)$ soldering form on $M(t)$ as in Section 2. (In terms

of Σ , we have $\sqrt{2}i\sigma_m = \tau^a \Sigma_{ab} q_m^b$ where q_m^b is the projection operator on $M(t)$. Here and in what follows, for simplicity of presentation, we ignore total divergences which are irrelevant to the expressions of the symplectic structure and the Hamiltonian.) The surface term can be re-expressed in terms of the $U(1)$ connection V as:

$$\frac{a_\Delta}{4\pi} \int dt \oint_{S_\Delta(t)} \text{Tr } A \wedge \mathcal{L}_l A = -\frac{a_\Delta}{2\pi} \int dt \oint_{S_\Delta} V \wedge \mathcal{L}_l V \quad (4.7)$$

From the Legendre transform (4.6) it is straightforward to obtain the phase-space description. Denote by M a generic leaf of the foliation. It is obvious that the dynamical fields are the pull-backs to M of pairs (A, Σ) .⁸ Note that there are no independent, surface degrees of freedom either at infinity or on the horizon: since all fields under consideration are smooth, by continuity, their values in the bulk determine their values on the boundary.⁹ In fact, as usual, the boundary conditions serve to *reduce* the number of independent fields on S_∞ and S_Δ . At infinity, the fields A, Σ on M must satisfy the asymptotic conditions (2.7); in particular, their limiting values are totally fixed. On the horizon, the pull-back of the connection to S_Δ is of a restricted form dictated by (3.4):

$$\overset{\leftarrow}{A_a}{}^{AB} \hat{=} -2iV_a i^{(A} o^{B)} + \beta_a o^A o^B \quad (4.8)$$

where β is determined by V via (3.7). Furthermore, the curvature \mathcal{F} of V is related to Σ via:

$$\mathcal{F} \hat{=} \frac{2\pi i}{a_\Delta} \underset{\Sigma}{\Sigma}{}^{AB} i_A o_B \quad (4.9)$$

Since $\beta = (i/r_\Delta)\bar{m}$, these two equations together with (3.7) imply that V is the only independent dynamical field on S_Δ .

The symplectic 1-form Θ is easily obtained from the Legendre transform. We have:

$$8\pi iG \Theta|_{(A,\sigma)}(\delta) = - \int_M \text{Tr } \delta A \wedge \Sigma + \frac{a_\Delta}{4\pi} \oint_S \text{Tr } \delta A \wedge A. \quad (4.10)$$

⁸From now on, in this section we will denote these pulled-back fields simply by A and Σ . The 4-dimensional fields on \mathcal{M} will carry a superscript 4 (e.g. 4A) to distinguish them from the 3-dimensional fields on M .

⁹By contrast, in quantum theory, the relevant histories are distributional and hence values of fields in the bulk do not determine their values on the boundary. We will see in [4] that it is this fact leads to quantum surface states which in turn account for the black hole entropy.

The symplectic structure Ω is just the exterior derivative of Θ :

$$\begin{aligned} & 8\pi iG \ \Omega|_{(A,\sigma)}(\delta_1, \delta_2) \\ &= \int_M \text{Tr} (\delta_1 A \wedge \delta_2 \Sigma - \delta_2 A \wedge \delta_1 \Sigma) - \frac{a_\Delta}{2\pi} \oint_S \text{Tr} \delta_1 A \wedge \delta_2 A \\ &= \int_M \text{Tr} (\delta_1 A \wedge \delta_2 \Sigma - \delta_2 A \wedge \delta_1 \Sigma) + \frac{a_\Delta}{\pi} \oint_S \delta_1 V \wedge \delta_2 V \end{aligned} \quad (4.11)$$

for any two tangent vectors $\delta_1 \equiv (\delta_1 A, \delta_1 \Sigma)$ and $\delta_2 \equiv (\delta_2 A, \delta_2 \Sigma)$ at (A, Σ) to the phase space. Finally, it is clear from the Legendre transform (4.6) that the fields $({}^4A.t), \vec{N}, N$ are Lagrange multipliers. The resulting constraints are the standard ones:

$$\mathcal{D}_a \tilde{\sigma}^a = 0, \quad \text{Tr } \tilde{\sigma}^a F_{ab} = 0, \quad \text{and} \quad \text{Tr } \tilde{\sigma}^a \tilde{\sigma}^b F_{ab} = 0, \quad (4.12)$$

where $\tilde{\sigma}^a$ is essentially the dual of the 2-form Σ_{ab} on M : $2\sqrt{2}\tilde{\sigma}^a = \tilde{\eta}^{abc}\Sigma_{bc}$ where $\tilde{\eta}$ is the metric independent Levi-Civita density on M . As usual, these form a set of first class constraints.

Recall that, in the Hamiltonian description, first class constraints generate gauge. Let us therefore examine the constraints one by one. Smearing the first (Gauss) constraint by a field $\lambda_A{}^B$ and integrating over M , we obtain a function on the phase space:

$$8\pi iG \ \mathcal{C}_\lambda(A, \Sigma) := \int_M \text{Tr} \lambda \mathcal{D}\Sigma \quad (4.13)$$

whose variation along a general vector δ at a point (A, Σ) yields

$$8\pi iG \ \delta \mathcal{C}_\lambda = \int_M \text{Tr} (\lambda \delta A \wedge \Sigma - \lambda \Sigma \wedge \delta A - \mathcal{D}\lambda \wedge \delta \Sigma) + \oint_{\partial M} \text{Tr} \lambda \delta \Sigma. \quad (4.14)$$

The question is if $\delta \mathcal{C}_\lambda$ is of the form $\Omega(\delta, \delta_\lambda)$ for some tangent vector δ_λ . If so, δ_λ would be the Hamiltonian vector field generated by the constraint functional \mathcal{C}_λ . Alternatively, since $\delta \mathcal{C}_\lambda$ vanishes for all vectors δ tangential to the constraint surface, δ_λ will be a degenerate direction of the pull-back of Ω to the constraint surface. Each of these properties implies that δ_λ would represent an infinitesimal gauge motion in the Hamiltonian theory.

A short calculation yields

$$\delta \mathcal{C}_\lambda = \Omega(\delta, \delta_\lambda) \quad (4.15)$$

for all tangent vectors δ , with

$$\delta_\lambda = (\mathcal{D}\lambda, [\Sigma, \lambda]), \quad (4.16)$$

provided: i) λ_A^B tends to zero at infinity (i.e., is $O(1/r)$), and, ii) has the form $i\lambda(i^A o_B + o^A i_B)$ on Δ . Note that the two conditions are necessary to ensure that δ_λ is a well-defined tangent vector to the phase space. Furthermore, (4.16) is precisely an internal $SL(2, C)$ rotation compatible with our boundary conditions. As one would have expected, the Hamiltonian theory tells us that these should be regarded as gauge transformations of the theory. Technically, the only non-trivial point is that, *because of the presence of the surface term in the symplectic structure, we do not have to require that λ_A^B should vanish on Δ for (4.15) to hold.* The Gauss constraint generates internal rotations which can be non-trivial on the horizon.

Next, let us consider the ‘diffeomorphism constraint’. The analysis [2] in the case without boundaries suggests that we consider the constraint function $C_{\vec{N}}$ defined by

$$8\pi iG \mathcal{C}_{\vec{N}}(A, \Sigma) := - \int_M \text{Tr} \left(\Sigma \wedge \vec{N} \cdot F \right) \quad (4.17)$$

where the smearing field \vec{N} is a suitable vector field on M . The variation of this function along an arbitrary tangent vector δ to the phase space at the point (A, Σ) yields,

$$\begin{aligned} 8\pi iG \delta \mathcal{C}_{\vec{N}} &= \int_M \text{Tr} \left(\delta A \wedge \mathcal{D}(\vec{N} \cdot \Sigma) - \vec{N} \cdot F \wedge \delta \Sigma \right) \\ &\quad - \oint_S \text{Tr} \left(\vec{N} \cdot \Sigma \wedge \delta A \right). \end{aligned} \quad (4.18)$$

It is easy to verify that

$$\delta \mathcal{C}_{\vec{N}} = \Omega(\delta, \delta_{\vec{N}}) \quad (4.19)$$

for all δ , with

$$\begin{aligned} \delta_{\vec{N}} &= \left(\vec{N} \cdot F, \mathcal{D}(\vec{N} \cdot \Sigma) \right) \\ &\equiv \left(\mathcal{L}_{\vec{N}} A - \mathcal{D}(\vec{N} \cdot A), \mathcal{L}_{\vec{N}} \Sigma - \vec{N} \cdot \mathcal{D}\Sigma - [\Sigma, \vec{N} \cdot A] \right), \end{aligned} \quad (4.20)$$

provided the smearing field \vec{N} satisfies the following properties: i) it vanishes (as $O(1/r)$) at infinity; and, ii) it is tangential to S . Again, note that the smearing field \vec{N} does *not* have to vanish on S ; it only has to be tangential to S . The diffeomorphisms generated by all such vector fields \vec{N} are to be regarded as ‘gauge’ transformations in this Hamiltonian theory. Note that asymptotic translations or rotations have well-defined action on the phase space but they are not generated by constraints and therefore are not regarded as gauge. This is the standard situation in the asymptotically flat context. On the internal boundary, diffeomorphisms which fail to be tangential to the boundary also do not correspond to gauge. This is not surprising since such diffeomorphisms do not even give rise to well-defined motions on the phase space.

Finally let us consider the scalar (or the Hamiltonian) constraint smeared by a lapse function N :

$$8\pi iG \mathcal{C}_N := i\sqrt{2} \int_M \text{Tr } N\sigma \wedge F \quad (4.21)$$

The analysis is completely parallel to that of the other two constraints. The result is:

$$\delta\mathcal{C}_N = \Omega(\delta, \delta_N) \quad (4.22)$$

for all δ , with

$$\delta_N = \left(\frac{N}{4} \epsilon^{bmn} [\Sigma_{mn}, F_{ab}] , \mathcal{D}_{[b} N \sigma_{c]} \right) , \quad (4.23)$$

provided the lapse N tends to zero both at infinity and on S . (Here ϵ^{abc} is the 3-form defined by the spatial soldering form σ and the square bracket denotes the commutator with respect to spinor indices.) In this case, (4.23) are precisely the Hamilton’s equations of motion for the basic canonical variables.

To summarize, the smearing fields λ, \vec{N}, N have to satisfy certain boundary conditions for the corresponding constraint functions to generate well-defined canonical transformations. Therefore, as in the case without internal boundaries [2], the constraint sub-manifold of the phase space is defined by the vanishing of the constraint functions $\mathcal{C}_\lambda, \mathcal{C}_{\vec{N}}, \mathcal{C}_N$ where the smearing fields satisfy these conditions. The corresponding canonical transformations represent gauge motions. In quantum theory, physical states are to be singled out by requiring that they

be annihilated by the quantum operators corresponding to *these* constraint functions.

4.3 Passage to real variables

As explained in section 2.2, at the present stage of development, quantization is fully manageable only in terms of real, SU(2) connections. Therefore, as in section 2.2 we will now carry out the canonical transformation to manifestly real variables, paying attention to the internal boundary Δ and taking in to account the presence of the boundary term in the symplectic structure.

Recall first that we can always express the connection A on M as $A_a = \Gamma_a - (i/\sqrt{2})K_a$ for some K_a^{AB} , where Γ_a^{AB} is the spin-connection compatible with the spatial soldering form σ . Furthermore, the fact that we are working with the real, Lorentzian theory implies that Γ is a real SU(2) connection and K_a is a real, su(2)-valued 1-form. (When equations of motion are satisfied, $K_{ab} := -\text{Tr} K_a \sigma_b$ has the interpretation of the extrinsic curvature of M .) Therefore, the symplectic 1-form Θ of (4.10) can be written as:

$$\begin{aligned} 8\pi iG\Theta(\delta) &= -\int_M \text{Tr} \delta A \wedge \Sigma + \frac{a_\Delta}{4\pi} \oint_S \text{Tr} \delta A \wedge A \\ &= \left(\frac{i}{\sqrt{2}}\right) \int_M \text{Tr} \delta K \wedge \Sigma - \int_M \text{Tr} \delta \Gamma \wedge \Sigma - \frac{a_\Delta}{2\pi} \oint_S \delta V \wedge V. \end{aligned} \quad (4.24)$$

Now, it is well-known [2] that the integrand of the second term is an exterior derivative of a 2-form which vanishes at infinity. Therefore, that term can be written [34] as an integral over the inner boundary S :

$$\int_M \text{Tr} \delta \Gamma \wedge \Sigma = -\frac{1}{2} \oint_S \text{Tr} \sigma \wedge \delta \sigma \quad (4.25)$$

Thus, $\Theta = \Theta^{(M)} + \Theta^{(S)}$, where

$$\begin{aligned} 8\pi G\Theta^{(M)}(\delta) &= \frac{1}{\sqrt{2}} \int_M \text{Tr} \delta K \wedge \Sigma, \\ 8\pi iG\Theta^{(S)}(\delta) &= \frac{1}{2} \oint_S \text{Tr} \sigma \wedge \delta \sigma + \frac{a_\Delta}{4\pi} \oint_S \text{Tr} \delta A \wedge A \end{aligned} \quad (4.26)$$

Using this fact, we will now show that, as one would expect, the symplectic structure is real.

For this, we first note that our boundary conditions imply that any tangent vector δ to our phase space has the following form when evaluated on S :

$$(\delta \underline{A}, \delta \underline{q}) \doteq \left(\left(\underline{\mathcal{D}}\lambda + \mathcal{L}_\xi \underline{A} \right), [\underline{q}, \lambda] + \mathcal{L}_\xi \underline{q} \right) \quad (4.27)$$

where $\lambda^{AB} = 2ih^{(A}o^{B)}$ for some real function h on S and ξ is a vector field on S satisfying $\oint \mathcal{L}_\xi^2 \epsilon = 0$. Next, recall that, since the symplectic structure Ω is the exterior derivative of Θ , we have

$$\Omega(\delta_1, \delta_2) = \mathcal{L}_{\delta_1} \Theta(\delta_2) - \mathcal{L}_{\delta_2} \Theta(\delta_1) - \Theta([\delta_1, \delta_2]) \quad (4.28)$$

for any two vector fields δ_1, δ_2 on the phase space. Using (4.27), one can now show that $\Omega^{(S)}$, obtained by substituting $\Theta^{(S)}$ for Θ in (4.28), vanishes. Hence $\Theta^{(S)}$ is closed and does not contribute to the symplectic structure. Since $\Theta^{(M)}$ is manifestly real, so is the symplectic structure.

Finally, as in Section 2.2, let us define real, SU(2)-valued forms

$$\begin{aligned} {}^\gamma A_a{}^{AB} &:= \Gamma_a{}^{AB} - \frac{\gamma}{\sqrt{2}} K_a{}^{AB} \\ {}^\gamma \Sigma_{ab}{}^{AB} &:= \frac{1}{\gamma} \Sigma_{ab}{}^{AB}. \end{aligned} \quad (4.29)$$

Since $\Theta^{(S)}$ is curl-free, it follows that the one form

$$\begin{aligned} 8\pi G {}^\gamma \Theta(\delta) &= \frac{\gamma}{\sqrt{2}} \int_M \text{Tr } \delta K \wedge {}^\gamma \Sigma - \int_M \text{Tr } \delta \Gamma \wedge {}^\gamma \Sigma \\ &\quad + \frac{a_\Delta}{4\pi\gamma} \oint_S \text{Tr } \delta {}^\gamma A \wedge {}^\gamma A \\ &= - \int_M \text{Tr } \delta {}^\gamma A \wedge {}^\gamma \Sigma + \frac{a_\Delta}{4\pi\gamma} \oint_S \text{Tr } \delta {}^\gamma A \wedge {}^\gamma A \end{aligned} \quad (4.30)$$

is also a symplectic potential for the total symplectic structure Ω . Hence we have:

$$\begin{aligned} 8\pi G \Omega|_{({}^\gamma A, {}^\gamma \Sigma)}(\delta_1, \delta_2) &= \int_M \text{Tr } (\delta_1 {}^\gamma A \wedge \delta_2 {}^\gamma \Sigma - \delta_2 {}^\gamma A \wedge \delta_1 {}^\gamma \Sigma) - \frac{a_\Delta}{2\pi\gamma} \oint_S \delta_1 {}^\gamma A \wedge \delta_2 {}^\gamma A \end{aligned} \quad (4.31)$$

$$= \int_M \text{Tr} (\delta_1 \gamma A \wedge \delta_2 \gamma \Sigma - \delta_2 \gamma A \wedge \delta_1 \gamma \Sigma) + \frac{a_\Delta}{\pi \gamma} \oint_S \delta_1 V \wedge \delta_2 V$$

Thus, as in section 2.2, we now have a phase space formulation in terms of manifestly real variables. The phase space variables $(\gamma A, \Sigma)$ are subject to boundary condition (2.11) at infinity. On the horizon boundary S , the independent part of γA is contained in a (γ -independent) U(1) connection V , whose curvature completely determines the pull-back to the boundary S via (3.12). The symplectic structure is given by (4.31). As mentioned above, this formulation in terms real variables is not necessary for any of the classical considerations —such as the laws of isolated horizon mechanics— but is needed, at this stage, for the passage to quantum theory [4].

5 Inclusion of hair: electric, magnetic and dilatonic charges

The boundary conditions at the horizon specified in section 3 allow a non-zero cosmological constant and matter fields are subject only to condition (v) in the main Definition. Once this condition is satisfied, the discussion of section 4 goes through and is completely insensitive to the details of matter fields. However, in a complete theory, we also need to ensure that the matter action is differentiable and work out the Hamiltonian framework for the matter sector as well. This can require imposition of additional boundary conditions on matter fields. In this section, we will incorporate Maxwell and dilatonic fields. The Einstein-Maxwell case is discussed in section 5.1 and the dilatonic case in section 5.2.

5.1 Maxwell Fields

Since we require that field equations hold on Δ , the gravitational boundary conditions already imply certain restrictions on the behavior of Maxwell fields there. Let us begin with these. As noted in section 3.3, boundary conditions imply that several components of the Ricci tensor vanish on Δ . In particular, since the expansion of l^a vanishes,

the Raychaudhuri equation implies that the Ricci tensor must satisfy $R_{ab}l^a l^b \cong 0$. Since $R_{ab} - \frac{1}{2}Rg_{ab} + \Lambda g_{ab} \cong 8\pi G T_{ab}$, the stress-energy tensor of the Maxwell field must satisfy $T_{ab}l^a l^b \cong 0$ in the Einstein-Maxwell theory.

To find implications of this condition for the electro-magnetic field \mathbf{F} , it is convenient to first recast it in the spinorial form. \mathbf{F} can be decomposed over the basis of 2-forms $\Sigma_{ab}, \bar{\Sigma}_{ab}$ (see Eq (A.2) in Appendix A) as:

$$-2\mathbf{F}_{ab} = \phi_{AB}\Sigma_{ab}^{AB} + \bar{\phi}_{A'B'}\bar{\Sigma}_{ab}^{A'B'}. \quad (5.1)$$

where the symmetric spinor field ϕ_{AB} is the Newman-Penrose representation of the self-dual Maxwell field [29]. It is straight forward to re-express the stress-energy tensor of the Maxwell field,

$$T_{ab} = \frac{1}{4\pi} \left(\mathbf{F}_{ac}\mathbf{F}_b{}^c - \frac{1}{4}g_{ab}\mathbf{F}_{mn}\mathbf{F}^{mn} \right), \quad (5.2)$$

in terms of ϕ_{AB} :

$$T_{ab} = -\frac{1}{2\pi}\phi_{AB}\bar{\phi}_{A'B'}\sigma_a^{AA'}\sigma_b^{BB'}. \quad (5.3)$$

The condition $T_{ab}l^a l^b \cong 0$ now translates to:

$$o^A o^B \phi_{AB} = 0, \quad (5.4)$$

Or, alternatively,

$$\phi_{AB} = -2\phi_1 i_{(AOB)} + \phi_2 o_{AOB} \quad (5.5)$$

for some (complex) fields ϕ_1, ϕ_2 on Δ [29]. Note that only the $\phi_0 := \phi_{AB}o^A o^B$ component —the “radiation part”— of the Maxwell field is required to be zero at Δ . Since it represent the “radiation field”, as one might intuitively expect, the condition $T_{ab}l^a l^b \cong 0$ simply states that there is no radiation field at Δ . It is straightforward to check that the vanishing of other components of the Ricci tensor does not lead to further restrictions of the Maxwell field at Δ .

Next, we have to ensure that conditions (v) in the Definition are satisfied. (v.a) is automatic because the Maxwell stress energy tensor satisfies the strong energy condition. To ensure that (v.b) is satisfied,

we will require that ϕ_1 is spherically symmetric on the preferred cross-sections. This restriction is physically motivated by the fact that, if the radiation field ϕ_0 were to vanish to the next order at Δ —i.e., if $n^a \nabla_a \phi_0 \hat{=} 0$ — then the Maxwell field equations at Δ imply that ϕ_1 is spherically symmetric on Δ . (Thus, in particular, if there is no electro-magnetic radiation in a neighborhood —however small— of Δ our restriction on ϕ_1 will be met.) Now ϕ_1 can be expressed in terms of the electric and magnetic charges, Q and P of the isolated horizon,

$$Q : \hat{=} -\frac{1}{4\pi} \oint_{S_v} {}^* \mathbf{F} \hat{=} \frac{1}{2\pi} \oint_S \operatorname{Re} \phi_1^2 \epsilon \quad (5.6)$$

$$P : \hat{=} -\frac{1}{4\pi} \oint_{S_v} \mathbf{F} \hat{=} \frac{1}{2\pi} \oint_{S_v} \operatorname{Im} \phi_1^2 \epsilon, \quad (5.7)$$

as follows:

$$\phi_1 \hat{=} \frac{2\pi}{a_\Delta} (Q + iP). \quad (5.8)$$

(Here S_v are the 2-spheres $v = \text{const}$ in the preferred foliation. The minus sign in front of the first integrals in 5.6 and 5.7 arise because we have oriented S_v such that the radial normal is in-going rather than outgoing.) In terms of Maxwell fields, this condition can be rewritten as:

$$\underline{\mathbf{F}}_{ab} \hat{=} \frac{4\pi iP}{a_\Delta} \underline{\Sigma}_{ab}^{AB} i_A o_B, \quad \text{and} \quad \underline{{}^* \mathbf{F}}_{ab} \hat{=} \frac{4\pi iQ}{a_\Delta} \underline{\Sigma}_{ab}^{AB} i_A o_B, \quad (5.9)$$

Note, however, that there is no restriction on ϕ_0 ; in particular, it need not be spherically symmetric.

Since $\mathbf{F} = d\mathbf{A}$, the magnetic charge P is independent of the 2-sphere S_v used in its evaluation; indeed we could have used a 2-sphere which does not belong to our preferred family. However, up to this point the electric charge Q (and hence ϕ_1 and $\underline{{}^* \mathbf{F}}$) can be a function of v . However, the field equations satisfied by \mathbf{F} —condition (iii) in the Definition— imply that Q is also independent of the choice of the 2-sphere cross-section: The full content of the field equations is

$$l^a \nabla_a \phi_0 \hat{=} 0 \quad \text{and} \quad l^a \nabla_a \phi_1 \hat{=} 0 \quad (5.10)$$

We are now ready to discuss the action. Recall that, in the gravitational case, we restricted ourselves to histories in which the area

of the horizon is a fixed constant a_Δ . For the same reasons, we will now restrict ourselves to histories for which the values of electric and magnetic charges on the horizon are fixed to Q_Δ and P_Δ respectively. To make the action principle well-defined, we need to impose suitable boundary conditions on the Maxwell fields. Conditions at infinity are the standard ones. (As in the gravitational case, to avoid repetition, we will specify them in the phase space framework which is of more direct interest in this series of papers.) To find boundary conditions on Δ , let us consider the standard electro-magnetic bulk action:

$$S_{\text{EM}} = -\frac{1}{16\pi} \int_{\mathcal{M}} \sqrt{-g} \mathbf{F}_{ab} \mathbf{F}^{ab} d^4x = \frac{1}{8\pi} \int_{\mathcal{M}} \mathbf{F} \wedge {}^* \mathbf{F}. \quad (5.11)$$

The numerical factor is adjusted in such a way that the total action $S_{\text{grav}} + S_{\text{EM}}$, yields Einstein equations $G_{ab} = 8\pi G T_{ab}$. Variation of S_{EM} yields

$$\delta \left(\frac{1}{8\pi} \int_{\mathcal{M}} \mathbf{F} \wedge {}^* \mathbf{F} \right) = -\frac{1}{4\pi} \int_{\mathcal{M}} \delta \mathbf{A} \wedge \delta \mathbf{F} + \frac{1}{4\pi} \int_{\partial \mathcal{M}} \delta \mathbf{A} \wedge {}^* \mathbf{F}. \quad (5.12)$$

As usual, the bulk term provides the equations of motion provided the surface term vanishes. The boundary term (5.12) vanishes at infinity due to the fall off conditions. However, when evaluated at the horizon, the boundary term (5.12) does not automatically vanish. Now, (5.5) implies that on Δ the pull-back of $\mathbf{F}_{ab} l^b$ vanishes. Therefore, on Δ , the boundary term in (5.12) reduces to

$$\int dv \oint_{S_v} \delta(\mathbf{A} \cdot l) {}^* F \quad (5.13)$$

where, as before, v is the affine parameter along the integral curves of l^a such that $v = \text{const}$ define the preferred foliation of Δ and S_v are the 2-spheres in this foliation. Now, since isolated horizons are to be thought of as “non-dynamical”, it is natural to work in a gauge in which $\mathcal{L}_l \underline{\mathbf{A}}_a \hat{=} 0$. Then, (5.5) implies that $\mathbf{A} \cdot l$ is constant on Δ . The form of the boundary term (5.13) suggests that we fix gauge so that $\mathbf{A} \cdot l$ is a *fixed* constant on our space of histories. The value of this constant is then determined by its standard value in the Reissner-Nordstrom solution:

$$\mathbf{A}_a l^a \hat{=} \frac{Q_\Delta}{r_\Delta}. \quad (5.14)$$

Then the boundary term arising in the variation of the action vanishes, i.e., the bulk action itself is differentiable and the action principle is well-defined. Note that the permissible gauge transformations are now restricted: If $\mathbf{A} \mapsto \mathbf{A} + df$, the generating function f has to (tend to 1 at infinity and) satisfy $l^a \partial_a f \hat{\equiv} 0$ on Δ . (For further discussion, see [6]).

We will conclude with a summary of the structure of the phase space of Maxwell fields. Fix a foliation of \mathcal{M} by a family of space-like 3-surfaces M_t (level surfaces of a time function t) which intersect Δ in the preferred 2-spheres. Denote by t^a the “time-evolution” vector field which is transversal to the foliation with affine parameter t which tends to a unit time-translation at infinity and to the vector field l^a on Δ . In terms of the lapse and shift fields N and N^a defined by t^a , the Legendre transform of the action yields:

$$S_{\text{EM}} = \frac{1}{4\pi} \int dt \int_{M_t} \left(\mathbf{E} \wedge (\mathcal{L}_t \mathbf{A}) - d(\mathbf{A} \cdot t) \wedge \mathbf{E} - (N \cdot \mathbf{F}) \wedge \mathbf{E} - \frac{N}{2} {}^{(*)}\mathbf{E} \wedge \mathbf{E} - \frac{N}{2} {}^{(*)}\mathbf{F} \wedge \mathbf{F} \right) \quad (5.15)$$

where where the 2-form \mathbf{E} is the pull-back to M of ${}^*\mathbf{F}$, ${}^*\mathbf{E}_a := \frac{1}{2} \epsilon_a^{bc} \mathbf{E}_{bc}$, and ${}^*\mathbf{F}_a := \frac{1}{2} \epsilon_a^{bc} \mathbf{F}_{bc}$. From (5.15), we can read off the symplectic structure and the Hamiltonian.

Thus, as usual, the phase space consists of pairs (\mathbf{A}, \mathbf{E}) on the 3-manifold M , subject to boundary conditions, where the 1-form \mathbf{A} is now the pull-back to M of the electro-magnetic 4-potential and the 2-form \mathbf{E} is the dual of the more familiar electric field vector density. These fields are subject to boundary conditions. On the horizon 2-sphere S , the pull-backs to S of conditions (5.9) must hold ensuring that these pull-backs of \mathbf{F} and \mathbf{E} are spherically symmetric. (Since $\mathbf{A} \cdot l$ appears as a Legendre multiplier in (5.15) condition (5.14) does not restrict the phase space variables at Δ .) At infinity, \mathbf{A}, \mathbf{E} are subject to the usual fall-off conditions. Consider first the case when the magnetic charge vanishes so that the vector potential \mathbf{A} is a globally defined 1-form on M . Then, as usual we require:

$$\mathbf{A} = O\left(\frac{1}{r^{1+\epsilon}}\right), \quad \text{and} \quad \mathbf{E} = O\left(\frac{1}{r^2}\right), \quad (5.16)$$

The case when the magnetic charge is non-zero is more subtle since the vector potential can no longer be specified globally. However, since we are considering histories with a *fixed* magnetic charge P_Δ we can reduce the problem to the one with zero magnetic charge. Fix, once and for all, a Dirac monopole potential \mathbf{A}_Δ with magnetic charge P_Δ and consider potentials \mathbf{A} on M of the type $\mathbf{A} = \mathbf{A}_\Delta + R$ where the remainder 1-form R has the fall-off $R = O(1/r^{1+\epsilon})$. For electric fields, we use the same fall-off as in (5.16). It is straightforward to verify that these boundary conditions on fields are preserved by evolution equations. Fields (\mathbf{A}, \mathbf{E}) satisfying these conditions constitute our phase space.

The symplectic structure on this Maxwell phase space can be read off from (5.15):

$$\Omega|_{(\mathbf{A}, \mathbf{E})}(\delta_1, \delta_2) = \frac{1}{4\pi} \int_M [\delta_1 \mathbf{E} \wedge \delta_2 \mathbf{A} - \delta_2 \mathbf{E} \wedge \delta_1 \mathbf{A}]. \quad (5.17)$$

(The asymptotic conditions ensure that the integrals converge.) As usual, there is one first class constraint, $d\mathbf{E} = 0$, which generates gauge transformations: Under the canonical transformation generated by $\int f d\mathbf{E}$, the canonical fields transform, as usual, via $\mathbf{A} \mapsto \mathbf{A} + df$ and \mathbf{E} remains unchanged. Note that our boundary conditions allow the generating function f to be non-trivial on the (intersection of M with) Δ ; the smeared constraint function is still differentiable. Thus, as in the gravitational case, the gauge degrees of freedom do *not* become physical in this framework.

Remarks.

a) It may seem that we could have avoided the fixing of $\mathbf{A} \cdot l$ on Δ and simply introduced instead a boundary term to the action, involving $\mathbf{A} \cdot l$. However, then the action would have failed to be gauge invariant under the local Maxwell gauge transformations.

b) The condition on sphericity of ϕ_1 is somewhat stronger than the condition on the stress-energy tensor $T_{ab}l^a l^b$ (which is satisfied if and only if $|\phi_1|^2$ is spherically symmetric). However, as pointed out above, the stronger condition is met if there is no electro-magnetic radiation ϕ_0 near Δ . Furthermore, when ϕ_1 is spherically symmetric, the smeared Gauss constraint is differentiable even when the generating function f is non-trivial on the horizon S so that local gauge transformations on the

horizon are regarded as ‘gauge’ also in the Hamiltonian framework. If we had imposed only the weaker condition, it would have been awkward to sort out the degrees of freedom in the Hamiltonian framework.

5.2 Dilatonic Couplings

In this subsection we add a dilatonic charge to the Einstein-Maxwell theory, i.e., consider a scalar field which is coupled non-minimally to the Maxwell field. (Further details can be found in [7].)

The dilatonic field theory has a bulk contribution to the action of the form,

$$S_{\text{Dil}} = -\frac{1}{16\pi} \int_{\mathcal{M}} \sqrt{-g} [2(\nabla\phi)^2 + e^{-2\alpha\phi} \mathbf{F}_{ab} \mathbf{F}^{ab}] d^4x \quad (5.18)$$

where α is a free parameter which governs the strength of the coupling of the dilaton to the Maxwell field. When $\alpha = 0$ we recover the Einstein-Maxwell-Klein-Gordon system, while for $\alpha = 1$ S_{Dil} is part of the low energy action of string theory. The total action is the dilatonic action plus the gravitational part considered in Section 3.

For the convenience of readers who may not be familiar with dilatonic gravity, we will first recall a few relevant facts. The standard equations of motion that follow from S_{Dil} are:

$$\nabla_a (e^{-2\alpha\phi} \mathbf{F}^{ab}) = 0, \quad (5.19)$$

$$\nabla^2 \phi + \frac{\alpha}{2} e^{-2\alpha\phi} \mathbf{F}^2 = 0, \quad (5.20)$$

$$R_{ab} = 2\nabla_a \phi \nabla_b \phi + 2e^{-2\alpha\phi} \mathbf{F}_{ac} \mathbf{F}_b{}^c - \frac{1}{2} g_{ab} e^{-2\alpha\phi} \mathbf{F}^2, \quad (5.21)$$

where $\mathbf{F}^2 = \mathbf{F}_{ab} \mathbf{F}^{ab}$. The first equation (5.19) can be rewritten as $d(e^{-2\alpha\phi} * \mathbf{F}) = 0$, so there is a conserved charge,

$$\tilde{Q} := \frac{1}{4\pi} \oint_S e^{-2\alpha\phi} * \mathbf{F}, \quad (5.22)$$

where S is any 2-sphere ‘containing the black hole’ (the integral is independent of the sphere as long as they are homologous). We will

refer to it as the *dilatonic charge* to distinguish it from the electric charge

$$Q_S := \frac{1}{4\pi} \oint_S {}^*F. \quad (5.23)$$

We will see that our boundary conditions directly imply that the electric charge is conserved along the horizon. Thus, by choosing S to be a 2-sphere cross-section of the horizon, we obtain a charge, Q_{bh} , which is intrinsically associated with the black hole. Next, note that Eq.(5.19) can be rewritten as $d^*F = 2\alpha d\phi \wedge {}^*F$. Therefore, we obtain a current three-form and a conserved quantity: Since

$$\int_M (d^*F - 2\alpha d\phi \wedge {}^*F) = 0 \quad (5.24)$$

for any (partial) Cauchy surface M , the difference

$$Q_\infty - Q_{bh} = \frac{\alpha}{2\pi} \int_M d\phi \wedge {}^*F \quad (5.25)$$

between the electric charge Q_∞ measured at spatial infinity and Q_{bh} evaluated at the intersection of M and Δ is also conserved.

Let us now impose boundary conditions to ensure that we have a well-defined action principle for dilatonic and Maxwell fields. At infinity we will impose the same boundary conditions for the Maxwell field as in Section 5.1. For the dilaton field we will require that it tend to a constant, ϕ_∞ , at infinity, and that its derivatives fall-off as $O(1/r^2)$. These conditions suffice for vanishing of the boundary terms at infinity that result from the variation of action.

Let us now turn to the boundary conditions at the horizon, starting with (v.a). We can read-off the stress-energy tensor from Eq (5.21) and verify that $T_{ab}l^a l^b \geq 0$. The Raychaudhuri equation now implies that $R_{ab}l^a l^b \geq 0$. Therefore, transvecting (5.21) with $l^a l^b$, we conclude that $\dot{\phi} := l^a \nabla_a \phi \geq 0$, and that the electro-magnetic field tensor is of the form (5.5). Next, it is simple to verify that the condition that $T_{ab}l^b$ be causal implies that $m^a \nabla_a \phi \geq 0$. Therefore dilaton field ϕ is constant on the horizon. It is easy to check that the constancy of ϕ is also sufficient to ensure that $T_{ab}l^b$ be causal. Next, consider condition (v.b). Since ϕ is constant on Δ , as in the Maxwell case, (v.b) can be satisfied by demanding that the component ϕ_1 of the Maxwell field be spherically

symmetric. Finally, note that since the dilaton is v independent on the horizon equations (5.5),(5.24) imply that $\int_{\Delta} d^* \mathbf{F} = 0$, so the electric charge Q_{bh} is independent of the 2-sphere of integration in (5.23). As before, we will restrict our histories to have specific values of area and of charges \tilde{Q} and Q_{bh} : $a_{\Delta}, \tilde{Q}_{\Delta}$ and Q_{Δ} . (For simplicity, in this section we have set the magnetic charge P to zero.)

We are now ready to analyze the differentiability of S_{Dil} . The variation of the action (5.18) yields two surface terms, one from the variation of the Maxwell field and the other from the variation of the dilaton field. Both vanish at infinity because of the standard boundary conditions. At the horizon, the first term gives:

$$\frac{e^{-2\alpha\phi}}{4\pi} \int_{\Delta} \delta \mathbf{A} \wedge {}^* \mathbf{F}. \quad (5.26)$$

As in section (5.1), we will gauge fix $\mathbf{A} \cdot l$ to its standard value in the static solution. Then $\delta \mathbf{A} \cdot l = 0$ and the integrand in the boundary term vanishes because the Maxwell field has the form (5.9).

Let us now consider the variation of S_{Dil} with respect to the dilaton field. The corresponding boundary term is

$$\frac{1}{4\pi} \int_{\Delta} {}^* d\phi \delta\phi. \quad (5.27)$$

This term also vanishes since $\dot{\phi}$ vanishes on Δ . Thus, with our boundary conditions, the bulk action S_{Dil} is differentiable by itself.

Next, let us examine the structure of the phase space of the theory. Consider first the term $S_{EM} = -\frac{1}{16\pi} \int d^4x \sqrt{-g} e^{-2\phi} \mathbf{F}^2$ in the action S_{Dil} . Performing the usual $3+1$ decomposition and defining a time translation vector field t^a that tends to l^a at the horizon, we find that the canonically conjugated momenta Π^a is given by:

$$\Pi^a = \frac{\sqrt{h}}{4\pi} e^{-2\alpha\phi} \mathbf{E}^a, \quad (5.28)$$

where h is the determinant of the metric h_{ab} on M and \mathbf{E}^a is the usual electric field $\mathbf{E}^a = g^{ab} \mathbf{F}_{cb} \mathbf{n}^c$. The action then reads,

$$S_{EM} = \frac{1}{4\pi} \int dt \int_{M_t} e^{-2\alpha\phi}$$

$$\cdot \left[\mathbf{E} \wedge (\mathcal{L}_t \mathbf{A}) - d(\mathbf{A} \cdot t) \wedge \mathbf{E} - (N \cdot \mathbf{F}) \wedge \mathbf{E} - \frac{N}{2} {}^{(*)}\mathbf{E} \wedge \mathbf{E} - \frac{N}{2} {}^{(*)}\mathbf{F} \wedge \mathbf{F} \right], \quad (5.29)$$

where, as before, $(*)$ denotes the 3-dimensional dual so that $(*)\mathbf{E}_a := \frac{1}{2}\epsilon_a{}^{bc}\mathbf{E}_{bc}$ and $(*)\mathbf{F}_a := \frac{1}{2}\epsilon_a{}^{bc}\mathbf{F}_{bc}$.

The $3 + 1$ decomposition of the term $S_\phi = -\frac{1}{8\pi} \int d^4x \sqrt{-g}(\nabla\phi)^2$, gives as the momentum conjugate to ϕ ,

$$\Pi = \frac{1}{4\pi} \sqrt{h} \mathbf{n}^a \nabla_a \phi, \quad (5.30)$$

where \mathbf{n}^a is the unit normal to M , and the action can be written as

$$S_\phi = \int dt \int_M d^3x \left[\Pi \dot{\phi} - \frac{2\pi N}{\sqrt{h}} \Pi^2 - N^a (\nabla_a \phi) \Pi + \frac{\sqrt{h}}{2\pi N} (N^a \nabla_a \phi)^2 + \frac{1}{8\pi} h^{ab} \nabla_a \phi \nabla_b \phi \right]. \quad (5.31)$$

Thus, the matter phase space consists of pairs (\mathbf{A}, Π) and (ϕ, Π) on M , satisfying our boundary conditions. The symplectic structure can be written as

$$\Omega|_{(\mathbf{A}, \Pi)}(\delta_1, \delta_2) = \frac{1}{4\pi} \int_M [\delta_1 \Pi^a \delta_2 \mathbf{A}_a - \delta_2 \Pi^a \delta_1 \mathbf{A}_a]. \quad (5.32)$$

for the Maxwell part and,

$$\Omega|_{(\phi, \Pi)}(\delta_1, \delta_2) = \frac{1}{4\pi} \int_M [\delta_1 \Pi \delta_2 \phi - \delta_2 \Pi \delta_1 \phi]. \quad (5.33)$$

for the dilaton. The gauge transformations in the Hamiltonian framework are the Maxwell $U(1)$ gauge rotations. The Hamiltonian can be read off from the Legendre transforms of the action given above.

6 Discussion

In this paper, we introduced the notion of a non-rotating isolated horizon, constructed an action which yields Einstein's equations for histories admitting these horizons, and obtained a Hamiltonian framework

in terms of real $\text{su}(2)$ -valued fields. We found that there is a surprising and interesting interplay between the space-time boundary conditions satisfied by isolated horizons Δ and the Chern-Simons theory for a $\text{U}(1)$ connection on Δ . The gravitational part of boundary conditions, the corresponding action and the phase space framework could be discussed in the general case, without committing oneself to specific matter fields. In addition, we discussed in detail the boundary conditions, action and the Hamiltonian formulation for Maxwell and dilaton fields. This detailed framework has five parameters:¹⁰ the cosmological constant Λ , the horizon area a_Δ , the electric and magnetic charges Q and P , and the dilaton charge \tilde{Q} . Since our primary motivation is to present a Hamiltonian framework that will serve as a point of departure for the entropy calculation of a quantum black hole in [4], in this paper we worked with sectors of the theory in which all these charges are fixed. A more general discussion is necessary to extend the laws of black hole mechanics to isolated horizons and is given in [6].

In the final phase space framework, the basic variables consist of pairs, $(\gamma A, \Sigma)$ of $\text{su}(2)$ -valued forms as in the simpler case without internal boundaries. However, they are now subject to boundary conditions not only at infinity but also on the internal boundary S representing the isolated horizon: on S , the independent information in $\underline{\mathcal{A}}$ is contained in the (γ independent) $\text{U}(1)$ connection V whose curvature \mathcal{F} is proportional to $\underline{\sum}^{AB} i_{AOB}$ (see (4.9)). Of particular interest is the interplay between various fields at S . In the gravitational sector, the boundary conditions fix neither the soldering form σ nor the connection A on S but rather a *relation* between them, namely (4.9). As a consequence, quantum theory will allow fluctuations in both these fields but they will be intertwined by the quantum analog of (4.9). In the matter sector, on the other hand, the dynamical fields on S are completely determined by the values of charges and geometry —more precisely, the field $\underline{\Sigma}$ — of S . This will turn out to be the key reason why the entropy depends only on the area a_Δ of S , independently of the matter

¹⁰In the standard treatments, one uses the ADM mass M as the fundamental parameter and expresses area a_Δ in terms of M and other charges. By contrast, we regard a_Δ as the basic parameter and M as a secondary quantity derived from Hamiltonian considerations [5, 6]. Thus, all our parameters—including charges in the dilatonic case— are defined directly at the horizon, without reference to infinity. This is particularly important for cosmological horizons where M may not even be defined! In this case, we will be still be able to account for the entropy [4].

content. Finally, we saw that the presence of the horizon modifies symplectic structure: in addition to the standard bulk term, there is now a surface term which coincides with the symplectic structure of a $U(1)$ Chern-Simons theory for V (see (4.31)). This feature will also play a key role in the quantization of the theory in [4].

The isolated horizons introduced here are special cases of Hayward's trapping horizons [23] in that we require that the expansion of l should vanish (and n to satisfy certain restrictions.) This condition on l was necessary for us because, for entropy considerations, we wish to focus on isolated horizons. While there are several similarities between the two sets of analyses, there are also important differences in the motivation and hence also in the subsequent developments. Hayward's papers discuss dynamical situations and, since he is specifically interested in characterizing black holes quasi-locally, he makes a special effort to rule out cosmological horizons. By contrast, we are primarily interested in the interface of general relativity, statistical mechanics and quantum field theory. Therefore, we focus on equilibrium situations and wish to incorporate not only black holes but also cosmological horizons (as well as other situations such as those depicted in figure 2) which are of importance in this context. Finally, because of our further restrictions on l and n , we were able to obtain an action principle and construct a detailed Hamiltonian framework which is necessary for quantization.

There are two directions in which our framework could be extended. First, one should weaken the boundary conditions and carry out the subsequent analysis to allow for rotation. As indicated in section 3.1, only conditions (iv.b) and (v.b) in the main Definition have to be weakened. Recent results of Lewandowski [26] have paved way for this task. Second, already in the non-rotating case, one could allow horizons which are distorted, e.g., because the presence of a cage around the black hole. It may not be possible to introduce an action principle and a complete Hamiltonian framework in this case since the matter fields involved are not fundamental. However, it should still be possible to introduce appropriate boundary conditions, work out their consequences and explore generalizations of the laws of black hole mechanics. The second type of extension involves going from general relativity to more general, higher derivative theories. In these cases, the boundary conditions at Δ would be the same but there would be some differences in their consequences arising from the differences in the field equations. The action

principle and the Hamiltonian framework would be significantly different. However, it should be possible to carry out the required extension in a tetrad framework [6]. This would provide an interesting generalization of Wald's analysis [37] based on Noether charges of stationary solutions.

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A Conventions

In this paper primed and unprimed upper case letters stand for $\text{SL}(2, C)$ spinor indices, lower case letters denote space-time tensor indices. (In sections on the phase space framework, the unprimed upper case letters denote the $\text{SU}(2)$ spinor indices and the lower case letters denote spatial indices.) The conventions are the same as in [2]. These differ somewhat from the Penrose-Rindler [29] conventions because while their metric has signature $+, -, -, -$ ours has signature $- , + , + , +$.

The soldering form (or, equivalently, the tetrad field) defines the metric via

$$g_{ab} = \sigma_a^{AA'} \sigma_{bAA'}. \quad (\text{A.1})$$

The soldering form σ is required to be anti-hermitian ($\bar{\sigma}_a^{AA'} = -\sigma_a^{A'A}$) so that g defined through (A.1) is a real Lorentzian metric of signature

($-+++$). The self-dual connection A defines a derivative operator D_a that operates on *unprimed* spinors and defines the connection 1-form A via $D_a \lambda_A = \partial_a \lambda_A + A_{aA}{}^B \lambda_B$.

The self-dual 2-forms Σ are defined by:

$$\Sigma_{ab}^{AB} = 2\sigma_{[a}^{AA'} \sigma_{b]}^B {}_{A'} = 2\sigma_a^{(A|A'|} \sigma_b^{B)} {}_{A'}. \quad (\text{A.2})$$

The spinorial equivalents of the null tetrad are given by

$$l^a = i o^A o^{A'} \sigma_{AA'}^a \quad (\text{A.3})$$

$$n^a = i i^A i^{A'} \sigma_{AA'}^a \quad (\text{A.4})$$

$$m^a = o^A i^{A'} \sigma_{AA'}^a \quad (\text{A.5})$$

$$\bar{m}^a = -i^A o^{A'} \sigma_{AA'}^a \quad (\text{A.6})$$

Our choice of orientation is as follows. The volume 4-form in \mathcal{M} is taken to be

$$\epsilon_{abcd} = 24il_{[a} n_b m_c \bar{m}_{d]}; \quad (\text{A.7})$$

the volume 3-form on M is

$$\epsilon_{abc} = \epsilon_{abcd} \tau^d; \quad (\text{A.8})$$

while that on Δ is

$${}^\Delta \epsilon_{abc} = -6in_{[a} m_b \bar{m}_{c]}. \quad (\text{A.9})$$

Finally, the area 2-form on S is assumed to be

$$\epsilon_{ab} = 2im_{[a} \bar{m}_{b]}, \quad (\text{A.10})$$

The Newman-Penrose components of the Weyl tensor are given by:

$$\begin{aligned} \Psi_0 &= \Psi_{ABCD} o^A o^B o^C o^D = -C_{abcd} l^a m^b l^c m^d \\ \Psi_1 &= \Psi_{ABCD} o^A o^B o^C i^D = i C_{abcd} l^a m^b l^c n^d \\ \Psi_2 &= \Psi_{ABCD} o^A o^B i^C i^D = C_{abcd} l^a m^b \bar{m}^c n^d \\ \Psi_3 &= \Psi_{ABCD} o^A i^B i^C i^D = -i C_{abcd} l^a n^b \bar{m}^c n^d \\ \Psi_4 &= \Psi_{ABCD} i^A i^B i^C i^D = -C_{abcd} \bar{m}^a n^b \bar{m}^c n^d, \end{aligned} \quad (\text{A.11})$$

where C_{abcd} is the Weyl tensor. Finally, the tetrad components of the Ricci tensor are given by:

$$\begin{aligned}
 \Phi_{00} &= \Phi_{ABA'B'} o^A o^B o^{A'} o^{B'} & = \frac{1}{2} R_{ab} l^a l^b \\
 \Phi_{01} &= \Phi_{ABA'B'} o^A o^B o^{A'} i^{B'} & = i \frac{1}{2} R_{ab} l^a m^b \\
 \Phi_{02} &= \Phi_{ABA'B'} o^A o^B i^{A'} i^{B'} & = -\frac{1}{2} R_{ab} m^a m^b; \\
 \Phi_{10} &= \Phi_{ABA'B'} o^A i^B o^{A'} o^{B'} & = -i \frac{1}{2} R_{ab} l^a \bar{m}^b; \\
 \Phi_{11} &= \Phi_{ABA'B'} o^A i^B o^{A'} i^{B'} & = \frac{1}{4} R_{ab} (l^a n^b + m^a \bar{m}^b); \quad (\text{A.12}) \\
 \Phi_{12} &= \Phi_{ABA'B'} o^A i^B i^{A'} i^{B'} & = i \frac{1}{2} R_{ab} n^a m^b; \\
 \Phi_{20} &= \Phi_{ABA'B'} i^A i^B o^{A'} o^{B'} & = -\frac{1}{2} R_{ab} \bar{m}^a \bar{m}^b; \\
 \Phi_{21} &= \Phi_{ABA'B'} i^A i^B o^{A'} i^{B'} & = -i \frac{1}{2} R_{ab} n^a \bar{m}^b; \\
 \Phi_{22} &= \Phi_{ABA'B'} i^A i^B i^{A'} i^{B'} & = \frac{1}{2} R_{ab} n^a n^b; \\
 && \quad (\text{A.13})
 \end{aligned}$$

(As usual, $\Phi_{ab} := \sigma_a^{AA'} \sigma_b^{BB'} \Phi_{AA'BB'}$ is the traceless part of the Ricci tensor: $-2\Phi_{ab} = R_{ab} - \frac{R}{4} g_{ab}$.)

Finally, the Newman-Penrose components of the Maxwell field are given by:

$$\begin{aligned}
 \phi_0 &= \phi_{AB} o^A o^B & = -i F_{ab} l^a m^b \\
 \phi_1 &= \phi_{AB} i^A o^B & = -\frac{1}{2} F_{ab} (l^a n^b - m^a \bar{m}^b) \\
 \phi_2 &= \phi_{ab} i^A i^B & = -i F_{ab} n^a \bar{m}^b
 \end{aligned} \quad (\text{A.14})$$

B Some consequences of boundary conditions

In this appendix we sketch proofs of assertions made in section 3.3. An alternate procedure, tailored to the Newman Penrose framework, and

other consequences of the boundary conditions which are not directly needed here can be found in [6].

1. To see that the Lie derivative of the induced metric on Δ with respect to l is zero, note first that the condition (iv.a) implies¹¹

$$\underline{D}_a o_A \hat{\equiv} -\alpha_a o_A \quad (\text{B.1})$$

for some one-form α_a on Δ . Now, using the expression (A.3) of l in terms of the spinors and the compatibility of σ and A implied by condition (iii) of the main Definition, one can easily find the expression for $\nabla_a l_b$ in terms of α_a :

$$\nabla_a l_b \hat{\equiv} -2U_a l_b, \quad (\text{B.2})$$

where, as in the main text, $U_a \hat{\equiv} \text{Re}\alpha_a$. (Conditions (iv.b) on n imply that $U_a = -(U \cdot l)n_a$.) Eq (B.2) implies that l is a geodesic vector field,

$$l^a \nabla_a l_b \hat{\equiv} -2(l^a U_a)l_b,$$

and that Lie derivative with respect to l of the metric induced on Δ vanishes:

$$\underline{\mathcal{L}} g_{ab} \hat{\equiv} 2\nabla_{(a} l_{b)} = 0. \quad (\text{B.3})$$

It also implies that l is twist-free, shear-free and expansion-free. In fact, the condition (iv.a) is equivalent to the requirement that l be geodesic, twist-free, torsion-free and expansion-free.

2. Next, let us show that the connection A on Δ is of the form (3.4). Eq (B.1) tells us a constraint on the connection imposed by the condition (iv.a) of the main Definition. Condition (iv.b) of the Definition provides a further restriction:

$$\underline{D}_a i_A \hat{\equiv} \alpha_a i_A + i f(v) \overline{m}_a o_A, \quad (\text{B.4})$$

where α_a is the same 1-form as in (B.1) (because the spin dyad is normalized $i^A o_A \hat{\equiv} 1$), and $f(v)$ is a function of v only as introduced by the boundary condition (iv.b). A general connection A_a^{AB} acting only on unprimed spinors can be decomposed into its $i^{(A} o^{B)}$, $i^A i^B$, $o^A o^B$

¹¹As in the main text, note that the pull-back operation is redundant for covariant fields which are defined intrinsically on Δ .

components. Then, in the gauge adapted to the dyad, $\partial_a i^A \hat{=} \partial_a o^A \hat{=} 0$, the connection has only $i^{(A} o^{B)}$, $o^A o^B$ components, as in (3.4):

$$\underline{\underline{A}}_a^{AB} \hat{=} -2\alpha_a i^{(A} o^{B)} - \beta_a o^A o^B \quad (\text{B.5})$$

It now follows immediately that the pull-back to Δ of the curvature is given by:

$$\underline{\underline{F}}_{ab}^{AB} = -4(\partial_{[a}\alpha_{b]}) i^{(A} o^{B)} - 2(\partial_{[a}\beta_{b]} - 2\alpha_{[a}\beta_{b]}) o^A o^B \quad (\text{B.6})$$

We will provide a proof of the assertions (3.7) and (3.8) on properties of U and β after discussing the properties of curvature tensors (see 5. below).

Finally, let us establish (3.9). We assume that the slice M intersects Δ in S such that on S the unit normal τ^a to M is given by $\tau^a = (l^a + n^a)/\sqrt{2}$. Therefore, using (B.1) and (B.4), we can compute the part $\underline{\underline{K}}_a^{AB}$ of the extrinsic curvature of M at points of S . One obtains:

$$\underline{\underline{K}}_a^{AB} = \frac{f(v)}{\sqrt{2}}(m_a i^A i^B + \bar{m}^a o^A o^B) \quad (\text{B.7})$$

Since the connection A on M is given by: $A_a^{AB} = \Gamma_a^{AB} - \frac{i}{\sqrt{2}}K_a^{AB}$, using the definition of V we have the required result:

$$\underline{\underline{V}}_a = -i\underline{\underline{\Gamma}}_a^{AB} i_A o_B. \quad (\text{B.8})$$

3. As is well-known, the full Riemann curvature can be decomposed into its (self- and antiself-dual) Weyl tensor, traceless Ricci tensor, and scalar curvature. In the spinorial notation, we have (see, e.g., [29]):

$$\begin{aligned} R_{AA'BB'CC'DD'} &= \Psi_{ABCD}\epsilon_{A'B'}\epsilon_{C'D'} + \bar{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD} \\ &\quad + \Phi_{ABC'D'}\epsilon_{A'B'}\epsilon_{CD} + \bar{\Phi}_{A'B'CD}\epsilon_{AB}\epsilon_{C'D'} \\ &\quad + \frac{R}{12}(\epsilon_{AC}\epsilon_{BD}\epsilon_{A'C'}\epsilon_{B'D'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'D'}\epsilon_{B'C'}). \end{aligned} \quad (\text{B.9})$$

The Weyl spinor Ψ_{ABCD} and the trace-free Ricci spinor $\Phi_{AA'BB'}$ can be further expanded in terms of their components in the spinor basis i^A, o^A defined in appendix A,

Let us consider the Ricci tensor. Since $T_{ab}l^a l^b \geq 0$ the Raychaudhuri equation for l^a implies $T_{ab}l^a l^b \hat{=} 0$, whence, via Einstein's equation which holds on Δ , we conclude $\Phi_{00} \hat{=} 0$. Since $T_{ab}l^b$ is causal and

$T_{ab}l^a l^b \hat{=} 0$, it follows that $T_{ab}m^a l^b \hat{=} 0$, whence we have: $\Phi_{10} \hat{=} 0$ and $\Phi_{01} \hat{=} 0$. These conditions can be summarized in the following equation:

$$\begin{aligned} \Phi_{ABA'B'} \hat{=} & 4\Phi_{11}i_{(A}o_{B)}i_{(A'}o_{B')} + \Phi_{22}o_A o_B o_{A'} o_{B'} + \Phi_{02}i_A i_B o_{A'} o_{B'} \\ & + \Phi_{20}o_A o_B i_{A'} i_{B'} - 2\Phi_{12}i_{(A}o_{B)} - 2\Phi_{21}o_A o_B i_{(A'}o_{B')}. \end{aligned} \quad (\text{B.10})$$

(We will show later that Φ_{20} and Φ_{02} also vanish.)

Let us now turn to the Weyl tensor. Using the equation (B.2) one gets:

$$\underbrace{R_{abcd}}_{} l^d = 2\underbrace{\nabla_{[a}\nabla_{b]}}_{} l_c = -2(\nabla_{[a}\mu_{b]})l_c \hat{=} 0, \quad (\text{B.11})$$

Transvecting this equation with appropriate vectors and using the fact that the trace of the Weyl tensor vanishes, we conclude that

$$\Psi_0 \hat{=} 0 \quad \text{and} \quad \Psi_1 \hat{=} 0. \quad (\text{B.12})$$

Next, using the boundary conditions (iv.a), (iv.b), one can calculate the derivative of n_a :

$$\underbrace{\nabla_a n_b}_{} = 2U_a n_b - 2f m_{(a} \overline{m}_{b)}. \quad (\text{B.13})$$

Hence, it follows that

$$\begin{aligned} \underbrace{R_{abcd}n^d}_{} &= 2\underbrace{\nabla_{[a}\nabla_{b]}}_{} n_c = -4f(\overline{m}_{[a} U_{b]} m_c + m_{[a} U_{b]} \overline{m}_c) \\ &\quad - (\partial_{[a} f)m_{b]} \overline{m}_c - (\partial_{[a} f)\overline{m}_{b]} m_c. \end{aligned} \quad (\text{B.14})$$

Transvecting this equation with suitable vectors and using the trace-free property of the Weyl tensor and the fact that we have set $f = 1/r_\Delta$, we conclude:

$$\text{Im}\Psi_2 \hat{=} 0, \quad \Psi_2 + \frac{R}{12} \hat{=} \frac{2U \cdot l}{r_\Delta}, \quad \text{and} \quad \Psi_3 - \Phi_{21} \hat{=} 0 \quad (\text{B.15})$$

Consequently, the Weyl spinor has the form:

$$\Psi_{ABCD} \hat{=} 6\Psi_2 i_{(A} i_B o_{C} o_{D)} - 4\Psi_3 i_{(A} o_B o_{C} o_{D)} + \Psi_4 o_A o_B o_{C} o_{D}. \quad (\text{B.16})$$

where Ψ_2 and Ψ_3 are subject to (B.15).

Next, Recall [2] the expression for F in terms of the self-dual part of the Riemann curvature:

$$F_{ab}^{AB} = -\frac{1}{4}R_{ab}^{cd}\Sigma_{ab}^{AB}. \quad (\text{B.17})$$

Using the decomposition (B.9) of the Riemann tensor, we obtain:

$$F_{ab}{}_{CD} \cong -\frac{1}{2}\Psi_{ABCD}\Sigma_{ab}^{AB} - \frac{1}{2}\bar{\Phi}_{A'B'CD}\bar{\Sigma}_{ab}^{A'B'} - \frac{R}{24}\Sigma_{ab}{}_{CD}. \quad (\text{B.18})$$

Finally, using the identity:

$$\underline{\Sigma_{ab}}^{AB} \cong 4i(o^A o^B)m_{[\underline{a}}\bar{m}_{b]} + 4io^A o^B n_{[\underline{a}}\bar{m}_{b]}, \quad (\text{B.19})$$

which follows from the definitions of the null tetrad, and equations (B.10) and (B.16), we can express the pull-back $\underline{F}_{ab}{}^{CD}$ of F to Δ as:

$$\underline{F}_{ab}{}^{CD} \cong \left(\Psi_2 - \Phi_{11} - \frac{R}{24} \right) \underline{\Sigma_{ab}}^{CD} - 2i(3\Psi_2 - 2\Phi_{11})o^C o^D n_{[\underline{a}}\bar{m}_{b]}. \quad (\text{B.20})$$

where we have used $\Psi_3 - \Phi_{21} = 0$. (Other Weyl and Ricci components do not appear because F is pulled-back to Δ .) This equation can also be rewritten in the following simpler form:

$$\underline{F}_{ab}^{AB} \cong \left[\left(\Psi_2 - \Phi_{11} - \frac{R}{24} \right) \delta_C^A \delta_D^B - \left(\frac{3\Psi_2}{2} - \Phi_{11} \right) o^A o^B i_C i_D \right] \underline{\Sigma_{ab}}^{CD}. \quad (\text{B.21})$$

Next, we will use the Bianchi identity $\mathcal{D} \wedge F \cong 0$ and the equation of motion $\mathcal{D} \wedge \Sigma \cong 0$ to extract further information about Ψ_2, Φ_{11} and R which appear in the above relation between \underline{F} and $\underline{\Sigma}$. Using (B.1) and (B.4), we obtain:

$$\begin{aligned} 0 &\cong \underline{\nabla}_{[a} \underline{F}_{bc]}^{AB} \\ &= \left[\underline{\nabla}_{[a} \left(\Psi_2 - \Phi_{11} - \frac{R}{24} \right) \delta_C^A \delta_D^B - \underline{\nabla}_{[a} \left(\frac{3\Psi_2}{2} - \Phi_{11} \right) o^A o^B i_C i_D \right. \\ &\quad \left. - 2 \left(\frac{3\Psi_2}{2} - \Phi_{11} \right) \beta_{[a} o^A o^B o_C i_D \right] \underline{\Sigma}_{bc]}^{CD}. \end{aligned} \quad (\text{B.22})$$

Transvecting this equation with $o_A o_B$, we get an identity. Transvecting it with $i_A o_B$ we conclude: $l^a \partial_a (\Psi_2 - \Phi_{11} - \frac{R}{24}) \hat{\equiv} 0$. Transvecting it with $i_A i_B$, we conclude that $\Psi_2 + \frac{R}{12}$ is spherically symmetric. Recall furthermore that by condition (v.b) in the main Definition, $T_{ab} l^a n^b$ is spherically symmetric. Using Einstein's equation on Δ we conclude that $(\Psi_2 - \Phi_{11} - \frac{R}{24})$ is spherically symmetric on Δ .

Finally, the pull-back of (B.20) to any 2-sphere in our preferred foliation yields:

$$\underline{\underline{F_{ab}}}^{CD} \hat{\equiv} \left(\Psi_2 - \Phi_{11} - \frac{R}{24} \right) \underline{\underline{\Sigma_{ab}}}^{CD} \quad (\text{B.23})$$

We can transvect this equation with $i_C o_D$ and integrate the result on a 2-sphere. The left side is then $-2\pi i$, the Chern number of the spin-bundle of the $U(1)$ connection, while the factor in the parenthesis on the right side, being constant, comes outside the integral and the integral itself then yields just ia_Δ . Hence, we have:

$$\Psi_2 - \Phi_{11} - \frac{R}{24} \hat{\equiv} - \frac{2\pi}{a_\Delta}. \quad (\text{B.24})$$

An alternate (non-spinorial) way to obtain this result is to note that the left side of (B.24) equals $-(^2R/4)$ where 2R is the scalar curvature of the 2-sphere cross-sections [29]. Then, using the Gauss Bonnet theorem, we obtain (B.24).

5. It only remains to show the properties (3.7) and (3.8) of μ and β and vanishing of Φ_{20} and Φ_{02} .

Eq (3.7) is an immediate consequence of the second equation in (B.15) and Einstein's equation on Δ . To obtain the first equation on β , let us transvect (B.6) with $i_A i_B$. Using (B.20), one can express the left side as

$$\underline{\underline{F_{ab}}}^{AB} i_A i_B \hat{\equiv} - 2i(\Psi_2 + \frac{R}{12}) n_{[a} \bar{m}_{b]}. \quad (\text{B.25})$$

The right side can be simplified by expanding α in terms of μ and V and using the expression (3.7) of μ :

$$-2(\partial_{[a} - 2\alpha_{[a}) \beta_{b]} = -2(\partial_{[a} - i\nu_{[a}) \beta_{b]} - 2i(\Psi_2 + \frac{R}{12}) n_{[a} \bar{m}_{b]} \quad (\text{B.26})$$

Hence we have the desired result:

$$(\partial_{[a} - i\nu_{[a}) \beta_{b]} \hat{=} 0 \quad (\text{B.27})$$

The second equation in (3.8) follows from (B.6) and (B.23).

Finally, transvect (B.18) with $l^a \bar{m}^b i_C i_D$ and (B.25) with $l^a \bar{m}^b$. Equating the two expressions, we conclude¹²: $\Phi_{20} \hat{=} 0$ and $\Phi_{02} \hat{=} 0$.

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