Adv. Theor. Math. Phys. **3** (2000) 495–565

Local Mirror Symmetry: Calculations and Interpretations

T.-M. Chiang,[a] A. Klemm,[b] S.-T. Yau,[a] and E. Zaslow[c]

[a]Department of Mathematics
Harvard University
Cambridge, MA 02138, USA
chi@math.harvard.edu
yau@math.harvard.edu

[b]School of Natural Sciences
IAS, Olden Lane
Princeton, NJ 08540, USA
klemm@ias.edu

[c]Department of Mathematics
Northwestern University
Evanston, IL 60208, USA
zaslow@math.nwu.edu

Abstract

We describe local mirror symmetry from a mathematical point of view and make several A-model calculations using the

---

e-print archive: http://xxx.lanl.gov/hep-th/9903053

mirror principle (localization). Our results agree with B-model computations from solutions of Picard-Fuchs differential equations constructed form the local geometry near a Fano surface within a Calabi-Yau manifold. We interpret the Gromov-Witten-type numbers from an enumerative point of view. We also describe the geometry of singular surfaces and show how the local invariants of singular surfaces agree with the smooth cases when they occur as complete intersections.

1 Introduction

"Local mirror symmetry" refers to a specialization of mirror symmetry techniques to address the geometry of Fano surfaces within Calabi-Yau manifolds. The procedure produces certain "invariants" associated to the surfaces. This paper is concerned with the proper definition and interpretation of these invariants. The techniques we develop are a synthesis of results of previous works (see [33], [42], [35], [37], [55]), with several new constructions. We have not found a cohesive explanation of local mirror symmetry in the literature. We offer this description in the hope that it will add to our understanding of the subject and perhaps help to advance local mirror symmetry towards higher genus computations.

Mirror symmetry, or the calculation of Gromov-Witten invariants in Calabi-Yau threefolds,[1] can now be approached in the traditional ("B-model") way or by using localization techniques. The traditional approach involves solving the Picard-Fuchs equations governing the behavior of period integrals of a Calabi-Yau manifold under deformations of complex structure, and converting the coefficients of the solutions near a point of maximal monodromy into Gromov-Witten invariants of the mirror manifold. Localization techniques, first developed by Kontsevich [44] and then improved by others [25], [50], offer a proof – without reference to a mirror manifold – that the numbers one obtains in this way are indeed the Gromov-Witten invariants as defined via the moduli space of maps.

[1] We restrict the term "mirror symmetry" to mean an equivalence of quantum rings, rather than the more physical interpretation as an isomorphism of conformal field theories.

Likewise, local mirror symmetry has these two approaches. One finds that the mirror geometry is a Riemann surface with a meromorphic differential. From this one is able to derive differential equations which yield the appropriate numerical invariants. Recall the geometry. We wish to study a neighborhood of a surface S in a Calabi-Yau threefold X, then take a limit where this surface shrinks to zero size. In the first papers on the subject, these equations were derived by first finding a Calabi-Yau manifold containing the surface, then finding its mirror and "specializing" the Picard-Fuchs equations by taking an appropriate limit corresponding to the local geometry. Learning from this work, one is now able to write down the differential equations directly from the geometry of the surface (if it is toric). We use this method to perform our B-model calculations.

We employ a localization approach developed in [50] for computing the Gromov-Witten-type invariants directly (the "A-model"). Since the adjunction formula and the Calabi-Yau condition of X tell us that the normal bundle of the surface is equal to the canonical bundle (in the smooth case), the local geometry is intrinsic to the surface. We define the Gromov-Witten-type invariants directly from K_S, following [25], [50]. We require S to be Fano (this should be related to the condition that S be able to vanish in X), which makes the bundle K_S "concave," thus allowing us to construct cohomology classes on moduli space of maps. We consider the numbers constructed in this way to be of Gromov-Witten type.

In section 2, we review the mirror principle and apply it to the calculation of invariants for several surfaces. In section 3, we give the general procedure for toric varieties. We then calculate the invariants "by hand" for a few cases, as a way of checking and illucidating the procedure. In section 5, we describe the excess intersection formula and show that the local invariants simply account for the effective contribution to the number of curves in a Calabi-Yau manifold due to the presence of a holomorphic surface.

In section 6, we develop all the machinery for performing B-model calculations without resorting to a specialization of period equations from a compact Calabi-Yau threefold containing the relevant local geometry. Actually, a natural Weierstrass compactification exists for toric Fano geometries, and its decompactification (the limit of large elliptic fiber) produces expressions intrinsic to the surface. In this sense, the end result makes no use of compact data. The procedure closely resembles the compact B-model technique of solving differential equations and taking combinations of solutions with different singular behaviors to produce a prepotential containing enumerative invariants as coefficients. Many examples are included.

In order to accommodate readers with either mathematical or physical backgrounds, we have tried to be reasonably self-contained and have included several examples written out in considerable detail. Algebraic geometers may find these sections tedious, and may content themselves with the more general sections (e.g., 3, 4.2, 6.3). Physicists wishing to get a feel for the mathematics of A-model computations may choose to focus on the examples of section 4.1.

2 Overview of the A-model

In this section, we review the techniques for calculating invariants using localization. We will derive the numbers and speak loosely about their interpretation, leaving more rigorous explanations and interpretations for later sections.

For smooth hypersurfaces in toric varieties, we define the Gromov-Witten invariants to be Chern classes of certain bundles over the moduli space of maps, defined as follows. Let $\overline{\mathcal{M}}_{0,0}(\vec{d};\mathbf{P})$ be Kontsevich's moduli space of stable maps of genus zero with no marked points. A point in this space will be denoted (C,f), where $f: C \to \mathbf{P}$, $\mathbf{P}$ is some toric variety, and $[f(C)] = \vec{d} \in H_2(\mathbf{P})$. Let $\overline{\mathcal{M}}_{0,1}(\vec{d};\mathbf{P})$ be the same but with one marked point. Consider the diagram

$$\overline{\mathcal{M}}_{0,0}(\vec{d};\mathbf{P}) \longleftarrow \overline{\mathcal{M}}_{0,1}(\vec{d};\mathbf{P}) \longrightarrow \mathbf{P},$$

where $\mathbf{P}$ is the toric variety in question,

$$ev : \overline{\mathcal{M}}_{0,1}(\vec{d};\mathbf{P}) \longrightarrow \mathbf{P}$$

is the evaluation map sending $(C,f,*) \mapsto f(*)$, and

$$\rho : \overline{\mathcal{M}}_{0,1}(\vec{d};\mathbf{P}) \longrightarrow \overline{\mathcal{M}}_{0,0}(\vec{d};\mathbf{P})$$

is the forgetting map sending $(C, f, *) \mapsto (C, f)$. Let Q be a Calabi-Yau defined as the zero locus of sections of a convex[2] bundle V over $\mathbf{P}$. Then $U_{\vec{d}}$ is the bundle over $\overline{\mathcal{M}}_{0,0}(\vec{d}; \mathbf{P})$ defined by

$$U_{\vec{d}} = \rho_* ev^* V.$$

The fibers of $U_{\vec{d}}$ over (C, f) are $H^0(C, f^*V)$. We define the Kontsevich numbers K_d by

$$K_d \equiv \int_{\overline{\mathcal{M}}_{0,0}(\vec{d}; \mathbf{P})} c(U_{\vec{d}}).$$

It is most desirable when $\dim \overline{\mathcal{M}}_{0,0}(\vec{d}; \mathbf{P}) = \mathrm{rank} U_{\vec{d}}$, so that K_d is the top Chern class.

The mirror principle is a procedure for evaluating the numbers K_d by a fancy version of localization. The idea, pursued in the next section, is as follows. When all spaces and bundles are torically described, the moduli spaces and the bundles we construct over them inherit torus actions (e.g., by moving the image curve). Thus, the integrals we define can be localized to the fixed point loci. As we shall see in the next section, the multiplicativity of the characteristic classes we compute implies relations among their restrictions to the fixed loci. The reason for this is that the fixed loci of degree γ maps includes stable curves constructed by gluing degree α and β maps, with $\alpha + \beta = \gamma$. One then constructs an equivariant map to a "linear sigma model," which is an easily described toric space. Indeed, the linear sigma model is another compactification of the smooth stable maps, which can be modeled as polynomial maps. We then push/pull our problem to this linear sigma model, where the same gluing relations are found to hold. The notion of Euler data is any set of characteristic classes on the linear sigma model obeying these relations. They are not strong enough to uniquely determine the classes, but as the equivariant cohomology can be modeled as polynomials, two sets of Euler data which agree upon restriction to enough points may be thought of as equivalent ("linked Euler data"). It is not difficult to construct Euler data linked to the Euler data of the characteristic classes in which we are interested. Relating the linked Euler data, and therefore solving the problem in terms of simply-constructed polynomial classes, is done via a mirror transform,

[2] "Convex" means that $H^1(C, f^*V) = 0$ for $(C, f) \in \overline{\mathcal{M}}_{0,0}(\vec{d}; \mathbf{P})$. For the simplest example, $\mathbf{P} = \mathbf{P}^4$ and $V = \mathcal{O}(5)$, as in the next subsection.

which involves hypergeometric series familiar to B-model computations. However, no B-model constructions are used. These polynomial classes can easily be integrated, the answers then related to the numbers in which we are interested by the mirror transform. This procedure is used to evaluate the examples in this section which follow, as well as all other A-model calculations.

Using the techniques of the mirror principle, we are able to build Euler data from many bundles over toric varieties. Typically, we have a direct sum of $\bigoplus_i \mathcal{O}(l_i)$ and $\bigoplus_j \mathcal{O}(-k_j)$ over $\mathbf{P}^n$, with $l_i, k_j > 0$. In such a case, if $\sum_i l_i + \sum_j k_j = n+1$ then we can obtain linked Euler data for the bundle $U_{\vec{d}}$ whose fibers over a point (f, C) in $\overline{\mathcal{M}_{0,0}}(d; \mathbf{P}^n)$ is a direct sum of $\bigoplus_i H^0(C, f^*\mathcal{O}(l_i))$ and $\bigoplus_j H^1(C, f^*\mathcal{O}(k_j))$. In this situation, the rank of the bundle (which is $\sum_i(dl_i+1) + \sum_j(dk_j-1)$) may be greater than the dimension D of $\overline{\mathcal{M}_{0,0}}(d; \mathbf{P}^n)$ (which is $D \equiv (n+1)(d+1)-4$). In that case, we compute the integral over moduli space of the Chern class $c_D(U_{\vec{d}})$. The interpretations will be discussed in the examples.

We begin with a convex bundle.

2.1 $\mathcal{O}(5) \to \mathbf{P}^4$

Recall that this is the classic mirror symmetry calculation. We compute this by using the Euler data $P_d = \prod_{j=1}^{5d}(5H - m)$ As the rank of U_d equals the dimension of moduli space, we take the top Chern class of the bundle and call this K_d. This has the standard interpretation: given a generic section with isolated zeros, the Chern class counts the number of zeros. If we take as a section the pull back of a quintic polynomial (which is a global section of $\mathcal{O}(5)$), then its zeros will be curves (C, f) on which the section vanishes identically. As the section vanishes along a Calabi-Yau quintic threefold, the curve must be mapped (with degree d) into the quintic – thus we have the interpretation as "number of rational curves." However, the contribution of curves of degree d/k, when k divides d, is also non-zero. In this case, we can compose any k–fold cover of the curve C with a map f of degree d/k into the quintic. This contribution is often called the "excess intersection." To calculate the contribution to the Chern class, we must look at how this space of k–fold covers of C (which, as $C \cong \mathbf{P}^1$ in the smooth case, is equal to

$\overline{\mathcal{M}_{0,0}}(k, \mathbf{P}^1)$) sits in the moduli space (i.e., look at its normal bundle). This calculation yields $1/k^3$, and so if n_d is the number of rational curves of degree d in the quintic, we actually count

$$K_d = \sum_{k|d} \frac{n_{d/k}}{k^3}. \tag{1}$$

This double cover formula will be discussed in detail in section 5.1.

In the appendix, several examples of other bundles over projective spaces are worked out. In the cases where the rank of U_d is greater (by n) than the dimension of moduli space, we take the highest Chern class that we can integate. The resulting numbers count the number of zeros of $s_0 \wedge \ldots \wedge s_n$, i.e., the places where $n+1$ generic sections gain linear dependencies. A zero of $s_0 \wedge \ldots \wedge s_n$ represents a point (C, f) in moduli space where $f(C)$ vanishes somewhere in the n-dimensional linear system of $s_0, \ldots, s_n$. See the appendix for details. We turn now to the study of some concave bundles.

2.2 $\mathcal{O}(-3) \to \mathbf{P}^2$

We think of $\mathcal{O}(-3)$ as $K_{\mathbf{P}^2}$, the canonical bundle. This case is relevant to Calabi-Yau manifolds containing projective surfaces. A tubular neighborhood of the surface is equivalent to the total space of the canonical bundle (by the adjunction formula and the Calabi-Yau condition $c_1 = 0$).

In this case, the rank of U_d (which is the bundle whose fiber over (C, f) is $H^1(C, f^*K_{\mathbf{P}^2})$) is equal to the dimension of moduli space, so we are computing the top Chern class $K_d = \int_{\overline{\mathcal{M}_{0,0}}(d,\mathbf{P}^2)} c_{3d-1}(U_d)$. From the K_d's we arrive at the following n_d's.

d	n_d
1	3
2	−6
3	27
4	−192
5	1695
6	−17064
7	188454
8	−2228160
9	27748899
10	−360012150

Table 1. Local invariants for $K_{\mathbf{P}^2}$.

The interpretation for the n_d's is not as evident as for positive bundles, since no sections of U_d can be pulled back from sections of the canonical bundle (which has no sections). Instead, we have the following interpretation.

Suppose the $\mathbf{P}^2$ exists within a Calabi-Yau manifold, and we are trying to count the number of curves in the same homology class as d times the hyperplane in $\mathbf{P}^2$. The analysis for the Calabi-Yau would go along the lines of the quintic above. However, there would necessarily be new families of zeros of your section corresponding to the families of degree d curves in the $\mathbf{P}^2$ within the Calabi-Yau. These new families would be isomorphic to $\overline{\mathcal{M}_{0,0}}(d, \mathbf{P}^2)$. On this space, we have to compute the contribution to the total Chern class. To do this, we would need to use the excess intersection formula. The result (see section 5.2) is precisely given by the K_d. Let us call once again n_d the integers derived from the K_d. Suppose now that we have two Calabi-Yau's, X_0 and X_1, in the same family of complex structures, one of which (say X_1 contains) a $\mathbf{P}^2$. Then the difference between $n_d(X_0)$ and $n_d(X_1)$ should be given by the n_d.

There are Calabi-Yau's, however, which generically contain a $\mathbf{P}^2$. The simplest examples are the following elliptic fibrations over a $\mathbf{P}^2$,
A.) the degree 18 hypersurface in denoted by
$\mathbf{P}_{6,9,1,1,1}[18]$, with $\chi = -540$, $h^{21} = 272$, $h^{11} = 2(0)$
B.) $\mathbf{P}_{3,6,1,1,1}[12]$, $\chi = -324$, $h^{21} = 165$, $h^{11} = 3(1)$
C.) $\mathbf{P}_{3,3,1,1,1}[9]$, $\chi = -216$, $h^{21} = 112$, $h^{11} = 4(2)$.
h^{11} contributions in brackets are non-toric divisors, in these cases they

correspond to additional components of the section, see below.

	A			B			C		
d_F	0	1	2	0	1	2	0	1	2
d_B									
0		540	540		216	324		162	162
1	3	−1080	143370	6	−432	10260	9	−324	3645
2	−6	2700	−574560	−12	1080	−41688	−18	810	−14904
3	27	−17280	5051970	54	−6912	378756	81	−5184	137781
4	−192	154440	−57879900	−384	61776	−4411260	−576	46332	−1617570

Table 2. Invariants of the three elliptic fibrations over $\mathbf{P}^2$ (B and F denote the class of a section and the elliptic fiber, respectively).

For such Calabi-Yau's the Gromov-Witten invariants of the homology class of the base would be a multiple of the invariants for K_{base}. In the above examples, we see from the first column the different multiples which arise as we are counting curves in the homology class of a curve which is dual to the hyperplane class of the base $\mathbf{P}^2$. This homology class sits inside a section of the elliptic fibration, and the multiplicities come from the fact that the A, B, C fibrations admit $1, 2, 3$ sections. For example, if we write the ambient toric variety for case C as $\mathbf{P}(\mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2} \oplus \mathcal{O}_{\mathbf{P}^2}(-3))$, a section is given by one of the three components of the vanishing locus of the coordinate on the fiber that transforms as $\mathcal{O}(-3)$ over $\mathbf{P}^2$.

Another interpretation of this number is as follows. The space $H^1(C, f^*K)$ represents obstructions to deformations of the curve C. Therefore, the top Chern class of the bundle whose fibers are $H^1(C, f^*K)$ represents the number of infinitesimal deformations in the family which represent finite deformations. Note this interpretation is equivalent to the one above. The numbers represent the effective number of curves of degree d in the Calabi-Yau "coming from" the $\mathbf{P}^2$.

This procedure can be performed for any Fano surface. The Hirzebruch (rational, ruled) surfaces are described in the Appendix 9. Next we discuss the general toric case.

3 The Mirror Principle for General Toric Manifolds

In this section, we review the mirror principle for computations of Gromov-Witten invariants of a toric variety. For a summary of what follows, we refer the reader to the start of section 2. Our treatment is somewhat more general than that of [50], as we consider general toric varieties, though we omit some proofs which will be included in [51] .

Throughout this section, we take our target manifold to be a smooth, toric and projective manifold $\mathbf{P}$. That is, we are interested in rational curves that map into $\mathbf{P}$. Let us write $\mathbf{P}$ as a quotient of an open affine variety:

$$\mathbf{P} = \frac{\mathbf{C}^{N_C} - \Delta}{G},$$

where $G \cong (\mathbf{C}^*)^{N_C - M}$. We can write the ith action of G as

$$(x_1, \dots, x_{N_C}) \to (\nu^{q_{i,1}} x_1, \dots, \nu^{q_{i,N_C}} x_{N_C}),$$

where ν is an arbitrary element of $\mathbf{C}^*$. There is a $T \equiv (S^1)^{N_C}$ action on $\mathbf{P}$ induced from its usual action on $\mathbf{C}^{N_C}$. This action has N_C fixed points which we denote by $p_1, \dots, p_{N_C}$. For example, for $\mathbf{P} = \mathbf{P}^4$, $\mathbf{T}$ is $(S^1)^5$ and the fixed points are the points with one coordinate nonvanishing.

The $\mathbf{T}$-equivariant cohomology ring can be obtained from the ordinary ring as follows. Write the ordinary ring as a quotient

$$\frac{\mathbf{Q}[B_1, \dots, B_{N_C}]}{I},$$

where B_i is the divisor class of $x_i = 0$ and I is an ideal generated by elements homogeneous in the B_i's. For example, for $\mathbf{P}^4$ we have the ring

$$\frac{\mathbf{Q}[B_1, \dots, B_5]}{(B_1 - B_2, B_1 - B_3, B_1 - B_4, B_1 - B_5, B_1 B_2 B_3 B_4 B_5)}.$$

Let $J_i, i = 1, \dots, M$ be the basis of nef divisors in $H^2(\mathbf{P}, \mathbf{Z})$. We can write the B_l's in terms of the J_k's:

$$B_i = \sum b_{ij} J_j.$$

The equivariant ring is then

$$H_{\mathbf{T}}(\mathbf{P}) = \frac{\frac{\mathbf{Q}[\kappa_1,\dots,\kappa_M,\lambda_1,\dots,\lambda_{N_C}]}{I_{\mathbf{T}}}}{},$$

where $I_{\mathbf{T}}$ is generated by $\sum q_{i,j}\lambda_j$ for $i = 1,\dots,N_C$ and the nonlinear relations in I with B_i replaced by $\sum b_{ij}\kappa_j - \lambda_i$. In the case of $\mathbf{P}^4$ this is

$$\frac{\mathbf{Q}[\kappa,\lambda_1,\dots,\lambda_5]}{(\prod_{i=1}^{5}(\kappa-\lambda_i),\sum\lambda_i)}.$$

Clearly, setting λ_i to zero in $H_{\mathbf{T}}(\mathbf{P})$ gives us the ordinary ring in which κ_j can be identified with J_j.

Having described the base and the torus action, we also need a bundle V to define the appropriate Gromov-Witten problem. For instance, if we are interested in rational curves in a complete intersection of divisors in $\mathbf{P}$, then V is a direct sum of the associated line bundles. For local mirror symmetry, we can also take a concave line bundle as a component of V. More generally, we take $V = V^+ \oplus V^-$, with V^+ convex and V^- concave.

Before proceeding to the next section, we introduce some notation for later use. Let F_j be the associated divisors of the line bundle summands of V, by associating each line bundle to a divisor in the usual way. We write F_j as greater or less than zero, depending on whether it is convex or concave. Homology classes of curves in $\mathbf{P}$ will be written in the basis H_j Poincaré dual to J_j. For instance, $\overline{\mathcal{M}}_{0,0}(\vec{d};\mathbf{P})$ is the moduli space of stable maps with image homology class $\sum d_iH_i$. Finally, x denotes a formal variable for the total Chern class.

3.1 Fixed points and a Gluing Identity

The pull-back of V to $\overline{\mathcal{M}}_{0,1}(\vec{d};\mathbf{P})$ by the evaluation map gives a bundle of the form $ev^*(V^+)\oplus ev^*(V^-)$. Then, in terms of the forgetful map from $\overline{\mathcal{M}}_{0,1}(\vec{d};\mathbf{P})$ to $\overline{\mathcal{M}}_{0,0}(\vec{d};\mathbf{P})$, we obtain a bundle on $\overline{\mathcal{M}}_{0,0}(\vec{d};\mathbf{P})$ $\rho_*ev^*(V^+)\oplus\mathcal{R}^1\rho_*ev^*(V^-)$. The latter is the obstruction bundle $U_{\vec{d}}$.

On $\overline{\mathcal{M}}_{0,0}(\vec{d};\mathbf{P})$, there is a torus action induced by the action on $\mathbf{P}$, i.e., by moving the image curve under the torus action. A typical fixed point of this action is $(f, \mathbf{P}^1)^3$, where $f(\mathbf{P}^1)$ is a $\mathbf{P}^1$ joining two $\mathbf{T}$-fixed points in $\mathbf{P}$.

Another type of fixed point we consider is obtained by gluing. Let $(f_1, C_1, x_1) \in \overline{\mathcal{M}}_{0,1}(\vec{r}; \mathbf{P})$ and $(f_2, C_2, x_2) \in \overline{\mathcal{M}}_{0,1}(\vec{d} - \vec{r}; \mathbf{P})$ be two fixed points. Then $f_1(x_1)$ is a fixed point of $\mathbf{P}$, i.e., one of the p_i's, say p_k. If $f_2(x_2)$ is also p_k, let us glue them at the marked points to obtain $(f, C_1 \cup C_2) \in \overline{\mathcal{M}}_{0,0}(\vec{d}; \mathbf{P})$, where $f|_{C_1} = f_1$, $f|_{C_2} = f_2$ and $f(x_1\, x_2) = p_k$. Clearly, $(f, C_1 \cup C_2)$ is a fixed point as (f_1, C_1, x_1) and (f_2, C_2, x_2) are fixed points. Let us denote the loci of fixed points obtained by gluing as above $FL(p_k, \vec{r}, \vec{d} - \vec{r})$. Over $C_1 \cup C_2$, there is an exact sequence for V:

$$0 \to f^*V \to f_1^*V \oplus f_2^*V \to V|_{f_1(x_1)=f_2(x_2)} \to 0. \tag{2}$$

The long exact cohomology sequence then gives us a gluing identity

$$\Omega^V_{\mathbf{T}} c_{\mathbf{T}}(U_{\vec{d}}) = c_{\mathbf{T}}(U_{\vec{r}}) c_{\mathbf{T}}(U_{\vec{d}-\vec{r}}), \tag{3}$$

where $\Omega^V_{\mathbf{T}} = c_{\mathbf{T}}(V^+)/c_{\mathbf{T}}(V^-)$ is the $\mathbf{T}$-equivariant Chern class of V.

This relation will generate one on the linear sigma model to which we now turn.

3.2 The spaces $M_{\vec{d}}$ and $N_{\vec{d}}$

Because $\overline{\mathcal{M}}_{0,0}(\vec{d}; \mathbf{P})$ is a rather unwieldy space, the gluing identity we found in the last section seems not to be useful. However, we will find, using the gluing identity, a similar identity on a toric manifold $N_{\vec{d}}$. We devote this section mainly to describing $N_{\vec{d}}$ and its relation to $\overline{\mathcal{M}}_{0,0}(\vec{d}; \mathbf{P})$.

First we consider $M_{\vec{d}} \equiv \overline{\mathcal{M}}_{0,0}((1, \vec{d}); \mathbf{P}^1 \times \mathbf{P})$. We will call π_1 and π_2 the projections to the first and second factors of $\mathbf{P}^1 \times \mathbf{P}$ respectively. Since π_2 maps to $\mathbf{P}$, one might consider a map from $M_{\vec{d}}$ to $\overline{\mathcal{M}}_{0,0}(\vec{d}; \mathbf{P})$ sending (f, C) to $(\pi_2 \circ f, C)$. However, this is not necessarily a stable map. If it is unstable, $\pi_2 \circ f$ maps some components of C to points, so

[3] We apologize for reversing notation from the previous section, and writing (f, C) instead of (C, f). This is to agree with [50] which we closely follow in this section.

if we let C' be the curve obtained by deleting these components, there is a map $\pi : M_{\vec{d}} \to \overline{\mathcal{M}}_{0,0}(\vec{d}; \mathbf{P})$ which sends (f, C) to $(\pi_2 \circ f, C')$.

Let us now recall some facts about maps from $\mathbf{P}^1$. A regular map to $\mathbf{P}$ is equivalently a choice of generic sections of $\mathcal{O}_{\mathbf{P}^1}(f^* B_i \cdot H_{\mathbf{P}^1})$, $i = 1, \dots, N_C$. For example, a map of degree d from $\mathbf{P}^1$ to $\mathbf{P}^4$ gives five generic sections of $\mathcal{O}_{\mathbf{P}^1}(d)$, i.e., five degree d polynomials. If one takes five arbitrary sections, constrained only by being not all identically zero, one gets a rational map instead. Generalizing this, arbitrary sections of $\mathcal{O}_{\mathbf{P}^1}(f^* B_i \cdot H_{\mathbf{P}^1})$ which are not in Δ give rational maps to $\mathbf{P}$. The space $N_{\vec{d}}$ is the space of all such maps with $f^*(J_i) = d_i J_{\mathbf{P}^1}$, where $J_{\mathbf{P}^1} \cdot H_{\mathbf{P}^1} = 1$. Explicitly, we can write it as a quotient space. Defining $D = \sum d_j H_j$, we have

$$N_{\vec{d}} = \frac{\oplus_i H^0(\mathbf{P}^1, \mathcal{O}(B_i \cdot D)) - \Delta}{G}. \tag{4}$$

There is a map $\psi : M_{\vec{d}} \to N_{\vec{d}}$, which we now describe. Take $(f, C) \in M_{\vec{d}}$ and decompose C as a union $C_0 \cup C_1 \cup \ldots \cup C_N$ of not neccesarily irreducible curves, so that C_j for $j > 0$ meets C_0 at a point, and C_0 is isomorphic to $\mathbf{P}^1$ under $\pi_1 \circ f$. Since $C_0 \cong \mathbf{P}^1$, $\pi_2 \circ f|_{C_0}$ can be regarded as a point in $N_{[\pi_2 \circ f(C_0)]}$, where $[\mu]$ denotes the homology class of μ. We can also represent $\pi_2 \circ f|_{C_j}$ for $j > 0$ by elements in $N_{[\pi_2 \circ f(C_j)]}$, except that since the maps are to have domain C_0, in this case we take the rational map from C_0 that vanishes only at $x_j = C_j \cap C_0$ and belongs to $N_{[\pi_2 \circ f(C_j)]}$.

Having now $N+1$ representatives, compose them via the map

$$N_{\vec{r}_1} \otimes N_{\vec{r}_2} \to N_{\vec{r}_1 + \vec{r}_2}$$

given by multiplying sections of $\mathcal{O}(B_i)$. The result, since $\sum_{i=0}^{N} [\pi_2 \circ f(C_i)] = D$, is a point in $N_{\vec{d}}$. Thus we have obtained a map from $M_{\vec{d}}$ to $N_{\vec{d}}$.

To illustrate, let us take the case of $\mathbf{P}^4$ again. Here (f, C) is a degree $(1, d)$ map. Let us decompose C as before into $C_0 \cup \ldots \cup C_N$, with $x_i = C_i \cap C_0$ and $\pi_1 \circ f|_{C_0}$ an isomorphism. The image of ψ is a rational morphism given as a map by $\pi_2 \circ f|_{C_0}$ except at the points x_i. At x_i, a generic hyperplane of $\mathbf{P}^4$ pulled back vanishes to the order given by the multiplicity of $[\pi_2 \circ f(C_i)]$ in terms of a generator.

So far we have discussed the spaces and the maps between them. We now briefly describe the torus actions they admit. Clearly, $M_{\vec{d}}$ has an $S^1 \times \mathbf{T}$ action induced from an action on $\mathbf{P}^1 \times \mathbf{P}$. In suitable coordinates, the S^1 action is $[w_0, w_1] \to [e^\alpha w_0, w_1]$.

Since sections of $\mathcal{O}_{\mathbf{P}^1}(1)$ is also a one-dimensional projective space, there is an S^1 action on $H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1))$. This induces an action on sections of $\mathcal{O}_{\mathbf{P}^1}(d)$. $N_{\vec{d}}$ is defined by the latter, so it admits an S^1 action. In addition, it has a $\mathbf{T}$-action induced from the action on $\mathcal{O}(B_i)$.

The map π is obviously $\mathbf{T}$-equivariant, since the $\mathbf{T}$-actions are induced from $\mathbf{P}$. It is shown in [51] that ψ is $(S^1 \times \mathbf{T})$-equivariant. Summarizing, we have the following maps:

$$N_{\vec{d}} \xleftarrow{\pi} M_{\vec{d}} \xrightarrow{\psi} \overline{\mathcal{M}}_{0,0}(\vec{d}; \mathbf{P}) \xleftarrow{\rho} \overline{\mathcal{M}}_{0,1}(\vec{d}; \mathbf{P}) \xrightarrow{ev} \mathbf{P}.$$

Pushing and pulling our problem to $N_{\vec{d}}$, we define

$$Q_{\vec{d}} = \psi_! \pi^* c_{\mathbf{T}}(U_{\vec{d}}).$$

3.3 Euler data

In this section we will derive from the gluing identity a simpler identity on $N_{\vec{d}}$. Recall that the gluing identity holds over fixed loci $FL(p_i, \vec{r}, \vec{d} - \vec{r}) \in \overline{\mathcal{M}}_{0,0}(\vec{d}; \mathbf{P})$. Therefore, an identity holds over $\pi^{-1}(FL(p_i, \vec{r}, \vec{d} - \vec{r}))$ in $M_{\vec{d}}$ under pull-back by π. We next turn to describing a sublocus of $\pi^{-1}(FL(p_i, \vec{r}, \vec{d} - \vec{r}))$ which, as we will see later is mapped by ψ to a fixed point in $N_{\vec{d}}$.

Let $F_{p_i, \vec{r}}$ denote the fixed point loci in $\overline{\mathcal{M}}_{0,1}(\vec{r}; \mathbf{P})$ with the marked point mapped to p_i. Let $(f_1, C_1, x_1) \in F_{p_i, \vec{r}}$ and $(f_2, C_2, x_2) \in F_{p_i, \vec{d} - \vec{r}}$ be two points. We define a point (f, C) in $M_{\vec{d}}$ as follows. For C we take $(C_0 \equiv \mathbf{P}^1) \cup C_1 \cup C_2$, with $C_0 \cap C_1 = x_1$ and $C_0 \cap C_2 = x_2$. For the map f, we define it by giving the projections $\pi_1 \circ f$ and $\pi_2 \circ f$. We require $\pi_1 \circ f(C_1) = 0$, $\pi_1 \circ f(C_2) = \infty$ and $\pi_1 \circ f|_{C_0}$ be an isomorphism. This "fixes" $\pi_1 \circ f$, since any other choice is related by an automorphism of the domain curve preserving x_1 and x_2, which is irrelevant in $M_{\vec{d}}$. We require $\pi_2 \circ f$ to map C_1 as f_1, C_2 as f_2 and C_0 to $f_1(x_1)$. Clearly, π maps (f, C) to a point in $FL(p_i, \vec{r}, \vec{d} - \vec{r})$. Let us denote the loci of such (f,C)'s by $MFL(p_i,\vec{r},\vec{d}-\vec{r})$. By construction, it is isomorphic to $F_{p_i,\vec{r}} \times F_{p_i,\vec{d}-\vec{r}}$.

We verify (f,C) is a fixed point. $f_i(C_i)$ and p_i are fixed in $\mathbf{P}$, so f is $\mathbf{T}$-fixed. The S^1-action fixes only 0 and ∞ on the first factor of $\mathbf{P}^1 \times \mathbf{P}$. Nevertheless, the point $(f, C_0 \cup C_1 \cup C_2)$ remains fixed under the S^1 action, as we need to divide out by automorphisms of C_0 preserving x_1 and x_2.

We now compare the maps just constructed with the fixed points of $N_{\vec{d}}$. It will be most convenient to do so by describing the latter in terms of rational morphisms. So take a point in $N_{\vec{d}}$, viewed as a rational morphism from $C_0 \equiv \mathbf{P}^1$. Let $x_1, \ldots, x_N$ be the points where it is undefined. At x_i, the chosen sections of $\mathcal{O}(B_j \cdot D)$ vanish to certain orders, including possibly zero. A generic section of $\mathcal{O}(J_j \cdot D) = \mathcal{O}(d_j)$ vanishes to order, say r_j at x_1. Any section of a line bundle $\mathcal{O}(L)$ pulled back by the map then vanishes at x_1 at least to order $L \cdot \sum r_j H_j$.

Therefore the rational morphism is equivalent to the data of a regular map from C_0 and a curve class for each bad point. The classes for the bad points indicate the multiplicity of vanishing of a generic section of a pulled-back line bundle. Altogether, the class of the image of the regular map and the curve classes we associate to the bad points sum to D, since a generic section of $\mathcal{O}(L)$ must have exactly $L \cdot D$ zeroes.

Now we can deduce the fixed points of $N_{\vec{d}}$. The $\mathbf{T}$-action moves the image of a rational morphism, whereas the S^1 action rotates the domain $\mathbf{P}^1$ about an axis joining 0 and ∞. So a fixed point is a rational map, undefined at 0 and ∞, whose image is a fixed point of $\mathbf{P}$. Let us denote them by $p_{i,\vec{r}}$, where p_i denotes a fixed point of $\mathbf{P}$, and $\sum r_i H_i$ determines orders of vanishing of pulled-back line bundles at the point $0 \in \mathbf{P}^1$. Clearly, ψ maps the fixed points in $M_{\vec{d}}$ discussed earlier to the fixed point $p_{i,\vec{r}}$.

We now use the Atiyah-Bott formula for localization to relate restrictions of $Q_{\vec{d}}$ to $p_{i,\vec{r}}$ (which we denote by $Q_{\vec{d}}(p_{i,\vec{r}})$) to $c_{\mathbf{T}}(\pi^* U_{\vec{d}})$. Explicitly,

$$Q_{\vec{d}}(p_{i,\vec{r}}) = \int_{N_{\vec{d}}} \phi_{p_{i,\vec{r}}} Q_{\vec{d}} = \int_{M_{\vec{d}}} \psi^*(\phi_{p_{i,\vec{r}}}) c_{\mathbf{T}}(\pi^*(U_{\vec{d}})).$$

where $\phi_{p_{i,\vec{r}}}$ is the equivariant Thom class of the normal bundle of $p_{i,\vec{r}}$

in $N_{\vec{d}}$. To evaluate the last integral, we need the equivariant euler class of the normal bundle of $MFL(p_i, \vec{r}, \vec{d}-\vec{r})$ in $M_{\vec{d}}$.

Since $MFL(p_i, \vec{r}, \vec{d}-\vec{r}) \cong F_{p_i,\vec{r}} \times F_{p_i,\vec{d}-\vec{r}}$, we have contributions from the normal bundles of $F_{p_i,\vec{r}} \in \overline{\mathcal{M}}_{0,1}(\vec{r}; \mathbf{P})$ and $F_{p_i,\vec{d}-\vec{r}} \in \overline{\mathcal{M}}_{0,1}(\vec{d}-\vec{r}; \mathbf{P})$. They are respectively $e(N(F_{p_i,\vec{r}}/\overline{\mathcal{M}}_{0,1}(\vec{r}; \mathbf{P})))$ and $e(N(F_{p_i,\vec{d}-\vec{r}}/\overline{\mathcal{M}}_{0,1}(\vec{d}-\vec{r}; \mathbf{P})))$. Points in $MFL(p_i, \vec{r}, \vec{d}-\vec{r})$ have domain of the form $C_0 \cup C_1 \cup C_2$, where $C_1 \cap C_0 = x_1$, and $C_2 \cap C_0 = x_2$. Now let $L_{\vec{r}}$ denote the line bundle on $\overline{\mathcal{M}}_{0,1}(\vec{r}; \mathbf{P})$ whose fiber at (f_1, C_1, x_1) is the tangent line at x_1. Then we can write the contributions from deforming x_1 and x_2 as $e(L_{\vec{r}} \otimes T_{x_1}C_0) = \alpha + c_1(L_{\vec{r}})$ and $\alpha + c_1(L_{\vec{d}-\vec{r}})$, respectively. In addition, automorphisms of C_0 which do not fix x_1 and x_2 need to be included. They can be shown to give weights of $T_{x_1}C_0$ and $T_{x_2}C_0$, so there is an extra factor of $(\alpha)(-\alpha)$. Finally, normal directions which move the image of the marked point from p_i have to be excluded, so we divide by the weights of $T_{p_i}\mathbf{P}$.

This yields, after using (3),

$$\Omega^V(p_i)Q_{\vec{d}}(p_{i,\vec{r}}) = \frac{-1}{\alpha^2} e(T_{p_i}\mathbf{P}) e(p_{i,\vec{r}}/N_{\vec{d}}) \sum_{F_{p_i,\vec{r}}} \int_{F_{p_i,\vec{r}}} \frac{\rho^* c_{\mathbf{T}}(U_{\vec{r}})}{e(N(F_{\vec{r}}))(\alpha + c_1(L_{\vec{r}}))}$$
$$\cdot \sum_{F_{p_i,\vec{d}-\vec{r}}} \int_{F_{p_i,\vec{d}-\vec{r}}} \frac{\rho^* c_{\mathbf{T}}(U_{\vec{d}-\vec{r}})}{e(N(F_{p_i,\vec{d}-\vec{r}}))(\alpha + c_1(L_{\vec{d}-\vec{r}}))}$$

We introduce some more notation. Let $\kappa_{j,\vec{d}}$ be the member of the $S^1 \times \mathbf{T}$ equivariant cohomology ring of $N_{\vec{d}}$ whose weight at the fixed point $p_{i,\vec{r}}$ is $\kappa_j(p_i) + r_j\alpha$. Clearly, $\kappa_{j,\vec{0}} \equiv \kappa_j$. The identity $e(T_{p_i}\mathbf{P})e(p_{i,\vec{r}}/N_{\vec{d}}) = e(p_{i,\vec{r}}/N_{\vec{r}})e(p_{i,\vec{0}}/N_{\vec{d}-\vec{r}})$ then implies

$$\Omega^V(p_i)Q_{\vec{d}}(p_{i,\vec{r}}) = \overline{Q_{\vec{r}}(p_{i,\vec{0}})}Q_{\vec{d}-\vec{r}}(p_{i,\vec{0}}). \tag{5}$$

Here the overbar - is an automorphism of the $S^1 \times \mathbf{T}$ equivariant cohomology ring with $\overline{\alpha} = -\alpha$ and $\overline{\kappa_{j,\vec{d}}} = \kappa_{j,\vec{d}}$. A sequence of equivariant cohomology classes satisfying (5) is called in [50] a set of Ω^V-Euler data.

3.4 Linked Euler data

If we knew the values of $Q_{\vec{d}}$ at all fixed points, we would also know $Q_{\vec{d}}$ as a class. Since we do not know this, we will use equivariance to compute $Q_{\vec{d}}$ at certain points, for example, those that correspond to the $\mathbf{T}$-invariant $\mathbf{P}^1$'s in $\mathbf{P}$. It turns out that this is also sufficient, as we will find Euler data which agree with $Q_{\vec{d}}$ at those points, and a suitable comparison between the two gives us the rest.

Before we begin the computation, we first describe a $\mathbf{T}$-equivariant map from $N_{\vec{0}} = \mathbf{P}$ to $N_{\vec{d}}$. Sections of $\mathcal{O}(B_i \cdot D)$ over $\mathbf{P}^1$ are polynomials in w_0 and w_1, where w_0 and w_1 are as before coordinates so that the S^1 action takes the form $[w_0, w_1] \to [e^{\alpha} w_0, w_1]$. Each polynomial contains a unique monomial invariant under the S^1 action. By sending the coordinates of a point to the coefficients of the invariant monomials, we hence obtain a map $I_{\vec{d}}$ from $N_{\vec{0}}$ to $N_{\vec{d}}$.

We begin with the case of a convex line bundle $\mathcal{O}(L)$, where L denotes the associated divisor. Let $(f, \mathbf{P}^1)$ be a point in $\overline{\mathcal{M}}_{0,0}(\vec{d}; \mathbf{P})$ with $f(\mathbf{P}^1)$ being a multiple of the $\mathbf{T}$-invariant $\mathbf{P}^1$ joining p_i and p_j in $\mathbf{P}$. The fiber of the obstruction bundle at $(f, \mathbf{P}^1)$ is $H^0(\mathcal{O}(L \cdot D))$, which is spanned in appropriate coordinates for the $\mathbf{P}^1$ by $u_0^k u_1^{L \cdot D - k}$, $k = 0, \ldots, L \cdot D$.

Choose a basis so that $u_1 = 0$ is mapped to p_i, and $u_0 = 0$ mapped to p_j. Since the section $u_0^{L \cdot D}$ does not vanish at $u_1 = 0$, its weight is equal to the weight of L at p_i, which we denote by $L(p_i)$. Similarly, $u_1^{L \cdot D}$ has weight $L(p_j)$. Hence the induced weight on u_1/u_0 is $(L(p_j) - L(p_i))/(L \cdot D)$, giving us the weights of all sections.

Since $(f, \mathbf{P}^1)$ is fixed by $\mathbf{T}$, the corresponding loci $\psi(\pi^{-1}((f, \mathbf{P}^1)))$ in $N_{\vec{d}}$ is, by equivariance, fixed by a dim$\mathbf{T}$ subgroup of $S^1 \times \mathbf{T}$. The points in $\psi(\pi^{-1}((f, \mathbf{P}^1)))$ represent a regular map from a $\mathbf{P}^1$ to the $\mathbf{T}$-invariant $\mathbf{P}^1$ joining p_i and p_j. Therefore any two points in $\psi(\pi^{-1}((f, \mathbf{P}^1)))$ differ by an automorphism of the domain. Explicitly, we can consider coordinates $[w_0, w_1]$ as before. Let η denote the point which sends $[w_0 = 1, w_1 = 0]$ to p_i and $[w_0 = 0, w_1 = 1]$ to p_j. Any other point, thought of as a map, factors via η by an automorphism $[w_0, w_1] \to [aw_0 + bw_1, cw_0 + dw_1]$. We can therefore find the relevant subgroup of $S^1 \times \mathbf{T}$ by choosing the element of S^1 to cancel the induced weight of

$\mathbf{T}$ on w_1/w_0. Explicitly, we have $\alpha = (L(p_i) - L(p_j))/(L \cdot D)$, so that the value of $Q_{\vec{d}}$ is

$$\prod_k (x + L(p_j) + k\alpha).$$

The weight of $\kappa_{i,\vec{d}}$ under $(S^1 \times \mathbf{T})/(\alpha - (L(p_i) - L(p_j))/(L \cdot D))$ at any point in $\psi(\pi^{-1}((f, \mathbf{P}^1)))$ is the same as its weight under $\mathbf{T}$ at the corresponding point obtained by setting w_0 to zero. For η, setting w_0 to zero gives a point in $N_{\vec{d}}$ which can be thought of as a rational map to p_j. Since $I_{\vec{d}}$ is equivariant, the weight of $\kappa_{i,\vec{d}}$ at this point is the same as the weight of κ_i at p_j. Thus, as $\sum l_i \kappa_i(p_j) = L(p_j)$, the value of $Q_{\vec{d}}$ is the same as the value of $P_{\vec{d}}$, where $P_{\vec{d}}$ is given by

$$\prod_k \left(x + \sum l_i \kappa_{i,\vec{d}} + k\alpha\right).$$

To simplify our computations, we take a partial nonequivariant limit by replacing κ_i with J_i. Then $P_{\vec{d}}$ reduces to $\tilde{\Gamma}(x, L, L \cdot D + 1, 0)$, where $\tilde{\Gamma}$ is defined as follows:

$$\tilde{\Gamma}(y, K, i, j) = \begin{cases} \prod_{k=j}^{i-1} (y + K + (k-1)\alpha) & \text{if } i > j \\ 1 & \text{if } i = j \\ \frac{1}{\prod_{k=i}^{j-1} (y + K + k\alpha)} & \text{if } i < j. \end{cases}$$

Similarly, the case of a concave line bundle gives $\tilde{\Gamma}(x, L, -L \cdot C, 1)$. For a direct sum, since the Chern class is multiplicative and the product of the cases just considered is a valid Euler data, the product gives us the appropriate value of $P_{\vec{d}}$.

Let us form a series:

$$HGA[Q](\vec{T}) = e^{-\sum T_i J_i/\alpha} \left(\sum_{D \in \overline{NE}(\mathbf{P}) - \vec{0}} I_D^*(Q_{\vec{d}}) \prod_i \tilde{\Gamma}(0, B_i, -B_i \cdot D, 0) e^{\sum T_i J_i \cdot D} + \Omega_{\mathbf{T}}^V \right), \tag{6}$$

where by $Q_{\vec{d}}$ we mean its partially nonequivariant limit, as described above, and $\overline{NE}(\mathbf{P})$ is the set of curve classes in $\mathbf{P}$ which have nonnegative intersection with the effective divisors of $\mathbf{P}$.

A similar series can be constructed from the partial nonequivariant limit of $P_{\vec{d}}$ as follows:

$$\begin{aligned} HGB[\vec{t}] =& e^{-\sum t_i J_i/\alpha}\Bigg(\sum_{D \in \overline{NE}(\mathbf{P})-\vec{0}} \prod_i \tilde{\Gamma}(0, B_i, -B_i \cdot D, 0) \\ &\cdot \prod_{i:F_i<0} \tilde{\Gamma}(x, F_i, -F_i \cdot D, 1) \\ &\cdot \prod_{i:F_i>0} \tilde{\Gamma}(x, F_i, F_i \cdot D + 1, 0) e^{J \cdot D} + \Omega^V_{\mathbf{T}} \Bigg). \end{aligned}$$

If we take the example of the quintic in $\mathbf{P}^4$, $\dim H_2(\mathbf{P}, \mathbf{Z}) = 1$, so we have

$$HGB[t] = e^{-J/\alpha} \left(\sum_{d>0} \frac{\prod_{m=1}^{5d}(x + 5H - m\alpha)}{\prod_{m=1}^{d}(x + H - m\alpha)^5} + 5H \right). \tag{7}$$

For local mirror symmetry, we can take $V = K_{\mathbf{P}^2}$, giving

$$HGB[t] = e^{-J/\alpha} \left(\sum_{d>0} \frac{\prod_{m=1}^{3d-1}(x - 3H + m\alpha)}{\prod_{m=1}^{d}(x + H - m\alpha)^3} - \frac{1}{3H} \right). \tag{8}$$

To compare $HGB[\vec{t}]$ and $HGA[\vec{T}]$, let us expand $HGB[\vec{t}]$ at large α, keeping terms to order $1/\alpha$. It is shown in [51] that $HGA[\vec{T}]$ has the form $\Omega^V_{\mathbf{T}}(1 - (\sum T_i J_i)/\alpha)$. Equating the two expressions gives us $\vec{T}$ in terms of $\vec{t}$. Then setting $HGA[\vec{T}(\vec{t})] = HGB[\vec{t}]$ gives us Q_D as a function of J_i and α, from which we may obtain K_D [51]:

$$\frac{(2 - \sum d_i t_i) K_D}{\alpha^3} = \int_{\mathbf{P}} e^{-\sum t_i J_i/\alpha} Q_{\vec{d}} \prod_j \tilde{\Gamma}(0, B_j, -B_j \cdot D, 0). \tag{9}$$

4 Explicit Verification Through Fixed-Point Methods

4.1 Some examples

Though the techniques of section one are extremely powerful, it is often satisfying – and a good check of one's methods – to do some computations by hand. In this section, we outline fixed point techniques for doing so, and walk through several examples. In this way, we have verified many of the results in the appendices for low degrees ($d = 1, 2, 3$). Readers familiar with such exercises may wish to skip to the next subsection.

All the bundles described in section 2 are equivariant with respect to the torus T-action which acts naturally on the toric manifold $\mathbf{P}$. Let $t \in T$ be a group element acting on $\mathbf{P}$. Then if $(C, f, *) \in \overline{\mathcal{M}}_{0,1}(\vec{d}; \mathbf{P})$ and $(C, f) \in \overline{\mathcal{M}}_{0,0}(\vec{d}; \mathbf{P})$ the induced torus action sends theses points to $(C, t \circ f, *)$ and $(C, t \circ f)$, respectively. The bundle actions are induced by the natural T-action on the canonical bundle, K.

Now that we understand the torus action, what are the fixed point theorems? First of all, we work in the realm of equivariant characteristic classes, which live in the equivariant cohomology ring $H_T^*(M)$ of a manifold, M. Let $\phi \in H_T^*(M)$ be an equivariant cohomology class. The integration formula of Atiyah and Bott is

$$\int_M \phi = \sum_P \int_P \left(\frac{i_P^* \phi}{e(\nu_P)} \right),$$

where the sum is over fixed point sets P, i_P is the embedding in M, and $e(\nu_P)$ is the Euler class of the normal bundle ν_P along P. For P consisting of isolated points and ϕ the Chern class (determinant), we get the ratio of the product of the weights of the T-action on the fibers at P (numerator) over the product of the weights of the T-action on the tangent bundle to M at P (denominator). One needs only to determine the fiber and tangent bundle at P and figure out the weights.

Let's start with degree one (K_1) for the quintic ($\mathcal{O}(5) \to \mathbf{P}^4$). Let $X_i \mapsto \alpha_i^{\lambda_i} X_i$, $i = 1, \ldots, 5$, be the $(\mathbf{C}^*)^5$ action on $\mathbf{C}^5$ ($\mathbf{P}^4 = \mathbf{P}(\mathbf{C}^5)$), where $\vec{\alpha} \in (\mathbf{C}^*)^5$ and the λ_i are the weights. The fixed curves are P_{ij}, where i, j run from 1 to 5 : $P_{ij} = \{X_k = 0, k \neq i, j\}$. Since f is a degree one map, we may equate $C \cong f(C)$ and the pull-back of $\mathcal{O}(5)$ is therefore equal to $\mathcal{O}(5)$ on C_e. Recall the bundle $\mathcal{O}(5)$ on $\mathbf{P}^1$. Its global sections are degree five polynomials in the homogeneous coordinates $[X, Y]$, so a convenient basis is $\{X^a Y^{5-a}, a = 0, \ldots, 5\}$ (or u^a in a local coordinate $u = X/Y$). The weights of these sections are $a\mu + (5-a)\nu$, if μ and ν are the weights of the torus $(\mathbf{C}^*)^2$ action on $\mathbf{C}^2$. The map $f : C_e \to \mathbf{P}^4$ looks like $[X, Y] \mapsto [\ldots, X, \ldots, Y, \ldots]$ with non-zero entries only in the ith and jth positions. Therefore, the weights of U_1 at the fixed point (C, f) are $a\lambda_i + (5-a)\lambda_j$, $a = 0 \ldots 5$. To take the top Chern class we take the product of these six weights.

We have to divide this product by the product of the weights of the normal bundle, which in this case (the image $f(C)$ is smooth) are the weights of $H^0(C, f^*N)$, where N is the normal bundle to $f(C)$. More generally, we take sections of the pull-back of $T\mathbf{P}^4$ and remove sections of T_C. The normal bundle of P_{ij} is equal to $\mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O}(1)$, each corresponding to a direction normal to $f(C)$ and each of which has two sections. Let $w = X_j/X_i$ be a coordinate along $C \cong P_{ij}$ on the patch $X_i \neq 0$. If $z_k = X_k/X_i$ is a local coordinate of $\mathbf{P}^4$, then $del_k \equiv \frac{\partial}{\partial z_k}$ is a normal vector field on C with weight $\lambda_k - \lambda_i$. $w\partial_k$ is the other normal vector field corresponding to the direction k, and has weight $\lambda_k - \lambda_j$. Summing over the $\begin{pmatrix} 5 \\ 2 \end{pmatrix} = 20$ choices of image curve P_{ij} gives us

$$K_1 = \sum_{(ij)} \frac{\prod_{a=0}^{5} [a\lambda_i + (5-a)\lambda_j]}{\prod_{k \neq i,j} (\lambda_k - \lambda_i)(\lambda_k - \lambda_j)} = 2875,$$

the familiar result.

Let's try degree two (K_2) for $K_{\mathbf{P}^2}$. The dimension of $\mathcal{M}_{0,0}(d; \mathbf{P}^2)$ is $3d - 1 = 5$ for $d = 2$. What are the fixed points? Well, the image of (C, f) must an invariant curve, so there are two choices for degree two. Either the image is a smooth $\mathbf{P}^1$ or the union of two $\mathbf{P}^1$'s. There are three fixed points on $\mathbf{P}^2$ and therefore $\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3$ invariant P_{ij}'s. If the image is a smooth $\mathbf{P}^1$, the domain curve may either be a smooth $\mathbf{P}^1$, in which case the map is a double cover (let's call this case 1a), or it may have two component $\mathbf{P}^1$'s joined at a node (let's call this case 1b). If the image has two components, the domain must as well. Let's call this case 2.

For case 1a, the situation is similar to the quintic case above. The tangent space to moduli space consists of sections of $H^0(C, f^*N)$, where N is the normal bundle to the image curve. That is, we take $H^0(C, f^*T\mathbf{P}^2)$ and delete those sections from $H^0(TC)$. Since $T\mathbf{P}^2|_{f(C)} \cong \mathcal{O}(2) \oplus \mathcal{O}(1)$ and f is a degree 2 map, we have $f^*T\mathbf{P}^2 \cong \mathcal{O}(4) \oplus \mathcal{O}(2)$, which has 5 + 3 = 8 sections. $TC \cong \mathcal{O}(2)$ has three sections, leaving us with five total. If $f(C) = P_{ij}$, then the $\mathcal{O}(2)$ sections are $\partial_k, w\partial_k, w^2\partial_k$, where $k \neq i, j$, w is the coordinate on C, and $z_m = X_m/X_i$ are inhomogeneous coordinates on $\mathbf{P}^2$. The degree two map is, in these coordinates, $w \mapsto (z_j = w^2, z_k = 0)$. Note that ∂_k is the only non-vanishing section at $w = 0$, and the others are obtained by successive multiplications by w. Notice that w inherits the weight $(\lambda_j - \lambda_i)/2$ by requiring equivariance. The weights are, so far, $\lambda_i - \lambda_k, (\lambda_i + \lambda_j)/2 - \lambda_k, \lambda_j - \lambda_k$. For the $\mathcal{O}(4)$ sections, the procedure is similar, only we must remove the weights $0, \pm(\lambda_i - \lambda_j)/2$, as these correspond to the tangent vectors $\partial_w, w\partial_w, w^2\partial_w$. We are left with $\pm(\lambda_j - \lambda_i)$ giving a total of five.

The weights of $H^1(C, f^*K_{\mathbf{P}^2})$ are easily calculated for the curve C by using Serre duality. That is, if one thinks (naively) of sections of a vector bundle E as elements of $\overline{\partial}$ cohomology, and recalling that the canonical bundle K is the bundle of holomorphic top forms, then $H^k(E)$ pairs with $H^{n-k}(E^* \otimes K)$ by wedging and contracting E with its dual E^*, then integrating. Thus, $H^k(E) \cong H^{n-k}(E^* \otimes K)^*$. For a curve, C, we have $H^1(f^*K_{\mathbf{P}^2}) \cong H^0(f^*K^{-1}_{\mathbf{P}^2} \otimes K_C)^*$. Let's compute. $K^{-1}_{\mathbf{P}^2} \cong \mathcal{O}(-3)$ as a bundle, and $K_C \cong \mathcal{O}_{\mathbf{P}^1}(-2)$, so $f^*K^{-1}_{\mathbf{P}^2} \otimes K_C \cong \mathcal{O}(2 \cdot (+3) - 2) = \mathcal{O}(4)$, and the five sections can be obtained in the usual way once we have a non-vanishing one at $w = 0$. Such a section is $\frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_k} \otimes dw$, and has weight $2\lambda_i - \lambda_j - \lambda_k + (\lambda_j - \lambda_i)/2$. In all, then the weights of $H^0(f^*K^{-1}_{\mathbf{P}^2} \otimes K_C)$ are $2\lambda_i - \lambda_j - \lambda_k + m(\lambda_j - \lambda_i)/2$, $m = 1, \ldots, 5$. For the dual space $H^1(f^*K_{\mathbf{P}^2})$ we must take the negatives of these weights. So much for case 1a.

Cases 1b and 2 the domain curves have two components (say, C_1 and C_2), so we must understand what is meant, for example, by $f^*T\mathbf{P}^2$ and TC in order to calculate the normal bundle. TC is locally free (like a vector bundle) everywhere except at the singularity. There, we require tangent vectors to vanish. The canonical bundle, K_C, is defined as a line bundle of holomorphic differentials, with the following construction at the singularity. Let $f(w)dw$ be a differential along C_1, and let $g(z)dz$ be a differential along C_2 where the singularity is taken to be $(z = 0) \sim (w = 0)$. To define a differential along the total space, we allow f and g to have up to simple poles at the origin, with the requirement that the total residue vanish: $\mathrm{res}_w f + \mathrm{res}_z g = 0$. What this does is serve as an identification, at the singularity, of the fibers of the canonical bundles of the two components. In this way, we arrive at a line "bundle." This canonical bundle, when restricted to a component C_i, looks like $K_{C_i}(p)$, where the "(p)" indicates twisting by the point, i.e., allowing poles.

At this point, we can proceed with the calculation. Consider case 2, for which the map f is a bijection. The sections of $H^1(f^*K_{\mathbf{P}^2}) \cong H^0(f^*(K_{\mathbf{P}^2})$ can be looked at on each component, where $K_C|_{C_i}$ is as above. Hence on C_i we have

$$\frac{1}{w}\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} \otimes dw, \qquad 1\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} \otimes dw, \qquad w\frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2} \otimes dw.$$

On the the other component, we have three analogous sections, but two with poles need to be identified, since they are related by the requirement of no total residue. Indeed, this identification is compatible with equivariance, since $\frac{1}{w}dw$ has zero weight. All in all, we have weights (recalling duality) $\lambda_j + \lambda_k - 2\lambda_i, \lambda_k - \lambda_i, \lambda_k - \lambda_j, \lambda_j - \lambda_i, \lambda_j - \lambda_k$.

The normal bundle to moduli space consists of sections of the pullback of tangent vectors on $\mathbf{P}^2$, less global tangent vectors on C. In addition, we include $T_pC_1 \otimes T_pC_2$, which is a factor corresponding to a normal direction in which the node is resolved [44] . Since the maps from components are degree one for these cases, we can take as sections of the normal bundles (two each) $\partial_w, z\partial_w$ and $\partial_z, w\partial_z$. Here we have identified the coordinate of the other component with the coordinate of $\mathbf{P}^2$ normal to the component. The $T_pC_1 \otimes T_pC_2$ piece gives $\partial_w \otimes \partial_z$. In total, the weights are $\lambda_i - \lambda_j, \lambda_k - \lambda_j, \lambda_i - \lambda_k, \lambda_j - \lambda_k, 2\lambda_i - \lambda_j - \lambda_k$.

One checks that the product of the numerator weights divided by the denominator weights is equal to -1. Since there are three graphs of this type, the total contribution to K_2 is -3. Graphs whose image is a single fixed $\mathbf{P}^1$ contribute $-21/8$,[4] giving $K_2 = -45/8$.

[4]Because we are considering integrals in the sense of orbifolds, we must divide out the contribution of each graph by the order of the automorphism group of the map. Automorphisms are maps $\gamma : C \to C$ such that $f \circ \gamma = f$. Cases 1a and 1b have $\mathbf{Z}_2$ automorphism groups.

In physics, local mirror symmetry is all that is needed to describe the effective quantum field theory from compactification on a Calabi-Yau manifold which contains a holomorphic surface, if we take an appropriate limit. In this limit, the global structure of the Calabi-Yau manifold becomes irrelevant (hence the term "local"), and we can learn about the field theory by studying the local geometry of the surface – its canonical bundle. We can therefore construct appropriate surfaces to study aspects of four-dimensional gauge theories of our choosing [35] . The growth of the Gromov-Witten invariants (or their local construction) in a specified degree over the base $\mathbf{P}^1$ is related to the Seiberg-Witten coefficient at that degree in the instanton expansion of the holomorphic prepotential of the gauge theory. For example, the holomorphic vanishing cycles of an A_n singularity fibered over a $\mathbf{P}^1$ give $SU(n+1)$ gauge theory (the McKay correspondence, essentially), and one can construct a Calabi-Yau manifold containing this geometry to check this [35], [42]. In this case, the local surface is singular, as it is several intersecting $\mathbf{P}^1$'s fibered over a $\mathbf{P}^1$ (for A_1 we can take two Hirzebruch surfaces intersecting in a common section. For this reason, it is important to understand the case where the surface is singular, as well. We will have more to say about this in section 6.

4.2 General procedure for fixed-point computations

Following [44] and [27], we can compute the weights of our bundles explicitly. Each connected component of the fixed point set is described by a graph, Γ, which is a collection of vertices, edges, and flags. The graph contains the data of the fixed map, which includes the image $\mathbf{P}^1$'s, the degrees of the maps to the fixed curves, and the way they are glued together.

Let us fix some notation. To each connected component of $f^{-1}(p)$, where p is a fixed point of $\mathbf{P}$, we have a vertex, v. We call $C_v = f^{-1}(p)$ the pre-image of p, and if $p = p_j$, we say that $i(v) = j$ (so i is a map from $\{vertices\}$ to $\{1 \ldots n+1\}$). Let $val(v)$ be the number of special (marked or nodal) points on C_v (for us equal to the number of edges with v as their vertex). The connected components of the pre-image of a fixed line P_{ij} are denoted C_e. An edge consists of C_e together with the data $i(e), j(e) \in \{1, \ldots, n+1\}$ encoding the image $f(C) = P_{i(e),j(e)}$, and d_e the degree of the map $f|_{C_e}$. If there is no confusion, we will write

i and j for $i(e)$ and $j(e)$. Note that in the case of higher genus maps, the genus $g > 0$ components of the domain curve must map to fixed points C_v as there are no invariant curves of higher genus. In particular, the C_e are all of genus zero. We call a pair (v, e) where C_v and C_e intersect non-trivially a "flag," F. For $F = (v, e)$, we define $i(F) = i(v)$.

The fixed point set corresponding to a graph Γ is then equal to a product over vertices of the moduli space of genus $g(v)$ curves with $val(v)$ marked points: $M_\Gamma = \prod_v \overline{M}_{g(v),val(v)}$.

The calculation of the weights along the fixed point sets follows from a simple, general observation. Given two varieties, Y_1 and Y_2, $X = Y_1 \cup Y_2$ may be singular, but we can construct the maps $Y_1 \cap Y_2 \rightrightarrows Y_1 \coprod Y_2 \to X$, from which we construct maps of sheaves of holomorphic functions:

$$\mathcal{O}_X \to \mathcal{O}_{Y_1} \oplus \mathcal{O}_{Y_2} \to \mathcal{O}_{Y_1 \cap Y_2}.$$

All maps are obtained from inclusions except the last map, which sends (f_1, f_2) to $f_1 - f_2$, so this sequence is exact.

For a graph with domain curve C which is equal to the union of all its components, things are simple because there are at most pairwise non-trivial intersections, those being points. Thus we have the sequence

$$0 \to \mathcal{O}_C \to \bigoplus_v \mathcal{O}_{C_v} \oplus \bigoplus_e \mathcal{O}_{C_e} \to \bigoplus_F \mathcal{O}_{x_F} \to 0, \tag{10}$$

where $x_F = C_v \cap C_e$ if $F = (v, e)$, and the second map sends $(g|_{C_V}, h|_{C_e})$ to $g - h$ on the point of intersection (if it exists).

We will use the long exact sequence associated to this short exact sequence in two ways. The fixed point formula tells us we need to compute the weights of our bundle U_d (whose fibers are $H^1(C, f^* K_{\mathbf{P}^2})$). When C is singular, we need to use the above sequence twisted by (or tensored by) $f^* K_{\mathbf{P}^2}$. Then using concavity of the canonical bundle, which states that $f^* K_{\mathbf{P}^2}$ has no global sections on C_e, the long exact sequence reads

$$\begin{aligned} 0 \to H^0(C, f^* K_{\mathbf{P}^2}) &\to \bigoplus_v H^0(C_v, f^* K_{\mathbf{P}^2}) \oplus \bigoplus_e H^0(C_e, f^* K_{\mathbf{P}^2}) \\ &\to \bigoplus_F K_{\mathbf{P}^2}|_{f(x_F)} \to H^1(C, f^* K_{\mathbf{P}^2}) \end{aligned}$$

$$\to \bigoplus_v H^1(C_v, f^*K_{\mathbf{P}^2}) \oplus \bigoplus_e H^1(C_e, f^*K_{\mathbf{P}^2}) \to 0. \tag{11}$$

The last term in the first line follows since x_F is a point, which is why the last term in the second line is zero. Note that $f^*K_{\mathbf{P}^2}$ is trivial as a bundle on C_v, since C_v is mapped to a point. However, this trivial line bundle has non-trivial weight equal to $\Lambda_i \equiv -3\lambda_i + \lambda_i + \lambda_j + \lambda_k$, where $i = i(F)$. This will affect equivariant Chern classes nontrivially. For example, $H^0(C_v, f^*K_{\mathbf{P}^2})$ is one-dimensional (constant section) with the same weight – let's call it $\mathbf{C}_{\Lambda_i}$. $H^1(C_v, f^*K_{\mathbf{P}^2})$ is thus equal to $H^1(C_v, \mathcal{O}) \otimes \mathbf{C}_{\Lambda_i}$. Also, since $H^1(C_v, \mathcal{O})$ are global holomorphic differentials, which may be integrated against cycles, we see that $H^1(C_v, \mathcal{O})$, as a bundle over the fixed point component $\overline{M}_{g(v),val(v)}$ in moduli space, is equal to the dual E^* of the rank $g(v)$ Hodge bundle, E. We are interested in $c_g(E^* \otimes \mathbf{C}_{\Lambda_i})$. The Chern character (not class) is well-behaved under tensor product, from which we can conclude [27]

$$c_g(E^* \otimes \mathbf{C}_{\Lambda_i}) = P_g(\Lambda_i, E^*) \equiv \sum_{r=0}^{g} \Lambda_i^r c_{g-r}(E^*),$$

where we have defined the polynomial $P_g(\Lambda_i, E^*)$

We know more about (11). $H^1(C_e, f^*K_{\mathbf{P}^2})$ can be computed exactly as in case 1a from the previous section, giving weights $\Lambda_i + m(\lambda_i - \lambda)j)$, $m = 1, \ldots, 3d_e - 1$. Also, $H^0(C_e, f^*K_{\mathbf{P}^2}) = 0$ by convexity, which tells us as well that $H^0(C, f^*K_{\mathbf{P}^2}) = 0$ (obvious if you think of the map). Therefore the map to flags on the first line of (11) is $1-1$, which is also obvious as it is restriction of constant sections (zero at a point iff the section is identically zero). Thus the weights from the top line which map into $H^1(C, f^*K_{\mathbf{P}^2})$ are $\prod_F \Lambda_{i(F)} / \prod_v \Lambda_{i(v)}$. Noting that there are $val(v)$ flags with v as their vertex, and combining with the weights from the middle term on the second line of (11), we have

$$\prod_v \Lambda_{i(v)}^{val(v)-1} P_{g(v)}(\Lambda_{i(v)}, E^*) \prod_e \left[\prod_{m=1}^{3d_e-1} \Lambda_i + m(\lambda_i - \lambda_j) \right]. \tag{12}$$

For the genus zero case, the polynomials involving the Hodge bundle disappear.

If we twist the sequence (10) by $f^*T\mathbf{P}^2$ we can deduce the information we need to compute $H^0(C, f^*T\mathbf{P}^2) - H^1(C, f^*T\mathbf{P}^2)$, which is most of what is needed to compute the virtual normal bundle to the fixed point locus.[5] However, a complete exposition for higher genus, where concavity or convexity is not enough to guarantee a smooth moduli space, is beyond the scope of this paper, and we refer the reader to the discussion in section four of [27] , with whose notation this paper is largely compatible. The genus zero case has been worked out in full by [44] (see the formula at the end of section 3.3.4).

The upshot is that we can determine all the weights and classes of the bundles restricted to the fixed point loci systematically. After dividing numerator (Chern class) by denominator (Euler class of normal bundle), one has polynomial class of degree equal to the dimension of the fixed locus. What's left is to integrate these classes over the moduli spaces of curves (not maps) $\overline{M}_{g(v),val(v)}$ at each vertex. These integrals obey famous recursion relations [68], which entirely determine them. A program for doing just this has been written by [18] . With this, and an algorithm for summing over graphs (with appropriate symmetry factors), one can completely automate the calculation of higher genus Gromov-Witten invariants. Subtleties remain, however, regarding multicovers [62], [38], [66].

5 Virtual Classes and the Excess Intersection Formula

One of the foundations of the theory of the moduli space of maps has been the construction of the virtual class [49], [6]. A given space of maps $\overline{\mathcal{M}}(\beta, X)$ may be of the wrong dimension, and the virtual class provides a way to correct for this. We consider the "correct dimension" to be one imposed either by physical theory, or by the requirement that the essential behavior of the moduli space be invariant under deformations of X (this may include topological deformations of X, or

[5]When $g \neq 0$, the moduli space of maps is not smooth (convexity/concavity is no longer valid), and one has to take care to define integration of forms in the expected ("top") dimension, as the moduli space will contain components other dimensions. To do so, one must define a cycle of the expected dimension – the virtual fundamental class ([6] , [49]). [27] proved that with these definitions, the Atiyah-Bott localization formulas continue to hold, with the normal bundle replaced by an appropriately defined virtual normal bundle.

just deformations of symplectic or almost complex structures). The virtual class is a class in the cohomology (or Chow) ring of $\overline{\mathcal{M}}(\beta, X)$; its principal properties are that it is a class in the cohomology ring of the expected dimension, and that numbers calculated by integrating over the virtual class are invariant under deformations of X. See [49], [6] for more exact and accurate statements. One main theme of this paper is to use the invariance under deformation to either calculate these numbers, or to explain the significance of a calculation.

The idea of a cohomology class (or cohomology calculation) which corrects for "improper behavior" has been around for a long time in intersection theory. One example is the excess intersection formula. If we attempt to intersect various classes in a cohomology ring, and if we choose representatives of those classes which fail to intersect transversely, the resulting dimension of the intersection may be too large. The excess intersection formula allows us to perform a further calculation on this locus to determine the actual class of the intersection. The purpose of this section is to describe the excess intersection formula for degeneracy loci of vector bundles, and to use this formula to evaluate or explain some mirror symmetry computations. In the cases we examine, the moduli space of maps to our space X can be given as the degeneracy locus of a vector bundle on a larger space of maps. In each case that these degeneracy loci are of dimension larger than expected, the virtual class will turn out to be the same as the construction given by the excess intersection formula. The virtual class and the excess intersection formula are both aspects of the single idea mentioned above, and have as common element in their construction the notion of the refined intersection class [21].

To a vector bundle E of rank r on a smooth algebraic variety X of dimension n, we associate the Chern classes, $c_j(E)$, $j = 0, \dots, r$, and also the total Chern class $c(E) = 1 + c_1(E) + c_2(E) + \cdots + c_r(E)$. These classes are elements of the cohomology ring of X. The class $c_j(E)$ represents a class of codimension j in X, and in particular the class $c_n(E)$ is a class in codimension n, and can be associated with a number. For any class α in the ring, the symbol $\int_X \alpha$ means to throw away all parts of α except those parts in degree n, and evaluate the number associated to those parts.

For a vector bundle E whose rank is greater or equal to the dimension n of X, we are often interested in calculating the number associated to $c_n(E)$, or in the previous notation, $\int_X c(E)$. One way to compute Chern classes is to realize them as degeneracy loci of linear combinations of sections. If E is of rank $r \geq n$, and we take $r-n+1$ generic global sections $\sigma_1, \ldots, \sigma_{r-n+1}$, the locus of points where $\sigma_1, \ldots, \sigma_{r-n+1}$ fail to be linearly independent represents the class $c_n(E)$. Often this way of interpreting the Chern classes is the one which has the most geometric meaning. The statement "generic" above means that if we carry out this procedure and find out that the degeneracy locus is of the correct dimension (that is: points), then the sections were generic enough.

Sometimes the sections we can get our hands on to try and calculate $\int_X c(E)$ with are not generic in this sense, and the degeneracy locus consists of some components which are positive dimensional. In this situation, the excess intersection formula tells us how to associate to each positive dimensional connected component of the degeneracy locus a number, called the "excess intersection contribution". This number is the number of points which the component "morally" accounts for. Part of the excess intersection theorem is the assertion that the sum of the excess intersection contributions over all the connected components of the degeneracy locus, and the sum of the remaining isolated points add up to $\int_X c(E)$. This corresponds to the invariance of numbers computed using the virtual class under deformations of the target manifold.

Let Y be one of the connected components described above. Let's assume for simplicity that Y is actually a submanifold of M. In this situation the excess intersection formula says that the excess intersection contribution of Y is

$$\int_Y \frac{c(E)}{c(N_{Y/M})}. \tag{13}$$

Here $N_{Y/M}$ is the normal bundle of Y in M, and the expression after the integral sign makes sense, since $c(N_{Y/M})$ is an element of a graded ring whose degree zero part is 1, and so $c(N_{Y/M})$ may be inverted in that ring.

5.1 Rational curves on the quintic threefold

As an example of an application of the excess intersection formula to explain the significance of a calculation, let us review the count of the rational curves on a quintic threefold as explained by Kontsevich [44]. Let $M_d = \mathcal{M}_{00}(d, \mathbf{P}^4)$ be the moduli space of maps of genus zero curves of degree d to $\mathbf{P}^4$. M_d is of dimension $5d+1$. Let U_d be the vector bundle on M_d whose fiber at any stable map (C,f) is $H^0(C, f^*\mathcal{O}_{\mathbf{P}^4}(5))$; this is a bundle of rank $5d+1$. The numbers $K_d = \int_{M_d} c(U_d)$ have been computed by mirror symmetry, and the first few are $K_1 = 2875$, $K_2 = 4876875/8$, and $K_3 = 8564575000/27$. To try and find a geometric interpretation of these numbers, we compute $\int_{M_d} c(U_d)$ by finding a global section of U_d and examining its degeneracy locus. Let F be a generic section of $\mathcal{O}_{\mathbf{P}^4}(5)$ on $\mathbf{P}^4$ which cuts out a smooth quintic threefold X. We pull F back to give us a global section of U_d, which we call σ_d. The degeneracy locus of σ_d in M_d consists of those maps (C,f) with $f(C)$ contained in this quintic threefold. This observation allows us to use the K_d to compute the number of rational curves of degree d on the quintic threefold X.

In degree 1 the degeneracy locus consists of one point for every line mapping into X, and so we see that $K_1 = 2875$ is the number of lines in a quintic threefold. In degree two, the degeneracy locus of σ_2 consists of an isolated point for every degree two rational curve in X, and 2875 positive dimensional loci, each one consisting of maps which map two to one onto a line in X. To calculate the actual number of degree two rational curves in X, we compute the excess intersection contribution of each of these positive dimensional components, and subtract from the previously computed total of 4876875/8. We now compute this excess intersection contribution.

For each line l in X, let Y_l be the submanifold of M_2 parameterizing two to one covers of l. The normal bundle of l in $\mathbf{P}^4$ is $N_{l/\mathbf{P}^4} = \mathcal{O}_l(1) \oplus \mathcal{O}_l(1) \oplus \mathcal{O}_l(1)$. A calculation on the tangent space of M_2 shows that the normal bundle N_{Y_l/M_2} is (at a map (C,f)) equal to $H^0(C, f^*N_{l/\mathbf{P}^4})$.

Since the line l is sitting in the quintic threefold X, its normal bundle maps naturally to the normal bundle of X in $\mathbf{P}^4$, with kernel the normal bundle of l in X. This gives us an exact sequence:

$$0 \longrightarrow \mathcal{O}_l(-1) \oplus \mathcal{O}_l(-1) \longrightarrow N_{l/\mathbf{P}^4} \longrightarrow \mathcal{O}_l(5) \longrightarrow 0.$$

Let V_2 be the bundle on Y_l whose fiber at a map (C, f) is $H^1(C, f^*\mathcal{O}_l(-1))$. The above short exact sequence on l gives us the sequence

$$0 \longrightarrow N_{Y_l/M_2} \longrightarrow E_2 \longrightarrow V_2 \oplus V_2 \longrightarrow 0$$

of bundles on M_2. The multiplicative properties of Chern classes in short exact sequences shows us that the excess contribution of Y_l is:

$$\int_{Y_l} \frac{c(E_2)}{c(N_{Y_l/M_2})} = \int_{Y_l} c(V_2)c(V_2).$$

This last number is the Aspinwall-Morrison computation of $1/d^3$, or in this case, $1/8$. This gives the number of actual degree two rational curves on a quintic threefold as $4876875/8 - 2875/8 = 609250$.

Under the assumption that each rational curve in X is isolated and smooth, then similar computations give the famous formula [11]

$$K_d = \sum_{k|d} \frac{n_{d/k}}{k^3}, \tag{14}$$

where n_d is the number of rational curves in degree d. A caveat: It has been shown that this assumption is false in at least one instance – in degree five, some of the rational curves are plane curves with six nodes [67]. This doesn't affect the computation until you try to calculate multiple cover contributions from these curves. For example,[6] in degree ten we have double covers of these nodal curves. The moduli space of double covers of a (once) nodal rational curve has two components: one being degree two maps to the normalization of the nodal curve, which contributes $1/8$ as for the smooth case; the other being a single point representing two disconnected copies of the normalization mapping down to the singular curve. If P, Q represent the points on the normalization which are to be identified for the nodal curve, there is a uniqe map from a domain curve with two components and one node, where the node is mapped to P on one copy and Q on the other (these points are identified). This double cover does not factor through the normalization. If we have n such curves, their double covers contribute $n/8 + n$. The integers n_d obtained from the formula (14) need to be shifted to have the proper enumerative interpretation ("experimentally," this shift is integral, though this has not been proven [15]).

[6]We thank N.C. Leung for describing this example to us.

5.2 Calabi-Yau threefolds containing an algebraic surface

Let us consider the situation where we have a Calabi-Yau threefold, X, in a toric variety, $\mathbf{P}$, and a smooth algebraic surface, B, contained within X :

$$B \subset X \subset \mathbf{P}. \tag{15}$$

We assume as well that B is a Fano surface so that X may be deformed so that B shrinks [59]. This is the scenario of interest to us in this paper. Now since there are holomorphic maps of many degrees into B, which therefore all lie within X, we will have an enormous degeneracy locus. If X is cut from some section s, then at degree β the whole space $\overline{\mathcal{M}}_{0,0}(\beta; B)$ will be a zero set of the pull-back section $\tilde{s}$.[7] Therefore, we will need to use the excess intersection formula to calculate the contribution of the surface to the Gromov-Witten invariants for X. From this, we will extract integers which account for the effective number of curves due to B.

Mapping tangent vectors, we have from (15) the following exact sequence: $0 \to N_{B/X} \to N_{B/\mathbf{P}} \to N_{X/\mathbf{P}} \to 0$. Note that $N_{B/X} = K_B$, by triviality of $\Lambda^3 TX$ and the exact sequence $TB \to TX \to N_{B/X}$. Therefore, we have

$$0 \longrightarrow K_B \longrightarrow N_{B/\mathbf{P}} \longrightarrow N_{X/\mathbf{P}} \longrightarrow 0.$$

Given $(C, f) \in \overline{\mathcal{M}}_{0,0}(\beta; B)$, we can pull back these bundles and form the long exact sequence of cohomology:

$$\begin{aligned} 0 \; &\longrightarrow H^0(C, f^*K_B) \longrightarrow H^0(C, f^*N_{B/\mathbf{P}}) \longrightarrow H^0(C, f^*N_{X/\mathbf{P}}) \longrightarrow \\ &\longrightarrow H^1(C, f^*K_B) \longrightarrow H^1(C, f^*N_{B/\mathbf{P}}) \longrightarrow .... \end{aligned}$$

Now $H^0(C, f^*K_B) = 0$ since B is Fano (its canonical bundle is negative), and $H^1(C, f^*N_{B/\mathbf{P}}) = 0$ when B is a complete intersection of (of nef divisors), which we assume. As a result, (16) becomes a

[7]Actually, β labels a class in X which may be the image of a number of classes in B. In such a case, our invariants are only sensitive to the image class, and represent a sum of invariants indexed by classes in B.

short exact sequence of the bundles over $\overline{\mathcal{M}}_{0,0}(\beta;B) \subset \overline{\mathcal{M}}_{0,0}(\beta;\mathbf{P})$ whose fibers are the corresponding cohomology groups. The bundle (call it U_β) with fiber $H^1(C, f^*K_B)$ is the one we use to define the local invariants. The bundle with fiber $H^0(C, f^*N_{B/\mathbf{P}})$ is $N_{\mathcal{M}(B)/\mathcal{M}(\mathbf{P})}$ (abbreviating the notation a bit). That with fiber $H^0(C, f^*N_{X/\mathbf{P}})$ is the one used to define the (global) Gromov-Witten invariants for X – call it E_β. Therefore, we have

$$0 \longrightarrow U_\beta \longrightarrow N_{\mathcal{M}(B)/\mathcal{M}(\mathbf{P})} \longrightarrow E_\beta \longrightarrow 0.$$

Now using (13) with $E = E_\beta$; $M = \overline{\mathcal{M}}_{0,0}(\beta;\mathbf{P})$; and $Y = \overline{\mathcal{M}}_{0,0}(\beta;B)$; the multiplicativity of the Chern class gives the contribution to the Gromov-Witten invariant of the threefold X from a surface $B \subset X$ is

$$K_\beta = \int_{\overline{\mathcal{M}}_{0,0}(\beta;B)} c(U_\beta), \tag{16}$$

which is what we have been computing.

Typically, the presence of a surface $B \subset X$ may not be generic, so that X can be deformed to a threefold X' not containing such a holomorphic surface. Let K_β^X be the Gromov-Witten invariant of X, and let $K_\beta^{X'}$ be the Gromow-Witten of X'. These are equal, as the Gromov-Witten invariant is an intersection independent of deformation: $K_\beta^X = K_\beta^{X'}$. For X' we have an enumerative interpretation[8] of $K_\beta^{X'}$ in terms of n'_β, the numbers of rational curves on X'. Let n_β be the numbers of rational curves on X, and let K_β be the integral in (16). For simplicity, let us assume that $\dim H_2(X') = 1$, so that degree is labeled by an integer: $\beta = d$. Then combining the enumerative interpretation with the interpretation of the excess intersection above, we find

$$K_d^{X'} = \sum_{k|d} n'_{d/k}/k^3 = \sum_{k|d} n_{d/k}/k^3 + K_d.$$

Subtracting, we find

$$K_d = \sum_{k|d} \delta n_{d/k}/k^3.$$

[8]Singular rational curves notwithstanding.

Here $\delta n = n' - n$ represents the effective number of curves coming from B. In the text, we typically write n for δn.

We therefore have an enumerative interpretation of the local invariants. After performing the $1/d^3$ reduction we get an integer representing an effective number of curves (modulo multiple covers of singular curves, which should shift these integers). We should note that one might ask about rational curves in the Calabi-Yau manifold which intersect our Fano surface. Such a situation would make for a more complicated degeneracy locus, but it turns out this situation does not arise. Indeed, if $C' \subset X$ is a holomorphic curve in X meeting B transversely, then $C' \cdot B > 0$ (strictly greater). However, for $C \subset B$, we have

$$C \cdot B = \int_D c_1(N_{B/X}) = \int_D c_1(K_B) < 0,$$

by the Fano condition. Therefore, C' cannot lie in the image of $H_2(B)$ in $H_2(X)$ – the only classes in which we are interested – and so our understanding of the numbers n_d is therefore complete.

5.3 Singular geometries

For physical applications, we will often want the surface B to be singular. For example, in order to geometrically engineer $SU(n+1)$ supersymmetric gauge theories in four dimensions, we consider the local geometry of an A_n singularity fibered over a $\mathbf{P}^1$. In fact, we take a resolution along each A_n fiber, so that the exceptional divisor over a point is a set of $\mathbf{P}^1$'s intersecting according to the Dynkin diagram of $SU(n+1)$. The total geometry of these exceptional divisors forms a singular surface, which is a set of $\mathbf{P}^1$ bundles over $\mathbf{P}^1$ (Hirzebruch surfaces) intersecting along sections. In [35] it is shown how the local invariants we calculate can be used to derive the instanton contributions to the gauge couplings. Roughly speaking, the number of wrappings of the $\mathbf{P}^1$ base determines the instanton number, while the growth with fiber degree of the number of curves with a fixed wrapping along the base determines the corresponding invariant.

It is clearly of interest, then, to be able to handle singular geometries. Actually, we will be able to do so without too much effort. Let us consider an illustrative example. Define the singular surface B' to be two $\mathbf{P}^2$'s intersecting in a $\mathbf{P}^1$. This can be thought of as a singular quadric surface, since it can be represented as the zero locus of the reducible degree two polynomial

$$XY = 0$$

in $\mathbf{P}^3$ with homogeneous coordinates $[X, Y, Z]$. The generic smooth quadric is a surface $B = \mathbf{P}^1 \times \mathbf{P}^1$. If we express B as a hypersurface in $\mathbf{P}^3$, we can define the local invariants (indexed only by the generator of $H_2(\mathbf{P}^3)$) of B as an intersection calculation in $\overline{\mathcal{M}}_{0,0}(d; \mathbf{P}^3)$ as follows. Define the bundle

$$E \equiv \mathcal{O}_{\mathbf{P}^3}(2) \oplus \mathcal{O}_{\mathbf{P}^3}(-2).$$

Then let $s_E = (s, 0)$ be a global section of E, where s is a quadric and 0 is the only global section of $\mathcal{O}_{\mathbf{P}^3}(-2)$. Note that, by design, E restricted to the zero set of s_E is equal to K_B. We now define a bundle over $\overline{\mathcal{M}}_{0,0}(d; \mathbf{P}^3)$ whose fibers over a point (C, f) are $H^0(C, f^*\mathcal{O}_{\mathbf{P}^3}(2)) \oplus H^1(C, f^*\mathcal{O}_{\mathbf{P}^3}(-2))$. We then compute the top Chern class of the bundle, which can be calculated as in the previous subsections in terms of the zero locus of $\tilde{s}_E$, which picks out maps into $B \cong \mathbf{P}^1 \times \mathbf{P}^1$. The calculation gives the usual local invariant for $\mathbf{P}^1 \times \mathbf{P}^1$, counting curves by their total degree $d = d_1 + d_2$, where d_i is the degree in $\mathbf{P}^1_i$, $i = 1, 2$. The reason is that $\mathcal{O}(2)|_B = N_{B/\mathbf{P}^3}$, so the contribution from this part to the total Chern class cancels with the normal bundle to the map. (The local invariants are listed in the first column of Table 7.)

Now note that this intersection calculation is independent of the section we use to compute it. In fact, if we use a reducible quadric whose zero locus is B', the calculation will reduce to one on the singular space $\overline{\mathcal{M}}_{0,0}(d; B')$. The excess intersection formula tells us exactly which class to integrate over this (singular) space. In fact, integration over the singular space is only defined via the virtual fundamental class – which is constructed to yield the same answer. In degree one, this can all be checked explicitly in this example [26] . The upshot is that as our calculations are independent of deformations, we can deform our singular geometries to do local calulations in a simpler setting. In fact, this makes intuitive physical sense: the A-model should be independent of deformations.

Another phenomenon that we note in examples is that the calculation of the mirror principle can be performed without reference to a specific bundle. In other words, the toric data defining any non-compact Calabi-Yau threefold works as input data. As a result, we can consider Calabi-Yau threefolds containing singular divisors and perform the calculation. For the example of A_2 fibered over a sphere, we get the numbers in Table 4. Though this technique has not yet been proven to work, it is tantalizing to guess that the whole machinery makes sense for any non-compact threefold, with intersections taking place in the Chow ring and with an appropriately defined prepotential.

In the next section, we will use the B-model to define differential equations whose solutions determine the local contributions we have been discussing.

6 Local Mirror Symmetry: The B-Model

In this section we describe the mirror symmetry calculation of the Gromov-Witten invariants for a $(n-1)$-dimensional manifold B with $c_1(B) > 0$. We first approach this by using mirror symmetry for a compact, elliptically fibered Calabi-Yau n-fold[9] $\hat{X}$ which contains B as a section, and taking then the volume of the fiber to infinity. If B is a Fano manifold or comes from a $(n-1)$-dimensional reflexive polyhedron a smooth Weierstrass Calabi-Yau manifold $\hat{X}$ with B as a section exists. Moreover the geometry of $\hat{X}$ depends only on B and therefore the limit can be described intrinsically from the geometry of B. This is an intermediate step. Later, we will define the objects relevant for the B-model calculation for B intrinsically, without referring to any embedding. Such an embedding, in fact, is in general not possible.

6.1 Periods and differential equations for global mirror symmetry

We briefly review the global case in the framework of toric geometry, following the ideas and notations of [2], [32]. According to [2], a mirror pair $(X, \hat{X})$ with the property $h_{p,q}(X) = h_{n-p,q}(\hat{X})$ can be represented

[9]We will state formulas for n-folds, when possible.

as the zero locus of the Newton polynomials[10] ($P = 0, \hat{P} = 0$) associated to a dual pair of reflexive $(n+1)$-dimensional polyhedra $(\Delta, \hat{\Delta})$. X is defined as hypersurface by the zero locus of

$$P = \sum_{\nu^{(i)}} a_i \prod_{j=1}^{n+1} X_j^{\nu_j^{(i)}} \tag{17}$$

in the toric ambient space $\mathbf{P}_{\Sigma(\hat{\Delta})}$, constructed by the complete fan $\Sigma(\hat{\Delta})$ associated to $\hat{\Delta}$. The sum (17) runs over r "relevant" points $\{\nu^{(i)}\} \in \Delta$, which do not lie on codimension one faces and with $\nu^{(0)}$ we denote the unique interior point in Δ. The a_i parametrize the complex structure deformations of X redundantly because of the induced $(\mathbf{C}^*)^{n+2}$ actions on the a_i, which compensate $X_i \to \lambda_i X_i$, $P \to \lambda_0 P$, such that $P = 0$ is invariant. Invariant complex structure coordinates are combinations

$$z_i = (-1)^{l_0^{(j)}} \prod_{j=0}^{r-1} a_j^{l_j^{(i)}}, \tag{18}$$

where the $l^{(j)}$, $j = 1, \ldots, k = r - (n+1)$ are an integral basis of linear relations among the extended "relevant" points $\bar{\nu}^{(i)} = (1, \nu^{(i)})$ with $\nu^{(i)} \in \mathrm{rel}(\Delta)$, i.e.,

$$\sum_{i=0}^{r-1} l_i^{(j)} \bar{\nu}^{(i)} = \vec{0}. \tag{19}$$

The $l^{(j)}$ have in the gauged linear sigma model [69] the rôle of charge vectors of the fields with respect to $U(1)^k$. Moreover, if the $l^{(j)}$ span a cone in the secondary fan of Δ, which correspond to a complete regular triangulation of Δ [23], then $l^{(j)}$ span the dual cone (Mori cone) to the Kähler cone of $\mathbf{P}_{\Sigma(\hat{\Delta})}$, which is always contained in a Kähler cone of $\hat{X}$ and $z_i = 0$ corresponds to a point of maximal unipotent monodromy, which by the mirror map [11], [32] corresponds to the large Kähler structure limit of $\hat{X}$.

The period integrals of X contain the information about the Gromov-Witten invariants of $\hat{X}$ (and vice versa). They are defined as integrals

[10]The generalization to complete intersections in toric ambient spaces is worked out in [32][4].

of the unique holomorphic $(n,0)$-form over the b_n $(= 2(h_{2,1}(X)+1)$ when $n=3)$ cycles Γ_i in the middle cohomology of X. The $(n,0)$-form is given by a generalization of Griffiths residue expressions [29]

$$\Omega = \frac{1}{(2\pi i)^{n+1}} \int_{\gamma_0} \frac{a_0 \omega}{P}, \qquad \text{with } \omega = \frac{\mathrm{dX}_1}{X_1} \wedge \ldots \wedge \frac{\mathrm{dX}_{\mathrm{n+1}}}{X_{n+1}}, \tag{20}$$

and γ_0 is a contour around $P=0$. General periods are then $\Pi_i(z_i) = \int_{\Gamma_i} \Omega$, and for a particular cycle[11] this leads to the following simple integral

$$\Pi_0(z_i) = \frac{1}{(2\pi i)^{n+1}} \int_{|X_i|=1} \frac{a_0 \omega}{P}. \tag{21}$$

Because of the linear relations among the points (19), the expression $\hat\Pi(a_i) = \frac{1}{a_0}\Pi(z_i)$ fulfills the differential identities

$$\prod_{l_i^{(k)}>0} \left(\frac{\partial}{\partial a_i}\right)^{l_i^{(k)}} \hat\Pi = \prod_{l_i^{(k)}<0} \left(\frac{\partial}{\partial a_i}\right)^{-l_i^{(k)}} \hat\Pi. \tag{22}$$

The fact that the $\bar\nu^{(i)}$ lie on a hyperplane, together with (19) imply the same numbers of derivatives on both sides of (22), assuring equality. Unlike the $\Pi(z_i)$ the $\hat\Pi(a_i)$ are however not well defined under the $\mathbf{C}^*$ action $P \to \lambda_0 P$ defined above. To obtain differential operators $\mathcal{L}_k(\theta_i, z_i)$ annihilating $\Pi_i(z_i)$ one uses (22), $[\theta_{a_i}, a_i^r] = r a_i^r$, and the fact that Π_i depends on the a_i only through the invariant combinations z_i. Here we defined logarithmic derivatives $\theta_{a_i} = a_i \frac{\partial}{\partial a_i}$, $\theta_i = z_i \frac{\partial}{\partial z_i}$.

Example. $\hat X$ is the degree 18 hypersurface in $\mathbf{P}(1,1,1,6,9)$, with Euler number -540 and $h_{1,1}(\hat X) = 2$ and $h_{2,1}(\hat X) = 272$. The toric data are

a.) $\hat\Delta = \mathrm{conv}\{[-6,-6,1,1],[-6,12,1,1],[0,0,-2,1],[0,0,1,-1],$
$[12,-6,1,1]\}$

b.) $\mathrm{rel}(\Delta) = \{[0,0,0,0];[1,0,2,3],[0,1,2,3],[-1,-1,2,3],[0,0,2,3],$
$[0,0,-1,0],[0,0,0,-1]\}$

[11]Which, when $n=3$ is dual to the S^3 which shrinks to zero at the generic point in the discriminant.

c.) triang $= \{[0,1,2,4,5],[0,1,3,4,5],[0,1,3,4,6],[0,2,3,4,5],$
$[0,1,2,4,6],[0,2,3,5,6],[0,1,2,5,6],$
$[0,1,3,5,6],[0,2,3,4,6]\}$

d.) $\mathcal{SRI} = \{x_1 = x_2 = x_3 = 0, x_4 = x_5 = x_6 = 0\}$

e.) $l^{(1)} = (-6; 0, 0, 0, 1, 2, 3), \qquad l^{(2)} = (0; 1, 1, 1, -3, 0, 0)$

f.) $J_1^3 = 9 \quad J_1^2 J_2 = 3 \quad J_1 J_2^2 = 1, \quad c_2 J_1 = 102, \quad c_2 J_2 = 36. \qquad (23)$

Here triang is a regular star triangulation of Δ, where the 4d simplices are specified by the indices of the points in rel(Δ). $\mathcal{SRI}$ denotes the Stanley Reisner Ideal. $J_i J_k J_l$ and $c_2 J_i$ are the triple intersection numbers and the evaluation of the second chern class on the forms J_i, i.e., $\int c_2 J_i$. Then by (17) X is given by

$$P = a_0 + X_3^2 X_4^3 \left(a_1 X_1 + a_2 X_2 + \frac{a_3}{X_1 X_2} + a_4 \right) + \frac{a_5}{X_3} + \frac{a_6}{X_4} = a_0 + \Xi$$

and the period (21) is easily integrated in the variables(18) $z_1 = \frac{a_4 a_5^2 a_6^3}{a_0^6}$, $z_2 = \frac{a_1 a_2 a_3}{a_4^3}$

$$\begin{aligned}\Pi_0 &= \frac{1}{(2\pi i)^4} \int \frac{1}{1 + \frac{1}{a_0}\Xi} \omega = \left[\sum_{n=0}^{\infty} \left(-\frac{\Xi}{a_0} \right)^n \right]_{\text{term constant in } X_i} \\ &= \left[\sum_{n=0}^{\infty} (-a_0)^{-n} \sum_{\nu_1 + \cdots + \nu_6 = n} \left(\frac{n!}{\nu_1!} \cdots \nu_6! \right) \right. \\ &\qquad \left. \cdot \, (a_1 X_1 X_3^2 X_4^3)^{\nu_1} \cdots \left(\frac{a_6}{X_4} \right)^{\nu_6} \right]_{\text{term constant in } X_i} \\ &= \sum_{r_1=0, r_2=0}^{\infty} \frac{\Gamma(6r_1 + 1)}{\Gamma(r_2+1)^3 \Gamma(r_1 - 3r_2 + 1) \Gamma(3r_1 + 1) \Gamma(2r_1 + 1)} z_1^{r_1} z_2^{r_2}.\end{aligned}$$

Likewise, it is easy to see that (22) leads to

$$\begin{aligned}\mathcal{L}_1 &= \theta_1(\theta_1 - 3\theta_2) - 12(6\theta_1 - 5)(6\theta_1 - 1) z_1 \\ \mathcal{L}_2 &= \theta_2{}^3 - (1 + \theta_1 - 3\theta_2)(2 + \theta_1 - 3\theta_2)(3 + \theta_1 - 3\theta_2) z_2 \,,\end{aligned} \qquad (24)$$

where we factored from the first operator a degree four differential operator. This is equivalent to discarding four solutions which have incompatible behavior at the boundary of the moduli space to be periods, while the remaining $(2h_{2,1}(X)+2)$ solutions can be identified with period integrals for X.

Note that in general at the point of maximal unipotent monodromy [11], [58], [32], [3] $z_i = 0$, $\Pi_0 = 1 + O(z)$ is the only holomorphic solution to the Picard-Fuchs system. Let us now[12] set $n = 3$. In general, there will be $h_{2,1}(X)$ logarithmic solutions of the form $\Pi_i = \frac{1}{2\pi i}\log(z_i)\Pi_0 + holom.$, and

$$t_i = \frac{\Pi_i(z)}{\Pi_0(z)} \tag{25}$$

defines affine complex structure parameters of X, which at $z_i = 0$ can be identified with the complexified Kähler parameters $t_i = iVol(C_i) + B(C_i)$ of $\hat{X}$, following [11]. The relation (25) is called the "mirror map" and in particular, in the limit $Vol(C_i) \to \infty$ one has

$$\log(z_i) \sim -Vol(C_i). \tag{26}$$

$h_{2,1}$ further solutions are quadratic, and one is cubic in the logarithm. These solutions are related to each other and to the quantum corrected triple intersection $c_{i,j,k}$ by special geometry, basically Griffith transversality[13] $\int(\partial_i\Omega)\wedge\Omega = \int(\partial_i\partial_j\Omega)\wedge\Omega = 0$. As a consequence, these quantities derive from a prepotential, which has the general form [11], [32] ($\mathrm{Li}_3(x) = \sum_{k=1}^{\infty}\frac{x^k}{k^3}$)

$$\begin{aligned}\mathcal{F} = {} & \frac{J_i\cdot J_k\cdot J_l}{6}t_it_jt_k + \frac{1}{24}c_2\cdot J_it_i - i\frac{\zeta(3)}{2(2\pi)^3}c_3 \\ & + \sum_{d_1,\dots,d_{h_{1,1}(\hat{X})}} N_{\vec{n}}\mathrm{Li}_3(q^{d_1}\cdots q^{d_{h_{1,1}(X)}})\end{aligned} \tag{27}$$

The relations are

$$\vec{\Pi} = \Pi_0\left(1, t_i, \partial_{t_i}\mathcal{F}, 2\mathcal{F} - \sum_i t_i\partial_{t_i}\mathcal{F}\right),$$

[12]Some aspects for the case of arbitrary n are discusses in [28], [41], [53].

[13]If n is even we get in general algebraic relations between the solutions *and* differential relations. The algebraic relations in the $K3$ case are well known, in the 3-fold case we have special geometry, for 4-folds the algebraic and differential relations can be found in [28], [53], [41].

$$c_{i,j,k} = \partial_{t_i}\partial_{t_j}\partial_{t_k}\mathcal{F} = J_i \cdot J_j \cdot J_k + \sum_{\{d_i\}} d_i d_j d_k N_{\vec{d}} \frac{\vec{q}^{\vec{d}}}{1 - \vec{q}^{\vec{d}}}, \tag{28}$$

where $\vec{q}^{\vec{d}} \equiv \prod_i q_i^{d_i}$ and the $N_{\vec{n}}$ are then the Gromov-Witten invariants of the mirror $\hat{X}$.

6.2 The limit of large elliptic fiber

We now identify the classes of the curves C_i and define the limit of large fiber volume. Then we will use (25) to translate that into a limit in the complex structure deformations parameters z_i of X. Note that in Batyrev's construction the points $\nu^{(i)} \in \mathrm{rel}(\Delta)$ correspond to monomials in P as well as to divisors D_i in $\mathbf{P}_{\Sigma(\Delta)}$ (in the example, $\mathbf{P}_{\Sigma(\Delta)} = \mathbf{P}(1,1,1,6,9)$) which intersect $\hat{X}$. Each $l^{(i)}$ defines a wall in the Kähler cone of $\hat{X}$ at which curves in the class $[C_i]$ vanish. Moreover, the entries $l_i^{(j)}$, $i = 1, \ldots, r$ are the intersection of these curves C_j with the restriction $\tilde{D}_i$ of the divisors D_i, $i = 1, \ldots, r$ to $\hat{X}$. From this information one can identify the classes $[C_i]$ in $\hat{X}$. It is convenient to use the Cox coordinate ring representation [14] $\mathbf{P}_{\Sigma(\Delta)} = \{C[x_1, \ldots x_r] \backslash \mathcal{SRI}\}/(\mathbf{C}^*)^k$, where the $\mathbf{C}^*$-actions are given by $x_i \to x_i(\lambda^{(j)})^{l_i^{(j)}}$ and $\mathcal{SRI}$ denotes the Stanley Reisner Ideal. In these coordinates D_i is simply given by $x_i = 0$ and the polynomial reads

$$\hat{P} = \sum_{i=0}^{\hat{r}-1} a_i \prod_{j=0}^{r-1} x_j^{\langle \nu^{(j)}, \hat{\nu}^{(i)} \rangle + 1} . \tag{29}$$

In the example,

$$\hat{P} = x_0(x_4^6 g_{18}(x_1, x_2, x_3) + x_4^4 x_5 f_{12}(x_1, x_2, x_3) + x_5^3 + x_6^2) \tag{30}$$

has a smooth Weierstrass form. We notice, taking into account (23) parts d and e, that D_4 meets $\hat{X}$ in a $\mathbf{P}^2$, the section of the Weierstrass form. As $[C_2] \cdot D_4 = -3$, C_2 must be contained in this $\mathbf{P}^2$, and from $C_2 \cdot D_i = 1$, $i = 1, 2, 3$, it follows that C_2 lies in that $\mathbf{P}^2$ with degree[14]

[14]As a consistency check, note that J_2 which is the dual divisor to C_2 must then have a component in the base P^2 and one in the fiber direction, hence on dimensional grounds it can at most intersect quadratically comp (23), part f.

1. $[C_1] \cdot \tilde{D}_4 = 1$, hence C_1 meets the section once and must be a curve in the fiber direction, whose volume goes to infinity in the large fiber limit. By this association $l^{(1)} = l^{(F)}$ and as $\log(z_F) \sim -Vol(C_F)$, $z_F = \frac{a_4 a_6^2 a_6^3}{a_0^6} \to 0$ is the correct complex structure limit. Next we pick the periods which stay finite in this limit – that is, whose cycle has still compact support. From (27), (28), and (23), part f, we see that the finite solutions are Π_0, $\Pi_0 t_2$ and $\Pi_0(\partial_1 - 3\partial_2)\mathcal{F}$. Moreover, as they do not contain $\log(z_1)$ terms they satisfy in this limit (and are in fact determined by) the specialization of (24) as $z_1 \to 0$:

$$\mathcal{L} = \theta^3 + 3z\theta(3\theta + 2)(3\theta + 1), \tag{31}$$

where we have put $z \equiv z_2$ and $\theta \equiv \theta_2$.

This differential equation comes from the relation of the points in the two dimensional face $\Delta_B = \mathrm{conv}\{\nu_1, \ldots, \nu_4\}$ in Δ. We call the Newton polynomial for this set of points P_B. We want to define a special limit of the finite $\Pi_i(z)$. The limit is $a_5, a_6 \to 0$ in P, which is compatible with $z_F = 0$. We define $W = X_3^2 X_4^3$ and $V = (X_3 X_4)^{-2}$. Then $Jac \sim X_3$, $\omega' = dW \wedge \frac{dV}{V} \wedge \frac{dX_1}{X_1} \wedge \frac{dX_2}{X_2}$, and $\omega'' = dW \wedge \frac{dX_1}{X_1} \wedge \frac{dX_2}{X_2}$. As $1/W$ is the non-compact direction, perpendicular to the compact plane, we remove the compactification point in the $1/W$-loop, which becomes open. Hence

$$\begin{aligned}
\Pi_0(z) &= \frac{c}{(2\pi)^4} \int_{\substack{|X_i|=1 \\ |V|=1}} \int_\epsilon^1 \frac{a_0}{W(a_0 + W P_B)} \omega' + O(a_5, a_6) \\
&\approx \frac{c}{(2\pi i)^3} \int_{|X_i|=1} \int_\epsilon^1 \frac{1}{W(1 + \frac{W P_B}{a_0})} \omega'' \\
&= \frac{c}{(2\pi i)^3} \int_{|X_i|=1} \int_\epsilon^1 W^{-1} \sum_{i=0}^{\infty} (-)^i \left(\frac{W P_B}{a_0}\right)^i \omega'' \\
&= -\log(\epsilon) - \frac{c}{(2\pi i)^3} \int_{|X_i|=1} \log(1 + \frac{P_B}{a_0}) \frac{dX_1}{X_1} \wedge \frac{dX_2}{X_2} \\
&= C - \frac{c}{(2\pi i)^3} \int_{|X_i|=1} \log(P_B') \frac{dX_1}{X_1} \wedge \frac{dX_2}{X_2},
\end{aligned}$$

where P_B' is P_B with rescaled a_i.

6.3 Local mirror symmetry for the canonical line bundle of a torically described surface

We will generalize the example of the previous section to the situation where one has as smooth Weierstrass form for the threefold over some base B. The known list of bases which lead to smooth Weierstrass forms are Fano varieties and torically described bases whose fans are constructed from the polyhedra Δ_B, which we display in figure 1. For these bases, we can demonstrate the property of admitting a smooth Weierstrass form by explicitly showing that total space constructed from $\Delta^{fibration} = \text{conv}\{(\nu_i^B, \nu^E), (0,0,-1,0), (0,0,0,-1)\}$ is smooth. Here ν_i^B runs over the 2-tuple of the coordinates of points in Δ_B and for ν^{E_k} one has the choice $(2,3)$, $(1,2)$, and $(1,1)$ for the E_8, E_7, E_6 respective fiber types described below.

For the smooth fibrations, all topological data of $\hat{X}$ are expressible from the base topology and we have a surjective map $i^* : H^{1,1}(X) \to H^{1,1}(B)$. Using the adjunction formula[15] one finds (here we understand that on the left side we integrate over $\hat{X}$ and on the right side over B)

$$\begin{gathered} c_3(\hat{X}) = -2hc_1(B)^2 \\ c_2(\hat{X})J_E = kc_2(B) + k\left(\frac{12}{k} - 1\right)c_1(B)^2, \quad c_2(\hat{X})J_i = 12kc_1(B)J_i \\ J_E^3 = kc_1^2(B), \quad J_E^2 J_i = kc_1(B)J_i, \quad J_E J_i J_k = kJ_iJ_k, \end{gathered}$$

Here J_E is a cohomology element supported on the elliptic fiber; its dual homology element is the base. The J_i are cohomology elements supported on curves in B, with homology dual curves in B together with their fibers. k is the "number" of sections for the various Weierstrass forms, i.e., 1 for the E_8 form $X_6(1,2,3)$, 2 for the E_7 form $X_4(1,1,2)$, 3 for the E_6 form $X_3(1,1,1)$ and 4 for the D_5 form $X_{2,2}(1,1,1,1)$; h is the dual Coxeter number associated with the groups: $h = 30, 18, 12, 8$, respectively.

From (27) and (28) we get that (were $c_1 = c_1(B)$)

$$\partial_{t_E}\mathcal{F} = t_E^2 c_1^2 + t_E \sum_i t_i(c_1 J_i) + t_i t_j (J_j J_i) + O(q)\partial_{t_i}\mathcal{F}$$

---

[15]This calculation arises in the F-theory context [20].

$$= t_E^2(c_1 J_i) + t_E \sum_j t_j (J_j J_i) + O(q)$$

and hence the unique finite combination in the $t_E \to i\infty$ limit is given by

$$\Pi_{fin} = \left(\partial_{t_E} - \sum_i x_i \partial_{t_i}\right) \mathcal{F}, \text{ with } \sum_i x_i (J_i J_j) = c_1 J_j \qquad (32)$$

We define new variables in the Mori cone $t_E = S$ and $t_i = \tilde{t}_i - x_i S$ such that $\Pi_{fin} = \partial_S \mathcal{F}|_{t_E \to i\infty}$, and by (28) we get a general form of the instanton expansion for the curves which live in the base

$$g_{i,k} := C_{S,i,k}|_{q_E=0} = J_i J_k + \frac{1}{2}(c_1 J_i + c_1 J_k) + \sum_{\{d_l\}} \left(\sum_{i=1}^{h^{1,1}(B)} -x_i d_i \right) d_j d_k N_{\vec{d}} \frac{q^{\vec{d}}}{1 - q^{\vec{d}}}, \qquad (33)$$

where the d_l run only over degrees of classes in the base. Note that $\mathcal{F}_{local} = \partial_S \mathcal{F}$ is a potential for the metric $g_{i,j}$,. For the polyhedra $2 - 14$ $\mathrm{Im}(g_{i,j})$ becomes in a suitable limit the exact gauge coupling for an $N = 2$ theory in 4 dimensions (Seiberg-Witten theory) [35] . Furthermore, all intersections in (33) are intersections in the two-dimensional base manifold and in this sense we have achieved the goal of formulating the mirror symmetry conjecture intrinsically from the geometric data of the base. It remains to construct a local Picard-Fuchs system which has $\mathcal{F}_{local} = \sum_{i,j=1}^{h^{1,1}(B)} (J_i J_j) \log z_i \log z_j + S_i \log z_i + S_0^{\cdot}$ as the unique solution quadratic in the logarithms.

As a generalization of the situation discussed in the last section, we propose the following data for local mirror symmetry: A convex $n-1$ dimensional polyhedron Δ_B, P_B its Newton polynomial,

$$\Pi_i(z_i) = \int_{\Gamma_i} \Omega,$$

$$\Omega = \int_{\gamma_0} \log(P_B)\omega : \gamma_0 : \text{contour around } P_B = 0, \quad \omega = \mathrm{d}X_1 X_1 \ldots \frac{\mathrm{d}X_{n-1}}{X_{n-1}}$$

$$z_i = \prod_{j=1}^{\#\nu} a_j^{l_j^{(i)}} : (\mathbf{C}^*)^{n+2} - \text{ invariant complex structure variables,}$$

$\{l^{(i)}\}$ a basis of linear relations among the $\bar{\nu}^{(i)} \in \bar{\Delta}_B = (1, \Delta_B)$, i.e., $\sum_{i=1}^{\#\nu} l_i^{(j)} \bar{\nu}^{(i)} = \vec{0}$ spanning the Mori cone of P_{Δ_B}.

The non-compact local geometry $T_{\Sigma(\Delta_B)}$ is the canonical bundle over $\mathbf{P}_{\Delta_B}$. It is described by the incomplete fan $\Sigma(\Delta_B)$, which is spanned in three dimensions ($n = 3$) by "extended" vectors $\bar{\nu}^{(i)} = (1, \nu^{(i)})$, with $\nu^{(i)} \in \Delta_B$ generate the lattice.

The local mirror geometry is in these cases given by an elliptic curve which is defined in coordinates as (29), w.r.t. to $(\Delta_B, \hat{\Delta}_B)$ and a meromorphic two-form Ω with nonvanishing residue. The number of independent cycles increases for the polyhedra $1 - 15$ with the number of nonvanishing residue, which in physics play the rôle of scale or mass parameters. This is in contrast to the situation in the next section, where the genus of the Riemann surface will increase and with it the number of double logarithmic solutions.

In particular, for the example discussed before we get from (29) that the mirror geometry is the elliptic curve given by the standard cubic in $\mathbf{P}^2$

$$P_B = a_1 x_1^3 + a_1 x_2^3 + a_3 x_3^3 + a_0 x_1 x_2 x_3.$$

An important intermediate step in the derivation of this form [2], which be useful later, is that the polynomial can also be represented by coordinates Y_i,

$$P_B = a_1 Y_1 + a_1 Y_2 + a_3 Y_3 + a_0 Y_0 \ , \tag{34}$$

with r relations $\prod_{i=1}^{\#\nu-1} Y_i^{l_i^{(k)}} = Y_0^{-l_0^{(k)}}$, $k = 1, \ldots, r$. The x_i are then introduced by an suitable étale map, here $Y_1 = x_1^3$, $Y_2 = x_2^3$, $Y_3 = x_3^3$ and $Y_0 = x_1 x_2 x_3$, which satisfy identically the relation(s).

The $\Pi_i(z)$ are now well-defined under $\mathbf{C}^*$-actions, up to a shift, and they satisfy directly

$$\prod_{l_i^{(k)}>0} \left(\frac{\partial}{\partial a_i}\right)^{l_i^{(k)}} \Pi = \prod_{l_i^{(k)}<0} \left(\frac{\partial}{\partial a_i}\right)^{-l_i^{(k)}} \Pi. \tag{35}$$

In particular $\Pi_0 = 1$ is always a solution. By the same procedure as indicated below (22), we now directly get the differential equation

(31). One can easily show that this differential equation has besides the constant solution a logarithmic and a double-logarithmic solution. The explicit form of the solutions can be given in general using the $l^{(i)}$, $i = 1, \dots m$ in specialized versions of the formulas which appeared in [32]:

$$\begin{aligned} \Pi_0(z) &= \sum_{\vec{n}} c(\vec{n}, \vec{\rho}) z_{\vec{n}}^{\vec{n}}|_{\vec{\rho}=0}, \quad c(\vec{n}, \vec{\rho}) = \frac{1}{\prod_i \Gamma(\sum_\alpha l_i^{(\alpha)}(n_\alpha + \rho_\alpha) + 1)}, \\ \Pi_i(z) &= \partial_{\rho_i} \Pi_0|_{\vec{\rho}=0}, \quad \Pi_{m+1} = \partial_S \mathcal{F} = \sum_{i,j} (J_i J_k) \partial_{\rho_i} \partial_{\rho_j} \Pi_0|_{\vec{\rho}=0}. \end{aligned} \tag{36}$$

The predictions for the local mirror symmetry are then obtained using (25) and (33).

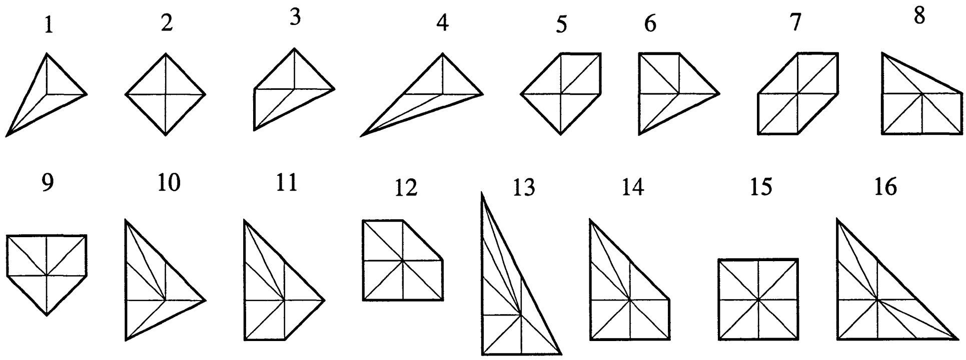

Figure 1: Reflexive Polyhedra (Δ_B) in two dimensions. $\hat{\Delta}_p = \Delta_{17-p}$ for $p = 1, \ldots, 6$. $\Delta_{7,8,9,10}$ are self-dual [43], [2][17]. Case one is the polyhedron representing the $\mathbf{P}^2$; two, three and four are the Hirzebruch surfaces $F_0 = \mathbf{P}^1 \times \mathbf{P}^1, F_1$ and F_2 and the others are various blow-ups of these cases. Note that $c_2 = \#2$ simpl. and $c_1^2 = 12 - \#2$ simpl. The labeling starts with 0 for the inner point. The point to its right is point 1 and the labels of the others increase counterclockwise.

Below we give further data for local mirror symmetry calculation for some[16] from of the polyhedra[17] in Fig. 1:

1. $l^{(1)} = (-3, 1, 1, 1), \quad \mathcal{C}_1 = 3J_1, \quad \mathcal{R} = J_1^2$
2. $l^{(1)} = (-2, 1, 0, 1, 0), \quad l^{(2)} = (-2, 0, 1, 0, 1),$
   $\mathcal{C}_1 = 2J_1 + 2J_2, \quad \mathcal{R} = J_1 J_2$
3. $l^{(1)} = (-2, 1, 0, 1, 0), \quad l^{(2)} = (-1, 0, 1, -1, 1),$
   $\mathcal{C}_1 = 3J_1 + 2J_2 \quad \mathcal{R} = J_1 J_2 + J_1^2$
4. $l^{(1)} = (-2, 1, 0, 1, 0), \quad l^{(2)} = (0, 0, 1, -2, 1),$
   $\mathcal{C}_1 = 4J_1 + 2J_2 \quad \mathcal{R} = J_1 J_2 + 2J_1^2$
5. $l^{(1)} = (-1, 1, -1, 1, 0, 0), \quad l^{(2)} = (-1, -1, 1, 0, 0, 1),$
   $l^{(3)} = (-1, 0, 1, -1, 1, 0)$

[16]In the cases we do not treat explicitly, the Mori cone is non-simplicial. This means that there are several coordinate choices for the large complex structure variables, which correspond to the simplical cones in a simplicial decomposition of the Mori cone. This is merely a technical complication. We checked that for the simplicial subcones we get consistent instanton expansions from (33).

[17]The polyhedra appeared only in the preprint version of [2] and are therefore reproduced here. We thank J. Stienstra for pointing out an omission in an earlier version.

$$\mathcal{C}_1 = 3J_1 + 2J_2 + 2J_3 \quad \mathcal{R} = J_1^2 + J_2J_1 + J_1J_3 + J_2J_3$$

6. $l^{(1)} = (-1,1,-1,1,0,0), \quad l^{(2)} = (0,0,0,1,-2,1),$
$l^{(3)} = (-1,0,1,-1,1,0)$
$\mathcal{C}_1 = 4J_3 + 2J_2 + 3J_1 \quad \mathcal{R} = J_2J_3 + 2J_3^2 + J_2J_1 + 2J_3J_1 + J_1^2$

11. $l^{(1)} = (-1,0,-1,0,0,1,1), \quad l^{(2)} = (0,0,1,0,1,-2,0),$
$l^{(3)} = (0,1,0,0,-2,1,0), \quad l^{(4)} = (0,0,1,1,0,0,-2),$
$\mathcal{C}_1 = 6J_1 + 4J_2 + 2J_3 + 3J_4$
$\mathcal{R} = 6J_1^2 + 4J_1J_2 + 2J_2^2 + 2J_1J_3 + J_2J_3 + 3J_1J_4 + 2J_2J_4 + J_3J_4J_4^2$

Here we use the short-hand notation $\mathcal{R} = \sum J_iJ_k \int_B J_iJ_k$ and $\mathcal{C}_1 = \sum_i J_i \int c_1(B)J_i$. Below are further Picard-Fuchs systems derived using (35).

$$\begin{aligned}
\mathcal{L}_1^{F_0} &= {\theta_1}^2 - 2z_1(\theta_1+\theta_2)(1+2\theta_1+2\theta_2)\\
\mathcal{L}_2^{F_0} &= {\theta_2}^2 - 2z_2(\theta_1+\theta_2)(1+2\theta_1+2\theta_2)\\
\mathcal{L}_1^{F_1} &= {\theta_2}^2 - z_1(\theta_2-\theta_1)(2\theta_1+\theta_2)\\
\mathcal{L}_2^{F_1} &= \theta_1(\theta_1-\theta_2) - z_2(2\theta_1+\theta_2)(1+2\theta_1+\theta_2)\\
\mathcal{L}_1^{F_2} &= \theta_1(\theta_1+\theta_2) - z_1 2\theta_1(2\theta_1+1)\\
\mathcal{L}_2^{F_2} &= \theta_1(\theta_1+\theta_2) - z_2 2\theta_2(2\theta_2+1)
\end{aligned}$$

In general, linear combinations of the $l^{(i)}$ may lead to to independent differential operators. For example, a complete system for the blown up F_2 (polyhedron 6) is obtained using in addition to the $l^{(i)}$, $i = 1,2,3$, the linear relations $l^{(2)} + l^{(3)}$ and

$$\begin{aligned}
\mathcal{L}_1 &= \theta_2(\theta_2-\theta_3+\theta_1) - (-2+2\theta_2-\theta_3)(-1+2\theta_2-\theta_3)z_1,\\
\mathcal{L}_2 &= (2\theta_2-\theta_3)(\theta_3-\theta_1) - (1+\theta_2-\theta_3+\theta_1)(-1+\theta_3+\theta_1)z_2,\\
\mathcal{L}_3 &= (-\theta_2+\theta_3-\theta_1)\theta_1 - (1+\theta_3-\theta_1)(-1+\theta_3+\theta_1)z_3,\\
\mathcal{L}_4 &= \theta_2(\theta_3-\theta_1) - (-1+2\theta_2-\theta_3)(-1+\theta_3+\theta_1)z_1z_2,\\
\mathcal{L}_5 &= -((2\theta_2-\theta_3)\theta_1) - (-2+\theta_3+\theta_1)(-1+\theta_3+\theta_1)z_2z_3.
\end{aligned}$$

Using similar arguments as in section four of the second reference in [32] and the calculation of toric intersections as described in [60], [22], [16], one can show that (35) and (36) implies the appearance of the intersection numbers in (33).

Concrete instanton numbers for $\mathbf{P}^2$ appear in Table 1; for K_{F_0}, K_{F_1} and K_{F_2} in the appendix; and for the canonical bundles over the geometry defined by the polyhedra P_5 and P_6 in Table 3 below.

$d_1 = 0$; d_2 \ d_3	0	1
0		1
1	1	0

$d_1 = 1$; d_2 \ d_3	0	1
0	1	−2
1	−2	3

$d_1 = 2$; d_2 \ d_3	0	1	2
0			
1		−4	5
2		5	−6

$d_1 = 3$; d_2 \ d_3	0	1	2	3
0				
1			−6	7
2		−6	35	−32
3		7	−32	27

$d_1 = 4$; d_2 \ d_3	0	1	2	3	4
0					
1				−8	9
2			−32	135	−110
3		−8	135	−400	286
4		9	−110	286	−192

$d_1 = 5$; d_2 \ d_3	0	1	2	3	4	5
0						
1					−10	11
2				−110	385	−288
3			−110	1100	−2592	1651
4		−10	385	−2592	5187	−3038
5		11	−288	1651	−3038	1695

Table 3: Invariants of dP_2(polyhedron 5). The invariants for $d_1 = 0$ sum to 2 and for all other degrees d_1 to zero. Note also that the invariants for the blow up of F_2 (polyhedon 6) are related to the above by $n^{(6)}_{k,i,i+j} = n^{(5)}_{k,i,j}$.

The K_{F_0} and K_{F_1} geometry[18] describes in the double scaling limit

[18]This is true also for K_{F_2}, which can be seen as specialization in the complex structure moduli space of the K_{F_0} case.

$N = 2$ $SU(2)$ Super-Yang-Mills theory [35]. Similarly in that limit the geometry of the canonical bundle over $\mathbf{P}_{\Delta_5}$ and $\mathbf{P}_{\Delta_6}$ describes $N = 2$ $SU(2)$ Super-Yang-Mills theory with one matter multiplet in the fundamental representation of $SU(2)$ [35].

6.4 Fibered A_n cases and more general toric grid diagrams

The fibered A_n geometry we will discuss here is motivated from physics [35]. Note that the complexified Kähler moduli of this geometry yield the vector moduli space for the Type II-A compactification and the electrically/magne tically charged BPS states come from even dimensional D-branes wrapping holomorphic curves/four-cycles. Mirror symmetry on the fiber relates it to the geometry considered in [39] for which the vector moduli space of a type II-B compactification emerges from its complex deformations.

The type II-A geometry arises when inside a Calabi-Yau space an A_n sphere tree is fibered over a $\mathbf{P}^1$. Again we consider the limit in which all other Kähler parameters of the threefold which do not control the sizes of the mentioned $\mathbf{P}^1$'s become large. In this case (33) becomes the exact gauge coupling of $SU(n+1)$ $N = 2$ Seiberg-Witten theory when one takes a double-scaling limit in which the size of the fiber $\mathbf{P}^1$ and the one of base $\mathbf{P}^1$ are taken small in a ratio described in [35]. We have already discussed the simplest cases: the Hirzebruch surfaces F_2, F_0 and F_1, which give rise to A_1 theory. Next we consider the A_n generalization of the F_2 case. We can describe it by $T_{\Sigma(\Delta_B)}$ as before (below (6.3)), but it clearly does not have the structure of a canonical bundle over a space.

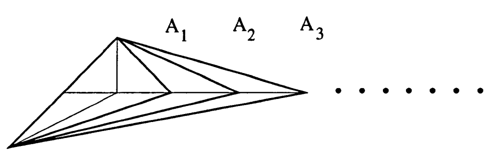

Figure 2: Toric diagrams (Δ_B) for the A_n singularity fibered over $\mathbf{P}^1$. Note that these diagrams have unique triangulations.

To obtain the local situation as a limit of a compact case, we can consider a polyhedron Δ defined analogous to $\Delta^{fibration}$ described at the beginning of subsection 6.3. It turns out that for A_2 (A_3) Δ becomes reflexive only after adding the point(s) $[1,0,1,1]$ ($[1,0,0,1],[2,0,1,2]$), giving us 2 (3) line bundles normal to the to the compact part of the local geometry. This is true more generally, i.e., one has n line bundles for A_n, but for large n it is not possible to embed the A_n singularity in a compact Calabi-Yau space. The n normal directions in the Kähler moduli space mean that the limit (32) leads to an n-dimensional linear space of finite double-logarithmic solutions whose coefficients reflect the directions in the normal Kähler moduli space in which the limit can be taken and for combinatorial reasons their number corresponds to the number of 'inner' points in the toric diagram. We denote a basis for these directions S_i, $i=1,\dots,n$. The n double-logarithmic solutions come as we will see, given the meromorphic differential (6.3), from a genus n Riemann surface for the local mirror. The Gromov-Witten invariants, on the other hand, should not depend on the direction of the limit we take to obtain Π_{fin} before matching it to the $N_{\vec{d}}$ in (33). They therefore become rather non-trivial invariants of the differential system (35). In the following, we explicitly describe for all n the solutions of this system corresponding to a preferred period basis, up to a choice of the double-logarithmic solutions.

The generators of linear relations are

$$\begin{array}{rl}
l^{(b)} & = (1,1,-2,\ 0,\ 0,\ 0,\ 0,\dots,\ 0,\ 0,\ 0,\ 0),\\
l^{(1)} & = (0,0,\ 1,-2,\ 1,\ 0,\ 0,\dots,\ 0,\ 0,\ 0,\ 0),\\
l^{(2)} & = (0,0,\ 0,\ 1,-2,\ 1,\ 0,\dots,\ 0,\ 0,\ 0,\ 0),\\
 & \qquad\qquad\qquad\quad \vdots \qquad\quad \vdots \\
l^{(n-1)} & = (0,0,\ 0,\ 0,\ 0,\ 0,\ 0,\dots,\ 1,-2,\ 1,\ 0),\\
l^{(n)} & = (0,0,\ 0,\ 0,\ 0,\ 0,\ 0,\dots,\ 0,\ 1,-2,\ 1).
\end{array}$$

Using the étale map $Y_0=zs$, $Y_1=\frac{s}{z}$, $Y_2=s$ and $Y_k=st^{k-2}$, $k=3,\dots,n+3$, solving $Y_0Y_1=Y_2^2$, $Y_iY_{i+2}=Y_{i+1}^2$, $i=2,\dots,n+1$ on the polynomial $P=\sum_{i=0}^{n+3}a_iY_i$ one gets [35]

$$P=sz+\frac{s}{z}+a_{n+1}s+a_nst+\ldots+b_0st^{n+1}\ . \tag{37}$$

This can be indentified, upon going to an affine patch $s=1$ after trivial redefinitions and the [35] limit, with the genus n $SU(n+1)$ curves for $N=2$ Super-Yang-Mills theory [40] .

To obtain the complete solutions of the period system to the local mirror geometry (37), we have to specify the classical intersection terms in the n double-logarithmic solutions $\partial_{S_i}\mathcal{F}$. Using (37),(35), the relation between the ideal of the principal part of the differential operator at the maximal unipotent point $z_i=0$ and the logarithmic solutions of GKZ-systems as in section 4 of [32], as well as some algebra, we arrive at the following structure of the general double-logarithmic solution for arbitrary n ($J_i \sim \log(z_i)$):

$$\mathcal{R} = \sum_{i=1}^{n} y_i J_i (J_b + \sum_{k=1}^{i} 2k J_k). \tag{38}$$

Here the coefficients of the y_i can be viewed as logarithmic terms of $\partial_{S_i}\mathcal{F}$. This leads by (36) to an explicit basis of solutions.

It remains to fix the x_i by requiring invariance of the $N_{\vec{d}}$ in (33) given the general solution (38) , which yields

$$x_i = M_{i,j} y_j, \; i,j = 1, \ldots, n \; , \tag{39}$$

where M is the Cartan matrix of A_n. Using this description, we can calculate the instantons for all A_n.

In the following we give some explicit numbers for A_2. We arrive at these same numbers from A-model techniques, even though this is a non-bundle case.

$d_b=0$	d_2	0	1
d_1			
0			-2
1		-2	-2

$d_b=1$	d_2	0	1	2	3	4	5	6
d_1								
1		-2	-2					
2		-4	-6	-6	$-2d_3-2$	..		
3		-6	-10	-12	-12	$-4d_3-6$	..	
4		-8	-14	-18	-20	-20	$-6d_3-12$	..
5		-10	-18	-24	-28	-30	-30	$-8d_3-20$

$d_b = 2$	d_2	0	1	2	3	4	5	6
d_1								
3		-6	-10	-12	-12	-10	$-4(d_3-1)-6$	..
4		-32	-70	-96	-110	-112	-126	-192
5		-110	-270	-416	-518	-576	-630	..

$d_b = 3$	d_2	0	1	2	3	4	5	6
d_1								
4		-8	-14	-18	-20	-20	-20	-18
5		-110	-270	-416	-518	-576	-630	..

Table 4. Gromov-Witten invariants for local A_2. For $d_3 > d_2$ we have $n_{1,d_2,d_3} = -2(d_2-1)d_3 - d_2(d_2-1)$.

For A_3:

$d_b = 0$	$[d_2, d_3]$	$[0,0]$	$[1,0]$	$[0,1]$	$[1,1]$
d_1					
0			-2	-2	-2
1		-2	-2		-2

$d_b = 1$	$[d_2, d_4]$	$[0,0]$	$[1,0]$	$[1,1]$	$[2,0]$	$[2,1]$	$[2,2]$	$[3,0]$	$[3,1]$	$[3,2]$
d_1										
1		-2	-2	-2						
2		-4	-6	-6	-6	-8	-6	-4	-6	-6
3		-6	-10	-10	-12	-16	-12	-12	-18	$-$

$d_b = 2$	$[d_2, d_3]$	$[0,0]$	$[1,0]$	$[1,1]$	$[2,0]$	$[2,1]$
d_1						
3		-6	-10	-10	-12	-16
4		-32	-70	-70	-96	$-$
5		-270	-110	$-$	$-$	$-$

Table 5. Gromov-Witten invariants for local A_3.

Note that, as expected, at $d_b = 0$ the only Gromov-Witten invariants occur at the degrees α^+ with $N_{\alpha^+} = -2$ where α^+ are the vector of positive roots in the Cartan-Weyl basis.

As a final example, we consider a toric grid diagram which admits a flop transition, describing in phase A two $\mathbf{P}^2$ connected by a $\mathbf{P}^1$. The local geometry is defined by the $\mathbf{C}^*$ operations generated by

$$
\begin{aligned}
l_A^{(1)} &= (\ 0,-1,\ 0,-1,\ 1,\ 1), & l_B^{(1)} &= (\ 0,\ 1,\ 0,\ 1,-1,-1)\\
l_A^{(2)} &= (\ 1,\ 1,\ 0,\ 1,\ 0,-3), & l_B^{(2)} &= (\ 1,\ 0,\ 0,\ 0,\ 1,-2)\\
l_A^{(3)} &= (\ 0,\ 1,\ 1,\ 1,-3,\ 0), & l_B^{(3)} &= (\ 0,\ 0,\ 1,\ 0,-2,\ 1)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{SRI}_A = \{&x_1 = x_2 = x_4 = 0, x_5 = x_6 = 0,\\
&x_2 = x_3 = x_4 = 0, x_1 = x_3 = 0,\\
&x_1 = x_5 = 0, x_3 = x_6 = 0\}\\
\mathcal{SRI}_B = \{&x_2 = x_4 = 0, x_1 = x_3 = 0,\\
&x_1 = x_5 = 0, x_3 = x_6 = 0\}
\end{aligned}
$$

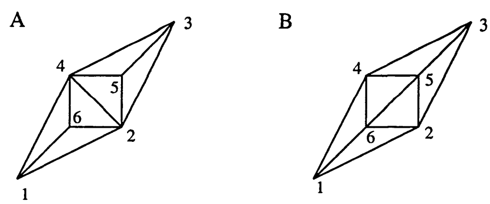

Figure 3: Two phases of a local Calabi-Yau manifold. Phase B is obtained by a flop transition.

and the indicated Stanley-Reisner ideal. As in the A_2 case the mirror geometry is given by a genus two Riemann surface. The space of double-logarithmic solutions is two dimensional and can be determined from (35). Local invariants follow then for the A and B phase via (33) from the following data

$$
\begin{aligned}
&\mathcal{R}_A = y_1 J_2^2 + y_2(J_3^2 - J_2^2), \quad x_1^A = y_1, \quad x_2^A = -3(y_1 - y_2), \quad x_3^A = -3y_2\\
&\mathcal{R}_B = y_1(2J_1J_2 + J_1J_3 + 2J_2^2 + J_3^2) + y_2(J_2^2 + J_1J_2)\\
&x_1^B = y_2, \quad x_2^B = -3y_1 - 2y_2, \quad x_3^B = -3y_1 - y_2\ .
\end{aligned}
$$

Below we list the Gromov-Witten invariants for the B phase of the $(\mathbf{P}^2, \mathbf{P}^2)$ diagram Figure 3. The instantons of the A phase are related to the one in the B phase by $n^A_{k,j,i} = n^B_{i+j-k,i,j}$. The only degree for which this formula does not apply is $n^A_{1,0,0} = 1$, which counts just the flopped $\mathbf{P}^1$.

$d_b = 0$	d_2	0	1
d_1			
0			−2
1		−2	−2

$d_b = 1$	d_2	0	1	2	3	4	5	6
d_1								
0		1	3	5	7	9	11	13
1		3	4	8	13	16	20	24
2		5	8	9	15	21	27	33
3		7	12	15	16	24	32	40
4		9	16	21	24	25	35	45
5		11	20	27	32	35	36	48

$d_b = 2$	d_2	0	1	2	3	4	5	6
d_1								
0				−6	−32	−110	−288	−644
1				−10	−70	−270	−770	−1820
2		−6	−10	−32	−126	−456	−1330	−3264
3		−32	−70	−126	−300	−784	−2052	−4928
4		−110	−270	−456	−784	−1584	−3360	−7260
5		−288	−770	−1330	−2052	−3360	−6076	−

$d_b = 3$	d_2	0	1	2	3	4	5
d_1							
0		0	0	0	27	286	1651
1		0	0	0	64	800	5184
2		0	0	25	266	1998	11473
3		27	64	266	1332	6260	26880
4		286	800	1998	6260	21070	70362

Table 6. Gromov-Witten invariants for the phase B in Fig. 3

From the examples treated so far it should be clear how to proceed for a general toric grid diagram with n inner points and m boundary points. After choosing a triangulation and a corresponding basis of the $m + n - 3$ linear relations $l^{(i)}$ one analyses the principal part of the differential system (35) to obtain a basis of the double-logarithmic solutions. This is the only additional information needed to specify the full set of the $2n+m-2$ solutions from (36) . The general structure of the solutions will be as follows. Besides the constant solution, we get for each of the n inner points of the toric diagram, whose total number equals the genus of the Riemann surface, one-single logarithmic solution and one double-logarithmic solution coming from the period integrals around a- and b-type cycles of the Riemann surface. From additional boundary points in the toric grid diagram beyond 3 we get additional single-logarithmic solutions which correpond to residues of the meromorphic form (6.3). Together with (25) the solutions determine (33) and hence the Gromov-Witten invariants, up to a choice of x_i and y_i, which represents a choice of the bases for the double-logarithmic solutions. Requiring the Gromov-Witten invariants to be independent of this choice gives a linear relation $x_i(y_j)$. This produces the exact vacuum solutions and the BPS counting functions for those five dimensional theories, as discussed in [46], which come from arbitrary grid diagrams [48].

6.5 Cases with constraints

The del Pezzo surfaces B_n can be constructed by blowing $\mathbf{P}^2$ up n times, $0 \leq n \leq 8$, in addition to $\mathbf{P}^1 \times \mathbf{P}^1$. As is well known, the case $n = 6$ can be represented as a cubic in $\mathbf{P}^3$ denoted by $X_3(1,1,1,1)$, and $n = 7,8$ are representable as degree four and six hypersurfaces in weighted projective spaces: $X_4(1,1,1,2)$ and $X_6(1,1,2,3)$, respectively. The $n = 5$ case can be represented as the degree (2,2) complete intersection in $\mathbf{P}^4$ $X_{2,2}(1,1,1,1,1)$. In addition, the quadric in $\mathbf{P}^3$, $X_2(1,1,1,1)$ is another representation of $\mathbf{P}^1 \times \mathbf{P}^1$. In the representation given for the non-local Calabi-Yau geometry as the canonical bundle over this del Pezzo surfaces below, the map $i^* : H^{1,1}(X) \to H^{1,1}(B)$ is not onto, as in the previous cases. As a consequence the Gromow-Witten invariants are a sum over curves with degree d_i in classes in $H^{1,1}(B)$ up to degree $d = \sum_i d_i$. These cases have been considered before [42], [47], [56], [55]
.

As all the weights in these representations are co-prime, the Kähler class associated to the hyperplane class of the ambient space is the only one which restricts to the surface (no exceptional divisors from the ambient space). We can recast the Chern classes of the surface and the of the canonical bundle over it in terms of this class J. This can be stated more generally for smooth complete intersections by the definition of the following formal weight or charge vectors $l^{(k)} = (d_{1(k)}, \ldots, d_{r(k)}|w_{1(k)}, \ldots, w_{s(k)})$, where $d_{i(k)}$ are the degree of the i'th polynomial, $i = 1, \ldots, p$, in the variables of the k'th weighted projective space, $k = 1, \ldots, s$. In terms of these we can express the total Chern class, by the adjunction formula, to obtain the following formal expansion

$$c = \frac{\prod_{k=1}^{s} \prod_{i=1}^{r(k)} (1 + w_{i,(k)} J_{(k)})}{\prod_{i=1}^{p} (1 + \sum_{k=1}^{s} d_{i,(k)} J_{(k)})} \frac{\prod_{i=1}^{p} \sum_{k=1}^{s} d_{j,(k)} J^{(k)}}{\prod_{k=1}^{s} \prod_{i=1}^{r(k)} w_{i,(k)}}. \tag{40}$$

Integrals over the top class for the non-compact case are formally defined by multiplying with the volume form of the normal bundle $\mathcal{V} = \prod_{k=1}^{t} \prod_{j=1}^{r} d_{j,(k)} J^{(k)}$ and picking the coefficient of $\prod_{k=1}^{t} J^{(s_{(k)}-1)}$. Similarly wedge products of c_2 with J and triple intersections are obtained.[19] We start by summarizing the weight or charge vectors for the non-compact case (the compact cases are obtained by deleting the last entry):

$$\begin{aligned} X_2(1,1,1,1) &: \; l = (-2|1,1,1,1,-1) \\ X_{2,2}(1,1,1,1,1) &: \; l = (-2,-2|1,1,1,1,1,-1) \\ X_3(1,1,1,1) &: \; l = (-3|1,1,1,1,-1) \\ X_4(1,1,1,2) &: \; l = (-4|1,1,1,2,-1) \\ X_6(1,1,2,3) &: \; l = (-6|1,1,2,3,-1). \end{aligned}$$

Using (40) we calculate for the cases in turn

$$\begin{aligned} \int J^3 &= -1,-4,-3,-2,-1 \\ \int J c_2 &= 2,-4,-6,-8,-10 \\ \int c_3 &= 4,16,24,36,60\ . \end{aligned}$$

The differential operators follow directly from (35) for the five cases they are

$$\mathcal{L}^{(1)} = \theta^3 - 4z(2\theta+1)^2\theta\ , \quad \mathcal{L}^{(2)} = \theta^3 + 4z(2\theta+1)^2\theta\ ,$$

[19]There are minor disagreements with the classical integrals equation (4.18) in [47] as well as with the normalization of the instanton numbers for the case referred to as "conifold" in [47] . These data should all follow from (40) .

$$\mathcal{L}^{(3)} = \theta^3 + 3z(3\theta+1)(3\theta+2)\theta \ , \quad \mathcal{L}^{(4)} = \theta^3 + 4z(4\theta+1)(4\theta+3)\theta \ ,$$
$$\mathcal{L}^{(5)} = \theta^3 + 12z(6\theta+1)(6\theta+5)\theta \ .$$

The principal discriminant appears in front of the highest derivative $\mathcal{L} = \Delta \frac{\mathrm{d}}{\mathrm{dz}}^3 + \ldots$, i.e., $\Delta = (1+az)$ with $a = -16, 16, 27, 64, 432$, respectively. The following properties concerning the exponents at the critical loci of the discriminants are common: the discriminat appears with Δ^r $r = -\frac{1}{6}$, as for the conifold, and the z^s appears with $s = -(12 + \int Jc_2)/12$. For these cases, the charge vectors and the topological data listed above give (using INSTANTON) the following invariants.

	$X_2(1,1,1,1)$		$X_{2,2}(1,1,1,1,1)$		$X_3(1,1,1,1)$	
d	rational	elliptic	rational	elliptic	rational	elliptic
1	−4	0	16	0	27	0
2	−4	0	−20	0	−54	0
3	−12	0	48	0	243	−4
4	−48	9	−192	5	−1728	−135
5	−240	136	960	−96	15255	−3132
6	−1356	1616	−5436	1280	−153576	62976
7	−8428	17560	33712	−14816	169086	−1187892
8	−56000	183452	−224000	160784	−20053440	21731112
9	−392040	1878664	1588160	−1688800	249740091	−391298442
10	−2859120	19027840	−11436720	−17416488	−3240109350	6985791864

	$X_4(1,1,1,2)$		$X_6(1,1,2,3)$	
d	rational	elliptic	rational	elliptic
1	56	0	252	−2
2	−272	3	−9252	762
3	3240	−224	848628	−246788
4	−58432	12042	−114265008	76413073
5	1303840	−574896	18958064400	−23436186174
6	−33255216	26127574	−3589587111852	7209650619780
7	930431208	−1163157616	744530011302420	−2232321201926988
8	−27855628544	51336812456	−165076694998001856	696061505044554012

Table 7: Gromov-Witten Invariants for local cases with constraints.

As expected from the Segre embedding of $\mathbf{P}^1 \times \mathbf{P}^1$ into $\mathbf{P}^3$ by the conic constraint, this case should correspond to the diagonal part of the local $\mathbf{P}^1 \times \mathbf{P}^1$ case, i.e., $\sum_{i+j=r} n_{i,j}^{P^1\times P^1} = n_r^{X_2(1,1,1,1)}$, which is indeed true. The Gromov-Witten invariants for the elliptic curves are calculated using the holomorphic anomaly of the topological B-model [7] . (In [38] some of them are checked using localisation.)

7 Discussion

We have established that mirror symmetry makes good sense in the local setting, with the enumerative invariants counting the effective contribution of the surface to the Gromov-Witten invariants of a would-be Calabi-Yau threefold which contains it. These invariants, defined and computed mathematically, are obtained through analyzing solutions to differential equations, as in the global case. As in previous works, we see the Seiberg-Witten curve arising from the B-model approach, if an N=2 gauge theory is geometrically engineered.

Several interesting observations were made along the way. We found, analyzing a reducible quadric, that singular surfaces pose no obstacle to defining the local invariants. Indeed the A-model should be independent of deformations; equivalently, calculations of Chern classes by sections are independent of the choice of section. Further, rather as the canonical bundle description breaks down for singular surfaces, we find in the fibered A_n examples (in which the fibered sphere-trees represent the singular surface) that the bundle structure does not appear to be necessary to proceed with the calculation. Heuristically, one can model not just the moduli space of maps as a projective variety, but in fact the whole vector bundle U_d. With intersections in the Chow ring and integration well-defined by virtue of a Thom class, the procedure seems to yield the correct results. This technique needs to be developed and made rigorous, but the numbers still agree with the B-model results in Table 4-5.

The recent work of Vafa and Gopakumar [66] introduces a new interpretation of these numbers and their analogues at higher genus. In particular, those authors count the contributions of BPS states (D-branes) in a fixed homology class (but not fixed genus) to the full string partition function, which is a sum over topological partition functions at all genera. Their calculations tell us how to organize the partition functions in order to extract integers, which represent BPS states corresponding to cohomology classes on the full moduli space of BPS states[20] and transforming under a certain $SU(2)$ action in a particular way. In genus zero, the contribution is equivalent to the Euler characteristic.

[20]This is the standard reduction to a supersymmetric sigma model on a moduli space.

At degree three in $\mathbf{P}^2$, for example, a smooth degree three polynomial is an elliptic curve, and the D-brane moduli space includes the choice of a $U(1)$ bundle over the curve, equivalently a point on the curve (if it is smooth). The choice of curves with points is shown in [66] to be a $\mathbf{P}^8$ bundle over $\mathbf{P}^2$, a space with Euler characteristic 27 (which is indeed n_3 for $\mathbf{P}^2$). The singular curves, however, should be accompanied by their compactified Jacobians as in [70], but these can in general no longer be equated with the curves themselves. Further, the compactified Jacobians of reducible curves (e.g., the cubic $XYZ = 0$ in $\mathbf{P}^2$) are particularly troublesome. Perhaps the non-compact direction in $K_{\mathbf{P}^2}$ leads to a resolution of these difficulties. It would be very interesting to mesh the Gromov-Witten and D-brane explanations of these local invariants.

Having extended traditional mirror symmetry to the non-compact case, one naturally asks whether other viewpoints of mirror symmetry make sense in the non-compact setting. Is there any kind of special-Lagrangian fibration? It is likely that, if so, the fibers would be decompactified tori, e.g., $S^1 \times S^1 \times \mathbf{R}$. [21] This would be an interesting venue to the conjectures of [63]. In particular, once the Kähler-Einstein metric of the surface is known, Calabi [8] has given a method to find a Ricci-flat metric on the total space of the canonical bundle. In [63] it is argued that not only should the total D-brane moduli space of the special-Lagrangian torus be the mirror manifold, but also that the metric of the mirror should be computable by an instanton expansion involving holomorphic discs bounding the torus. The local setting of a degenerate fiber, such as has been studied and given an explicit metric in [61], may prove an illustrative starting point (though several of us have been unable to crack this example).

What about the categorical mirror symmetry conjecture of Kontsevich? Unfortunately, few explicit descriptions are known of the derived categories of coherent sheaves over non-compact spaces (or any spaces, for that matter). In two dimensions, however, the recent work of [34] gives a description of the derived category over resolutions of A-D-E singularities in two dimensions. The fibered versions of these spaces are just what we consider in this paper. It would be extremely interesting to calculate Fukaya's category in these examples, especially as

[21] Recently, [52] has found such a fibration for canonical bundles of projective spaces.

we currently have a real dearth testing grounds for Kontsevich's ideas.

We feel that the local setting may be the best place for gleaning what's really at work in mirror symmetry and tying together our still fragmented understanding of this subject.

8 Acknowledgements

We are indebted to B. Lian, for explaining extensions of the mirror principle, and for helping us at various stages of this project; and to M. Roth, for explaining many algebro-geometric constructions and for his involvement at the early stages of our work. We thank them, as well as T. Graber, R. Pandharipande, and C. Vafa for many helpful discussions. The work of T.-M. Chiang and S.-T. Yau is supported in part by the NSF grant DMS-9709694; that of A. Klemm in part by a DFG Heisenberg fellowship and NSF Math/Phys DMS-9627351 and that of E. Zaslow in part by DE-F602-88ER-25065.

9 Appendix: Examples (A-Model)

9.1 $\mathcal{O}(4) \to \mathbf{P}^3$

This case is very similar, only we note that the rank of U_d is $4d+1$ and the dimension of moduli space is $4d$. Thus we must take the Chern class c_{4d} integrated over the moduli space. Taking the next-to-top Chern class has the following interpretation. Instead of just counting the zeros of a section, s_0, we take two sections s_0 and s_1 and look at the zeros of $s_0 \wedge s_1$, i.e., we look for points where the two sections are not linearly independent. The number of such points also has the following interpretation. Look at the $\mathbf{P}^1$ linear system generated by s_0 and s_1. If $s_0 \wedge s_1$ has a zero at a point (C, f) in moduli space, then some section $as_0 + bs_1$ vanishes identically on $f(C)$, i.e., $f(C)$ maps to the zero locus of $as_0 + bs_1$. Therefore, the interpretation of the next-to-top Chern class is as the number of rational curves in any member of the linear system generated by two linearly independent sections.

Of course, in this problem, we must account for multiple covers as above. After doing so, the numbers n_d which we get are as follows:

d	n_d
1	320
2	5016
3	144192
4	5489992
5	247634752
6	12472771416
7	678995926336
8	39156591980232
9	2360737982593344
10	147445147672907352

Table 8. Gromov-Witten invariants for a K_3 surface inside a Calabi-Yau threefold.

Practically speaking, sections are simply quartic polynomials, and the zero loci are quartic K3 surfaces. Therefore, we are counting the number of rational curves in a pencil of quartic K3 surfaces. We may wish to compare our results with a Calabi-Yau manifold admitting a K3 fibration. Of course, the number of curves will depend on the nature of the fibration. Our count pertains to a trivial total family, i.e., the zero locus of a polynomial of bi-degree $(1,4)$ in $\mathbf{P}^1 \times \mathbf{P}^3$. The degree 8 hypersurface Calabi-Yau manifold in $\mathbf{P}_{2,2,2,1,1}$ is a pencil of quartic surfaces fibered over $\mathbf{P}^1$ in a different way, though the counting differs only by a factor of two. Specifically, if we look at the Gromov-Witten invariants in the homology class of d times the fiber (for this example $h^{11} = 2$, one class coming from the $\mathbf{P}^1$ base, one from the projective class of the K3 fiber), we get twice the numbers computed above.

9.2 $\mathcal{O}(3) \to \mathbf{P}^2$

In this example, the rank of U_d is now $3d+1$ which is two greater than the dimension of $\mathcal{M}_{00}(d, \mathbf{P}^2)$, so we must take $c_{(\mathrm{top}-2)}(U_d)$. The interpretation is similar to the case of $\mathcal{O}(4) \to \mathbf{P}^3$. We count the number of rational curves in a two-dimensional family of cubic curves generated by three linearly independent sections of $\mathcal{O}(3)$ (cubics).

The numbers n_d are: $n_1 = 21, n_2 = 21, n_3 = 18, n_4 = 21, n_5 = 21, n_6 = 18$, and so on, repeating these three values (as far as we have computed them explicitly). Some of the numbers can be easily verified. For example, the space of cubics on $\mathbf{P}^2$ (three variables) forms (modulo scale) a $\mathbf{P}^9$, while conics form a $\mathbf{P}^5$ and lines form a $\mathbf{P}^2$. In order for a cubic curve to admit a line, the polynomial must factor into a linear polynomial times a conic. To count the number of cubics in a $\mathbf{P}^2$ family which do so, we must look at the intersection of $\mathbf{P}^2 \subset \mathbf{P}^2$ with the image V of $m : \mathbf{P}^2 \times \mathbf{P}^5 \to \mathbf{P}^9$, which is just the map of multiplication of polynomials. The Poincaré dual of the $\mathbf{P}^2$ family is just H^7, where H is the hyperplane class. Therefore, we wish to compute $\int_V H = \int_{\mathbf{P}^2 \times \mathbf{P}^5} m^*(H)$. Now the map m is linear in each of the coefficients (of the line and the polynomial of the conic), so we have $m^*(H) = H_1 + H_2$, where the H_i are the hyperplane classes in $\mathbf{P}^2$ and $\mathbf{P}^5$, respectively. The integral just picks up the coefficient of $H_1^2 H_2^5$ in $(H_1 + H_2)^7$, which is 21. Note that the same analysis applies to n_2, since we have already computed the number of cubics factoring into a conic (times a line).

To compute n_3 one needs more information about the discriminant locus of the $\mathbf{P}^2$ family. Similar calculations have been done in [10], where the authors consider a Calabi-Yau manifold which is a fibered by elliptic curves over a two-dimensional base. The numbers differ from the ones we have computed, since the fibration structure is different. Nevertheless, the repeating pattern of three numbers survives.

9.3 K_{F_n}

"Local mirror symmetry" of canonical bundle of Hirzebruch surfaces can also be computed. The results are as follows:

d_B \ d_F	0	1	2	3	4	5	6
0		−2	0	0	0	0	0
1	−2	−4	−6	−8	−10	−12	−14
2	0	−6	−32	−110	−288	−644	−1280
3	0	−8	−110	−756	−3556	−13072	−40338
4	0	−10	−288	−3556	−27264	−153324	−690400
5	0	−12	−644	−13072	−153324	−1252040	−7877210
6	0	−14	−1280	−40338	−690400	−7877210	−67008672

Table 9. Invariants of $K_{\mathbf{F}_0}$ (B and F denote the $\mathbf{P}^1$'s).

d_B \ d_F	0	1	2	3	4	5	6
0		−2	0	0	0	0	0
1	1	3	5	7	9	11	13
2	0	0	−6	−32	−110	−288	−644
3	0	0	0	27	286	1651	6885
4	0	0	0	0	−192	−3038	−25216
5	0	0	0	0	0	1695	35870
6	0	0	0	0	0	0	−17064

Table 10. Invariants of $K_{\mathbf{F}_1}$ (B and F denote the base and fiber class respectively).

The numbers for $d_B = d_F$ in the above table are the same as that for $K_{\mathbf{P}^2}$. As $\mathbf{F}_1$ is the blowup of $\mathbf{P}^2$ at a point and the homology class of a line in $\mathbf{P}^2$ pulled back to $\mathbf{F}_1$ is $B + F$, this is what we expect.

d_B \ d_F	0	1	2	3	4	5	6
0		-2	0	0	0	0	0
1	$-1/2$	-2	-4	-6	-8	-10	-12
2	0	0	0	-6	-32	-110	-288
3	0	0	0	0	-8	-110	-756
4	0	0	0	0	0	-10	-288
5	0	0	0	0	0	0	-12
6	0	0	0	0	0	0	0

Table 11. Invariants of $K_{\mathbf{F}_2}$ (B and F denote the base and fiber class respectively).

We do not understand the result of $-1/2$ above, but it reflects the fact that the moduli space of stable maps into the base, which is a curve of negative self-intersection, is not convex. Therefore the A-model calculation is suspect and we consider the B-model result of 0 for this invariant to be the right answer. (For higher degree maps into the base, the A- and B-model results agree again.)

References

[1] P. Aspinwall and R. Donagi, *The Heterotic String, the Tangent Bundle, and Derived Categories*, hep-th/9806094.

[2] V. Batyrev, *Dual Polyhedra and Mirror Symmetry for Calabi-Yau Hypersurfaces in Toric Varieties*, J. Algebraic Geom., **3** (1994), 493-535.

[3] V. Batyrev, *Variations of the Mixed Hodge Structure of Affine Hypersurfaces in Algebraic Tori*, Duke Math. Jour., **69(2)** (1993), 349.

[4] V. Batyrev and L. Borisov, *On Calabi-Yau Complete Intersections in Toric Varieties*, in "Higher-dimensional Complex Varieties", (Trento, 1994), 39-65, de Gruyter (Berlin), 1996.

[5] K. Becker, M. Becker, D.R. Morrison, H. Ooguri, Y. Oz, and Z. Yin, *Supersymmetric Cycles in Exceptional Holonomy Manifolds and Calabi-Yau Fourfolds*, Nucl. Phys., **B480** (1996), 225-238.

[6] K. Behrend and B. Fantechi, *The Intrinsic Normal Cone*, math.AG/9601010.

[7] M. Bershadsky, S. Cecotti, H. Ooguri, C. Vafa (with an appendix by S. Katz), *Holomorphic Anomalies in Topological Field Theories*, Nucl. Phys., **B405** (1993), 279-304.

[8] E. Calabi, *Métriques Kählèriennes et Fibrés Holomorphiques*, Ann. Scient. Éc. Norm. Sup., **12** (1979), 269-294.

[9] P. Candelas, A. Font, S. Katz, and D.R. Morrison, *Mirror Symmetry for Two Parameter Models – II*, Nucl. Phys., **B429** (1994), 626-674.

[10] P. Candelas, X. de la Ossa, A. Font, S. Katz, and D.R. Morrison, *Mirror Symmetry for Two Parameter Models – I*, Nucl.Phys., **B416** (1994), 481-538.

[11] P. Candelas, X.C. De La Ossa, P.S. Green, L. Parkes, *A Pair of Calabi-Yau Manifolds as an Exactly Soluble Superconformal Theory*, Nucl.Phys., **B359** (1991), 21-74.

[12] S. Cecotti and C. Vafa, *Topological Anti-Topological Fusion*, Nucl. Phys., **B367** (1991), 359-461.

[13] S. Cecotti and C. Vafa, *On Classification of $N = 2$ Supersymmetric Theories*, Commun. Math. Phys., **158** (1993), 569-644.

[14] D. Cox, *The Homogeneous Coordinate Ring of a Toric Variety*, J. Alg. Geom., **4** (1995) 17; math.AG/9206011.

[15] D.A. Cox and S. Katz, *Mirror Symmetry and Algebraic Geometry*, AMS, Providence, RI USA, 1999, to appear.

[16] V.I. Danilov, *The Geometry of Toric Varieties*, Russian Math. Surveys,**33** (1978), 97.

[17] B. Dubrovin, *Geometry of 2D Topological Field Theories,* Lecture Notes in Math., **1620** (1996), 120-348.

[18] C. Faber, *Algorithm for Computing Intersection Numbers on Moduli Spaces of Curves, with an Application to the Class of the Locus of the Jacobians*, math.AG/9706006.

[19] For several articles on the subject of helices, see Rudakov, A. N. et al, *Helices and Vector Bundles.* Seminaire Rudakov, London Mathematical Society Lecture Note Series, **148**, Cambridge University Press (Cambridge), 1990.

[20] R. Friedman, J. Morgan, E. Witten, *Vector Bundles and F Theory*, Commun. Math. Phys., **187** (1997), 679-743, Section 7; `hep-th 9701162`.

[21] W. Fulton, *Intersection Theory*, Second edition, Springer-Verlag (Berlin), 1998.

[22] W. Fulton, *Introduction to Toric Varieties*, Princeton Univ. Press, **131**, (Princeton), 1993.

[23] I.M. Gel'fand, M. Kapranov, A. Zelevinsky, *Multidimensional Determinants, Discriminants and Resultants*, Birkhäuser (Boston), 1994;
L. Billera, J. Filiman and B. Sturmfels, *Constructions and Complexity of Secondary Polytopes*, Adv. Math., **83** (1990), 155-17.

[24] Geometry and Analytic Theory of Frobenius Manifolds, math.AG/9807034.

[25] A. Givental, *A Mirror Theorem for Toric Complete Intersections*, Topological Field Theory, Primitive Forms and Related Topics (Kyoto, 1996), Prog. Math., **160**, 141-175; math.AG/9701016.

[26] T. Graber, *private communication.*

[27] T. Graber and R. Pandharipande, *Localization of Virtual Classes*, math.AG/9708001.

[28] B.R. Greene, D.R. Morrison, M.R. Plesser, *Mirror Manifolds in Higher Dimension*, Commun. Math. Phys., **173** (1995), 559-598.

[29] P. Griffiths, *On the Periods of certain Rational Integrals*, Ann. Math., **90** (1969), 460.

[30] D. Guzzetti, *Stokes Matrices and Monodromy for the Quantum Cohomology of Projective Spaces*, preprint SISSA 87/98/FM.

[31] F.R. Harvey and H.B. Lawson, *Calibrated Geometries*, Acta Math., **148** (1982), 47;
F.R. Harvey, *Spinors and Calibrations,* Academic Press, New York, 1990.

[32] S. Hosono, A. Klemm, S. Theisen and S.T. Yau, *Mirror Symmetry, Mirror Map and Applications to Calabi-Yau Hypersurfaces*, Commun. Math. Phys., **167** (1995) 301-350, `hep-th/9308122`; and *Mirror Symmetry, Mirror Map and Applications to Calabi-Yau Hypersurfaces*, Nucl. Phys., **B433** (1995), 501-554, `hep-th/9406055`.

[33] S. Kachru, A. Klemm, W. Lerche, P. Mayr, and C. Vafa, *Nonperturbative Results on the Point Particle Limit of N=2 Heterotic String*, Nucl. Phys., **B459** (1996), 537.

[34] M. Kapranov and E. Vasserot, *Kleinian Singularities, Derived Categories, and Hall Algebras*, math.AG/9812016.

[35] S. Katz, A. Klemm, and C. Vafa, *Geometric Engineering of Quantum Field Theories*, Nucl. Phys., **B497** (1997), 173-195.

[36] S. Katz, A. Klemm, and C. Vafa, *Geometric Engineering of Quantum Field Theories*, Nucl. Phys., **B497** (1997), 173-195, `hep-th/9609239`; and
A. Klemm, P. Mayr, and C. Vafa, *BPS States of Exceptional Non-Critical Strings*, `hep-th/9607139`.

[37] S. Katz, P. Mayr, and C. Vafa, *Mirror Symmetry and Exact Solution of 4D N=2 Gauge Theories - I*, Adv. Theor. Math. Phys., **1** (1997), 53-114.

[38] A. Klemm, *Mirror Symmetry at Higher Loops: A Precision Test for M-theory/IIA Duality*, to appear.

[39] A. Klemm, W. Lerche, P. Mayr, C. Vafa, and N. Warner, *Selfdual Strings and N=2 Supersymmetric Field Theory*, Nucl. Phys., **B477** (1996), 746-766, `hep-th/9604034`.

[40] A. Klemm, W. Lerche, S. Theisen, and S. Yankielowicz, *Simple Singularities and N=2 Supersymmetric Yang-Mills Theory*, Phys. Lett., **B344** (1995), 169;
A. Klemm, W. Lerche, S. Theisen, *Nonperturbative Effective Actions of $N = 2$ Supersymmetric Gauge Theories*, J. Mod. Phys., **A11** (1996), 1929-1974.

[41] A. Klemm, B. Lian, S.-S. Roan, S.-T. Yau. *Calabi-Yau fourfolds for M- and F-Theory compactifications*, Nucl.Phys., **B518** (1998), 515-574.

[42] A. Klemm, P. Mayr, and C. Vafa, *BPS States of Exceptional Non-Critical Strings*, Nucl. Phys., **B** (Proc. Suppl.), **58** (1997), 177-194; hep-th/9607139.

[43] R.J. Koelman, *A criterion for the ideal of a projectively embedded toric surface to be generated by quadrics*, Beiträge Algebra Geom., **34**(**1**) (1993), 57-62.

[44] M. Kontsevich, *Enumeration of Rational Curves via Torus Actions*, in "The Moduli Space of Curves", Dijkgraaf et al eds., Progress in Mathematics, **129**, Birkhäuser (Boston), 1995.

[45] M. Kontsevich, *Homological Algebra of Mirror Symmetry*, Proceedings of the 1994 International Congress of Mathematicians, **I**, Birkäuser, Zürich, 1995, p. 120; math.AG/9411018.

[46] A. Lawrence and N. Nekrasov, *Instanton sums and five-dimensional theories*, Nucl. Phys., **B513** (1998), 93.

[47] W. Lerche, P. Mayr, and N.P. Warner, *Non-Critical Strings, Del Pezzo Singularities and Seiberg-Witten Curves*, hep-th/9612085.

[48] N.C. Leung and C. Vafa, *Branes and Toric Geometry*, Adv. Theor. Math. Phys., **2** (1998), 91-118, hep-th/9711013.

[49] J. Li and G. Tian, *Virtual Moduli Cycles and Gromov-Witten Invariants of Algebraic Varieties*, J. Amer. Math. Soc., **11**(**1**) (1998), 119-174.

[50] B. Lian, K. Liu, and S.-T. Yau, *Mirror Principle I*, Asian J. of Math., **1**(**4**) (1997), 729-763; math.AG/9712011.

[51] B. Lian, K. Liu, and S.-T. Yau, *Mirror Principle II*, in preparation.

[52] P. Lu, *Special Lagrangian Tori on a Borcea-Voisin Threefold*, math.DG/9902063.

[53] P. Mayr, $N = 1$ *Superpotentials and Tensionless Strings on Calabi-Yau Four-Folds*, Nucl.Phys., **B494** (1997), 489-545.

[54] R. McLean, *Deformations of Calibrated Submanifolds*, Duke preprint 96-01: www.math.duke.edu/preprints/1996.html.

[55] J.A. Minahan, D. Nemechansky, C. Vafa, and N.P. Warner, *E-Strings and* $N = 4$ *Topological Yang-Mills Theories*, Nucl. Phys., **B527** (1998), 581-623.

[56] J.A. Minahan, D. Nemechansky and N.P. Warner, *Partition function for the BPS States of the Non-Critical* E_8 *String*, Adv. Theor. Math.Phys., **1** (1998), 167-183; `hep-th/9707149` and *Investigating the BPS spectrum of the Non-Critical* E_n *strings*, Nucl.Phys., **B508** (1997), 64-106; `hep-th/9705237`.

[57] D. Morrison, *The Geometry Underlying Mirror Symmetry*, math.AG/9608006;
M. Gross and P. Wilson, *Mirror Symmetry via 3-tori for a Class of Calabi-Yau Threefolds*, to appear in Math. Ann., math.AG/9608009;
B. Acharya, *A Mirror Pair of Calabi-Yau Fourfolds in Type II String Theory*, Nucl. Phys., **B524** (1998), 283-294, `hep-th/9703029`;
N.C. Leung and C. Vafa, *Branes and Toric Geometry*, Adv. Theor. Math. Phys., **2** (1998), 91-118, `hep-th/9711013`;
N. Hitchin, *The Moduli Space of Special Lagrangian Submanifolds*, math.DG/9711002.

[58] D. Morrison, *Where is the large Radius Limit?* Int. Conf. on Strings 93, Berkeley; `hep-th/9311049`.

[59] D.R. Morrison and C. Vafa, *Compactifications of F-Theory on Calabi-Yau threefolds – II*, Nucl. Phys., **B476** (1996), 437-469; `hep-th/9603161`.

[60] T. Oda, *Convex Bodies and Algebraic Geometry, An Introduction to the Theory of Toric Varieties*, Ergennisse der Mathematik und ihrer Grenzgebiete, 3; Folge, Bd., **15**, Springer-Verlag (Berlin), 1988.

[61] H. Ooguri and C. Vafa, *Summing up Dirichlet Instantons*, Phys. Rev. Lett., **77** (1996), 3296-3298.

[62] R. Pandharipande, *Hodge Integrals and Degenerate Contributions*, math.AG/9811140.

[63] A. Strominger, S.-T. Yau, and E. Zaslow, *Mirror Symmetry is T-Duality*, Nuclear Physics, **B479** (1996), 243-259; hep-th/9606040.

[64] C. Vafa, *Gas of D-Branes and Hagedorn Density of BPS States*, Nucl. Phys., **B463** (1996), 415-419; hep-th/9511088.

[65] C. Vafa, *Extending Mirror Conjecture to Calabi-Yau with Bundles*, hep-th/9804131.

[66] C. Vafa and R. Gopakumar, *M-theory and Topological Strings-I & II*, hep-th/9809187, hep-th/9812127.

[67] I. Vainsencher, *Enumeration of n-fold Tangent Hyperplanes to a Surface*, J. Algebraic Geom., **4** 1995, 503-526, math.AG/9312012.

[68] E.Witten, *Two-dimensional Gravity and Intersection Theory on Moduli Space*, Surveys in Differential Geometry, **1** (1991), 243-310.

[69] E. Witten, *Phases of $N = 2$ Theories in Two Dimensions*, Nucl. Phys., **B403** (1993), 159.

[70] S.-T. Yau and E. Zaslow, *BPS States, String Duality, and Nodal Curves on K3*, Nucl. Phys., **B471** (1996), 503-512; hep-th/9512121.

[71] S.-T. Yau and E. Zaslow, *BPS States as Symplectic Invariants from String Theory*, in "Geometry and Physics", Proceedings of the Special Session on Geometry and Physics, Aarhus, Denmark, 1996.

[72] E. Zaslow, *Solitons and Helices: The Search for a Math-Physics Bridge*, Commun. Math. Phys., **175** (1996), 337-375.