Standard-model bundles

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Abstract

We describe a family of genus one fibered Calabi-Yau threefolds
with fundamental group \( \mathbb{Z}/2 \). On each Calabi-Yau \( Z \) in the family we
exhibit a positive dimensional family of Mumford stable bundles whose
symmetry group is the Standard Model group \( SU(3) \times SU(2) \times U(1) \)
and which have \( c_3 = 6 \). We also show that for each bundle \( V \) in our
family, \( c_2(Z) - c_2(V) \) is the class of an effective curve on \( Z \). These
conditions ensure that \( Z \) and \( V \) can be used for a phenomenologically
relevant compactification of Heterotic M-theory.

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1 Introduction

In this paper we construct a particular class of bundles with constrained Chern classes on certain non-simply connected Calabi-Yau threefolds. These bundles are instrumental in deriving the Standard Model of particle physics in the context of the Heterotic M-theory [DOPWb]. Bundles of this type have been the subject of active research for quite some time [TY87], [Kac95], [PR99], [ACK], [DOPWc], [Tho]. In contrast with the classical constructions [TY87], [Kac95], [PR99], where the bundles obtained are associated with the tangent bundle of the Calabi-Yau manifold and tend to be rigid, our examples are independent of the geometry of the tangent bundle and vary in families. In particular we construct infinitely many positive dimensional families of bundles which are suitable for phenomenologically relevant compactifications of Heterotic M-theory. Our construction takes place entirely within the realm of algebraic geometry. The physical implications of our results are discussed in the companion paper [DOPWb], which also contains a summary of the construction written for physicists. In the remainder of this introduction we give a brief overview of the physical motivation for our work, followed by an outline of the actual geometric construction.

The search for exotic principal bundles on Calabi-Yau threefolds is motivated by string theory. To compactify the $E_8 \times E_8$-Heterotic string to four dimensions one prescribes:

- a Calabi-Yau 3-fold $Z$;
- a Ricci flat Kähler metric on $Z$ with a Kähler form $\omega$;
- an $\omega$-instanton $\mathcal{E} \to Z$ with a structure group $E_8 \times E_8$.

The Hermit-Einstein connection on $\mathcal{E}$ is a vacuum of the Heterotic string theory. The moduli space of $\mathcal{E}$'s is a subspace of of the moduli space of vacua for the Heterotic string. In view of the Uhlenbeck-Yau theorem [UY86] every such $\mathcal{E}$ can be identified with an algebraic $E_8 \times E_8$-bundle on $Z$ which is Mumford polystable with respect to the polarization $\omega$. In view of this theorem one can use algebraic geometry to study the moduli space of Heterotic vacua.

The type of bundles $\mathcal{E}$ allowed in a Heterotic compactification is restricted in physics in three ways:

(Supersymmetry preservation) $\mathcal{E}$ has to be Mumford polystable.
\textbf{(Anomaly cancellation)} \( c_2(\mathcal{E}) = c_2(Z) \).

\textbf{(Gauge symmetries)} If the compactification of the Heterotic string has a group of symmetries \( G \subset E_8 \times E_8 \), then the structure group of \( \mathcal{E} \) can be reduced to the centralizer \( G' \) of \( G \) in \( E_8 \times E_8 \). Furthermore the corresponding \( G' \) bundle \( \mathcal{E}_{G'} \to Z \) should also be supersymmetric and anomaly-free.

Using these three principles one can look for special compactifications of the Heterotic string that reproduce in their low energy limits well understood and experimentally confirmed quantum field theories. Of particular interest are compactifications that will lead to the Standard Model of particle physics. For such compactifications one imposes two additional requirements on the triple \((Z, \mathcal{E}, \omega)\):

\textbf{(Standard Model gauge symmetries)} The group \( G \) of symmetries of \( \mathcal{E} \), i.e. the centralizer inside \( E_8 \times E_8 \) of a minimal subgroup \( G' \subset E_8 \times E_8 \) to which the structure group of \( \mathcal{E} \) reduces, is \( G = U(1) \times SU(2) \times SU(3) \).

\textbf{(3-generations condition)}

\[
\chi(\text{ad}(\mathcal{E})) = \frac{c_3(\text{ad}(\mathcal{E}))}{2} = 3.
\]

Triples \((Z, \omega, \mathcal{E})\) satisfying the five conditions above are hard to come by and tend to be rigid (see e.g. [Kac95]). However the recent advances in string theory prompted by the ground breaking work of Hořava-Witten [HW96b, HW96a, Wit96] on orbifold compactifications of M-theory, allow for a significant relaxation of the anomaly cancellation condition. This leads to two essential simplifications. First, it turns out that using the Hořava-Witten mechanisms one can suppress completely one copy of \( E_8 \) in the structure group of \( \mathcal{E} \). Secondly it was argued in [DLOW99, DOW99, ACK] that one can use M-theory 5-branes to relax the equality in the anomaly cancellation condition to an inequality. This leads to the following purely mathematical problem.

\textbf{Main Problem.} Find a smooth Calabi-Yau 3-fold \((Z, \omega)\) and a reductive subgroup \( G' \subset E_8 \) so that

\( \triangleright \) the centralizer \( G \) of \( G' \) in \( E_8 \) is a group isogenous to \( SU(3) \times SU(2) \times U(1) \);
there exists an ω-stable $G'_C$-bundle $V \to Z$ so that

- $c_1(V) = 0$,
- $c_2(Z) - c_2(V)$ is the class of an effective reduced curve on $Z$,
- $c_3(V) = 6$.

Here the Chern classes of $V$ are calculated in the adjoint representation of $E_8$ considered as a representation of $G'$. In fact for the physics applications it suffices for $G$ to contain a group isogenous to $SU(3) \times SU(2) \times U(1)$ as a direct summand.

The groups $G' \subset E_8$ whose centralizer contains $SU(3) \times SU(2) \times U(1)$ as a direct summand can be classified. It turns out that there are no connected subgroups $G'$ with $Z_{E_8}(G') = SU(3) \times SU(2) \times U(1)$. The stability assumption on $V$ guarantees that the the structure group of $V$ can not be reduced to a proper subgroup of $G'_C$. Therefore the structure group of the associated $\pi_0(G'_C)$-bundle $V \times_{G'_C} \pi_0(G'_C)$ can not be reduced to a proper subgroup of $\pi_0(G'_C)$. Since $V \times_{G'_C} \pi_0(G'_C)$ is a Galois cover of $Z$ with Galois group $\pi_0(G'_C)$, this just means that there should be a surjective homomorphism $\pi_1(Z) \to \pi_0(G'_C)$ and so we are forced to work with a non-simply connected $Z$.

Some possible choices for $G'$ are: $SU(3) \times (Z/6)$, $SU(4) \times (Z/3)$ and $SU(5) \times (Z/2)$. The corresponding centralizers are isogenous to $(SU(3) \times SU(2) \times U(1)) \times U(1) \times U(1)$, $(SU(3) \times SU(2) \times U(1)) \times U(1)$ and $SU(3) \times SU(2) \times U(1)$. When $G'_0$, the connected component of the identity in $G'$, is a classical group, it turns out that the Chern classes of $V$ in the fundamental representation of $G'_0$ coincide with the Chern classes of $V$ in the adjoint representation of $E_8$.

In this paper we explain how to build a big family of solutions of the 
Main Problem
above for $G' = SU(5) \times (Z/2)$.

For concreteness we look for $Z$'s with $\pi_1(Z) = Z/2$. Let $V$ be an $SL(5, \mathbb{C}) \times (Z/2)$-bundle on such a $Z$. Then $V$ splits as a product of a rank five vector bundle and the unique non-trivial local system on $Z$ with monodromy $Z/2$. Pulling back this vector bundle to the universal cover $X$ of $Z$ we get a rank five vector bundle on $X$ which is invariant under the action of $\pi_1(Z)$ on $X$. Conversely every $\pi_1(Z)$-equivariant vector bundle $V \to X$ descends to a vector bundle on $Z$. Thus, in order to solve the Main
**Problem**, it suffices to construct a quadruple \((X, \tau_X, H, V)\) such that the following conditions hold:

(Z/2) \(X\) is a smooth Calabi-Yau 3-fold and \(\tau_X : X \to X\) is a freely acting involution. \(H\) is a fixed ample line bundle (Kähler structure) on \(X\).

(S) \(V\) is an \(H\)-stable vector bundle of rank five on \(X\).

(I) \(V\) is \(\tau_X\)-invariant.

(C1) \(c_1(V) = 0\).

(C2) \(c_2(X) - c_2(V)\) is effective.

(C3) \(c_3(V) = 12\).

Since we need a mechanism for constructing bundles on \(X\), we will choose \(X\) to be elliptically fibered and use the so-called *spectral construction* \([FMW97, FMW99, Don97, BJPS97]\) to produce bundles on \(X\). Note that the spectral construction applies only to elliptic fibrations, i.e., genus one fibrations with a section. This is the reason we build an equivariant \(V\) on \(X\) rather than obtaining directly \(V\) on \(Z\). In general, there are two ways in which the spectral construction can be modified to work on genus one fibrations such as \(Z\). One is to work with a spectral cover in the Jacobian fibration of \(Z\) and an abelian gerbe on it. The other route (which is the one we chose) is to work with equivariant spectral data on the universal cover of \(Z\). Note that there are higher algebraic structures involved in both approaches: the stackiness of the first approach is paralleled by complicated group actions on the derived category in the second.

More specifically we take \(X\) to be a Calabi-Yau of Schoen type \([Sch88]\), i.e., a fiber product of two rational elliptic surfaces \(B\) and \(B'\) over \(\mathbb{P}^1\), both in the four dimensional family described in \([DOPWa]\). The rank five bundle \(V\) is built as an extension of two vector bundles \(V_2\) and \(V_3\) of ranks two and three respectively. Each of these is manufactured by the spectral construction. Alternatively \(V\) may be viewed as a bundle corresponding to spectral data with a reducible spectral cover. Our preliminary research of this problem (some of which is recorded in \([DOPWc]\)) showed that bundles corresponding to smooth spectral covers are unlikely to satisfy all of the above conditions. In fact, for the Calabi-Yau’s we consider, one can show
rigorously (see Remark 2.3) that $V$'s coming from smooth spectral covers can never satisfy (I), (C1) and (C3) at the same time.

The paper is organized as follows. Section 2 describes the construction of $X$ and lists the geometric constraints on the spectral data which will ensure the validity of (I). Section 3 deals with the actual construction. We describe $V_2$ and $V_3$ in terms of their spectral data. The data for each $V_i$ involves a spectral curve $C_i$ in the surface $B$, a line bundle $N_i$ on $C_i$, another line bundle $L_i$ on the surface $B'$, and some optional parameters. The effect of taking these additional parameters to be non-zero is interpreted in section 3.2 as a series of Hecke transforms. The freedom to perform these Hecke transforms gives us at the end of the day infinitely many families of bundles. In section 4 we explain how the geometric information about the action of the spectral involution, obtained in [DOPWa, Theorem 7.1], takes care of condition (I). A delicate point here is that we need two genericity assumptions on $C_i$. The first one is that $C_i$ is finite over the base of the elliptic fibration on $B$. The second assumption is that $\text{im}[\text{Pic}(B) \to \text{Pic}(C_i)]$ is Zariski dense in $\text{Pic}^0(C_i)$. In sections 4.2 and 4.3 we check these two assumptions in the special case that is ultimately utilized in the construction of $V$. In section 5 we translate the remaining conditions into a sequence of rather tight numerical inequalities. In Section 5.4 we show how the latter can be solved. In Section 6 we summarize the construction of $(X, \tau_X, H, V)$ and give an estimate on the dimension of the moduli space of such quadruples. Finally in Appendix A we have gathered some basic facts on Hecke transforms of vector bundles which are used in Section 3.

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2 Elliptic Calabi-Yau threefolds with free involutions

Our goal is to construct special $SU(5)$ bundles on smooth Calabi-Yau 3-folds with fundamental group $\mathbb{Z}/2$. We construct our Calabi-Yau 3-fold $Z$ as the quotient of a smooth Calabi-Yau 3-fold $X$ by a freely acting involution $\tau_X : X \to X$. Our $X$ will be elliptic and the elliptic fibration will be preserved by $\tau_X$, so that $Z$ will still have a genus one fibration. This enables us to apply the spectral construction to produce bundles.

The manifold $X$ is constructed as the fiber product $B \times_{\mathbb{P}^1} B'$ of two rational elliptic surfaces $B$ and $B'$ which live in the four dimensional family described in [DOPWa, Section 4]. For the first surface $B$ we use the notation from [DOPWa]. In particular we have $\beta : B \to \mathbb{P}^1$, $\epsilon, \zeta : \mathbb{P}^1 \to B$ and the involutions $\alpha_B, \tau_B : B \to B$ and $\tau_{\mathbb{P}^1} : \mathbb{P}^1 \to \mathbb{P}^1$. We use the same symbols with primes for the corresponding objects on $B'$.

We choose an isomorphism of $\mathbb{P}^1$ with $\mathbb{P}^1'$ which identifies $\tau_{\mathbb{P}^1}$ with $\tau_{\mathbb{P}^1}'$ and sends $0 \in \mathbb{P}^1$ to $\infty' \in \mathbb{P}^1'$ and $\infty \in \mathbb{P}^1$ to $0' \in \mathbb{P}^1'$. With this convention we will make no distinction between $\mathbb{P}^1$ and $\mathbb{P}^1'$ from now on.

Define $X := B \times_{\mathbb{P}^1} B'$. For a generic choice of $B$ and $B'$ this $X$ will be smooth. It is an elliptic 3-fold in two ways: via its projections $\pi : X \to B'$ and $\pi' : X \to B$. Since most of our analysis will involve the elliptic fibers we will work with the elliptic structure $\pi : X \to B'$ in order to avoid cumbersome notation. By construction the discriminant of $\pi$ is in the linear system $\beta^* O_{\mathbb{P}^1}(12) = -12 K_{B'}$ and so $X$ is a Calabi-Yau 3-fold.

For the zero section of $\pi$ we take the section $\sigma : B' \to X$ corresponding to $\epsilon : \mathbb{P}^1 \to B$. Let $\alpha_X := \alpha_B \times_{\mathbb{P}^1} \tau_{B'}$ and let $\tau_X := \tau_B \times_{\mathbb{P}^1} \tau_{B'}$. Since the fixed points of $\tau_B$ and $\tau_{B'}$ sit over $\infty$ and $0$ respectively, we conclude that $\tau_X$ acts freely on $X$. In particular the quotient $Z := X/\tau_X$ is non-singular. We claim that $Z$ is in fact a Calabi-Yau. This is equivalent to saying that $\tau_X$ preserves the holomorphic 3-form on $X$. Indeed, $\tau_X$ acts on $H^0(X, \Omega^3_X)$ as multiplication by a number $\lambda \in \mathbb{C}^\times$. Since the fiber $f_0 \times f_0'$ of $X \to \mathbb{P}^1$ is stable under $\tau_X$ it suffices to compute the action of $\tau_X$ on $H^0(f_0 \times f_0', \Omega^3_X)$. But

$$\Omega^3_{X|f_0 \times f_0'} = (K_{f_0} \boxtimes K_{f_0'}) \otimes T_0^* \mathbb{P}^1,$$

and $\tau_X$ acts as $\tau_{B|f_0}$ on $K_{f_0}$, $\tau_{B'|f_0'}$ on $K_{f_0'}$ and as $\tau_{\mathbb{P}^1}$ on $T_0^* \mathbb{P}^1$. Since $\tau_{B|f_0}$ is a translation on $f_0$, it acts on $H^0(f_0, K_{f_0})$ as $+1$. Since $\tau_{B'|f_0'}$ and $\tau_{\mathbb{P}^1}$ each have a fixed point, they act as $-1$ on $H^0(f_0', K_{f_0'})$ and $T_0^* \mathbb{P}^1$ respectively.
Hence \( \lambda = 1 \cdot (-1) \cdot (-1) = 1 \).

The vector bundles on \( Z \) can be interpreted as \( \tau_X \)-invariant vector bundles on \( X \). To construct vector bundles on \( X \) we will exploit the fact that \( X \) is an elliptically fibered 3-fold and so we can manufacture bundles by using a relative Fourier-Mukai transform.

Concretely, let \( \mathcal{P}_X \rightarrow X \times B' \) be the Poincare sheaf corresponding to the section \( \sigma \). That is \( \mathcal{P}_X \) is the rank one torsion-free sheaf given by

\[
\mathcal{P}_X = \mathcal{O}_{X \times B'}(\Delta - \sigma \times B' X - X \times B' \sigma - m^*c_1(B')) = p_{13}^*\mathcal{P}_B,
\]

where \( m : X \times B' \rightarrow B' \) and \( p_{13} : X \times B' = B \times_{\mathbb{P}^1} B' \times_{\mathbb{P}^1} B \rightarrow B \times_{\mathbb{P}^1} B \) are the natural projections. As in [DOPWa, Section 6], one argues that \( \mathcal{P}_X \) defines an autoequivalence (see [BM, Theorem 1.2]) of \( D^b(X) \):

\[
FM_X : D^b(X) \longrightarrow D^b(X)
\]

\[
\mathcal{F} \longmapsto R^\bullet p_{1*}(p_2^*\mathcal{F} \otimes \mathcal{P}_X).
\]

If \( V \rightarrow X \) is a vector bundle of rank \( r \) which is semistable and of degree zero on each fiber of \( \pi : X \rightarrow B' \), then its Fourier-Mukai transform \( FM_X(V)[1] \) is a torsion sheaf of pure dimension two on \( X \). The support of \( FM_X(V)[1] \) is a surface \( i_\Sigma : \Sigma \hookrightarrow X \) which is finite of degree \( r \) over \( B' \). Furthermore \( FM_X(V) \) is of rank one on \( \Sigma \). In fact, if \( \Sigma \) is smooth, then \( FM_X(V)[1] = i_\Sigma^*L \) is just the extension by zero of some line bundle \( L \in \text{Pic}(\Sigma) \). Conversely if \( \mathcal{N} \rightarrow X \) is a sheaf of pure dimension two which is flat over \( B' \), then \( FM_X(\mathcal{N}) \) is a vector bundle on \( X \) of rank equal to the degree of \( \text{supp}(\mathcal{N}) \) over \( B' \) and whose first Chern class is vertical (for the projection \( \pi : X \rightarrow B' \)). This correspondence between vector bundles on \( X \) and sheaves on \( X \) supported on finite covers of \( B' \) is commonly known as the spectral construction and has been extensively studied in the context of Weierstrass elliptic fibrations [FMW97, FMW99, Don97, BJPS97]. The torsion sheaf \( \mathcal{N} \) on \( X \) is called spectral datum and the surface \( \Sigma = \text{supp}\mathcal{N} \) is called a spectral cover.

Since our elliptic Calabi-Yau \( X \) is not Weierstrass we briefly describe how the spectral construction works (at least for generic spectral data) on \( X \) and how it interacts with the involution \( \tau_X \). First we need to understand the action of \( FM_X \) on line bundles on \( X \). Note that since \( X = B \times_{\mathbb{P}^1} B' \) is a fiber product we have \( \text{Pic}(X) = (\text{Pic}(B) \times \text{Pic}(B'))/\text{Pic}(\mathbb{P}^1) \). In particular, every line bundle on \( X \) can be written as \( L \boxtimes L' := \pi'^*L \otimes \pi^*L' \) for some \( L \rightarrow B \) and \( L' \rightarrow B' \).

**Lemma 2.1.** For every line bundle \( \mathcal{L} = L \boxtimes L' \) on \( X \), the actions of the Fourier-Mukai transform and of the spectral involution are given by:
(a) $FM_X(L) = FM_X(L \boxtimes L') = \pi^*FM_B(L) \otimes \pi^*L' = FM_B(L) \boxtimes L'$.

(b) $T_X(L) := (FM_X^{-1} \circ \tau^*_X \circ FM_X)(L) = \pi^*(T_B(L)) \otimes \pi^*(\tau^*_B L') = T_B(L) \boxtimes \tau^*_B L'$.

Proof. Part (b) is an obvious consequence of part (a). To prove part (a) we will use the identification $X \times_{B'} X = B \times_{B'} B' \times_{B'} B$. In terms of this identification we have:

$FM_X(L) = Rp_{23*}(p_{12}^* L \otimes P_X)$
$= Rp_{23*}(p_{12}^*(L \boxtimes L') \otimes p_{13}^* P_B)$
$= Rp_{23*}(\pi_1^* L \otimes \pi_2^* L' \otimes p_{13}^* P_B)$
$= Rp_{23*}(p_{13}^*(\pi_1^* L \otimes P_B) \otimes p_{23}^*(\pi_2^* L'))$
$= Rp_{23*}(p_{13}^*(\pi_1^* L \otimes P_B)) \otimes \pi^* L'$.

Here $\pi_1, \pi_2$ and $\pi_3$ are the natural projections of $B \times_{B'} B' \times_{B'} B$ onto $B$, $B'$ and $B$ respectively, $p_1 : B \times_{B'} B \rightarrow B$ are the projections on the two factors, and in the last identity we have used the projection formula for $p_{23}$.

Now using the base change property for the fiber square

$$
\begin{array}{ccc}
X \times_{B'} X & \xrightarrow{p_{23}} & B' \times_{B'} B \\
\downarrow p_{13} & & \downarrow \pi' \\
B \times_{B'} B & \xrightarrow{p_2} & B
\end{array}
$$

we get $Rp_{23*}p_{13}^* = \pi'^*Rp_{2*}$ and so

$FM_X(L) = \pi'^*(Rp_{2*}(p_{1}^* L \otimes P_B)) \otimes \pi^* L' = FM_B(L) \boxtimes L'$.

The lemma is proven. □

Let now $i_\Sigma : \Sigma \hookrightarrow X$ be a surface which is finite and of degree $r$ over $B'$. Then for any line bundle $L \in \text{Pic}(X)$ the torsion sheaf $N := i_{\Sigma*}i_\Sigma^*L$ has a resolution by global line bundles. Namely

$$
0 \rightarrow L(-\Sigma) \rightarrow L \rightarrow N \rightarrow 0.
$$

In particular $N$ is quasi-isomorphic to the two-step complex of line bundles $[L(-\Sigma) \rightarrow L]$ on $X$ and so the actions of $FM_X$ and $T_X$ on $N$ can be computed via the formulas in Lemma 2.1. Specifically we have:

Lemma 2.2. Let $L = L \boxtimes L'$ be a global line bundle on $X$ and let $i_\Sigma : \Sigma \hookrightarrow X$ be a surface finite over $B'$. Let $N = i_{\Sigma*}i_\Sigma^*L$ be such that $V = FM_X(N)$ is a rank $r$ vector bundle on $X$ with $c_1(V) = 0$. Then $\tau^*_X V \cong V$ if and only if the following three conditions

\[ \text{Lemma 2.2.} \]
\[ \alpha_X(\Sigma) = \Sigma; \]
\[ \tau_{B'}^*L' \cong L'; \]
\[ T_B^*L \cong L. \]

are satisfied.

**Remark 2.3.** Notice that the \( \tau_X \) invariance of \( V \) amounts to two separate conditions on the spectral data. The first is that the spectral surface \( \Sigma \) has to be invariant under the involution \( \alpha_X \). This condition is relatively easy to satisfy. It just means that \( \Sigma \) is pulled back from the quotient \( X/\alpha_X \). The second condition requires the \( \tau_B \) invariance of \( L' \) and the \( T_B \) invariance of \( L \).

In fact, the formulas in [DOPWa, Table 3] (written in terms of the basis of \( H^2(B,\mathbb{Z}) \) described in [DOPWa, Section 4.2]) show that \( L \in \text{Pic}(B) \otimes \mathbb{Q} \) will be \( T_B \)-invariant if and only if \( L \) is in the affine subspace

\[ \frac{1}{2}e_1 + \text{Span}_\mathbb{Q}(f, e_9, e_4 - e_5, e_4 - e_6, 3\ell - 2(e_4 + e_5 + e_6) - 3e_7, \ell - e_7 - 2e_8), \]

which does not intersect \( \text{Pic}(B) \subset \text{Pic}(B) \otimes \mathbb{Q} \). This implies that \( V \) can not be \( \tau_X \)-invariant if \( N = i_{\Sigma}^*i_{\Sigma}^*\mathcal{L} \) for some global \( \mathcal{L} \in \text{Pic}(X) \). For \( \Sigma \) smooth and very ample the Lefschetz hyperplane section theorem asserts that every \( N \) comes from a global \( \mathcal{L} \) and hence one is forced to work with singular or non-very ample surfaces \( \Sigma \).

### 3 The construction

#### 3.1 The basic construction

In this section we describe in detail our method of constructing \( \tau_X \)-invariant vector bundles on \( X \).

In order to circumvent the difficulty pointed out in Remark 2.3 we build our rank five bundle \( V \) on \( X \) not directly by the spectral construction but as an extension

\[ 0 \to V_2 \to V \to V_3 \to 0. \]
Here $V_i$, $i = 2, 3$ is a rank $i$ bundle on $X$ which is $\tau_X$-invariant and satisfies some strong numerical conditions which will be discussed in the next section. In addition, we will see that the stability condition on $V$ amounts to the extension being non split.

Each $V_i$ is produced by an application of the spectral construction on $X$ with a reducible spectral cover and a line bundle on it which is not the restriction of a global line bundle on $X$. Define $V_i$ from its spectral data as follows:

- Let $C_i$ be a curve in the linear system $|\mathcal{O}_B(ie + k_i f)|$ where $k_i$ is an integer. Let $\Sigma_i := C_i \times \mathbb{P}^1 B'$. Recall that $\beta' : B' \to \mathbb{P}^1$ has two $I_2$ fibers $f_1', f_2'$. Let $F_j, j = 1, 2$ be the corresponding fibers of $B$; note that, because of the way we glued the $\mathbb{P}^1$ bases, these are not the reducible fibers $f_j$ of $B$. Let

$$\{p_{ijk}\}_{k=1}^i := C_i \cap F_j.$$ 

Then $\Sigma_i \to C_i$ is an elliptic surface having $2i$ fibers of type $I_2$: $(n_j' \cup o_j') \times \{p_{ijk}\}$ where $j = 1, 2$ and $k = 1, \ldots, i$.

- Define $V_i$ as

$$V_i = FM_X \left((\Sigma_i, (\pi|_{\Sigma_i})^* N_i \otimes \pi^* L_i \otimes \mathcal{O}_{\Sigma_i} \left( - \sum \{p_{ijk}\} \times (a_{ij k} n_j' + b_{ij k} o_j') \right)) \right),$$

where $L_i$ is a line bundle on $B'$, $N_i$ is a line bundle on the curve $C_i$ and the optional parameters $a_{ij k}, b_{ij k}$ are integers.

Note that there is a redundancy in our choices, because $n_j' + o_j' = f_j'$ is a pullback from $\mathbb{P}^1$, and so can be absorbed in $L_i$. In particular, we can always arrange for all the coefficients $a_{ij k}, b_{ij k}$ to be non-negative. Also, without a loss of generality we may assume that for any given $j$ we have $a_{ij k} \cdot b_{ij k} = 0$ for all $i, k$. With this convention, we have an alternative description of $V_i$:

Put

$$\tilde{W}_i := V_i \otimes \pi^* L_i^{-1}.$$ 

It turns out that the bundle $\tilde{W}_i$ can be constructed directly. Consider the vector bundle $W_i$ on $B$, built by the spectral construction as $W_i := FM_B(C_i, N_i)$. Then $\tilde{W}_i$ is obtained from the vector bundle $\pi'^* W_i$ by $a_{ij k}$ successive Hecke transforms along the divisors $\{p_{ijk}\} \times n_j'$ and $b_{ij k}$ successive Hecke transforms along the divisors $\{p_{ijk}\} \times o_j'$. The center of each Hecke is
a line bundle on the surface $F_j \times n'_j$ or $F_j \times o'_j$. (For the definition and basic properties of Hecke transforms see appendix A).

In fact, $V$ itself could be built by applying the spectral construction on $X$ to the reducible spectral cover $\Sigma_2 \cup \Sigma_3$ and an appropriately chosen sheaf on it. However the construction with extensions is technically easier because it allows us to avoid dealing with sheaves on singular surfaces. This approach is a variation on the method employed by Richard Thomas in [Tho].

**Remark 3.1.** Observe that in the definition of $C_i$ we could have taken the linear system more generally to be of the form $|\mathcal{O}_B(ie + k_1f + \eta_i)|$ where $k_1$ is an integer and $\eta_i \in \text{Pic}(B)$ is a class perpendicular to $e$ and $f$. If we impose the condition $c_1(W_i) = 0$, then the classes $\eta_i$ are forced to be zero by the Riemann-Roch formula. However the introduction of the $L_i$'s gives us the extra freedom of working with $W_i$'s that have arbitrary vertical $c_1$. We will not exploit this extra freedom but we expect that many examples exist which are similar to ours but have $\eta_i \neq 0$.

Since the Hecke interpretation of $\tilde{W}_i$ will be important in determining the invariance properties of $V$ and in implementing the numerical constraints, we proceed to spell it out explicitly.

### 3.2 Reinterpretation via Hecke transforms

Recall from section 2 that $X = B \times_{\mathbb{P}^1} B'$ fits into a commutative diagram of projections

\[
\begin{array}{ccc}
\pi' & \downarrow & \pi \\
X & \rightarrow & X \\
\beta' & \downarrow & \beta \\
B' & \leftarrow & B \\
\mathbb{P}^1 & \downarrow & \mathbb{P}^1 \\
& & \\
\end{array}
\]

Let $C \subset B$ be a smooth connected curve in the linear system $|\mathcal{O}_B(re + kf)|$. Let $\mathcal{N} \in \text{Pic}^d(C)$ and let $W := FM_B((C, \mathcal{N}))$ be the corresponding rank $r$ vector bundle on $B$.

Consider $\Sigma = \pi'^{-1}(C) = C \times_{\mathbb{P}^1} B'$ and the line bundle $\mathcal{L} = (\pi'|_{\Sigma})^*\mathcal{N} = \mathcal{N} \boxtimes \mathcal{O}_{B'}$ on $\Sigma$. 

Let \( f'_j = n'_j \cup o'_j, F_j = \beta^{-1}(\beta'(f'_j)), j = 1, 2 \) be as above. We assume that \( C \) is general enough so that the intersections \( C \cap F_j, j = 1, 2 \) are transversal.

Let \( p \in C \cap F_j \) and let \( a \) be a non-negative integer. Define
\[
W[a, p] := FM_X((\Sigma, L(-a(\{p\} \times n'_j)))).
\]
Consider the divisor \( D = F_j \times n'_j \subset X \) and the line bundles
\[
\mathcal{O}_{F_j}(p - e) \boxtimes \mathcal{O}_{n'_j}(2a) \in \text{Pic}(D),
\]
where \( a \in \mathbb{Z} \) and by an abuse of notation \( e \) denotes the point of intersection of the curves \( e, F_j \) in \( B \). With this notation we have

**Lemma 3.2.** Fix \( a \geq 0 \).

(i) There is a canonical surjective map \( W[a, p]|_D \to \mathcal{O}_{F_j}(p - e) \boxtimes \mathcal{O}_{n'_j}(2a) \)
which fits in a short exact sequence of vector bundles on \( D \)
\[
(\psi_{a+1}) \quad 0 \to K_a \to W[a, p]|_D \to \mathcal{O}_{F_j}(p - e) \boxtimes \mathcal{O}_{n'_j}(2a) \to 0.
\]

(ii) For \( a = 0 \) we have \( W[0, p] = \pi'^* W \) and for \( a \geq 1 \)
\[
W[a, p] = \text{Hecke}_{(\psi_a)^{-1}} \circ \text{Hecke}_{(\psi_{a-1})^{-1}} \circ \ldots \circ \text{Hecke}_{(\psi_1)^{-1}}(\pi'^* W).
\]

**Proof.** We will prove the lemma by induction in \( a \). By definition we have \( W[0, p] = \pi'^* W \) which takes care of the base of the induction. Assume that \( W[i, p] = \text{Hecke}_{(\psi_i)^{-1}}(W[i - 1, p]) \) for all \( 0 < i < a \). Consider the short exact sequence of sheaves on \( \Sigma \):

\[
(3.2) \quad 0 \to L(-a(\{p\} \times n'_j)) \to L(-(a - 1)(\{p\} \times n'_j)) \to L(-(a - 1)(\{p\} \times n'_j))|_{\{p\} \times n'_j} \to 0.
\]
We have \( L|_{\{p\} \times n'_j} = ((\pi'_\Sigma)^*\mathcal{N})|_{\{p\} \times n'_j} = \mathcal{O}_{\{p\} \times n'_j} \). Also \( \{p\} \times n'_j \) is a component of an \( I_2 \) fiber of the elliptic surface \( \pi'_\Sigma : \Sigma \to C \) and so \( \mathcal{O}_{\{p\} \times n'_j} \). Let it denote the natural inclusions. Then if we extend each of the sheaves in the sequence (3.2) by zero we obtain a short exact sequence of sheaves on \( X \):

\[
(3.4) \quad 0 \to i_{*\Sigma}L(-a(\{p\} \times n'_j)) \to i_{*\Sigma}L(-(a - 1)(\{p\} \times n'_j)) \to i_{*\Sigma}\mathcal{O}_{n'_j}(2(a - 1)) \to 0.
\]

Applying \( FM_X \) to (3.4) we get
\[
(3.5) \quad 0 \to W[a, p] \to W[a - 1, p] \to FM_X(i_{*}\mathcal{O}_{n'_j}(2(a - 1))) \to 0.
\]
By the definition of $FM_X$ we have
\[
FM_X(\iota_*\mathcal{O}_{n_j'}(2(a-1))) = Rp_2^*(p_1^*(\iota_*\mathcal{O}_{n_j'}(2(a-1)) \otimes \mathcal{P}_X),
\]
where $p_1, p_2: X \times B' \to X$ are the natural projections.

If we use the identification $X \times B' X \cong B \times_{\mathbb{P}1} B \times_{\mathbb{P}1} B'$, then the projection $p_i: X \times B' \to X$ gets identified with the projection $p_i: B \times_{\mathbb{P}1} B \times_{\mathbb{P}1} B' \to B \times_{\mathbb{P}1} B'$ and $\mathcal{P}_X = p_1^*\mathcal{P}_B$. In particular in terms of the identification $X \times B' X \cong B \times_{\mathbb{P}1} B \times_{\mathbb{P}1} B'$ we see that $p_1^*(\iota_*\mathcal{O}_{n_j'}(2(a-1)))$ is supported on the surface $\{p\} \times F_j \times n_j' \subset B \times_{\mathbb{P}1} B \times_{\mathbb{P}1} B'$ is precisely
\[
pr_{n_j'}^*\mathcal{O}_{n_j'}(2(a-1)) \otimes \mathcal{P}_X|_{\{p\} \times F_j \times n_j'} = pr_{n_j'}^*\mathcal{O}_{n_j'}(2(a-1)) \otimes pr_{F_j}^*\mathcal{P}_B|_{\{p\} \times F_j}
= pr_{n_j'}^*\mathcal{O}_{n_j'}(2(a-1)) \otimes pr_{F_j}^*\mathcal{O}_{n_j'}(p-e).
\]
Also for the restricted map $p_2|_{\{p\} \times F_j \times n_j'}: \{p\} \times F_j \times n_j' \to X$ we get
\[
P_2|_{\{p\} \times F_j \times n_j'} = p_2|_{\{p\} \times F_j \times n_j'} = i_D,
\]
and hence
\[
FM_X(\iota_*\mathcal{O}_{n_j'}(2(a-1))) = Rp_2^*(p_1^*(\iota_*\mathcal{O}_{n_j'}(2(a-1)) \otimes \mathcal{P}_X)
= i_D^*(\mathcal{O}_{F_j}(p-e) \boxtimes \mathcal{O}_{n_j'}(2(a-1))).
\]
which combined with (3.5) concludes the proof of the lemma. \qed

If now $b \geq 0$ is another integer we may consider also the vector bundle
\[
W\{b,p\} = FM_X((\Sigma, \mathcal{L}(-b(\{p\} \times o_j')))).
\]
In exactly the same way we see that $W\{0\} = \pi^*W$, that for every $b \geq 1$ there is a canonical exact sequence
\[
(\phi_{b+1}) \quad 0 \to M_b \to W\{b,p\}|_{\{F_j\} \times o_j'} \to \mathcal{O}_{F_j}(p-e) \boxtimes \mathcal{O}_{o_j'}(2b) \to 0,
\]
and that
\[
W\{b,p\} = Hecke_{(\phi_a)}^- \circ Hecke_{(\phi_{a-1})}^- \circ \ldots \circ Hecke_{(\phi_1)}^-(\pi^*W).
\]
For future reference we record

**Corollary 3.3.** The Chern classes of $W[a,p]$ and $W\{b,p\}$ are given by
\[
\begin{align*}
ch(W[a,p]) &= \pi^*ch(W) - d\pi^*n_j - d^2(f \times pt); \\
ch(W\{b,p\}) &= \pi^*ch(W) - b\pi^*o_j - b^2(f \times pt)
\end{align*}
\]
Proof. Clearly it suffices to prove the corollary for $W[a, p]$. By Lemma 3.2 we have short exact sequences

$$0 \to W[n, p] \to W[n - 1, p] \to i_D*\psi_n \to 0$$

for all $n \geq 1$. Here we have slightly abused the notation by writing $\psi_a$ for the middle term of the short exact sequence $(\psi_a)$. Hence $ch(W[n, p]) = ch(W[n - 1, p]) - ch(i_D*\psi_n)$ and so

$$ch(W[a, p]) = \pi^*ch(W) - \sum_{n=1}^{a} ch(i_D*\psi_n).$$

Using Grothendieck-Riemann-Roch we calculate

$$ch(i_D*\psi_n) = i_D*(ch(\psi_n)td(D))td(X)^{-1}$$
$$= i_D* \left( 1 + \psi_n + \frac{\psi_n^2}{2} \right)(1 + F_j \times pt)(1 - (f \times pt + pt \times f'))$$
$$= i_D*((1 + 2(n - 1)F_j \times pt)(1 + F_j \times pt)(1 - (f \times pt + pt \times f'))$$
$$= i_D*(1 + (2n - 1)F_j \times pt)(1 - (f \times pt + pt \times f'))$$
$$= D + (2n - 1)(f \times pt) = \pi^*n_j + (2n - 1)(f \times pt).$$

Consequently

$$ch(W[a, p]) = \pi^*ch(W) - \sum_{n=1}^{a} (\pi^*n_j + (2n - 1)(f \times pt)$$
$$= \pi^*ch(W) - a\pi^*n_j - a^2(f \times pt).$$

The corollary is proven. \square

Finally, we are ready to give the Hecke interpretation of $\hat{W}_i = V_i \otimes \pi^*L_i^{-1}$. Recall that

$$\hat{W}_i = F M_X \left( \left( \Sigma_i, L_i \left( - \sum \{p_{ijk} \} \times (a_{ijk}n_j + b_{ijk}o_j) \right) \right) \right)$$

where $a_{ijk}, b_{ijk}$ are non-negative integers satisfying $a_{ijk}b_{ijk} = 0$. Since Hecke transforms whose centers have disjoint supports obviously commute, we see from the above discussion that

$$\hat{W}_i = W[a_{i11}, p_{i11}]\{b_{i11}, p_{i11}\}a_{i12}, p_{i12}\{b_{i12}, p_{i12}\} \cdots a_{i1i}, p_{i1i}\{b_{i1i}, p_{i1i}\}. $$
4 Invariant spectral data

In this section we examine the conditions for $V$ to be $\tau_X$-invariant. It is easy to reduce this, first to invariance of the $V_i$, then to invariance of the $W_i$. Indeed, assume that the bundles $V_i$ are $\tau_X$-invariant, and choose liftings of the $\tau_X$ action to the $V_i$. The space $\text{Ext}^1(V_3, V_2)$ parameterizing all extensions is a direct sum of its invariant and anti-invariant subspaces. So if $\text{Ext}^1(V_3, V_2) \neq 0$ we also have an extension which is either invariant or anti-invariant. Finally, changing the lifted action which is either invariant or anti-invariant. Recently, changing the lifted action of $\tau_X$ on one of the $V_i$ interchanges invariants with anti-invariants, so we are done.

Since $V_i = \widetilde{W}_i \otimes \pi^* L_i$, we have

$$\tau_X^*(V_i) = \tau_X^*(\widetilde{W}_i) \otimes \pi^* \tau_B^* L_i.$$ 

So it suffices to have a $\widetilde{W}_i$ which is $\tau_X$-invariant and an $L_i$ which is $\tau_B^*$-invariant. From [DOPWa, Table 1] we know that there is a 6-dimensional lattice of $\tau_B^*$-invariant classes on $B'$, so we have lots of possibilities for the $L_i$. Now $\widetilde{W}_i$ is a Hecke transform of $\pi^* (W_i)$, so we want $W_i$ to be $\tau_B^*$-invariant and the center of the Hecke transform to be $\tau_X$-invariant. Above we took the support of this Hecke to be an arbitrary collection of components of the surfaces $F_j \times f'_j$ for $j = 1, 2$. It can be seen from [DOPWa, Table 1] together with the expression [DOPWa, Formula (4.2)] for the components of the $I_2$-fibers of $B$ in terms of our basis, that the action of $\tau_B^*$ interchanges $\alpha_1$ with $\alpha_2$ and $\alpha_2$ with $\alpha_1$. Therefore the condition for $\tau_X$-invariance of the center of the Hecke transform becomes $a_{i1k} = b_{i2k}$ and $a_{i2k} = b_{i1k}$. Because of the redundancy in our choices we are free to take $a_{i2k} = b_{i1k} = 0$ and $a_{i1k} = b_{i2k} \geq 0$.

Finally we have to find the conditions that will ensure the $\tau_B^*$-invariance of $W_i$.

4.1 The $\tau_B^*$-invariance of $W_i$

Throughout this subsection we work with a spectral curve $C_i$ in the linear system $|i e + k_i f|$, $i = 2$ or 3, which is finite over $\mathbb{P}^1$, and a line bundle $N_i \in \text{Pic}(C_i)$. The $\tau_B^*$-invariance of $W_i = F_M B((C_i, N_i))$ is equivalent to the $T_B$ invariance of $i_{C_i*} N_i$.

In [DOPWa, Proposition 7.7] we saw that for any curve $C \subset B$ which is finite over $\mathbb{P}^1$ and for any line bundle $N$ on $B$ the image $T_B(i_{C*} N)$ is
again a sheaf of the form $i_{\alpha_B(C)*}(?)$ for some line bundle $? \in \text{Pic}(\alpha_B(C))$. Therefore $T_B$ induces a well defined map $T_C: \text{Pic}(C) \to \text{Pic}(\alpha_B(C))$. Due to this, the $\tau_B$-invariance of $FM_B((C,N))$ is equivalent to the following two conditions:

\begin{align}
(4.1) & \quad C = \alpha_B(C) \\
(4.2) & \quad N = T_C(N).
\end{align}

**Lemma 4.1.** The linear system $|re + kf|$ contains smooth $\alpha_B$-invariant curves if $r = 3$, $k \geq 3$ or if $r = 2$ and $k \geq 2$ is even.

**Proof.** First of all, from the explicit equations of a spectral curve [FMW97] and Bertini’s theorem, it is easy to see that the general curve $C$ in the linear system $|re + kf|$ will be smooth as long as $k \geq r > 1$. The same kind of analysis allows one to understand the $\alpha_B$-invariant members of these linear systems as well. Indeed, recall (see e.g. [FMW99]) that for every $a \geq 0$ we have an isomorphism $\beta_a O_B(\alpha e) = O_{P^1} \oplus O_{P^1}(-2) \oplus \ldots O_{P^1}(-a)$. In particular, by the projection formula we get isomorphisms

\[
\begin{align*}
H^0(B, O_B(e)) &= H^0(P^1, O_{P^1}) \\
H^0(B, O_B(2e + 2f)) &= H^0(P^1, O_{P^1}(2)) \oplus H^0(P^1, O_{P^1}) \\
H^0(B, O_B(3e + 3f)) &= H^0(P^1, O_{P^1}(3)) \oplus H^0(P^1, O_{P^1}(1)) \oplus H^0(P^1, O_{P^1}).
\end{align*}
\]

Let $X \in H^0(B, O_B(2e + 2f)), Y \in H^0(B, O_B(3e + 3f))$ and $Z \in H^0(B, O_B(e))$ be the preferred sections corresponding to the generator of the piece $H^0(P^1, O_{P^1})$ under the above decompositions. Note that in terms of the sections $x \in H^0(P, O_P(1) \otimes p^*O_{P^1}(2), y \in H^0(P, O_P(1) \otimes p^*O_{P^1}(3)$ and $z \in H^0(P, O_P(1)$, which were used in [DOPWa, Section 3.2] to define the Weierstrass model of $B$, we have $x|_{W_\beta} = XZ, y|_{W_\beta} = Y, z|_{W_\beta} = Z^3$.

With this notation the isomorphism

\[
H^0(O_{P^1}(k)) \oplus H^0(O_{P^1}(k - 2) \oplus \ldots \oplus H^0(O_{P^1}(k - r)) \to H^0(B, O_B(re + kf))
\]

is given explicitly by the formula

\[
(a_k, a_{k-2}, \ldots, a_{k-r}) \mapsto (\beta^*a_k)Z^r + (\beta^*a_{k-2})XZ^{r-2} + (\beta^*a_{k-3})YZ^{r-3} + \ldots.
\]

In particular the curves $C_2$ and $C_3$ can be identified with the zero loci of

\begin{align}
(4.3) & \quad (\beta^*a_{k_2})Z^2 + (\beta^*a_{k_2-2})X \quad \text{and} \quad (\beta^*a_{k_3})Z^3 + (\beta^*a_{k_3-2})XZ + (\beta^*a_{k_3-3})Y,
\end{align}

respectively.
Now recall, that in [DOPWa, Section 3.2] we identified $\alpha_B$ with the involution induced from $\tau_{P|W_\beta}$ and that $\tau_P$ acts trivially on the sections $x$, $y$, $z$. In view of the comparison formulas $x|_{W_\beta} = XZ$, $y|_{W_\beta} = Y$, $z|_{W_\beta} = Z^3$, this implies that $\alpha^*_B(X) = X$, $\alpha^*_B(Y) = Y$ and $\alpha^*_B(Z) = Z$. Here the lifting of the action of $\alpha_B$ to an action on line bundles of the form $O_B(re + kf)$ is chosen in the way described in [DOPWa, Section 3.2]. In particular $\alpha^*_B(\beta^* s) = \beta^*(\tau_{P_1}^* s)$ for any section $s \in O_{P_1}(k)$.

Since $\alpha^*_B$ acts linearly on the projective space $|re + kf|$ it follows that $\alpha_B$ will preserve a divisor $C \in |re + kf|$ if and only if $$C \in \mathbb{P}(H^0(B, O_B(re + kf))^+) \cup \mathbb{P}(H^0(B, O_B(re + kf)^-) \subset |re + kf|,$$
where $H^0(B, O_B(re + kf))^\pm$ denote the $\pm 1$ eigenspaces of $\alpha^*_B$ acting on $H^0(B, O_B(re + kf))$. Therefore we see that for each $i$ there are two families of $\alpha_B$-invariant $C_i$'s, each parameterized by a projective space. In particular we will have $\alpha_B(C_i) = C_i$ if and only if all the coefficients in the polynomial expressions (4.3) are simultaneously $\tau_{P_1}$-invariant or simultaneouly $\tau_{P_1}$-anti-invariant. Now the Bertini theorem immediately implies that we can find a smooth $C_3$, which is preserved by $\alpha_B$ as long as $k_3 \geq 3$ and we can find a smooth $C_2$, which is preserved by $\alpha_B$ as long as $k_2 \geq 2$ and $k_2$ is even. □

**Remark 4.2.** Unfortunately, when $k_2$ is odd the linear systems $|2e + k_2f|^\pm$ will each have a fixed component and so all the $\alpha_B$-invariant curves $C_2$ will be reducible. In order to see this consider the homogeneous coordinates $(t_0 : t_1)$ on $\mathbb{P}^1$ which were used in [DOPWa, Section 3.2] to define the standard action of $\tau_{P_1}$. In other words $(t_0 : t_1)$ are such that $\tau_{P_1}^* (t_0) = t_0$, $\tau_{P_1}^* (t_1) = -t_1$ and $0 = (1 : 0)$ and $\infty = (0 : 1)$. Now it is clear that if $a$ is a $\tau_{P_1}$-invariant homogeneous polynomial in $t_0$ and $t_1$ of odd degree, then $a$ is divisible by $t_0$. Similarly if $a$ is a $\tau_{P_1}$-anti-invariant polynomial of odd degree, then $a$ is divisible by $t_1$. In particular, since $k_2$ and $k_2 - 2$ have the same parity we see that for $k_2$ odd, the fiber $f_\infty$ is the fixed component of the linear system $|2e + k_2f|^+$ and the fiber $f_0$ is the fixed component of the linear system $|2e + k_2f|^-$. 

**Lemma 4.3.** Let $C$ be an $\alpha_B$-invariant curve in $|re + kf|$ which is finite over $\mathbb{P}^1$, and assume that $i^*_C \text{Pic}(B)$ is dense in $\text{Pic}^0(C)$. Then for every $d \in \mathbb{Z}$ there exist line bundles $N \in \text{Pic}^d(C)$, s.t. $T_C(N) = N$. 


Proof. The morphism \( T_C : \text{Pic}(C) \to \text{Pic}(C) \) is given explicitly by the formula:

\[
T_C(N) = \alpha_C^*(N) \otimes \mathcal{O}_C(e_9 - e_1 + f),
\]

where \( \alpha_C = \alpha_{B|C} \).

Indeed, by part (b) of [DOPWa, Proposition 7.7] this formula holds for all line bundles \( N \in \text{Pic}(i_C^* \text{Pic}(B)) \). By the density assumption it holds for all \( N \in \text{Pic}^0(C) \). But applying \( T_C \) to the short exact sequence

\[
0 \to N(-p) \to N \to \mathcal{O}_p \to 0
\]

we find \( T_C(N(-p)) = T_C(N)(-\alpha_C(p)) \), so the formula extends to all components of \( \text{Pic}(C) \).

Thus a point \( x \in \text{Pic}^0(C) \) will be fixed under \( T_C \) if and only if

\[
x - \alpha_C^*(x) = \mathcal{O}_C(e_9 - e_1 + f).
\]

This equation is consistent exactly when

\[
\mathcal{O}_C(e_9 - e_1 + f) \in \text{im} \left[ \text{Pic}^0(C) \xrightarrow{\alpha_C^* - \text{id}} \text{Pic}^0(C) \right].
\]

Since \( \alpha_C \) has fixed points on \( C \), it follows [Mum74] that \( \text{im}(\alpha_C^* - \text{id}) = \ker(\alpha_C^* + \text{id}) \). But from [DOPWa, Table 1 and Formula (4.2)] we see that

\[
(\alpha_B^* + \text{id})(e_9 - e_1 + f) = 2e_9 + 2f + e_7 - \ell = o_1 + o_2.
\]

Since \( o_1 \) and \( o_2 \) do not intersect \( C \), this implies that \( \mathcal{O}_C(e_9 - e_1 + f) \) is \( \alpha_C^* \)-anti-invariant. Hence there is a translate of of \( \text{Pic}^0(C/\alpha_C) \) in \( \text{Pic}^0(C) \) consisting of solutions of (4.5).

Shifting by an arbitrary multiple of an \( \alpha_{C_i} \)-fixed point, we see that there are \( T_{C_i} \)-fixed points in \( \text{Pic}^d(C_i) \) for every \( d \).

\( \square \)

In view of this lemma it only remains to check the density of \( i_{C_i}^* \text{Pic}(B) \) in \( \text{Pic}^0(C_i) \). We do this only in the cases \( (i = 2, k_2 = 3) \) and \( (i = 3, k_3 = 6) \), which are the cases needed in section 5. Unfortunately the statement of Lemma 4.3 does not directly apply to the first of these cases (see Remark 4.2), so we will treat it separately next.
4.2 Invariance for $k_2 = 3$

By Remark 4.2, the general $\alpha_B$-invariant curve $C_2$ in the linear system $|2e + 3f|$ is of the form $C_2 = \overline{C}_2 + F$, where $\overline{C}_2$ is a smooth curve in the linear system $|2e + 2f|$ and $F$ denotes one of the elliptic curves $f_0$, $f_\infty$. Assume that $\mathcal{N}_2 \to C_2$ is a line bundle and let $\overline{\mathcal{N}}_2 = \mathcal{N}_2 \otimes \mathcal{O}_{\overline{C}_2}$ be its restriction to $\overline{C}_2$. We know that $W_2 := \mathcal{F}M_B(i_{\overline{C}_2*}\overline{\mathcal{N}}_2)$ is a vector bundle. We want $W_2 := \mathcal{F}M_B(i_{C_2*}\mathcal{N}_2)$ to be a vector bundle too.

**Lemma 4.4.** $W_2$ is a vector bundle if and only if $\deg(\mathcal{N}_2[F]) = 1$.

**Proof.** We have a short exact sequence of torsion sheaves on $B$

$$0 \to i_{F*}(\mathcal{N}_2[F](-D)) \to i_{C_2*}\mathcal{N}_2 \to i_{\overline{C}_2*}\overline{\mathcal{N}}_2 \to 0,$$

where $D \subset F$ is the effective divisor $D = \overline{C}_2 \cap F$. Let $G := \mathcal{N}_2[F](-D)$. Since $G$ is a line bundle on the fiber $F$ we have $\mathcal{F}M_B(i_{F*}G) = i_{F*}(\mathcal{F}M_F(G))$, where $\mathcal{F}M_F : D^b(F) \to D^b(F)$ is the Fourier-Mukai transform with respect to the Poincare bundle $P_{B[F \times F]}$. If we apply $\mathcal{F}M_B$ to the above exact sequence, we will get the long exact sequence of cohomology sheaves

$$0 \longrightarrow \mathcal{H}^0(i_{F*}\mathcal{F}M_F(G)) \longrightarrow \mathcal{H}^0\mathcal{F}M_B(i_{C_2*}\mathcal{N}_2) \longrightarrow \overline{W}_2 \longrightarrow \mathcal{H}^1(i_{F*}\mathcal{F}M_F(G)) \longrightarrow \mathcal{H}^1\mathcal{F}M_B(i_{C_2*}\mathcal{N}_2) \longrightarrow 0.$$

Since we want the line bundle $\mathcal{N}_2 \to C_2$ to be chosen so that $\mathcal{H}^1\mathcal{F}M_B(i_{C_2*}\mathcal{N}_2) = 0$ and $W_2 = \mathcal{H}^0\mathcal{F}M_B(i_{C_2*}\mathcal{N}_2)$ is a rank two vector bundle on $B$, we must have $\mathcal{H}^0(i_{F*}\mathcal{F}M_F(G)) = 0$ and $\mathcal{H}^1(i_{F*}\mathcal{F}M_F(G))$ must be a line bundle on $F$ such that there exists a surjection $W_2[F] \to \mathcal{H}^1(i_{F*}\mathcal{F}M_F(G))$. This can only happen if $G$ has degree $-1$ on $F$. 

We note that in the situation of the lemma $W_2$ fits in a short exact sequence

$$0 \to W_2 \to \overline{W}_2 \to i_{F*}(G^\vee) \to 0,$$

where $G$ is defined in the proof of the lemma. Indeed, the proof of Lemma 4.4 gave us a short exact sequence

$$0 \to W_2 \to \overline{W}_2 \to i_{F*}(\mathcal{F}M_F(G)[1]) \to 0.$$

But every line bundle of degree $-1$ on an elliptic curve $F$ is of the form $\mathcal{O}_F(-p)$ for some point $p \in F$. Applying $\mathcal{F}M_F$ to the short exact sequence

$$0 \to \mathcal{O}_F(-p) \to \mathcal{O}_F \to \mathcal{O}_p \to 0$$
we see that $\text{FM}_F(G) = G^\vee[-1]$ for any line bundle of degree $-1$ on $F$.

Given $(\overline{C}_2, \overline{N}_2)$ Lemma 4.4 produces a 2-parameter family of vector bundles $W_2$. Indeed, let $G \to F$ be any line bundle of degree $-1$. Consider the semi-stable bundle $\overline{W}_{2|F}$ on $F$. Since generically $\overline{C}_2$ intersects $F$ at two distinct points we will have $\overline{W}_{2|F} = A \oplus A^\vee$, where $A$ is a non-trivial line bundle of degree zero on $F$. Therefore $h^0(F, A^\vee \otimes G^\vee) = h^0(F, A \otimes G^\vee) = 1$ and so we have unique (up to scale) maps $A \to G^\vee$ and $A^\vee \to G^\vee$. Also since the degree one line bundles $A^\vee \otimes G^\vee$ and $A \otimes G^\vee$ are not isomorphic, it follows that their unique (up to a scale) sections vanish at different points on $F$. Hence we get a one parameter family of surjective maps of vector bundles $A \oplus A^\vee \to G^\vee$. The corresponding Hecke transform of $\overline{W}_2$:

$$W_2 = \ker[\overline{W}_2 \to i_{F*}(A \oplus A^\vee) \to i_{F*}(G^\vee)]$$

is a rank two vector bundle on $B$ which is the Fourier-Mukai image of a line bundle $N_2$ on $C_2 = \overline{C}_2 \cup F$. In particular $W_2$ is a deformation of a rank two vector bundle $W$ corresponding to a spectral datum $(C, N)$ where $C$ is a smooth (but non-invariant) curve in the linear system $|2e + 3f|$ and $N$ is a line bundle on $C$. Even though this $W$ can not be $\tau_B$ invariant, this shows that as far as Chern classes are concerned the bundle $W_2$ behaves as a bundle corresponding to a smooth spectral cover.

We are now ready to analyze the $\tau_B$-invariance properties of $W_2$. First of all, since $\overline{C}_2$ is smooth Lemma 4.3 applies modulo the following density statement:

**Lemma 4.5.** $i_{\overline{C}_2*} \text{Pic}(B)$ is Zariski dense in $\text{Pic}^0(\overline{C}_2)$.

**Proof.** The curves $\overline{C}_2$ have genus 2 and in the linear system $|2e + 2f|^+$ we have a degenerate curve consisting of the zero section $e$ taken with multiplicity two and of two fibers of $\beta$ which are exchanged by $\alpha_B$. In particular the Jacobian of this degenerate curve is just the product of the two fibers. But the Mordell-Weil group of $B$ has rank 6 and in particular we get elements of infinite order in the general fiber of $\beta$ which are restrictions of global line bundles. By continuity this implies that for a general $\overline{C}_2 \in |2e + 2f|^+$ we can find both $\alpha_B$-invariant and $\alpha_B$-anti-invariant line bundles on $B$ that restrict to elements of infinite order in $\text{Pic}^0(\overline{C}_2)$. Finally $\alpha_B$ has two fixed points on a general $\overline{C}_2$ and so $g(\overline{C}_2/\alpha_{\overline{C}_2}) = 1$ and $\dim \text{Prym}(\overline{C}_2, \alpha_{\overline{C}_2}) = 1$. Hence $i_{\overline{C}_2*} \text{Pic}(B)$ is dense in $\text{Pic}^0(\overline{C}_2)$.

We now reach the main point of this subsection:
Proposition 4.6. For every integer $d$ there exists a $\tau_B$-invariant vector bundle $W_2$ whose spectral data $(C_2, N_2 \in \text{Pic}(C_2))$ deforms flatly to a smooth curve in $B$ and a line bundle of degree $d$ on it.

Proof. We have a two parameter family of $\alpha_B$-invariant curves $\overline{C}_2$. By Lemmas 4.3 and 4.5, there is a one parameter family of $\mathbf{T}_{\overline{C}_2}$-invariant line bundles $N_2$ on each. Altogether we get a three parameter family of $\tau_B$-invariant bundles $\overline{W}_2$. We have seen above that each of these gives rise to a two parameter family of bundles $W_2$. We will check now that in each such two parameter family there is a finite number (in fact, four) of $\tau_B$ invariant $W_2$.

Indeed for every such $\overline{W}_2$ we must look for a $\tau_B$-invariant Hecke transform $W_2$. For this we need to ensure that $G^\vee$ is preserved by $\tau_B|F$ and that the map $\overline{W}_2 \to i_F^*(G^\vee)$ is $\tau_B$-equivariant. We have two possibilities: $F = f_0$ or $F = f_\infty$. Recall that $\tau_B|f_0 = t_{\zeta(0)}$ and $\tau_B|f_\infty = t_{\zeta(\infty)} \circ (-1)|f_\infty$ and that $\zeta(0) \in f_0$ and $\zeta(\infty) \in f_\infty$ are non-trivial points of order two. In particular $\tau_B|f_0$ does not fix any line bundle of degree one on $f_0$ and $\tau_B|f_\infty$ fixes precisely four such bundles, namely the four square roots of the degree two line bundle $\mathcal{O}_{f_\infty}(e(\infty) + \zeta(\infty))$.

Choose now $F = f_\infty$ and $G^\vee$ to be one of the four square roots of $\mathcal{O}_{f_\infty}(e(\infty) + \zeta(\infty))$. Choose a non-zero map $s : A \to G^\vee$. Then $\tau_B^*|f_\infty s : A^\vee \to G^\vee$ is also a non-zero map and so, as before, $s \oplus \tau_B^*|f_\infty s : A \oplus A^\vee \to G^\vee$ is surjective. Using this map as the center of a Hecke transform, we get a $\tau_B$-invariant $W_2$. □

4.3 Invariance for $k_3 = 6$

Let $W_3 = FM_B(i_{C_3*}N_3)$ for some curve in $|3e + 6f|$. As we saw above, in this case, we can chose $C_3$ to be smooth and preserved by $\alpha_B$ and so in order to find $\tau_B$-invariant $W_3$'s we only need to show that for a general $C_3 \in |3e + 6f|$ the image $i_{C_3*} \text{Pic}(B)$ will be Zariski dense in $\text{Pic}^0(C_3)$.

The Zariski closure of the image $i_{C_3*} \text{Pic}(B)$ varies lower-semi-continuously with $C_3$, so it suffices to exhibit one good $C_3$. Our $C_3$ will be reducible, consisting of a generic $\alpha_B$-invariant curve $\overline{C}_2$ in the linear system $|2e + 2f|$, plus the zero section $e$, plus two generic fibers $\phi_1$ and $\phi_2$, plus their images $\phi_3 := \alpha_B(\phi_1)$ and $\phi_4 := \alpha_B(\phi_2)$.
The arithmetic genus of \( C_3 \) is 13. The 13-dimensional \( \text{Pic}^0(C_3) \) is a \((\mathbb{C}^\times)^7\) extension of the six dimensional abelian variety \( A := \text{Pic}^0(\overline{C}_2) \times \prod_{i=1}^4 \text{Pic}^0(\phi_i) \). So our density statement follows from the following two lemmas.

**Lemma 4.7.** For a generic choice of \( C_2, \phi_1, \phi_2 \), the image of \( i_{C_3*} \text{Pic}^0(B) \) in \( A \) is Zariski dense.

**Lemma 4.8.** For a generic choice of \( C_2, \phi_1, \phi_2 \), no proper subgroup of \( \text{Pic}^0(C_3) \) surjects onto \( A \).

**Proof of Lemma 4.7.** Under the genericity assumption in the hypothesis of the lemma, it is clear that there are no isogenies among \( \phi_1, \phi_2 \), and the two elliptic curves \( \text{Pic}^0(\overline{C}_2/\alpha_{\overline{C}_2}) \) and \( \text{Prym}(\overline{C}_2, \alpha_{\overline{C}_2}) \). So it suffices to prove density separately in each of the two dimensional abelian varieties \( \phi_1 \times \phi_3 \), \( \phi_2 \times \phi_4 \) and \( \text{Pic}^0(\overline{C}_2) \). Density in \( \text{Pic}^0(\overline{C}_2) \) was already proved in Lemma 4.5 and density in say \( \phi_1 \times \phi_3 \) was established during the proof of that result. The lemma is proven.

**Proof of Lemma 4.8.** Let \( B \) be a \( \mathbb{C}^\times \)-extension of an abelian variety \( A \). Such a \( B \) determines a line bundle \([B] \in \text{Pic}^0(A)\). A proper subgroup of \( B \) surjecting onto \( A \) will exist if and only if \([B]\) is torsion.

Similarly, our \((\mathbb{C}^\times)^7\)-extension \( \text{Pic}^0(C_3) \) will contain a proper subgroup surjecting onto \( A \) if and only there is a non-zero character \( \chi : (\mathbb{C}^\times)^7 \to \mathbb{C}^\times \) such that the associated line bundle \( L_\chi := \text{Pic}^0(C_3) \times_\chi \mathbb{C} \) over \( A \) is torsion.

Therefore it suffices to find seven characters \( \chi_1, \ldots, \chi_7 \) of \((\mathbb{C}^\times)^7\) such that the associated line bundles are linearly independent over \( \mathbb{Q} \). For this we will need an intrinsic description of the character lattice \( \Lambda \) of \((\mathbb{C}^\times)^7\) in terms of the geometry of the curve \( C_3 \). The singular set of \( C_3 \) is \( S = \{s_{ij} | i = 1, 2, 3, 4 \text{ and } j = 1, 2, 3\} \), where \( \phi_i \cap \overline{C}_2 = \{s_{ij}\}_{j=1}^2 \), and \( s_{i3} = \phi_i \cap e \). Here the singular points are labelled so that \( \alpha_B(s_{1j}) = s_{3j} \) and \( \alpha_B(s_{2j}) = s_{4j} \). Now the lattice \( \Lambda \) is explicitly described as:

\[
\Lambda = \text{ker}[\mathbb{Z}^S \to \pi_0(C_3 - S)].
\]
Here the map \( Z^S \to \pi_0(C_3 - S) \) sends the characteristic function \( \epsilon_{ij} \) of \( s_{ij} \) to the difference \( \phi_i - C_2 \) for \( j = 1, 2 \) and to \( \phi_i - e \) for \( j = 3 \). Our seven characters \( \chi_k \) are the following seven elements in \( Z^S \):

\[
\begin{align*}
\chi_1 &= \epsilon_{11} + \epsilon_{31} - \epsilon_{12} - \epsilon_{32}, \\
\chi_2 &= \epsilon_{21} + \epsilon_{41} - \epsilon_{22} - \epsilon_{42}, \\
\chi_3 &= \epsilon_{11} + \epsilon_{33} - \epsilon_{13} - \epsilon_{31}, \\
\chi_4 &= \epsilon_{21} + \epsilon_{43} - \epsilon_{23} - \epsilon_{41}, \\
\chi_5 &= \epsilon_{11} + \epsilon_{32} - \epsilon_{12} - \epsilon_{31}, \\
\chi_6 &= \epsilon_{21} + \epsilon_{42} - \epsilon_{22} - \epsilon_{41}, \\
\chi_7 &= \epsilon_{21} + \epsilon_{41} - \epsilon_{11} - \epsilon_{31} + \epsilon_{13} + \epsilon_{33} - \epsilon_{23} - \epsilon_{43}.
\end{align*}
\]

To prove the independence of the \( L_{\chi_k} \) over \( \mathbb{Q} \) it suffices to prove the independence of the restrictions \( \lambda_k := L_{\chi_k} | \prod_{i=1}^4 \phi_i \). We represent a degree zero line bundle \( \lambda \) on \( \prod_{i=1}^4 \phi_i \) by a four-tuple \( (\lambda^i \in \text{Pic}^0(\phi_i))_{i=1}^4 \). In this notation the \( \lambda_k \)'s become:

\[
\begin{align*}
\lambda_1 &= (a, 0, a, 0), \\
\lambda_2 &= (0, b, 0, b), \\
\lambda_3 &= (c, 0, -c, 0), \\
\lambda_4 &= (0, d, 0, -d), \\
\lambda_5 &= (a, 0, -a, 0), \\
\lambda_6 &= (0, b, 0, -b), \\
\lambda_7 &= (-c, d, -c, d),
\end{align*}
\]

where

\[
\begin{align*}
a &= \mathcal{O}_{\phi_1}(s_{11} - s_{12}) \in \text{Pic}^0(\phi_1) \cong \text{Pic}^0(\phi_3), \\
b &= \mathcal{O}_{\phi_2}(s_{21} - s_{22}) \in \text{Pic}^0(\phi_2) \cong \text{Pic}^0(\phi_4), \\
c &= \mathcal{O}_{\phi_1}(s_{11} - s_{13}) \in \text{Pic}^0(\phi_1) \cong \text{Pic}^0(\phi_3), \\
d &= \mathcal{O}_{\phi_2}(s_{21} - s_{23}) \in \text{Pic}^0(\phi_2) \cong \text{Pic}^0(\phi_4).
\end{align*}
\]

So we only need to show the linear independence over \( \mathbb{Q} \) of \( a, c \in \text{Pic}^0(\phi_1) \) and similarly for \( b, d \in \text{Pic}^0(\phi_2) \). This however is obvious from the fact that \( c, d \) are univalued as functions on the base \( \mathbb{P}^1 \), whereas \( a, b \) are two-valued. The lemma is proven. \( \square \)

5 Numerical conditions

Our goal is to construct a stable rank five holomorphic vector bundle on the Calabi-Yau manifold \( Z := X/\tau_X \) which has a trivial determinant, three generations and an anomaly which can be absorbed into \( M5 \)-branes. In
terms of $X$ this amounts to finding a rank five vector bundle $V \to X$ so that:

(S) $V$ is a stable vector bundle.

(I) $V$ is $\tau_X$-invariant.

(C1) $c_1(V) = 0$.

(C2) $c_2(X) - c_2(V)$ is effective.

(C3) $c_3(V) = 12$.

We will construct a whole family of $V$'s satisfying these conditions. As explained in section 3, each $V$ will be constructed as an extension

$$0 \to V_2 \to V \to V_3 \to 0,$$

where the $V_i$'s have special form. In fact, as we argued in section 3, in order to satisfy the condition (I) it is sufficient to take

$$V_i = FM_X \left( \Sigma_i, (\pi|_{\Sigma_i})^* N_i \otimes \pi^* L_i \otimes O_{\Sigma_i} \left( \sum_{k=1}^{i} a_{ik} \left( \{ p_{i1k} \} \times n'_i + \{ p_{i2k} \} \times o'_2 \right) \right) \right)$$

with $a_{ik}$ being positive integers, $\Sigma_i = \pi^* C_i$ for some smooth curve $C_i \subset B$ satisfying (4.1) and $N_i \in \text{Pic}^{d_i}(C_i)$ satisfying (4.2). In fact, we do not need $C_i$ to be smooth; it is sufficient for the pair $(C_i, N_i)$ to be deformable to a pair $(C'_i, N'_i)$ with $C'_i$ being smooth but not necessarily $\alpha_{B}$-invariant. Furthermore, we showed in section 3, that such $(C_i, N_i)$ do exist and move in positive dimensional families, at least for specific values of $k_i$. From now on we will assume that $C_i$ and $N_i$ are chosen so that they are deformable to a smooth pair and that (4.1) and (4.2) hold.

Next we will rewrite the conditions (S) and (C1-3) as a sequence of numerical constraints on the numbers $k_i$, $d_i$ and on the line bundles $L_i$. 
5.1 The Chern classes of $V$

There are several ways of calculating the Chern classes of the bundles $V_i$. One possibility is to use the cohomological Fourier-Mukai transform on $X$. To avoid long and cumbersome calculations of $fm_X$ we choose a slightly different approach which utilizes the details of the geometric structure of $V$.

Recall that in section 3 we gave an alternative description of the bundles $V_i$ as

$$V_i = \widetilde{W}_i \otimes \pi^*L_i,$$

where $\widetilde{W}_i$ is the result of $a_{ik}$, $k = 1, \ldots i$ Hecke transforms of the bundle $\pi^*W_i$, where $W_i = FM_B((C_i,N_i))$. Due to Corollary 3.3 and the identity (3.6) we have

$$ch(\widetilde{W}_i) = \pi^*ch(W_i) - \left( \sum_{k=1}^{i} a_{ik} \right) \pi^*(n_1' + o_2') - 2 \left( \sum_{k=1}^{i} a_{ik}^2 \right) (f \times pt).$$

Next we need the following

**Lemma 5.1.** Let $C \subset B$ be a smooth curve in the linear system $|O_B(re + mf)|$ and let $N \in Pic^d(C)$. Let $W = FM_B((C,N))$. Then

$$ch(W) = r + \left( d + \left( \frac{r+1}{2} \right) - rm - r \right) f - m \cdot pt.$$

**Proof.** The bundle $W$ is defined as the Fourier-Mukai transform of the spectral datum $(C,N)$ on $B$. Explicitly this means that $W = FM_B(i_{C*}N)$, where $i_C : C \hookrightarrow B$ is the inclusion map. In particular

$$ch(W) = fm_B(ch(i_{C*}N))$$

and so it suffices to calculate $ch(i_{C*}N)$.

By Groethendieck-Riemann-Roch theorem we have

$$ch(i_{C*}N) = (i_{C*}(ch(N)td(C)))td(B)^{-1}$$

$$= (i_{C*}(1 + (d + 1 - g) \cdot pt)) \cdot \left( 1 + \frac{1}{2} f + pt \right)^{-1}$$

$$= ((re + mf) + (d + 1 - g) \cdot pt) \cdot \left( 1 - \frac{1}{2} f - pt \right)$$

$$= (re + mf) + \left( d + 1 - g - \frac{r}{2} \right) \cdot pt.$$
Also by adjunction we have \( 2g - 2 = C \cdot (K_B + C) = 2rm - r^2 - r \). Hence

\[
ch(i_{C*N}) = (re + mf) + \left( d + \left( \frac{r + 1}{2} \right) - rm - \frac{r}{2} \right) \cdot pt.
\]

Finally using [DOPWa, Table 2] we get

\[
ch(W) = fm_B(ch(i_{C*N}));
\]

\[
= fm_B \left( (re + mf) + \left( d + \left( \frac{r + 1}{2} \right) - rm - \frac{r}{2} \right) \cdot pt \right)
\]

\[
= r \left( 1 - \frac{1}{2} f \right) + m \cdot (pt) + \left( d + \left( \frac{r + 1}{2} \right) - rm - \frac{r}{2} \right) \cdot f
\]

\[
= r + \left( d + \left( \frac{r + 1}{2} \right) - rm - r \right) f - m \cdot pt.
\]

The lemma is proven. \( \square \)

**Remark 5.2.** Since the Chern classes are topological invariants, the conclusion of the previous lemma still holds even if \( C \) is singular, as long as the sheaf \( i_{C*N} \) deforms flatly to a line bundle on some smooth spectral curve in the same linear system. In particular it will hold for the \( W_2 \) from section 4.2.

Going back to the calculation of \( ch(V) \) let us write \( S_i^\alpha \) for the Newton sums

\[
S_i^\alpha = \sum_{k=1}^{i} a_{ik}^\alpha.
\]

In this notation we need to calculate the product

\[
ch(V_i) = (\pi' \cdot ch(W_i) - S_i^1 \pi^* (n_1' + o_2') - 2S_i^2 (f \times pt)) \cdot ch(\pi^* L_i).
\]

But

\[
ch(\pi^* L_i) = \pi^* ch(L_i) = 1 + \pi^* L_i + \frac{L_i^2}{2} (f \times pt),
\]

and so

- \( \pi^* (n_1' + o_2') \cdot ch(\pi^* L_i) = \pi^* (n_1' + o_2') + (L_i \cdot (n_1' + o_2')) \cdot (f \times pt) \);

- \( (f \times pt) \cdot ch(\pi^* L_i) = f \times pt \);
Lemma 5.1 yields

\[ \pi^* \text{ch}(W_i) \cdot \text{ch}(\pi^* L_i) = \]
\[ = \pi^* \left( i + \left( d_i + \frac{i+1}{2} \right) - k_i i - i \right) \cdot f - k_i \cdot pt \right) \cdot \left( 1 + \pi^* L_i + \frac{L_i^2}{2} (f \times pt) \right) \]
\[ = i + \pi^* \left( iL_i + \left( d_i + \frac{i+1}{2} \right) - k_i i - i \right) f' + \]
\[ + \left[ \left( iL_i^2 + \left( d_i + \frac{i+1}{2} \right) - k_i i - i \right) \cdot (L_i \cdot f') \right] (f \times pt) - k_i (pt \times f') \]
\[ - k_i (L_i \cdot f') pt. \]

Combining these formulas we get formulas for \( \text{ch}(V_2) \) and \( \text{ch}(V_3) \):

\[
\text{ch}(V_2) = 2 + \pi^* (2L_2 + (d_2 - 2k_2 + 1)f' - S_2^1 (n_1' + o_2')) + (L_2^2 + (d_2 - 2k_2 + 1)(L_2 \cdot f') - S_2^1(L_2 \cdot (n_1' + o_2')) - 2S_2^2) \cdot (f \times pt) - k_2 (pt \times f') \]
\[ - k_2 (L_2 \cdot f') pt. \]

and similarly

\[
\text{ch}(V_3) = 3 + \pi^* (3L_3 + (d_3 - 3k_3 + 3)f' - S_3^1(n_1' + o_2')) + \left( \frac{3}{2}L_3^2 + (d_3 - 3k_3 + 3)(L_3 \cdot f') - S_3^1(L_3 \cdot (n_1' + o_2')) - 2S_3^2 \right) \cdot (f \times pt) \]
\[ - k_3 (pt \times f') \]
\[ - k_3 (L_3 \cdot f') pt. \]

Together these formulas give

\[
\text{ch}(V) = 5 +
 + \pi^* (2L_2 + 3L_3 + (d_2 + d_3 - 2k_2 - 3k_3 + 4)f' - (S_2^1 + S_3^1)(n_1' + o_2')) + [(L_2^2 + (3/2)L_3^2 + (d_2 - 2k_2 + 1)(L_2 \cdot f') + (d_3 - 3k_3 + 3)(L_3 \cdot f') \]
\[ - (S_2^1L_2 + S_3^1L_3) \cdot (n_1' + o_2') - 2(S_2^2 + S_3^2))(f \times pt) \]
\[ - (k_2 + k_3)(pt \times f')) \]
\[ - ((k_2L_2 + k_3L_3) \cdot f') pt. \]

Therefore, taking into account that \( c_2(X) = 12(f \times pt + pt \times f') \) we see that the conditions \((\text{C1-3})\) translate into the following numerical constraints:

\[(\#\text{C1}) \ 2L_2 + 3L_3 = (S_2^1 + S_3^1)(n_1' + o_2') - (d_2 + d_3 - 2k_2 - 3k_3 + 4)f'. \]
Our next task is to express the stability of $V$ in a numerical form.

5.2 Stability of $V$

We need to make sure that the bundle $V$ is Mumford stable with respect to some ample class $H \in H^2(X, \mathbb{Z})$. Recall (see e.g. [FMW99, Section 7]) that a polarization $H \in H^2(X, \mathbb{Z})$ is called suitable if up to an overall scale the components of the fibers of $\pi : X \to B'$ have sufficiently small volume. Starting with any polarization $H_0 \in H^2(X, \mathbb{Z})$ we can construct a suitable polarization by fixing some polarization $h' \in H^2(B', \mathbb{Z})$ and taking

$$H := H_0 + n \cdot \pi^* h',$$

with $n \gg 0$. As explained in [FMW99, Theorem 7.1] for a suitable $H$ every vector bundle on $X$ which comes from a spectral cover will be $H$-stable on $X$. From now on we will always assume that $H = H_0 + n \cdot \pi^* h'$ is chosen to be suitable. For a torsion free sheaf $\mathcal{F}$ on $X$ denote by $\mu_H(\mathcal{F})$ the $H$-slope of $\mathcal{F}$, i.e. $\mu_H(\mathcal{F}) = (c_1(\mathcal{F}) \cdot H^2) / \text{rk}(\mathcal{F})$. By repeating the argument in the proof of [FMW99, Theorem 7.1] one gets the following lemma whose proof is left to the reader.

**Lemma 5.3.** The bundle $V$ constructed in the previous section is $H$-stable if and only if

(i) The extension

$$0 \to V_2 \to V \to V_3 \to 0,$$

is non-split.

(ii) $\mu_H(V_2) < \mu_H(V) = 0$.

Next we express both these conditions into a numerical form. Note that an extension

$$0 \to V_2 \to V \to V_3 \to 0$$
will be non-split if and only if it corresponds to a non-zero element in
\( \text{Ext}^1(V_3, V_2) = H^1(X, V_3^\vee \otimes V_2) \). Thus we only need to ensure that \( H^1(X, V_3^\vee \otimes V_2) \neq 0 \). We have the following

**Lemma 5.4.** For \( V_2 \) and \( V_3 \) as above one has \( H^1(X, V_3^\vee \otimes V_2) \neq 0 \) if \( L_2 \cdot f' > L_3 \cdot f' \).

**Proof.** We are assuming that the \( W_i \)'s deform to vector bundles on \( B \) coming from smooth spectral covers. So by the upper-semi-continuity of \( H^1(X, V_2^\vee \otimes V_3) \) it is enough to prove the lemma for \( W_i \)'s arising from smooth spectral covers.

Let \( L = L_3^{-1} \otimes L_2 \). Then

\[
V_3^\vee \otimes V_2 = \widetilde{W}_3^\vee \otimes \widetilde{W}_2 \otimes \pi^* L.
\]

To calculate \( H^1(X, \widetilde{W}_3^\vee \otimes \widetilde{W}_2 \otimes \pi^* L) \) we use the Leray spectral sequence for the projection \( \pi : X \to B' \). It yields an exact sequence of vector spaces

\[
\begin{array}{c}
0 \rightarrow H^1(B', \pi_* (\widetilde{W}_3^\vee \otimes \widetilde{W}_2) \otimes L) \rightarrow H^1(X, V_3^\vee \otimes V_2) \\
H^0(B', R^1 \pi_* (\widetilde{W}_3^\vee \otimes \widetilde{W}_2) \otimes L) \rightarrow H^2(B', \pi_* (\widetilde{W}_3^\vee \otimes \widetilde{W}_2) \otimes L).
\end{array}
\]

By construction \( \widetilde{W}_2 \) and \( \widetilde{W}_3 \) are vector bundles on \( X \) coming from the spectral construction applied to line bundles on smooth spectral covers, which are also finite over \( B' \). In particular the restriction of \( \widetilde{W}_3^\vee \otimes \widetilde{W}_2 \) to the general fiber of \( \pi \) is regular and semistable of degree zero. Hence for general \( \widetilde{W}_2 \) and \( \widetilde{W}_3 \) we have

- \( \pi_* (\widetilde{W}_3^\vee \otimes \widetilde{W}_2) = 0 \);
- \( R^1 \pi_* (\widetilde{W}_3^\vee \otimes \widetilde{W}_2) \) is supported on a curve in \( B' \) and is a line bundle on its support.

Therefore

\[
H^1(X, V_3^\vee \otimes V_2) = H^0(B', R^1 \pi_* (\widetilde{W}_3^\vee \otimes \widetilde{W}_2) \otimes L).
\]

To calculate the latter space notice that \( R^1 \pi_* (\widetilde{W}_3^\vee \otimes \widetilde{W}_2) \) is supported on the curve

\[
\pi((-1)^X \Sigma_3 \cap \Sigma_2) \subset B'.
\]

Since \( \Sigma_i = \pi'^* C_i \), this implies

\[
(-1)^X \Sigma_3 \cap \Sigma_2 = \prod_{t \in (-1)^X C_3 \cap C_2} \{ t \} \times f'.
\]
Without a loss of generality we may assume that all \( t \in (-1)^B_c C_3 \cap C_2 \) project to distinct points in \( \mathbb{P}^1 \) under the map \( \beta : B \to \mathbb{P}^1 \). Consequently

\[
R^1 \pi_* (\tilde{W}_3^\vee \otimes \tilde{W}_2) = \oplus_{t \in (-1)^B_c C_3 \cap C_2} \Phi_t,
\]

where \( \Phi_t \) is a line bundle on the curve \( \pi(\{t\} \times f') = f'_{\beta(t)} \). The line bundle \( \Phi_t \) depends only on the the restriction of \( \tilde{W}_3^\vee \otimes \tilde{W}_2 \) to the surface \( f_{\beta(t)} \times f'_{\beta(t)} \).

Let \( \text{pr}_1 : f_{\beta(t)} \times f'_{\beta(t)} \to f_{\beta(t)} \) and \( \text{pr}_2 : f_{\beta(t)} \times f'_{\beta(t)} \to f'_{\beta(t)} \) denote the natural projections. Then we have

\[
\Phi_t = R^1 \text{pr}_2^* ((\tilde{W}_3^\vee \otimes \tilde{W}_2)|_{f_{\beta(t)} \times f'_{\beta(t)}}) = R^1 \text{pr}_2^* (\pi_1^* (W_3^\vee \otimes W_2))
\]

\[
= R^1 \text{pr}_2^* (\pi_1^* \mathcal{O}_{f_{\beta(t)}}) = H^1 (f_{\beta(t)}, \mathcal{O}_{f_{\beta(t)}}) \otimes \mathcal{O}_{f'_{\beta(t)}}.
\]

In other words \( H^1 (X, V_3^\vee \otimes V_2) \neq 0 \) if and only if \( L_{|f_{\beta(t)}'} \) is effective for some \( t \in (-1)^B_c C_3 \cap C_2 \). For this it suffices to have \( L \cdot f' > 0 \), and it is necessary to have \( L \cdot f'' \geq 0 \).

However we saw in the previous section that the condition (C1) implies \( 2L_2 \cdot f' + 3L_3 \cdot f'' = 0 \). If we assume that \( L \cdot f'' = 0 \), then we will get \( L_2 \cdot f' = L_3 \cdot f'' = 0 \) which contradicts (C3). Thus \( H^1 (X, V_3^\vee \otimes V_2) \neq 0 \) iff \( L \cdot f' > 0 \). The lemma is proven. \( \square \)

Expressing the slope condition in numerical terms is completely straightforward. In the previous section we showed that

\[
c_1 (V_2) = \pi^* (2L_2 + (d_2 + 1 - 2k_2) f' - S_2^1 (n_1' + h')).
\]

Hence \( \mu_H (V_2) < 0 \) if and only if

\[
\pi^* (2L_2 + (d_2 + 1 - 2k_2) f' - S_2^1 (n_1' + h')) \cdot H^2 < 0.
\]

It is more convenient to rewrite this as a condition on the surface \( B' \). Recall that we take \( H \) to be of the form

\[
H = H_0 + n \cdot \pi^* h'
\]

where \( h' \in H^2 (B', \mathbb{Z}) \) is some polarization and \( n \gg 0 \). Since \( X = B \times \mathbb{P}^1 B' \) we see that any polarization \( H_0 \) on \( X \) can be written as

\[
H_0 = \pi^* h_0 + \pi^* h'_0
\]

for some polarizations \( h_0 \in H^2 (B, \mathbb{Z}) \) and \( h'_0 \in H^2 (B', \mathbb{Z}) \). In particular

\[
H^2 = (\pi^* h_0 + \pi^* h'_0)^2 + 2n (\pi^* h_0 + \pi^* h'_0) \pi^* h' + n^2 \pi^* (h'^2)
\]

\[
= (h_0^2) \cdot (\{pt\} \times f') + 2 (\pi^* h'_0 + n \cdot \pi^* h') \pi^* h_0 + \pi^* ((h'_0 + n \cdot h')^2).
\]
By the projection formula we get

\[ \mu_H(V_2) = \pi^*(2L_2 + (d_2 + 1 - 2k_2)f' - S_2^1(n'_1 + o'_2)) \cdot H^2 \]

\[ = (2L_2 + (d_2 + 1 - 2k_2)f' - S_2^1(n'_1 + o'_2)) \cdot \pi_*(H^2) \]

\[ = (2L_2 + (d_2 + 1 - 2k_2)f' - S_2^1(n'_1 + o'_2)) \]

\[ \cdot ((h'_0)f' + 2(h'_0 + n \cdot h') \pi_* \pi^* h_0) \]

\[ = (2L_2 + (d_2 + 1 - 2k_2)f' - S_2^1(n'_1 + o'_2)) \cdot ((h'_0)f' + 2(h_0 \cdot f) h'_0) \]

\[ + 2(h_0 \cdot f)n(2L_2 + (d_2 + 1 - 2k_2)f' - S_2^1(n'_1 + o'_2)) \cdot h'. \]

To derive the last identity we used (3.1) to write

\[ \pi_* \pi^* h_0 = \beta^* \beta_* h_0 = (h_0 \cdot f) \cdot 1 + mf' \in H^* (B', \mathbb{Z}), \]

with \( m \) being a positive integer. This implies that \( \alpha \cdot \pi_* \pi^* h_0 = (h_0 \cdot f)(\alpha \cdot 1) + m(\alpha \cdot f') \) for any cohomology class \( \alpha \) on \( B' \). In particular \( pt \cdot \pi_* \pi^* h_0 = (h_0 \cdot f) pt \) and hence the above formula.

In conclusion, we see that for \( n \gg 0 \) we have

\[ \mu_H(V_2) < 2(h_0 \cdot f)n(2L_2 + (d_2 + 1 - 2k_2)f' - S_2^1(n'_1 + o'_2)) \cdot h', \]

and so \( \mu_H(V_2) < 0 \) provided that

\[ (2L_2 + (d_2 + 1 - 2k_2)f' - S_2^1(n'_1 + o'_2)) \cdot h' < 0. \]

We are now ready to list all the conditions on \( V \) in a numerical form.

### 5.3 The list of constraints

In the previous two sections we translated the conditions (S), (I), (C1-3) into a set of numerical conditions. Together those read:

\[(\#Se)\] \( L_2 \cdot f' > L_3 \cdot f' \).

\[(\#Ss)\] \( (2L_2 + (d_2 + 1 - 2k_2)f' - S_2^1(n'_1 + o'_2)) \cdot h' < 0 \) for some ample class \( h' \in \text{Pic}(B') \).

\[(\#I)\] \( \tau_B^* L_i = L_i \), for \( i = 2, 3 \).

\[(\#C1)\] \( 2L_2 + 3L_3 = (S_2^1 + S_3^1)(n'_1 + o'_2) - (d_2 + d_3 - 2k_2 - 3k_3 + 4)f' \).

\[(\#C2f)\] \( k_2 + k_3 \leq 12 \).
(\#C2\ f') \ L_2^2 + (3/2)L_3^2 + (d_2 - 2k_2 + 1)(L_2 \cdot f') + (d_3 - 3k_3 + 3)(L_3 \cdot f') - (S_2^2 L_2 + S_3^1 L_3)(n_1' + d_2') - 2(S_2^2 + S_3^2) \geq -12.

(\#C3) \ k_2(L_2 \cdot f') + k_3(L_3 \cdot f') = -6.

Observe that these conditions already constrain severely the possible values of \( k_2, k_3, L_2 \cdot f' \) and \( L_3 \cdot f' \). Indeed, intersecting both sides of (\#C1) with the curve \( f' \subset B' \) we see that \( 2(L_2 \cdot f') + 3(L_3 \cdot f') = 0 \). Recall also that we showed in section 4 that for the existence of smooth curves \( C_2 \) and \( C_3 \) one needs to take \( k_2 \geq 2 \) and \( k_3 \geq 3 \). Thus the integers \( k_2, k_3, L_2 \cdot f' \) and \( L_3 \cdot f' \) should satisfy:

- \( L_2 \cdot f' > L_3 \cdot f' \)
- \( k_2 \geq 2 \) and \( k_3 \geq 3 \);
- \( k_2 + k_3 \leq 12 \);
- \( 2(L_2 \cdot f') + 3(L_3 \cdot f') = 0 \);
- \( k_2(L_2 \cdot f') + k_3(L_3 \cdot f') = -6 \).

Solving these, we find the following finite list of values for \( k_2, k_3, L_2 \cdot f' \) and \( L_3 \cdot f' \).

<table>
<thead>
<tr>
<th>( k_2 )</th>
<th>( k_3 )</th>
<th>( L_2 \cdot f' )</th>
<th>( L_3 \cdot f' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>9</td>
<td>-6</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>3</td>
<td>-2</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>18</td>
<td>-12</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>6</td>
<td>-4</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>9</td>
<td>-6</td>
</tr>
</tbody>
</table>

Table 1: Possible values of \( k_2, k_3, L_2 \cdot f', L_3 \cdot f' \)

5.4 Some solutions

In this section we show that the numerical constraints (\#) can all be satisfied. In fact, we find infinitely many solutions of (\#). These represent an infinite sequence of moduli spaces (of arbitrarily large dimension) of all possible \( V \)'s.
Fix $k_2$ and $k_3$ from the values in Table 1. For such a choice the corresponding numbers $L_2 \cdot f'$ and $L_3 \cdot f'$ are the solutions to the linear system
\[
\begin{pmatrix}
2 & 3 \\
k_2 & k_3
\end{pmatrix}
\begin{pmatrix}
L_2 \cdot f' \\
L_3 \cdot f'
\end{pmatrix}
=
\begin{pmatrix}
0 \\
-6
\end{pmatrix}.
\]
Thus in terms of $k_2$ and $k_3$ we have
\[
L_2 \cdot f' = \frac{18}{2k_3 - 3k_2}, \quad L_3 \cdot f' = \frac{-12}{2k_3 - 3k_2}.
\]
Put $k = 2k_3 - 3k_2$. Express $L_i$, $i = 2, 3$ in terms of the standard classes on $B'$ as follows
\[
L_2 = \frac{9}{k}(e' + \zeta') + x_2 f' + y_2(n'_1 + o'_2) + 3M
\]
\[
L_3 = \frac{6}{k}(e' + \zeta') + x_3 f' + y_3(n'_1 + o'_2) - 2M.
\]
The most general way to make the $L_i$'s satisfy (#I) together with (5.2) is to take $M$ to be $\tau_B$-invariant and perpendicular to $e' + \zeta'$, $f'$ and $n'_1 + o'_2$. This follows from the fact that the intersection form on $H^2(B', \mathbb{Z})$ is non-degenerate on the span of $e' + \zeta'$, $f'$ and $n'_1 + o'_2$. This is evident from Table 2

<table>
<thead>
<tr>
<th></th>
<th>$e' + \zeta'$</th>
<th>$f'$</th>
<th>$n'_1 + o'_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e' + \zeta'$</td>
<td>-2</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$f'$</td>
<td>2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$n'_1 + o'_2$</td>
<td>2</td>
<td>0</td>
<td>-4</td>
</tr>
</tbody>
</table>

Table 2: Intersection pairing on $e' + \zeta'$, $f'$ and $n'_1 + o'_2$

The condition (#C1) translates into
\[
\begin{align*}
2x_2 + 3x_3 &= -(d_2 + d_3 - 2k_2 - 3k_3 + 4) \\
2y_2 + 3y_3 &= S_2^1 + S_3^1.
\end{align*}
\]
Using Table 2 we compute
\[
L_2^2 = -\frac{162}{k^2} - 4y_2^2 + 9M^2 + \frac{36}{k}(x_2 + y_2)
\]
\[
L_3^2 = -\frac{72}{k^2} - 4y_3^2 + 4M^2 - \frac{24}{k}(x_3 + y_3)
\]
\[
(n'_1 + o'_2) \cdot L_2 = +\frac{18}{k} - 4y_2
\]
\[
(n'_1 + o'_2) \cdot L_3 = -\frac{12}{k} - 4y_3.
\]
So the condition \((\#C2f')\) becomes

\[
\frac{-270}{k^2} - 4y_2^2 - 6y_3^2 + 15M^2 + \frac{36}{k}(x_2 + y_2 - x_3 - y_3)
\]

\[
+ \frac{1}{k}(18d_2 - 12d_3 - 36k_2 + 36k_3 - 18)
\]

\[
- S_2^1 \left( \frac{18}{k} - 4y_2 \right) - S_3^1 \left( \frac{12}{k} - 4y_3 \right) - 2(S_2^2 + S_3^2) \geq -12.
\]

We eliminate \(d_3\) using (5.3) and complete the squares involving \(y_2\) and \(y_3\), to find:

\[
\frac{-135}{k^2} - 4u_2^2 - 6u_3^2 + 15M^2 + ((S_2^1)^2 - 2S_2^2) + \left( \frac{2}{3}(S_3^1)^2 - 2S_3^2 \right)
\]

\[+
\frac{30}{k}(2x_2 + d_2 - 2k_2 + 1) \geq -12,
\]

where

\[
u_2 = y_2 - \frac{1}{2} \left( \frac{9}{k} + S_2^1 \right)
\]

\[
u_3 = y_3 - \frac{1}{3} \left( -\frac{9}{k} + S_3^1 \right).
\]

Implementing the second condition in (5.3) we get \(2u_2 + 3u_3 = 0\). Introduce new variables

\[
u := 2u_2 = -3u_3
\]

\[
x := 2x_2 + d_2 - 2k_2 + 1.
\]

Substituting back into the expressions for \(L_2\) and \(L_3\) we get

\[
L_2 = \frac{9}{k}(\epsilon' + \zeta') + \frac{1}{2}(x - d_2 + 2k_2 - 1)f'
\]

\[+
\frac{1}{2}\left( u + \frac{9}{k} + S_2^1 \right)(n_1' + o_2') + 3M
\]

\[\tag{5.4}
L_3 = -\frac{6}{k}(\epsilon' + \zeta') + \frac{1}{3}(-x - d_3 + 3k_3 - 3)f'
\]

\[+
\frac{1}{3}\left( -u - \frac{9}{k} + S_3^1 \right)(n_1' + o_2') - 2M.
\]

Similarly for the conditions \((\#C2f')\) and \((\#Ss)\) we get

\[
\frac{5}{3}u^2 - 15M^2 \leq 12 - \frac{135}{k^2} + \frac{30}{k}x + ((S_2^1)^2 - 2S_2^2) + \left( \frac{2}{3}(S_3^1)^2 - 2S_3^2 \right),
\]

\(\tag{5.5}
\)
and

\[(5.6) \quad \left( \frac{18}{k} (e' + \zeta') + x f' + \left( u + \frac{9}{k} \right) (n_1' + o_2') + 6M \right) \cdot h' < 0. \]

respectively.

We will use the flexibility we have in choosing \( M \) to show that (5.5) and (5.6) have solutions that lead to integral coefficients in (5.4). The key observation here is that since \( \text{span}(e' + \zeta', f', n_1' + o_2') \subset H^2(B', \mathbb{Z}) \) contains an ample class, the Hodge index theorem implies that \( M^2 \leq 0 \). Therefore, one expects that there will be non-effective admissible \( M \)'s which will make (5.6) easier to satisfy.

Note that the means inequality implies that \((S_2^1)^2 - 2S_2^2 \leq 0\) with equality if and only if all the \( a_{2k} \)'s are equal to each other. Similarly \((2/3)(S_3^1)^2 - 2S_3^2 \leq 0\) with equality if and only if all the \( a_{3k} \)'s are equal to each other. In particular, for any choice of numbers \( u, x, k_2, k_3, a_{ik} \) which satisfies (5.5), the numbers \( u, x, k_2, k_3 \) will satisfy

\[(5.7) \quad \frac{5}{3} u^2 \leq 12 + 15M^2 - \frac{135}{k^2} + \frac{30}{k} x \]

as well.

But from Table 1 we see that \( k = 2k_3 - 3k_2 > 0 \) for all admissible values of \( k_2 \) and \( k_3 \). Combined with the fact that \( e' + \zeta' \) is an effective curve this implies

\[\left( x f' + \left( u + \frac{9}{k} \right) (n_1' + o_2') + 6M \right) \cdot h' < 0. \]

On the other hand \( f' = n_j' + o_j' \) for \( j = 1, 2 \) and so \( f' \cdot h' > n_1' \cdot h' \) and \( f' \cdot h' > o_1' \cdot h' \). Thus it suffices to check that

\[(5.8) \quad \left( \left( x + u + \frac{9}{k} \right) f' + 6M \right) \cdot h' < 0. \]

To make things more concrete recall that the only conditions that we need to impose on \( M \) are that \( M \) should be \( \tau_{B'} \)-invariant and that \( M \) should be perpendicular to \( \text{span}(e' + \zeta', f', n_1' + o_2') \). From [DOPWa, Table 1] we see that the classes \( e_4' - e_5', 3\ell' - 2(e_4' + e_5') - 3e_7' \) constitute a rational basis of the space of such \( M \)'s. Let us choose for example \( M \) to be of the form

\[ M = z(e_4' - e_5') \]
for some integer $z$. With this choice (5.7) and (5.8) become

$$\frac{5}{3} u^2 + 30z^2 - \frac{30}{k} x + \frac{135}{k^2} - 12 \leq 0,$$

and

$$\left( x + u + \frac{9}{k} \right) f' + 6z (e'_4 - e'_5) \cdot h' < 0$$

respectively.

Consider next the class $\gamma := (x+u+9/k) f' + 6z(e'_4 - e'_5)$. Since the Kähler cone is dual to the Mori cone we know that an ample class $h'$ with $\gamma \cdot h' < 0$ will exist as long as $\gamma$ is not effective. But $\gamma$ satisfies $\gamma \cdot e'_4 = x + u + 9/k - 6z$ and so if $6z > x + u + 9/k$ we will have $\gamma \cdot e'_4 < 0$. Under this assumption we have two alternatives: either $\gamma$ is not effective or $\gamma - e'_4$ is effective. However we have $(\gamma - e'_4) \cdot f' = -1$ and $f'$ moves, so $\gamma - e'_4$ and hence $\gamma$ can not be effective.

In other words, as a first check for the consistency of the inequalities (5.5) and (5.6) it suffices to make sure that in the 3-space with coordinates $(x, u, z)$ one can find points between the plane

$$6z = x + u + \frac{9}{k}$$

and the paraboloid

$$\frac{5}{3} u^2 + 30z^2 - \frac{30}{k} x + \frac{135}{k^2} - 12 = 0.$$

If we use the equation of the plane to eliminate $x$ and substitute the result in (5.9) we obtain the quadratic inequality

$$\frac{5}{3} \left( u + \frac{9}{k} \right)^2 + 30 \left( z - \frac{3}{k} \right)^2 - 12 \leq 0,$$

which always has solutions regardless of the value of $k$.

To find an actual solution we will choose a particular value for $k$. By examining Table 1 we see that the possible values of $k = 2k_3 - 3k_2$ are $1, 2, 3$ and $6$. Furthermore, since all the coefficients in (5.4) must be integers, $k$ has to divide gcd$(6, 9) = 3$ i.e. we may have either $k = 1$ (which corresponds to $k_2 = 3$ and $k_3 = 5$) or $k = 3$ (which corresponds to $k_2 = 3$ and $k_3 = 6$. For concreteness we choose the second case, i.e.

$$k_2 = 3, \quad k_3 = 6, \quad k = 3.$$
Note that the geometry of this case has already been carefully analyzed in sections 4.2 and 4.3. In particular we showed that for these values of \( k_i \) there are spectral pairs \((C_i, N_i)\) which are deformable to smooth pairs and which satisfy (4.1) and (4.2).

To minimize (5.11) we will take

\[
\begin{align*}
u &= -3, & z &= 1, & x &= 5,
\end{align*}
\]

where the value of \( x \) is chosen to satisfy (5.10).

We then have

\[
L_2 = 3 (e' + \zeta') + \frac{1}{2} (4 - d_2) f' + \frac{1}{2} (6 + S_2^1) (n'_1 + o'_2) + 3 (e'_4 - e'_5)
\]

\[
L_3 = -2 (e' + \zeta') + \frac{1}{3} (16 - d_3) f' + \frac{1}{3} (-6 + S_3^1) (n'_1 + o'_2) - 2 (e'_4 - e'_5).
\]

We see that all the coefficients in (5.12) will be integral as long as: \( d_2 \) is even, \( d_3 \equiv 1 \pmod{3} \), \( S_2^1 \) is even and \( S_3^1 \) is divisible by 3.

The inequality (5.5) now reads

\[
-2 \leq ((S_2^1)^2 - 2S_2^2) + \left(\frac{2}{3}(S_2^1)^2 - 2S_2^2\right)
\]

and the inequality (5.6) reads

\[
(6 (e' + \zeta') + 5f' + 6 (e'_4 - e'_5)) : h' < 0
\]

Note that (5.14) does not involve the numbers \( a_{ik} \) and so all the restrictions on the \( a_{ik} \)’s come from (5.13) and from the integrality of (5.12).

In view of the discussion about the means inequality above we see that (5.13) will be automatically satisfied if we take all \( a_{2k} \)’s to be equal to a fixed integer \( a_2 \geq 0 \) and all \( a_{3k} \)’s to be equal to another fixed integer \( a_3 \geq 0 \). Moreover with such a choice we clearly have \( S_2^1 = 2a_2 \) and \( S_3^1 = 3a_3 \) and so we have infinitely many possibilities for the numbers \( a_{ik} \). Since \( d_2 \) and \( d_3 \) are unconstrained except for the conditions \( d_2 \equiv 0 \pmod{2} \) and \( d_3 \equiv 1 \pmod{3} \) we see that all the conditions (S), (I) and (C1-3) will be satisfied if we can prove the following:

**Claim 5.5.** There exists an ample class \( h' \in \text{Pic}(B') \) satisfying (5.14).

**Proof.** As explained above the existence of \( h' \) is equivalent to showing that the class

\[
6(e' + \zeta') + 5f' + 6(e'_4 - e'_5) = 6(e'_1 + e'_3) + 5f' + 6(e'_4 - e'_5)
\]
is not in the Mori cone of $B'$. First consider the class $\xi' := e'_4 - e'_5 + e'_0 + f' \in \text{Pic}(B')$. We have $\xi'^2 = -1$ and $\xi' \cdot f' = 1$. So $\xi'$ is an exceptional class on $B'$. It is well known [DPT80] that on a general rational elliptic surface every exceptional class is effective and is a section. Since our $B'$ is not generic we can’t use this statement to conclude that $\xi'$ is the class of a section. However we have

**Lemma 5.6.** The divisor $\xi'$ satisfies

$$\mathcal{O}_{B'}(\xi') = c_1([e'_4] - [e'_5]).$$

In particular $\xi'$ is effective and is a section of $\beta' : B' \to \mathbb{P}^1$.

**Proof.** Let $\xi'$ be the section of $B'$ for which $[\xi'] = [e'_4] - [e'_5]$, that is $c_1([e'_4] - [e'_5]) = \mathcal{O}_{B'}(\xi')$. Since the group law on the general fiber $f'_t$, $t \in \mathbb{P}^1$ is defined in terms of the Abel-Jacobi map and since we have taken $e'_0(t) \in f'_t$ to be the neutral element for the group law it follows that

$$\mathcal{O}_{B'}(\xi' - e'_0)|_{f'_t} = (c_1([e'_4] - [e'_5]) \otimes \mathcal{O}_{B'}(-e'_0))|_{f'_t} = \mathcal{O}_{B'}(e'_4 - e'_5)|_{f'_t}$$

for the general $t \in \mathbb{P}^1$. Therefore the line bundle $\mathcal{O}_{B'}(\xi' + e'_5 - e'_4 - e'_0)$ must be a combination of vertical divisors on $B'$, i.e. we can write

$$\xi' = e'_4 - e'_5 + e'_0 + a \cdot f' + b \cdot n'_1 + c \cdot n'_2$$

for some integers $a$, $b$ and $c$. By [DOPWa, Formula (4.2)] we have $e'_4 \cdot n'_i = e'_5 \cdot n'_i = 0$ and $e'_0 \cdot n'_i = 1$ for $i = 1, 2$. Consider the $I_2$ fiber $n'_1 \cup o'_1$ of $B'$. The smooth part $(n'_1 \cup o'_1)^\# := (n'_1 \cup o'_1) - (n'_1 \cap o'_1)$ of this fiber is an abelian group isomorphic to $\mathbb{Z}/2 \times \mathbb{C}^\times$ with $n'_1 - (n'_1 \cap o'_1)$ being the connected component of the identity. By definition the section $\xi'$ intersects $n'_1 \cup o'_1$ at a point which is the difference of the points $e'_4 \cdot (n'_1 \cup o'_1) = e'_4 \cdot o'_1$ and $e'_5 \cdot (n'_1 \cup o'_1) = e'_5 \cdot o'_1$ in the group law of $(n'_1 \cup o'_1)^\#$. Since these two points belong to the same component of $(n'_1 \cup o'_1)^\#$ and the group of connected components of $(n'_1 \cup o'_1)^\#$ is $\mathbb{Z}/2$ it follows that $\xi'$ intersects $n'_1 \cup o'_1$ at a point in $n'_1$, i.e. $\xi' \cdot n'_1 = 1$. Similarly $\delta' \cdot n'_2 = 1$. Therefore, intersecting both sides of (5.15) with $n'_1$ and $n'_2$ we get $1 = 1 + b$ and $1 = 1 + c$ respectively. Thus $b = c = 0$. Finally from the fact that $\xi'^2 = -1$ we compute that $a = 1$ and so $\xi' = \xi'$. The lemma is proven. □

In view of the previous lemma we have a section $\xi'$ of $\beta' : B' \to \mathbb{P}^1$ and we need to show that the class

$$\mu := 6e'_4 + 6\xi' - f' \in \text{Pic}(B')$$

is not in the Mori cone of $B'$. 
Assume that $\mu$ is in the Mori cone. Note that by the definition of $\xi'$ we have $e_1' \cdot f' = \xi' \cdot f' = 1$, $e_2' \cdot f' = \xi_2' = -1$ and $e_1' \cdot \xi' = 1$. Then $\mu \cdot \xi' = -1$ and so $\mu - \xi' = 6e_1' + 5\xi' - f'$ will also have to be in the Mori cone. But now $(\mu - \xi') \cdot e_1' = -2$ and so $\mu - 2e_1' - \xi'$ will be in the Mori cone. Intersecting with $\xi'$ again we get $(\mu - 2e_1' - \xi') \cdot \xi' = -2$ and so continuing iteratively we conclude that $-\xi - f$ must be in the Mori cone which is obviously false. This shows that $\mu$ is not in the Mori cone of $B'$ and so $h'$ ought to exist.

For completeness we will identify an explicit ample class $h'$ on $B'$ with $\mu \cdot h' < 0$. We will look for $h'$ of the form

\[ h' = af' + be_1' + c\xi' \]

and will try to adjust the coefficients $a$, $b$ and $c$ so that $h'$ is ample and $\mu \cdot h' < 0$. First we have the following

**Lemma 5.7.** The divisor class $h' = af' + be_1' + c\xi'$ is ample provided that $a$, $b$ and $c$ are positive and $a > |b - c|$.

**Proof.** Assume that $a$, $b$ and $c$ are positive and $a > |b - c|$. By the Nakai-Moishezon criterion for ampleness [Har77, Theorem 1.10], $h'$ will be ample if $h'^2 > 0$ and if $h' \cdot C > 0$ for every irreducible curve $C \subset B'$.

Let $C \subset B'$ be an irreducible curve. Then we have two possibilities: either $C$ is a component of a fiber of $\beta' : B' \to \mathbb{P}^1$, or $\beta' : C \to \mathbb{P}^1$ is a finite map. If $\beta' : C \to \mathbb{P}^1$ is finite and $C \neq \xi', e_1'$, then $C \cdot f' > 0$, $C \cdot e_1' > 0$ and $C \cdot \xi' \geq 0$. In particular the fact that $a$, $b$ and $c$ are positive implies that $C \cdot h' > 0$. Hence $h'$ will be ample if we can show that the intersections $h'^2$, $h' \cdot f'$, $h' \cdot n_1'$, $h' \cdot o_1'$, $h' \cdot e_1'$ and $h' \cdot \xi'$ are all positive. For this we calculate

\[
\begin{align*}
h' \cdot e_1' &= a + c - b \\
h' \cdot \xi' &= a + b - c \\
h' \cdot f' &= b + c \\
h' \cdot n_1' &= h' \cdot n_2' = c \\
h' \cdot o_1' &= h' \cdot o_2' = b \\
h'^2 &= -2b^2 - c^2 + 2ab + 2ac + 2bc = b(a + c - b) + c(a + b - c) + ab + ac,
\end{align*}
\]

which are manifestly positive provided that $a > |b - c|$. The lemma is proven. □

Now we see that the condition $\mu \cdot h' < 0$ translates into $12a < b + c$, i.e we may take $h' = 25f' + 144e_1' + 168\xi'$. This finishes the proof of the claim. □
6 Summary of the construction

In this section we recapitulate the main points of the construction. Recall that we want to build a quadruple \((X, H, \tau_X, V)\) satisfying:

\((\mathbb{Z}/2)\) \(X\) is a smooth Calabi-Yau 3-fold and \(\tau_X : X \to X\) is a freely acting involution. \(H\) is a fixed Kähler structure (ample line bundle) on \(X\)

\((S)\) \(V\) is an \(H\)-stable vector bundle of rank five on \(X\).

\((I)\) \(V\) is \(\tau_X\)-invariant.

\((C1)\) \(c_1(V) = 0\).

\((C2)\) \(c_2(X) - c_2(V)\) is effective.

\((C3)\) \(c_3(V) = 12\).

The construction is carried out in several steps.

6.1 The construction of \((X, \tau_X)\)

\(X\) is built as the fiber product of two rational elliptic surfaces of special type.

6.1.1. Building special rational elliptic surfaces. Let \(\Gamma_1 \subset \mathbb{P}^2\) be a nodal cubic with a node \(A_8\). Choose four generic points on \(\Gamma_1\) and label them \(A_1, A_2, A_3, A_7\). Let \(\Gamma \subset \mathbb{P}^2\) be the unique smooth cubic which passes through \(A_1, A_2, A_3, A_7, A_8\) and is tangent to the line \(\langle A_7 A_i \rangle\) for \(i = 1, 2, 3\) and 8. Consider the pencil of cubics spanned by \(\Gamma_1\) and \(\Gamma\). All cubics in this pencil pass through \(A_1, A_2, A_3, A_7, A_8\) and are tangent to \(\Gamma\) at \(A_8\). Let \(A_4, A_5, A_6\) be the remaining three base points, and let \(B\) denote the blow-up of \(\mathbb{P}^2\) at the points \(A_i, i = 1, 2, \ldots, 8\) and the point \(A_9\) which is infinitesimally near \(A_8\) and corresponds to the line \(\langle A_7 A_8 \rangle\).

The pencil becomes the anti-canonical map \(\beta : B \to \mathbb{P}^1\) which is an elliptic fibration with a section. The map \(\beta\) has two reducible fibers \(f_i = n_i \cup o_i, i = 1, 2\) of type \(I_2\). We denote by \(e_i, i = 1, \ldots, 7\) and \(e_9\) the exceptional divisors corresponding to \(A_i, i = 1, \ldots, 7\) and \(A_9\), and by \(e_8\) the reducible divisor \(e_9 + n_1\). The divisors \(e_i\) together with the pullback \(\ell\) of a class of a line from \(\mathbb{P}^2\) form a standard basis in \(H^2(B, \mathbb{Z})\).
The surface $B$ has an involution $\alpha_B$ which is uniquely characterized by the properties: $\alpha_B$ commutes with $\beta$, $\alpha_B$ induces an involution on $\mathbb{P}^1$, and $\alpha_B$ fixes the proper transform of $\Gamma$ pointwise.

Choosing $e_9$ as the zero section of $\beta$, we can interpret any other section $\xi$ as an automorphism $t_\xi : B \to B$ which acts along the fibers of $\beta$. The automorphism $\tau_B = t_{e_1} \circ \alpha_B$ is again an involution of $B$ which commutes with $\beta$, induces the same involution on $\mathbb{P}^1$ as $\alpha_B$ and has four isolated fixed points sitting on the same fiber of $\beta$.

The special rational elliptic surfaces form a four dimensional irreducible family. Their geometry was the subject of [DOPWa].

6.1.2. Building $(X, \tau_X)$. Choose two special rational elliptic surfaces $\beta : B \to \mathbb{P}^1$ and $\beta' : B' \to \mathbb{P}^1$ so that the discriminant loci of $\beta$ and $\beta'$ in $\mathbb{P}^1$ are disjoint, $\alpha_B$ and $\alpha_{B'}$ induce the same involution on $\mathbb{P}^1$ and the fix loci of $\tau_B$ and $\tau_{B'}$ sit over different points in $\mathbb{P}^1$. The fiber product $X := B \times_{\mathbb{P}^1} B'$ is a smooth Calabi-Yau 3-fold which is elliptic and has a freely acting involution $\tau_B \times \tau_{B'}$ and another (non-free) involution $\alpha_X := \alpha_B \times \alpha_{B'}$. For concreteness we fix the elliptic fibration of $X$ to be the projection $\pi : X \to B'$ to $B'$.

The Calabi-Yau's form a nine dimensional irreducible family.

6.1.3. Building $H$. Choose any ample divisor $H_0$ on $X$ and take $H = H_0 + n \cdot \pi^* h'$ for some positive integer $n$. Then the divisor $H$ will be ample as long as $h'$ is ample on $B'$ and $n \gg 0$.

Choose $h' = 25f' + 144e'_1 + 168\xi'$ with $\xi'$ being the unique section of $\beta' : \mathbb{P}^1 \to B'$ satisfying $[e'_4] - [e'_5] = [\xi']$. The divisor class $h' \in \text{Pic}(B')$ is ample on $B'$ by Lemma 5.7.

6.2 The construction of $V$

The bundle $V$ is build as a non-split extension

$$0 \to V_2 \to V \to V_3 \to 0$$

of two $\tau_X$-invariant stable vector bundles $V_2$ and $V_3$ of ranks 2 and 3 respectively.

Each $V_i$ is constructed via the spectral cover construction on $X$.

6.2.1. Building $V_2$ and $V_3$. Choose curves $\overline{C}_2, C_3 \subset B$, so that
• $\bar{C}_2 \in |O_B(2e_9 + 2f)|$ and $C_3 \in |O_B(3e_9 + 6f)|$.

• $\bar{C}_2$, $C_3$ are $\alpha_B$-invariant.

• $\bar{C}_2$ and $C_3$ are smooth and irreducible.

Set $C_2 := \bar{C}_2 + f_{\infty}$ where $f_{\infty}$ is the smooth fiber of $\beta$ containing the fixed points of $\tau_B$.

The space of such $\bar{C}_2$'s is an open set (see section 4) in $\mathbb{P}(H^0(B, O_B(2e_9 + 2f))^+)$, where $H^0(B, O_B(2e_9 + 2f))^+$ denote the spaces of invariants/anti-invariants for the $\alpha_B$ action on $H^0(B, O_B(2e_9 + 2f))$. Using the explicit equations (4.3) of the spectral curves we easily see that all such $C_2$ form a 2 dimensional irreducible family. The space of permissible $C_3$'s is an open subset in the disjoint union of the projective spaces $\mathbb{P}(H^0(B, O_B(3e_9 + 6f))^+)$, which have dimensions 8 and 6.

Fix an even integer $d_2$ and an integer $d_3$ satisfying $d_3 \equiv 1 \pmod{3}$.

Choose line bundles $N_2$ and $N_3$ on $C_2$ and $C_3$ respectively, which satisfy

• $N_i \in \text{Pic}^{d_i}(C_i)$ for $i = 1, 2$,

• $N_i = T_{C_i}(N_i) := \alpha^*_i|_{C_i} N_i \otimes O_{C_i}(e_1 - e_9 + f)$ for $i = 1, 2$.

As explained in sections 4.2 and 4.3, such $N_2$'s and $N_3$'s are parameterized by abelian subvarieties of $\text{Pic}^{d_2 - 1}(\bar{C}_2)$ and $\text{Pic}^{d_3}(C_3)$ of dimensions equal to the genera of the quotient curves $\bar{C}_2/\alpha_{C_2}$ and $C_3/\alpha C_3$ respectively. Thus there is a one dimensional space of $N_2$'s and a six dimensional space of $N_3$'s.

Let $\Sigma_i = C_i \times_{\mathbb{P}^1} B'$ for $i = 1, 2$. Recall that $\beta' : B' \rightarrow \mathbb{P}^1$ has two $I_2$ fibers $f_1'$ and $f_2'$. Let $F_1$, $F_2$ be the corresponding (smooth) fibers of $\beta : B \rightarrow \mathbb{P}^1$. Let $C_i \cap F_j = \{p_{ijk}\}_{k=1}^i$ for $i = 2, 3$, $j = 1, 2$. Then $\Sigma_i \rightarrow C_i$ is an elliptic surface having $2i$ fibers of type $I_2$: $\{p_{ijk}\} \times f_j'$. Also $\Sigma_i \subset X$ and the natural projection $\pi|_{\Sigma_i} : \Sigma_i \rightarrow B'$ is finite of degree $i$.

Fix non-negative integers $a_2$ and $a_3$. Define

$$V_i = FM_X \left( \left( \Sigma_i, (\pi|_{\Sigma_i})^* N_i \otimes O_{\Sigma_i} \left( -a_i \sum_{k=1}^i (\{p_{i1k}\} \times n_1' + \{p_{i2k}\} \times o_2') \right) \right) \right) \otimes \pi^* L_i,$$

where $L_2$ and $L_3$ are the line bundles

$$L_2 = 3(e_1' + e_4' - e_5' + e_9') + \frac{1}{4}(4 - d_2)f' + (3 - a_2)(n_1' + o_2')
L_3 = -2(e_1' + e_4' - e_5' + e_9') + \frac{1}{3}(16 - d_3)f' + (-2 + a_3)(n_1' + o_2'),$$
6.2.2. Building V. Take $V$ to be a non-split extension of $V_2$ by $V_3$ which is $\tau_X$-invariant. As explained in section 5.2, the space of all such extensions of $V_2$ by $V_3$ is the union of projective spaces $\mathbb{P}(H^1(X, V_3^\vee \otimes V_2)^+) \cup \mathbb{P}(H^1(X, V_3^\vee \otimes V_2)^-)$. $H^1(X, V_3^\vee \otimes V_2)\pm$ denote the invariants/anti-invariants for the $\tau_X$ action on $H^1(X, V_3^\vee \otimes V_2)$.

Furthermore, it is shown in section 5.2 that
\[
\dim H^1(X, V_3^\vee \otimes V_2) \geq (((-1)^B C_3) \cdot C_2) \cdot (L_2^i \cdot f' - L_3^i \cdot f') = 150,
\]
and from the explicit description of $H^1(X, V_3^\vee \otimes V_2)$ we see that the $\pm$ decomposition breaks this as $150 = 70 + 80$, so the dimension of the admissible extensions of $V_2$ by $V_3$ is at least 79.

In other words, for a fixed $(X, \tau_X, H)$ as above we find infinitely many components of the moduli space of $V$'s satisfying (S), (I) and (C1-3). Each component corresponds to a choice of the integers $a_2, a_3, d_2$ and $d_3$ and has dimension $2 + 8 + 1 + 6 + 79 = 96$.

Appendix A Hecke transforms

In this appendix we review the definition and some basic properties of the Hecke transforms (aka 'elementary modifications') of vector bundles along divisors. For more details the reader may wish to consult [Mar82, Mar87], [Fri98].

A.1 Definition and basic properties

Let $X$ be a smooth complex projective variety. Let $i : D \hookrightarrow X$ be a divisor with normal crossings.

Let $E \to X$ be a vector bundle and let $(\xi)$ be a short exact sequence of vector bundles on $D$ of the form
\[
(\xi) : 0 \to F \to E|_D \to G \to 0.
\]

There are two Hecke transforms $\text{Hecke}^{\pm}_{\xi}(E)$ attached to the pair $(E, (\xi))$. 

Definition A.1.

(i) The down-Hecke transform of $E$ along $(\xi)$ is the coherent sheaf $\text{Hecke}_{-\xi}(E)$ defined by the exact sequence

$$0 \rightarrow \text{Hecke}_{-\xi}(E) \rightarrow E \rightarrow i_*G \rightarrow 0.$$

(ii) The up-Hecke transform of $E$ along $(\xi)$ is the coherent sheaf $\text{Hecke}_{+\xi}(E) = (\text{Hecke}_{-\xi}(E^\vee))^\vee$

The first properties of the Hecke transforms are given by the following two lemmas.

Lemma A.2.

(a) The sheaves $\text{Hecke}_{-\xi}(E)$ and $\text{Hecke}_{+\xi}(E)$ are locally free.

(b) The up-Hecke transform $\text{Hecke}_{+\xi}(E)$ of $E$ along $(\xi)$ fits in the exact sequence

$$0 \rightarrow E \rightarrow \text{Hecke}_{+\xi}(E) \rightarrow i_*F \otimes \mathcal{O}_X(D) \rightarrow 0.$$

(c) $\text{Hecke}_{-\xi}(E)|_D$ and $\text{Hecke}_{+\xi}(E)|_D$ are furnished with natural exact sequences

$$(\xi^-) : 0 \rightarrow G(-D) \rightarrow \text{Hecke}_{-\xi}(E)|_D \rightarrow F \rightarrow 0,$$

$$(\xi^+) : 0 \rightarrow G \rightarrow \text{Hecke}_{+\xi}(E)|_D \rightarrow F(D) \rightarrow 0.$$

(d) $\text{Hecke}_{-\xi}(\bullet)$ and $\text{Hecke}_{+\xi}(\bullet)$ are mutually inverse in the sense that

$$\text{Hecke}_{-\xi}(\text{Hecke}_{+\xi}(E)) = E, \quad \text{Hecke}_{+\xi}(\text{Hecke}_{-\xi}(E)) = E.$$

(e) $\text{Hecke}_{-\xi}(E) = \text{Hecke}_{+\xi}(E)(-D).$
Proof. The proof of (a) is straightforward. For the proof of (b) recall that by definition the dual bundle $\text{Hecke}^+_{(\xi)}(E)^\vee$ fits in the short exact sequence of sheaves

$$0 \rightarrow \text{Hecke}^+_{(\xi)}(E)^\vee \rightarrow E^\vee \rightarrow i_*(F^\vee) \rightarrow 0.$$ 

Application of $\text{Hom}_{O_X}(\bullet, O_X)$ combined with the fact that $i_*(F^\vee)$ is torsion yields the long exact sequence

$$\cdots \rightarrow \text{Hom}_{O_X}(E^\vee, O_X) \rightarrow \text{Hom}_{O_X}(\text{Hecke}^+_{(\xi)}(E)^\vee, O_X) \rightarrow \text{Ext}^1_{O_X}(i_*(F^\vee), O_X) \rightarrow \cdots$$

Furthermore $E^\vee$ and $O_X$ are both locally free and hence every extension of $E^\vee$ by $O_X$ splits locally yielding $\text{Ext}^1_{O_X}(E^\vee, O_X) = 0$. Thus we obtain the exact sequence

$$0 \rightarrow E \rightarrow \text{Hecke}^+_{(\xi)}(E) \rightarrow \text{Ext}^1_{O_X}(i_*(F^\vee), O_X) \rightarrow 0.$$ 

To calculate $\text{Ext}^1_{O_X}(i_*(F^\vee), O_X)$ consider the ideal sequence of the divisor $D$:

$$0 \rightarrow O_X \rightarrow O_X(D) \rightarrow O_D(D) \rightarrow 0.$$ 

After applying $\text{Hom}_{O_X}(i_*(F^\vee), \bullet)$ and taking into account that $O_X$ and $O_X(D)$ are locally free sheaves, we get the exact sequence

$$0 \rightarrow \text{Hom}_{O_X}(i_*(F^\vee), O_D D) \rightarrow \text{Ext}^1_{O_X}(i_*(F^\vee), O_X) \rightarrow \text{Ext}^1_{O_X}(i_*(F^\vee), O_X(D)) \rightarrow \cdots$$

To understand the map

(A.1) $\text{Ext}^1_{O_X}(i_*(F^\vee), O_X) \rightarrow \text{Ext}^1_{O_X}(i_*(F^\vee), O_X(D))$,

consider a point $p \in D \subset X$. Let $R := O_{X,p}$ and let $t \in R$ be a local equation of $D$ around $p$. Let $M$ be the finitely generated $R/tR$ module whose sheafification gives $F^\vee \rightarrow D$ in a neighborhood of $p$.

An element $(\alpha) \in \text{Ext}^1_{O_X}(i_*(F^\vee), O_X)_p$ of the stalk of $\text{Ext}^1_{O_X}(i_*(F^\vee), O_X)$ at $p$ is an extension of $R$-modules of the form

$$(\alpha) : 0 \rightarrow R \rightarrow A \rightarrow M \rightarrow 0,$$

where $M$ is given its $R$-module structure via $R \rightarrow R/tR$. 
The image $(\beta) \in Ext^1_{O_X}(i_*(F^\vee), O_X(D))_p$ of $(\alpha)$ under the map (A.1) is just the pushout of the extension $(\alpha)$ via the homomorphism

$$R \longrightarrow \frac{1}{t} R.$$ 

That is, there is a commutative diagram

\[
\begin{array}{c}
(\alpha) & 0 & \longrightarrow & R & \longrightarrow & A & \longrightarrow & M & \longrightarrow & 0 \\
&&& \downarrow & & \downarrow & & \downarrow & & \\
(\beta) & 0 & \longrightarrow & \frac{1}{t} R & \longrightarrow & B & \longrightarrow & M & \longrightarrow & 0
\end{array}
\]

and $B = (A \oplus \frac{1}{t} R)/R$.

On the other hand, since $tR$ annihilates $M$ we have $\pi(tx) = t\pi(x) = 0$, for all $x \in A$. In particular the map

$$s : \ A \oplus \frac{1}{t} R \longrightarrow \frac{1}{t} R$$

$$x \oplus \frac{t}{t} \longmapsto \frac{tx}{t} + \frac{t}{t},$$

is well defined and descends to $B$ to a map splitting the exact sequence

$$0 \longrightarrow \frac{1}{t} R \longrightarrow B \longrightarrow M \longrightarrow 0.$$ 

Therefore the map

$$Ext^1_{O_X}(i_*(F^\vee), O_X) \longrightarrow Ext^1_{O_X}(i_*(F^\vee), O_X(D))$$

is the zero map and we get an isomorphism

$$Hom_{O_X}(i_*(F^\vee), i_*O_D \otimes O_X(D)) \cong Ext^1_{O_X}(i_*(F^\vee), O_X)$$

which concludes the proof of the lemma.

There is a natural symmetry between the up and down Hecke transforms. If $X, D, E$ and $(\xi)$ are as above, then we can form the dual exact sequence

$$(\xi^\vee) : 0 \longrightarrow G^\vee \longrightarrow E_D^\vee \longrightarrow F^\vee \longrightarrow 0,$$

and the up and down Hecke transforms of $E^\vee$ along $(\xi^\vee)$. The relation with the Hecke transforms of $E$ is given by the following lemma.
Lemma A.3.

\[ \text{Hecke}^*(E)^Y \cong \text{Hecke}^*(E^Y) \]

\[ \text{Hecke}^{-}(E)^Y \cong \text{Hecke}^{+}(E^Y) \]

**Proof.** Clear. \(\square\)

A.2 Geometric interpretation - flips

Let \((E, \xi)\) be as in Section A.1 and let \(\tau \to \mathbb{P}(E)\) be the relatively ample tautological line bundle. Denote by \(Y := \text{Bl}_{\mathbb{P}(E)} \mathbb{P}(E)\) the blow-up of \(\mathbb{P}(E)\) along \(\mathbb{P}(F)\).

Let \(p : Y \to \mathbb{P}(E)\) be the blow-up morphism and let \(\mathcal{E} \subset Y\) be the exceptional divisor. The image of \(Y\) under the full linear system \(p^*\tau \otimes \mathcal{O}_Y(-\mathcal{E})\) is again a projective bundle \(\mathbb{P}(E') \to X\). We have the following diagram

\[
\begin{array}{ccc}
\tau & \xrightarrow{p} & Y \\
\downarrow & & \downarrow \quad \quad \downarrow \\
\mathbb{P}(E) & \xrightarrow{f} & X \\
\end{array}
\]

where \(\tau \to \mathbb{P}(E)\) and \(\tau' \to \mathbb{P}(E')\) are relatively ample line bundles having the properties

\[ f_*\tau = E^Y \]

\[ f'_*\tau' = E'^Y \]

\[ p^*\tau = p^*\tau \otimes \mathcal{O}_Y(-\mathcal{E}) \]

To identify \(E'\) in terms of Hecke transforms consider the ideal sequence of \(\mathcal{E}\):

\[ 0 \to \mathcal{O}_Y(-\mathcal{E}) \to \mathcal{O}_Y \to \mathcal{O}_\mathcal{E} \to 0. \]
Tensoring by \( p^*\tau \) we get

\[
\text{(A.2)} \quad 0 \rightarrow p'^*\tau' \rightarrow p^*\tau \rightarrow p^*\tau \otimes O_\mathcal{E} \rightarrow 0.
\]

Let \( \pi : Y \rightarrow X \) be the composition \( \pi = f \circ p = f' \circ p' \). Consider the \( \pi \) direct image of (A.2):

\[
0 \rightarrow \pi_*p'^*\tau' \rightarrow \pi_*p^*\tau \rightarrow \pi_*(p^*\tau \otimes O_\mathcal{E}) \rightarrow R^1\pi_*p'^*\tau' \rightarrow \ldots
\]

Observe first that every fiber of \( \pi \) is either a projective space or has two irreducible components (meeting transversally) each of which is a projective space. Furthermore \( p'^*\tau' \) restricted on a component \( \mathcal{P} \) of the fiber is either \( O_p(1) \) or \( O_p \) and hence by Serre’s vanishing theorem doesn’t have higher cohomology. Thus by the base change and cohomology theorem \( R^1\pi_*p'^*\tau' = 0 \). Next

\[
\pi_*p'^*\tau' = f'^*f_*p'^*\tau' = f'^*\tau' = E'\vee.
\]

Here we used that \( p' : Y \rightarrow \mathbb{P}(E') \) has connected fibers.

Similarly \( \pi_*p^*\tau = E\vee \) and we get

\[
0 \rightarrow E'\vee \rightarrow E\vee \rightarrow \pi_*(p^*\tau \otimes O_\mathcal{E}) \rightarrow 0.
\]

But \( \pi_*(p^*\tau \otimes O_\mathcal{E}) = \iota_*\bar{f}_*(\gamma|_\mathcal{P}(F)) \) where \( \bar{f} : \mathbb{P}(F) \rightarrow D \) is the natural projection. Hence we get the short exact sequence

\[
0 \rightarrow E'\vee \rightarrow E\vee \rightarrow \iota_*F\vee \rightarrow 0
\]

and thus

\[
E' = \text{Hecke}^+_\mathcal{E}(E).
\]

### A.3 An example

Let \( X \) and \( D \) be as before. One has the short exact sequence:
(D) : \quad 0 \longrightarrow N_D^\vee X \longrightarrow \Omega^1_{X|D} \longrightarrow \iota_* \Omega^1_D \longrightarrow 0 \\
\mathcal{O}_D(-D)

We can form the up and down Hecke transforms of $\Omega^1_X$ along (D).

**Lemma A.4.** Denote by $\Omega^1_X(\log D)$ the sheaf of one forms on $X$ with logarithmic poles along $D$. Then the up and down Hecke transforms of $\Omega^1_X$ along (D) can be identified as follows

\[ \text{Hecke}^+_{(D)}(\Omega^1_X) = \Omega^1_X(\log D) \]
\[ \text{Hecke}^-_{(D)}(\Omega^1_X) = \Omega^1_X(\log D) \otimes \mathcal{O}_X(-D). \]

**Proof.** To prove the first equality observe that $\Omega^1_X(\log D)$ fits in the residue sequence

\[ 0 \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X(\log D) \longrightarrow \iota_* \mathcal{O}_D \longrightarrow 0, \]

where the map $\Omega^1_X(\log D) \rightarrow \iota_* \mathcal{O}_D$ is given by the residue along $D$. On the other hand, according to Lemma A.2 we have an exact sequence

\[ 0 \longrightarrow \Omega^1_X \longrightarrow \text{Hecke}^+_{(D)}(\Omega^1_X) \longrightarrow \iota_* \mathcal{O}_D \longrightarrow 0 \]

and it is easy to check that the two extension classes coincide. \(\square\)

**References**


