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D-branes, open string vertex operators, and Ext groups

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Abstract

In this note we explicitly work out the precise relationship between Ext groups and massless modes of D-branes wrapped on complex submanifolds of Calabi-Yau manifolds. Specifically, we explicitly compute the boundary vertex operators for massless Ramond sector states, in open string B models describing Calabi-Yau manifolds at large radius, directly in BCFT using standard methods. Naively these vertex operators are in one-to-one correspondence with certain sheaf cohomology groups (as is typical for such vertex operator calculations), which are related to the desired Ext groups via spectral sequences. However, a subtlety in the physics of the open string B model has the effect of physically realizing those spectral sequences in BRST cohomology, so that the vertex operators are actually in one-to-one correspondence with Ext group elements. This gives an extremely concrete physical test of recent proposals regarding the relationship between derived categories and D-branes. We check these results extensively in numerous examples, and comment on several related issues.

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1 Introduction

Recently it has become fashionable to use derived categories as a tool to study D-branes wrapped on complex submanifolds of Calabi-Yau spaces. Derived categories are now believed to have a direct physical interpretation, via a number of rather formal arguments. (See [1, 2, 3, 4] for an incomplete list of early references on this subject.)

One prediction of this derived categories program is that massless states of open strings between D-branes wrapped on complex submanifolds of Calabi-Yau spaces should be related to certain mathematical objects known as Ext groups. For those readers not familiar with such technology, Ext groups are analogous to cohomology groups, and are defined with respect to two coherent sheaves. The usual notation is

$$\mathrm{Ext}^n_X\left(\mathcal{S}_1, \mathcal{S}_2\right)$$

where $\mathcal{S}_1$, $\mathcal{S}_2$ are two coherent sheaves on X and n is an integer. Phrased in this language, if we have one D-brane wrapped on a complex submanifold $i : S \hookrightarrow X$ with holomorphic vector bundle $\mathcal{E}$ on S and another D-brane wrapped on a complex submanifold $j : T \hookrightarrow X$ with holomorphic vector bundle $\mathcal{F}$ on T, then the prediction in question is that massless states of open strings between these D-branes should be counted by groups denoted

$$\mathrm{Ext}^*_X\left(i_*\mathcal{E}, j_*\mathcal{F}\right)$$

and

$$\mathrm{Ext}^*_X\left(j_*\mathcal{F}, i_*\mathcal{E}\right)$$

(depending upon the orientation of the open string).

This mathematically natural prediction has been checked in a number of special cases. For example, in the trivial case that both branes are wrapped on the entire Calabi-Yau, the result is easily checked to be true. See [5] for a discussion of the special case $\widetilde{\mathbf{C}^3/\mathbf{Z}_3}$. Also see [6] for a self-consistency test of this hypothesis in the special case of the quintic. Additional special cases have also been checked [7].

Also note this prediction is closely analogous to some well-known results in heterotic compactifications. In heterotic compactifications involving gauge sheaves that are not bundles [8], it has been shown [9, 10] that massless modes are counted by Ext groups, replacing the sheaf cohomology groups that count the massless modes when the gauge sheaves are honest bundles

[11]. Similarly, in the trivial case that the wrapped D-branes are wrapped on the entire Calabi-Yau, the Ext groups reduce to sheaf cohomology. The prediction that states are counted by Ext groups is equivalent to the statement that for more general brane configurations than the trivial one, sheaf cohomology is replaced by Ext groups, which is certainly what happened in heterotic compactifications.

However, there has not, to our knowledge, been any systematic attempt to check directly in BCFT that open string states between appropriate D-branes are always related to Ext groups. In particular, the general correspondence between BCFT vertex operators and Ext group elements does not exist in the literature. Moreover, were it not for the fact that this classification of massless states is a prediction of a currently fashionable research program, the claim might sound somewhat suspicious. For example, typically relations between physics and algebraic geometry rely crucially on supersymmetry. Yet, the proposed classification of massless states in terms of Ext groups is needed to hold for non-BPS brane configurations, as well as BPS configurations.

In this paper, we shall begin to fill this gap in the literature. Using standard well-known methods, we explicitly compute, from first-principles, the spectrum of (BRST-invariant) vertex operators corresponding to massless Ramond sector states in open strings connecting D-branes wrapped on complex submanifolds of Calabi-Yau spaces at large radius, and explicitly relate those vertex operators to appropriate Ext group elements, for all possible configurations of complex submanifolds (both BPS and non-BPS).

Although the methods involved are standard, we do find some interesting physical subtleties in the open string case. In the closed string case, such vertex operators are in one-to-one correspondence with bundle-valued differential forms, and so the spectrum of BRST-invariant vertex operators is expressed in terms of a cohomology theory of such bundle-valued differential forms, known as sheaf cohomology[1]. For example, in heterotic strings with holomorphic gauge bundle $\mathcal{E}$, some of the massless modes are counted by [11]

$$H^n\left(X, \Lambda^m \mathcal{E} \right).$$

[1]Sheaf cohomology is defined for more general sheaves than merely bundles. Only in the special case that the sheaves in question are locally-free, *i.e.* that they correspond to bundles, does sheaf cohomology have a de-Rham-type description in terms of differential forms. We will only be interested in sheaf cohomology valued in bundles, not more general sheaves, and so for the purposes of making this paper more accessible to a physics audience, we will not distinguish between cohomology theories of bundle-valued differential forms and more general sheaf cohomology.

For another example, in the closed string B model [12], there is a one-to-one correspondence between (BRST-invariant) vertex operators and the sheaf cohomology groups

$$H^n\,(\,X,\,\Lambda^m TX\,)\,.$$

In the case at hand, a naive analysis of the massless Ramond sector states in such open strings yields a counting in terms of sheaf cohomology groups, and not Ext groups. Although we show that the sheaf cohomology groups in question are always related mathematically to Ext groups via spectral sequences, these spectral sequences are often nontrivial – although sheaf cohomology can be used to determine Ext groups, a given Ext group element need not be in one-to-one correspondence with any sheaf cohomology group element. A more careful analysis reveals a physical subtlety that has the effect of realizing the spectral sequences physically in terms of BRST cohomology, so the spectrum of massless Ramond sector states is, in fact, in one-to-one correspondence with Ext group elements.

In the process of working out correspondences to Ext groups, we also run across some interesting new physics. For example, we find evidence for a conjecture that for branes wrapped on two intersecting complex submanifolds S and T, if either of the line bundles on $S \cap T$

$$\Lambda^{top} N_{S\cap T/T}$$
$$\Lambda^{top} N_{S\cap T/S}$$

(constructed from the normal bundles to $S \cap T$ in T, S, respectively) are nontrivial, then the brane configuration cannot be BPS. Furthermore, in order to get a correspondence with Ext groups in this case, we will see that the open string boundary vacua must be interpreted as sections of line bundles over $S \cap T$.

Most of the paper is organized into a set of case-by-case studies of intersecting branes of increasing complexity. In section 2, we begin by discussing the relationship between open string boundary vertex operators and Ext groups in the simplest case, namely that in which the complex submanifolds on which the branes are wrapped are the same submanifold. The vertex operators for this particular case already exist in the literature, although their relationship to Ext groups does not seem to have been previously discussed. The vertex operators are naively counted by certain sheaf cohomology groups, which are related to Ext groups via a spectral sequence. We discuss the spectral sequence in detail, and include an example in which this spectral sequence is nontrivial, in the sense that the unsigned sum of the number of boundary vertex operators is *not* the same as the unsigned sum of the dimensions of the Ext groups. We describe the physical subtlety that alters the vertex operator analysis, and show how, in fact, the spectral sequence is realized physically in BRST cohomology. Thus, we see explicitly that there is a one-to-one correspondence between massless Ramond sector states (properly counted) and Ext group elements.

In section 3 we consider the relationship between boundary vertex operators and Ext groups in the next simplest case, namely when one submanifold is itself a submanifold of the other: $T \subseteq S$. We discuss the boundary vertex operators and the spectral sequence relating the vertex operators to Ext groups. We conjecture that the spectral sequence is realized physically as a modification of BRST cohomology, as happened in the case of parallel coincident branes. We also examine a naive problem with Serre duality that crops up when the line bundle $\Lambda^{top} N_{T/S}$ is nontrivial, a puzzle that is resolved in the next section. We also note in this section that the degree of the Ext group as it arises in algebraic geometry can differ from the charge of the vertex operators (as used to determine the type of resulting massless fields).

In section 4 we consider the general case of two intersecting complex submanifolds S and T, which need not be parallel. After disposing with the technical complications introduced by having branes at angles, we find boundary vertex operators and spectral sequences that generalize the results of sections 2 and 3. However, in the general case there is an extremely interesting complication that did not appear previously. Previously there was always a spectral sequence relating the boundary vertex operators to Ext groups. However, in the general case, we find that in order for such a relationship to exist, the boundary vacua must necessarily be sections of certain line bundles over the intersection $S \cap T$. This proposal resolves apparent difficulties with Serre duality (including the difficulty first seen in section 3), and is consistent with anomalies, as we discuss extensively.

In section 5 we very briefly dispose of the case of nonintersecting branes. In section 6 we briefly begin to describe how one can see Ext groups of complexes, not just individual torsion sheaves, in a special simple case. (More extensive effort will be delayed to later publications.) Finally, in appendix A we give mathematical derivations of the spectral sequences that play an important role in the text.

In passing, note that we are primarily concerned with only writing the spectrum of massless Ramond sector open string states in a more elegant fashion. Such analysis does not require the target brane worldvolume theory to be well-behaved; all we are doing is calculating part of the tree-level open string spectrum. For example, if the brane worldvolume theory contains a tachyon, then it is unstable; however, as we are merely rewriting the spectrum of string tree-level massless Ramond sector boundary states, such tachyons would not affect our calculations. Similarly, our calculations are insensitive to any anomalies in the target worldvolume theory. Again, we are merely rewriting part of the string tree-level open string spectrum; whether the target worldvolume theory has tachyons or anomalies certainly has a tremendous impact on the resulting physics, but does not alter the string tree-level open string spectrum.

In the remainder of this paper, we shall make the following assumptions. First, all calculations are performed at large-radius (closely analogous to the original heterotic vertex operators calculated in [11]). Second, we only consider branes wrapped on (smooth) complex submanifolds of a Calabi-Yau, whose intersections are again smooth submanifolds. Thus, we are interested in counting massless Ramond sector states, or equivalently, B-twisted topological field theory states on the boundary of the open string. Third, we shall assume throughout this paper that the B field vanishes identically. Nonzero B fields play an interesting and important role in D-branes. If we turn on a B field, the mathematical analysis can be handled using derived categories of twisted sheaves. Since the complications introduced are not relevant to the main point of this paper, we content ourselves to set the B field to zero. Fourth, we shall only consider cases in which the gauge 'sheaf' on the brane worldvolume is an honest bundle; we shall not attempt to study more general sheaves on the worldvolume of the brane. Finally, there are no antibranes in this paper, only branes.

2 Parallel coincident branes on $S \hookrightarrow X$

2.1 Basic analysis

First, we shall compute the massless Ramond sector spectrum of open strings between two D-branes on the same complex submanifold S of a Calabi-Yau manifold X, with inclusion $i : S \hookrightarrow X$. We shall assume one of the branes has gauge fields described by a holomorphic bundle $\mathcal{E}$, and the other has gauge fields described by a holomorphic bundle $\mathcal{F}$. Our methods are standard and well-known in the literature; see for example [11] for a closely related computation of massless states in heterotic string compactifications and [12] for another closely related computation of vertex operators in the closed string B model.

Following the conventions of [12], the bulk action can be written in the form

$$\frac{1}{2}g_{i\bar{\jmath}}\partial\phi^i\overline{\partial}\phi^{\bar{\jmath}} + \frac{1}{2}g_{i\bar{\jmath}}\partial\phi^{\bar{\jmath}}\overline{\partial}\phi^i + ig_{i\bar{\jmath}}\psi_-^{\bar{\jmath}}D_z\psi_-^i + ig_{i\bar{\jmath}}\psi_+^{\bar{\jmath}}D_{\bar{z}}\psi_+^i + R_{i\bar{\imath}j\bar{\jmath}}\psi_+^i\psi_+^{\bar{\imath}}\psi_-^j\psi_-^{\bar{\jmath}} \tag{1}$$

where

$$\begin{aligned} \psi_\pm^{\bar{\imath}} &\in \Gamma\left(\phi^* T^{0,1}X\right), \\ \psi_+^i &\in \Gamma\left(K \otimes \phi^* T^{1,0}X\right), \\ \psi_-^i &\in \Gamma\left(\overline{K} \otimes \phi^* T^{1,0}X\right) \end{aligned}$$

and with BRST transformations

$$\begin{aligned} \delta\phi^i &= 0, \\ \delta\phi^{\bar{\imath}} &= i\alpha\left(\psi_+^{\bar{\imath}} + \psi_-^{\bar{\imath}}\right), \\ \delta\psi_+^i &= -\alpha\partial\phi^i, \\ \delta\psi_+^{\bar{\imath}} &= -i\alpha\psi_-^{\bar{\jmath}}\Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{m}}\psi_+^{\bar{m}}, \\ \delta\psi_-^i &= -\alpha\overline{\partial}\phi^i, \\ \delta\psi_-^{\bar{\imath}} &= -i\alpha\psi_+^{\bar{\jmath}}\Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{m}}\psi_-^{\bar{m}}. \end{aligned}$$

Following [12], we define

$$\begin{aligned} \eta^{\bar{\imath}} &= \psi_+^{\bar{\imath}} + \psi_-^{\bar{\imath}}, \\ \theta_i &= g_{i\bar{\jmath}}\left(\psi_+^{\bar{\jmath}} - \psi_-^{\bar{\jmath}}\right), \\ \rho_z^i &= \psi_+^i, \\ \rho_{\bar{z}}^i &= \psi_-^i, \end{aligned}$$

and it is easy to calculate that in the absence of background gauge fields, the boundary conditions deduced from (1) are

$$\delta\eta^{\bar{\imath}} = \delta\theta_i = 0.$$

Along Neumann directions,

$$\psi_+^i|_{\partial\Sigma} = \psi_-^i|_{\partial\Sigma}$$

so we see that $\theta_i = 0$ for i an index along a Neumann direction, and similarly, along Dirichlet directions,

$$\psi_+^i|_{\partial\Sigma} = -\psi_-^i|_{\partial\Sigma}$$

so $\eta^{\bar{\imath}} = 0$ for i an index along Dirichlet directions.

In writing the boundary conditions above, we have neglected two important subtleties, one mathematical, and the other physical:

1. First, although as C^∞ bundles $TX|_S \cong TS \oplus N_{S/X}$ globally on S, as holomorphic bundles $TX|_S \not\cong TS \oplus N_{S/X}$ in general. On any one (sufficiently small) complex-analytic local neighborhood U one can find complex-analytic coordinates such that $TX|_S|_U \cong TS|_U \oplus N_{S/X}|_U$, and such a choice of coordinates is implicit in writing the local-coordinate expressions for the boundary conditions given above. However, because $TX|_S$ does not split globally on S, it is not quite correct to say that $\theta_i = 0$ for directions "normal" to S implies that the θ's couple to $N_{S/X}$, as one would naively believe. However, in practice this subtlety has no meaningful impact, as later we will see explicitly that it does not alter our conclusions.

2. A second subtlety arises from physics, and is due to the fact that along Neumann directions, the Chan-Paton factors twist the boundary conditions (see *e.g.* [13]), so that in fact $\theta_i = (\text{Tr } F_{i\bar{\jmath}})\, \eta^{\bar{\jmath}}$. We will (eventually) see that taking into account that Chan-Paton-induced twist has the effect of physically realizing spectral sequences discussed below in terms of BRST cohomology, so that the massless Ramond sector states are in one-to-one correspondence with Ext group elements.

For the moment, we shall ignore both these subtleties, and will return to them later after completing a more naive analysis.

Now, let us construct boundary vertex operators corresponding to massless Ramond sector states (ignoring, for the moment, the subtleties mentioned above). These are vertex operators that, in an infinite strip, would be placed in the infinite past, or alternatively, if one conformally maps to an upper half plane with different boundary conditions for $x > 0$ and $x < 0$, these are vertex operators that would be placed on the boundary at $x = 0$. The calculational method we shall use is a simple extrapolation of Born-Oppenheimer-based methods discussed in, for example, [11, 12]. Since we are working in a Ramond sector, the worldsheet bosons and fermions contribute equally and oppositely to the normal ordering constant, so massless states are constructed by acting on the vacuum with zero modes. Also, since we are dealing with zero modes of strings, the Chan-Paton factors appear as nothing more than indices on the vertex operators, as in [16].

For D-branes wrapped on the same complex submanifold $S \hookrightarrow X$, as discussed above, we have boundary vertex operators

$$b^{\alpha\beta j_1 \cdots j_m}_{\bar{\imath}_1 \cdots \bar{\imath}_n}(\phi_0)\, \eta^{\bar{\imath}_1} \cdots \eta^{\bar{\imath}_n} \theta_{j_1} \cdots \theta_{j_m}$$

(where α, β are Chan-Paton indices). Because of the boundary conditions, the θ indices are constrained to only live along directions normal to S, and the η indices are constrained to only live along directions tangent to S. Also, because of boundary conditions the ϕ zero modes ϕ_0 are constrained to only map out S. Also note that we are implicitly using θ and η to denote zero modes of both fields. These vertex operators are in one-to-one correspondence with bundle-valued differential forms living on S, and their BRST cohomology classes are identified with the (sheaf cohomology) group

$$H^n \left(S, \mathcal{E}^\vee \otimes \mathcal{F} \otimes \Lambda^m N_{S/X} \right) \tag{2}$$

where $N_{S/X}$ is the normal bundle to S in X.

Note in passing that this calculation is very similar to several other closed string calculations, where analogous results are obtained. For example, closely related computations in heterotic string compactifications with holomorphic gauge bundle $\mathcal{E}$ show there are massless states counted by [11, section 3]

$$H^n \left(X, \, \Lambda^m \mathcal{E} \right),$$

and in the closed string B model [12], vertex operators are counted by the sheaf cohomology groups

$$H^n \left(X, \, \Lambda^m TX \right).$$

Readers not familiar with the techniques being used may find sheaf cohomology unfamiliar, but in fact sheaf cohomology is nearly ubiquitous in these sorts of vertex operator computations.

The boundary vertex operators we have described above are not new to this paper; the same vertex operators are also described in, for example, [14, section 6.4] or more recently [15]. However, neither vertex operators for more general brane configurations (in which both sides of the open strings are not on the same submanifold) nor the relationship of these vertex operators to Ext groups have been discussed previously in the literature, and these topics will occupy the bulk of our attention in this paper.

We should also take a moment to speak to potential boundary corrections to the BRST operator. In the vertex operator analysis above, we implicitly assumed that the BRST operator on the boundary is the same as the restriction of the bulk BRST operator to the boundary. We claim that, modulo covariantizations, this is a reasonable assumption. Two general remarks should be made to clarify this matter further.

- First, from the Chan-Paton terms [16]

$$\int \left(\phi^* A \; - \; i \eta^{\bar{\imath}} F_{\bar{\imath} j} \rho^j \right)$$

we find that the Noether charge associated with the BRST operator picks up a term proportional to $A_{\bar{\imath}}\eta^{\bar{\imath}}$, where A is the Chan-Paton gauge field. This term merely serves to covariantize the BRST operator. After all, the BRST operator essentially acts as $\overline{\partial}$, but for fields coupling to bundles, one must add a connection term. Thus, adding contributions from the Chan-Paton action to the Noether current for the boundary BRST operator merely serves to covariantize the BRST operator.

- Second, in [4, section 2.4], certain additional boundary-specific terms added to the BRST operator played an important role. These terms arose after deforming the action, modelling giving a nonzero vacuum expectation value to a tachyon in a brane-antibrane system. Here, at no point will we consider deformations of the action. Thus, no boundary-specific contributions to the BRST operator of the form used in [4] will appear here.

Thus, in our analysis, the boundary BRST operator will always be the restriction of the bulk operator to the boundary (modified by covariantization with respect to the Chan-Paton gauge fields).

Serre duality acts to swap open string states of the form (2) with those of open strings of the opposite orientation. To see this, a useful identity is, for any complex bundle $\mathcal{G}$,

$$\Lambda^n \mathcal{G} \cong \left(\Lambda^{r-n}\mathcal{G}^\vee\right) \otimes \left(\Lambda^r \mathcal{G}\right) \tag{3}$$

where $r = \text{rank } \mathcal{G}$, so as $\Lambda^{top} N_{S/X} \cong K_S$, we see that Serre duality implies

$$H^n\left(S, \mathcal{E}^\vee \otimes \mathcal{F} \otimes \Lambda^m N_{S/X}\right) \cong H^{s-n}\left(S, \mathcal{F}^\vee \otimes \mathcal{E} \otimes \Lambda^{r-m} N_{S/X}\right)^*$$

where $s = \dim S$ and $r = \text{rk } N_{S/X}$. Also note that the boundary operator of maximal charge that corresponds to the holomorphic top form $\omega_{i_1 \cdots i_n} dz^{i_1} \wedge \cdots \wedge dz^{i_n}$ of the Calabi-Yau always exists in this case (assuming $\mathcal{E} = \mathcal{F}$ and suppressing Chan-Paton indices) and is given simply by

$$\overline{\omega}_{\bar{\imath}_1 \cdots \bar{\imath}_s}{}^{j_{s+1} \cdots j_n} \, \eta^{\bar{\imath}_1} \cdots \eta^{\bar{\imath}_s} \theta_{j_{s+1}} \cdots \theta_{j_n}$$

where $s = \dim S$ and in this one equation $n = \dim X$. Later, when considering more general boundary conditions, we shall find cases in which Serre duality is no longer an involution of the boundary vertex operator spectrum, and in such cases, a maximal-charge vertex operator corresponding to the holomorphic top form of the Calabi-Yau will no longer exist.

In the literature, it is frequently asserted that open string modes are in one-to-one correspondence with global Ext groups between torsion sheaves representing the D-branes. In the present case, this would be the claim that the open string modes are in one-to-one correspondence with elements of

$$\mathrm{Ext}^p_X\left(i_*\mathcal{E}, i_*\mathcal{F}\right)$$

where $i_*\mathcal{E}$ and $i_*\mathcal{F}$ are sheaves supported on $S \hookrightarrow X$, identically zero away from S, that look like the bundles $\mathcal{E}$ and $\mathcal{F}$ over S.

By contrast to the assertions quoted above, our naive description of the open string boundary vertex operators is in terms of bundle-valued differential forms which lead to (2) rather than Ext groups. However, that is not to say they are unrelated to Ext groups; they do determine Ext groups mathematically via the spectral sequence

$$E_2^{p,q} : H^p\left(S, \mathcal{E}^\vee \otimes \mathcal{F} \otimes \Lambda^q N_{S/X}\right) \Longrightarrow \mathrm{Ext}^{p+q}_X\left(i_*\mathcal{E}, i_*\mathcal{F}\right) \tag{4}$$

(See appendix A for a derivation.)

Earlier we mentioned that our boundary conditions were slightly oversimplified, in that along Neumann directions, the θ_i do not vanish, but rather obey $\theta_i = (\mathrm{Tr}\ F_{i\bar{\jmath}})\, \eta^{\bar{\jmath}}$ [13], something we have not taken into account so far in our description of massless Ramond sector states. We shall see shortly that when this complication is taken into account, one finds that the spectral sequence above is realized physically via BRST cohomology, and so the massless Ramond sector states are actually in one-to-one correspondence with Ext group elements, and not sheaf cohomology groups.

For the moment, we shall check our vertex operator analysis in some simple examples in which the spectral sequence is trivial. After that, we shall work through the subtleties in the boundary conditions mentioned above.

2.2 Examples

We shall check our vertex operator counting in the following two extreme cases:

1. Branes wrapping a Calabi-Yau
2. Points on Calabi-Yau manifolds

First, consider branes wrapping an entire Calabi-Yau, *i.e.*, $S = X$. In this case, $N_{S/X} = 0$, so the spectral sequence degenerates to give

$$Ext^n_X\left(\mathcal{E}, \mathcal{F}\right) = H^n\left(X, \mathcal{E}^\vee \otimes \mathcal{F}\right)$$

and the massless Ramond sector boundary states are of the form

$$b^{\alpha\beta}_{\bar{\imath}_1\cdots\bar{\imath}_n}\eta^{\bar{\imath}_1}\cdots\eta^{\bar{\imath}_n}.$$

These boundary states are well-known (see for example [14]), and their low-energy interpretation is a function of their $U(1)$ charge. For example, from the charge zero operator $b^{\alpha\beta}(\phi_0)$ one can construct a conformal dimension one operator $\exp(-\phi)b^{\alpha\beta}\psi^\mu$, where ϕ is the bosonized superconformal ghost and ψ^μ a worldsheet fermion transforming as a spacetime vector. Such charge zero operators correspond in this fashion to low-energy gauge fields. A charge one operator, say $b^{\alpha\beta}_{\bar{\imath}}\eta^{\bar{\imath}}$, corresponds to a low-energy spacetime scalar, with vertex operator of the form $\exp(-\phi)b^{\alpha\beta}_{\bar{\imath}}\eta^{\bar{\imath}}$, which is a conformal weight one operator transforming as a spacetime scalar. As this story is well-known, we shall not belabor the point further.

Next, we shall consider branes 'wrapped' on points on Calabi-Yau threefolds. Consider for example N D3-branes at a point S on a Calabi-Yau threefold X. For notational brevity, define $\mathcal{E} = \mathcal{O}^{\oplus N}$. The nonzero sheaf cohomology groups are

$$\begin{aligned} H^0\left(S, \mathcal{E}^\vee \otimes \mathcal{E}\right) &= \mathbf{C}^{N^2}, \\ H^0\left(S, \mathcal{E}^\vee \otimes \mathcal{E} \otimes N_{S/X}\right) &= \mathbf{C}^{3N^2}, \\ H^0\left(S, \mathcal{E}^\vee \otimes \mathcal{E} \otimes \Lambda^2 N_{S/X}\right) &= \mathbf{C}^{3N^2}, \\ H^0\left(S, \mathcal{E}^\vee \otimes \mathcal{E} \otimes \Lambda^3 N_{S/X}\right) &= \mathbf{C}^{N^2} \end{aligned}$$

determining

$$\mathrm{Ext}^n_X\left(i_*\mathcal{E}, i_*\mathcal{E}\right) = \left\{ \begin{array}{ll} \mathbf{C}^{N^2} & n = 0, 3, \\ \mathbf{C}^{3N^2} & n = 1, 2. \end{array} \right.$$

The first and last sheaf cohomology groups are Serre dual, and correspond to open strings of opposite orientation; the second and third groups are also Serre dual. Thus, we need only consider the first two groups. The first group describes states of $U(1)$ charge zero, and so correspond in the low-energy theory to components of a $U(N)$ gauge field. The second describes states of $U(1)$ charge one, and so correspond in the low-energy theory to three adjoint-valued fields. Thus, we recover the expected field content for D3-branes at a point on a Calabi-Yau threefold.

In these examples there was a natural correspondence between the degree of the Ext group and the $U(1)$ charge, as correlated with the type of low-energy field (vector, scalar, *etc*). However, in later sections we shall see explicitly that unfortunately this correspondence cannot hold in general.

2.3 First subtlety: mathematics

We mentioned earlier that there were two subtleties in the boundary conditions. The first subtlety described is, on its face, an obscure mathematical point. Namely, although for C^∞ bundles, $TX|_S \cong TS \oplus N_{S/X}$, this is not true for holomorphic bundles in general. As a result, the interpretation of boundary conditions such as $\theta_i = 0$ is somewhat subtle.

We will see that this subtlety, on its own, has little real effect. Its proper understanding does not alter the naive conclusion above, that massless Ramond sector states appear to be counted by sheaf cohomology groups, and not Ext groups. In order to see explicitly that the massless Ramond sectors states are actually counted by Ext groups, we shall have to use the second subtlety mentioned. However, although this subtlety will not have a significant impact on the results, its proper understanding will play a significant role in the physical realization of the spectral sequence discussed earlier, and for that reason we shall discuss it in detail.

In general, globally on S, $TX|_S$ is merely an extension of $N_{S/X}$ by TS:

$$0 \longrightarrow TS \longrightarrow TX|_S \longrightarrow N_{S/X} \longrightarrow 0.$$

One simple example in which $TX|_S$ does not split involves conics C in $\mathbf{P}^2$. There, $TC = \mathcal{O}(2)$, $T\mathbf{P}^2|_C = \mathcal{O}(3) \oplus \mathcal{O}(3)$, and $N_{C/\mathbf{P}^2} = \mathcal{O}(4)$, so clearly $T\mathbf{P}^2|_C \not\cong N_{C/\mathbf{P}^2} \oplus TC$; rather, $T\mathbf{P}^2|_C$ is merely an extension of $\mathcal{O}(4)$ by $\mathcal{O}(2)$. Although globally one cannot split $TX|_S$ holomorphically, in any one sufficiently small complex-analytic local coordinate patch, one can arrange for $TX|_S$ to split, and boundary conditions such as $\theta_i = 0$ are implicitly written in such special coordinates.

What effect does this subtlety have? First, note that if we are working in a special case in which $TX|_S$ *does* split, *i.e.*, in special cases in which $TX|_S = N_{S/X} \oplus TS$ holomorphically globally on S, then the analysis of the previous section goes through without a hitch. If $TX|_S$ splits globally on S, then the local-coordinate expression $\theta_i = 0$ does indeed imply that the θ's couple to $N_{S/X}$, and the previous analysis is unchanged.

If $TX|_S \not\cong TS \oplus N_{S/X}$, then the analysis is more complicated, but the result is the same. For simplicity let us consider vertex operators with a single θ, naively corresponding to sheaf cohomology valued in $N_{S/X}$. Since $TX|_S \not\cong TS \oplus N_{S/X}$, it is no longer true that $\theta_i = 0$ implies that the θ couple to $N_{S/X}$. After all, under a change of coordinates, $N_{S/X}$ will mix with TS, and so the condition $\theta_i = 0$ for Neumann directions is will not be invariant under holomorphic coordinate changes. Rather, the θ merely couple to

$TX|_S$, but are constrained such that in certain special complex-analytic local coordinates, some of the θ vanish. In particular, sheaf cohomology valued in $N_{S/X}$ can no longer be translated directly into vertex operators.

We can deal with this more complicated scenario as follows. Although sheaf cohomology valued in $N_{S/X}$ can not be used to write down vertex operators immediately, we can lift differential forms valued in $N_{S/X}$ to differential forms valued in $TX|_S$, and we *can* write down vertex operators associated to those $TX|_S$-valued differential forms, since the θ_i couple to $TX|_S$.

Now, something interesting happens when we demand BRST invariance of those newly-minted $TX|_S$-valued forms; namely, they need no longer be $\overline{\partial}$-closed[2], yet they still define BRST-invariant states.

Mathematically, we are taking advantage of a commuting diagram which we shall write schematically as

$$\begin{array}{ccccc} \mathcal{A}^{0,n}(TS) & \longrightarrow & \mathcal{A}^{0,n}(TX|_S) & \longrightarrow & \mathcal{A}^{0,n}(N_{S/X}) \\ \downarrow \overline{\partial} & & \downarrow \overline{\partial} & & \downarrow \overline{\partial} \\ \mathcal{A}^{0,n+1}(TS) & \longrightarrow & \mathcal{A}^{0,n+1}(TX|_S) & \longrightarrow & \mathcal{A}^{0,n+1}(N_{S/X}) \end{array}$$

where $\mathcal{A}^{0,n}$ denotes differential $(0,n)$ forms, horizontal arrows are induced by the short exact sequence above, the rows are exact, and we have suppressed the factors $\mathcal{E}^\vee \otimes \mathcal{F}$ throughout. The image under $\overline{\partial}$ is a higher-degree $TX|_S$-valued form, and from commutativity of the diagram above, that higher-degree form is the image of a TS-valued form.

Technically what we are doing is realizing the coboundary map in the long exact sequence of sheaf cohomology

$$H^n\left(S, \mathcal{E}^\vee \otimes \mathcal{F} \otimes N_{S/X}\right) \longrightarrow H^{n+1}\left(S, \mathcal{E}^\vee \otimes \mathcal{F} \otimes TS\right)$$

induced by the short exact sequence

$$0 \longrightarrow TS \longrightarrow TX|_S \longrightarrow N_{S/X} \longrightarrow 0.$$

In algebraic topology, such a map is known as the Bockstein homomorphism. We started with a $N_{S/X}$-valued form, and created a TS-valued form of higher degree. The physical vertex operators are defined by the $TX|_S$-valued differential forms appearing in the first intermediate step.

[2]In the special case that the $TX|_S$ splits globally on S, *i.e.* $TX|_S = N_{S/X} \oplus TS$, then it is possible to generate a closed $TX|_S$-valued differential form from any closed $N_{S/X}$-valued differential form. When $TX|_S$ does not so split, this is not always possible.

Now, how can this be BRST invariant, as claimed? We started with $N_{S/X}$-valued sheaf cohomology, lifted the coefficients to $TX|_S$ to create differential forms that we could associate to vertex operators, and then argued that $\overline{\partial}$ of those differential forms gives $\overline{\partial}$-closed TS-valued differential forms of one higher degree. But in order to be BRST invariant, our vertex operators (associated to $TX|_S$-valued forms) must be annihilated by $\overline{\partial}$.

The answer is in the boundary conditions $\theta_i = 0$ (for Neumann directions). These boundary conditions annihilate TS-valued forms. Thus, since the image of our vertex operators under $\overline{\partial}$ is TS-valued, our vertex operators are closed under the BRST transformation.

Now, the reader might well ask, why we went to the trouble of working through these details. What we have concluded, after considerable effort, is that even though $TX|_S \not\cong TS \oplus N_{S/X}$ holomorphically on S, the massless Ramond sector states are nevertheless counted by the sheaf cohomology groups

$$H^n\left(S, \mathcal{E}^\vee \otimes \mathcal{F} \otimes \Lambda^m N_{S/X}\right)$$

(at least so long as we are dealing with the boundary condition described as $\theta_i = 0$ for Neumann directions). The vertex operators are slightly more complicated to express than one would have naively thought, but at the end of the day, we do not seem to have learned anything significantly new.

The reason we went to this trouble is that this complication will play an important role when unraveling the next subtlety, involving the altered boundary condition $\theta_i = (\text{Tr } F_{i\overline{\jmath}})\, \eta^{\overline{\jmath}}$. The coboundary map discussed in detail above will form half of the differential of the spectral sequence. As here, vertex operators will be associated to $TX|_S$-valued differential forms created by lifting $N_{S/X}$-valued $\overline{\partial}$-closed differential forms, and the BRST operator will act as $\overline{\partial}$ on those $TX|_S$-valued forms. Just as here, the result will be a TS-valued form, at which point we can apply the boundary condition on the θ's. Unlike the present case, the boundary condition will not annihilate any TS-valued θ's, so demanding BRST-invariance of our $TX|_S$-valued forms will give an additional condition that will be equivalent to being in the kernel of the differential of the spectral sequence.

2.4 Second subtlety: physics

The second subtlety we mentioned previously was that along Neumann directions, in suitable local complex-analytic coordinates, it is not true that $\theta_i = 0$, but rather $\theta_i = (\text{Tr } F_{i\overline{\jmath}})\, \eta^{\overline{\jmath}}$. We shall deal with this subtlety in this section. We shall find that this subtlety effectively alters the BRST cohomology in such a way that the spectral sequence discussed earlier is realized directly in BRST cohomology. Thus, the spectrum of massless Ramond sector states is counted directly by Ext groups, instead of sheaf cohomology.

We begin this section by discussing the nontriviality of the spectral sequence, followed by a detailed discussion of the differentials of the spectral sequence. Finally, we describe explicitly how those differentials are realized physically.

2.4.1 Nontriviality of the spectral sequence

We have argued that, after making a slight simplification of the boundary conditions, massless Ramond sector states in open strings are in one-to-one correspondence with sheaf cohomology groups, related to Ext groups via a spectral sequence. We shall argue shortly that this spectral sequence is realized physically after taking into account the correct boundary conditions, but before we work through those details, we shall discuss the spectral sequence in greater detail.

In particular, in this subsection we shall discuss the nontriviality of the spectral sequence, because if the spectral sequence were always trivial, then there would be little point in worrying about it. In general, spectral sequences lose information – the fact that the obvious bigrading structure of the sheaf cohomology groups reduces to a unigraded structure is one indication of this loss of information. However, in the explicit examples we have computed above, it was the case that the spectral sequence was trivial, in the sense that the dimension of an Ext group was the same as the sum of the dimensions of the sheaf cohomology groups feeding into it.

If it were always the case that the spectral sequence were trivial, *i.e.*, if it were always the case that the number of independent sheaf cohomology group elements was the same as the number of independent Ext group elements, then our point that formally spectral sequences lose information would seem rather moot, and discussions of physical realizations of the spectral sequence would be rather pointless.

However, in general, the spectral sequence relating the sheaf cohomology groups to Ext groups is not trivial – the number of independent boundary vertex operators is not the same as the number of Ext group generators. Thus, the map from boundary vertex operators to Ext group elements is not invertible, in the strongest sense of the term. On the other hand, the signed sum of dimensions of sheaf cohomology groups will always be the same as the signed sum of the dimensions of the Ext groups. This will be the case in general, as $d_r^{p,q}$ maps $E_2^{p,q}$ to $E_2^{p+r,q-r+1}$ and so will always increase the charge of an operator by 1. Thus the spectral sequence will always cancel out vertex operators in pairs, with differing sign in the index.

One example in which this spectral sequence is nontrivial is as follows. Let X be a K3-fibered Calabi-Yau threefold, and let S be a smooth K3 fiber. Assume further that S contains a $C \simeq \mathbf{P}^1$ which is rigid in X, having normal bundle $N_{C/X} \simeq \mathcal{O}_C(-1) \oplus \mathcal{O}_C(-1)$. This typically implies that the bundle $\mathcal{O}_S(C)$ itself does not deform to first order as S moves in the fibration, but let's add that as another explicit assumption.

Let $\mathcal{E} = \mathcal{F} = \mathcal{O}_S(C)$. We claim that this gives an example with a nontrivial spectral sequence.

First note that the sheaf $i_* \mathcal{O}_S(C)$ does not deform, not even to first order. To see this, first observe that the support S of $i_* \mathcal{O}_S(C)$ can only deform in the given fibration; but then we have assumed that $i_* \mathcal{O}_S(C)$ does not deform in the fibration so the sheaf $i_* \mathcal{O}_S(C)$ does not deform in X in any way whatsoever.

Next we note that $\mathrm{Ext}^1(i_* \mathcal{O}_S(C), i_* \mathcal{O}_S(C))$ is the space of first order deformations of the sheaf $i_* \mathcal{O}_S(C)$, which we have just shown is 0.

But the spectral sequence (4) has the nontrivial terms

$$E_2^{0,1} = H^0(S, \mathcal{E}^\vee \otimes \mathcal{F} \otimes N_{S/X}) = H^0(S, \mathcal{O}) = \mathbf{C}$$

and

$$E_2^{2,0} = H^2(S, \mathcal{E}^\vee \otimes \mathcal{F}) = H^2(S, \mathcal{O}) = \mathbf{C}.$$

It is immediate to additionally check that $E_2^{0,0} = E_2^{2,1} = \mathbf{C}$ and all other terms in the spectral sequence are 0.

The spectral sequence (4) has a differential $d_2^{0,1} : E_2^{0,1} \to E_2^{2,0}$ which we will argue is nontrivial. Since $E_2^{p,q} = 0$ for $p \geq 3$, all the differentials $d_r^{p,q}$ vanish for $r \geq 3$. So $d_2^{0,1}$ is the only differential in the spectral sequence that could be nonzero. If it were zero, then we would compute $\mathrm{Ext}^1(i_* \mathcal{O}_S(C), i_* \mathcal{O}_S(C)) \neq 0$, a contradiction. So $d_2^{0,1}$ is nonzero. In particular we have an open string mode corresponding to an element of $H^0(S, N_{S/X})$ which does *not* parametrize a nonzero Ext element.

As an explicit example, consider $X = P(1,1,2,2,2)[8]$ considered in [17]. The K3 fibration comes from the map $X \to \mathbf{P}^1$ sending $(x_1, \ldots, x_5)$ to

(x_1, x_2), and the general K3 fiber is identified with a degree 4 K3 hypersurface in $\mathbf{P}^3$. While a general degree 4 surface contains no lines, it was argued that if X has general moduli, then exactly 640 of these degree 4 K3 surfaces contain a line, and that these are rigid in X. Furthermore, still choosing X to have general moduli, we can assume that the general K3 fiber has Picard number at least 1. Recall [18, Pp. 590–94] that the moduli space M_2 of quartic K3 surfaces with Picard number at least 2 is a generically smooth divisor in the moduli space M_1 of quartic K3's with Picard number 1. Given X, we get a map $\psi_X : \mathbf{P}^1 \to M_1$ sending a point to the K3 fiber it parameterizes. It is easy to check that if X is general, then ψ_X meets M_2 transversally at smooth points. This is enough to guarantee that $\mathcal{O}_S(C)$ does not deform to first order as S moves in the K3 fibration.

2.4.2 Details of the differentials

We have just argued that the spectral sequence relating sheaf cohomology to Ext groups is nontrivial in general, so it is very important to check that that spectral sequence really is realized physically. Before we describe the physical realization of the spectral sequence, we will first describe the differentials in more detail.

Consider the special case of an open string connecting a D-brane to itself. In this case, we have the same Chan-Paton gauge field on either side of the open string. In this case, the level two differential

$$d_2 : \; H^0\left(S, \mathcal{E}^\vee \otimes \mathcal{E} \otimes N_{S/X}\right) \;\mapsto\; H^2\left(S, \mathcal{E}^\vee \otimes \mathcal{E}\right)$$

is realized mathematically by the composition

$$H^0\left(S, \mathcal{E}^\vee \otimes \mathcal{E} \otimes N_{S/X}\right) \longrightarrow H^1\left(S, \mathcal{E}^\vee \otimes \mathcal{E} \otimes TS\right) \longrightarrow H^2\left(S, \mathcal{E}^\vee \otimes \mathcal{E}\right).$$

The first map in the composition is the coboundary map

$$H^0\left(S, \mathcal{E}^\vee \otimes \mathcal{E} \otimes N_{S/X}\right) \longrightarrow H^1\left(S, \mathcal{E}^\vee \otimes \mathcal{E} \otimes TS\right)$$

in the long exact sequence of sheaf cohomology induced by the tensor product of the short exact sequence

$$0 \longrightarrow TS \longrightarrow TX|_S \longrightarrow N_{S/X} \longrightarrow 0 \tag{5}$$

with $\mathcal{E}^\vee \otimes \mathcal{E}$. We discussed this coboundary map in detail in section 2.3. Recall from section 2.3 this coboundary map vanishes if $TX|_S \cong TS \oplus N_{S/X}$ globally on S; it is only nontrivial if $TX|_S$ does not split globally. Put another way, if $TX|_S$ splits globally, then the spectral sequence is trivial.

The second map in the composition is much easier to describe. It involves contracting the TS indices on the trace of the curvature form of the connection on the bundle. In other words, if θ_i schematically indicates a TS direction, then the second map in d_2 involves the replacement

$$\theta_i \mapsto (\mathrm{Tr}\, F_{i\bar{\jmath}})\, d\bar{z}^{\bar{\jmath}}. \tag{6}$$

The close relationship between the expression above and the altered boundary conditions induced as in [13] is no accident, and forms the heart of the physical realization of the spectral sequence.

In principle, the higher differentials are constructed from the same ingredients. For example, let us consider

$$d_3 : E_3^{0,2} \longrightarrow E_3^{3,0}.$$

Note that $E_3^{0,2}$ consists of the part of $E_2^{0,2} = H^0(S, \mathcal{E}^\vee \otimes \mathcal{E} \otimes \Lambda^2 N_{S/X})$ that is annihilated by d_2. Consider the short exact sequence

$$0 \to N_{S/X} \otimes TX \to N_{S/X} \otimes TX|_S \to N_{S/X} \otimes N_{S/X} \to 0 \tag{7}$$

obtained from (5) by tensoring with $N_{S/X}$. In this case, d_2 acts by combining the coboundary map of (7) with the replacement (6). In other words, to see the action of d_2, lift the $\Lambda^2 N_{S/X}$-valued zero form to a $N_{S/X} \otimes TX|_S$-valued form, then apply $\overline{\partial}$ and commutativity of (2.3) to get a $N_{S/X} \otimes TS$-valued $(0,1)$-form. Finally apply (6) to get a $N_{S/X}$-valued two-form. Here we have viewed $\Lambda^2 N_{S/X}$ as the subbundle of antisymmetric elements of $N_{S/X} \otimes N_{S/X}$. The part of $H^0(S, \mathcal{E}^\vee \otimes \mathcal{E} \otimes \Lambda^2 N_{S/X})$ in the kernel of this map is $E_3^{0,2}$. The differential d_3 acts on $E_3^{0,2}$ by lifting the $\Lambda^2 N_{S/X}$-valued form to a $\Lambda^2 TX|_S$-valued form, applying $\overline{\partial}$, and contracting both of the resulting TS indices with the curvature, using (6). The resulting indices correspond to TS rather than merely $TX|_S$ by the assumption that our $\Lambda^2 N_{S/X}$-valued section is in the kernel of d_3.

2.4.3 Physical realization of the spectral sequence

So far in our analysis of massless Ramond sector states, we have assumed that along Neumann directions, $\theta_i = 0$. However, strictly speaking this is only the case when one has trivial Chan-Paton gauge fields. As noted many years ago in *e.g.* [13], in the presence of nontrivial Chan-Paton gauge fields, Neumann boundary conditions are twisted by the curvature of the gauge field. For example, for worldsheet scalars, ordinarily the Neumann boundary conditions state that

$$\partial_n X = 0$$

where n denotes the direction normal to the boundary. If the Chan-Paton factors have nontrivial curvature, this condition is modified to become

$$\partial_n X^\mu \;=\; (\mathrm{Tr}\; F^\mu_\nu)\; \partial_t X^\nu.$$

Although this twisting of boundary conditions seems to have been largely ignored in most discussions of the open string B model, it has played an important role elsewhere in physics recently (see *e.g.* [19]).

Let us carefully consider how this modifies our analysis of massless Ramond sector states. Boundary conditions for worldsheet fields coupling to directions normal to the brane are unchanged, as otherwise it would be impossible to make sense of the Chan-Paton action. Boundary conditions for worldsheet fields coupling to directions parallel to the brane, however, are changed as above. One still has constant bosonic maps, as the modified boundary conditions above only couple to derivatives of the worldsheet bosons. The boundary conditions on the worldsheet fermions can now be written in suitable local coordinates as

$$\theta_i \;=\; (\mathrm{Tr}\; F_{i\overline{\jmath}})\; \eta^{\overline{\jmath}}. \tag{8}$$

In general, if the Chan-Paton factors on either side of the open string are different, then these boundary conditions will change the fermion moding, and hence change the fermion zero-mode structure. Put another way, the effect of these boundary conditions is very closely analogous to the effect of having branes at angles, as discussed in [20].

We shall only consider the special case that the Chan-Paton gauge fields on either side of the open string are identical, *i.e.*, that the open string is connecting a D-brane to itself. This corresponds to the 'dipole string' case discussed in [13]. In this special case, one has the same fermion zero modes as assumed previously, so the analysis is very similar to that discussed so far, except that the θ_i no longer couple to $N_{S/X}$. Instead, because the θ_i parallel to the brane can be nonzero, the θ_i merely couple to $TX|_S$.

As before, boundary vertex operators should be of the general form

$$b^{\alpha\beta j_1 \cdots j_m}_{\overline{\imath}_1 \cdots \overline{\imath}_n}(\phi_0)\, \eta^{\overline{\imath}_1} \cdots \eta^{\overline{\imath}_n} \theta_{j_1} \cdots \theta_{j_m}$$

(where α, β are Chan-Paton indices). Now, however, there are subtleties in the interpretation. The θ_i couple to $TX|_S$, not $N_{S/X}$, and in principle θ_i parallel to the brane are related to the $\eta^{\overline{\jmath}}$ by the boundary conditions. In the special case that $TX|_S \cong N_{S/X} \oplus TS$ holomorphically, we can simply ignore the θ_i parallel to the brane, use only θ_i normal to the brane to construct the vertex operators above, and immediately recover a classification in terms of sheaf cohomology. In this same case, the spectral sequence relating sheaf cohomology to Ext groups is trivial. When the spectral sequence is nontrivial, $TX|_S \not\cong N_{S/X} \oplus TS$, but rather is merely an extension of $N_{S/X}$ by TS. In this case, although locally in coordinate patches one can distinguish θ_i parallel to the brane from θ_i normal to the brane, globally along S one cannot make such a distinction.

Let us assume that $TX|_S \not\cong N_{S/X} \oplus TS$, and work out how to describe the vertex operators. For simplicity, for the moment we shall only consider vertex operators of the form

$$b^{\alpha\beta j}(\phi_0)\theta_j. \tag{9}$$

We shall argue that computing BRST cohomology is equivalent to evaluating the spectral sequence.

First, because the θ_i couple to $TX|_S$ and not $N_{S/X}$, we cannot associate elements of $H^0(\mathcal{E}^\vee \otimes \mathcal{E} \otimes N_{S/X})$ with the vertex operator (9). Also, because the θ_i "parallel" to the brane (not a well-defined notion when $TX|_S$ does not split holomorphically) are related to the $\eta^{\overline{\jmath}}$ by the curvature of the Chan-Paton factors, we cannot merely claim that the BRST cohomology is simply $H^0(\mathcal{E}^\vee \otimes \mathcal{E} \otimes TX|_S)$. Instead, let us proceed more carefully. We can manufacture an operator that is 'close' to being BRST-closed by starting with an element of $H^0(\mathcal{E}^\vee \otimes \mathcal{E} \otimes N_{S/X})$, and lifting the coefficients to $TX|_S$ to recover a (not-necessarily-closed) differential form valued in $TX|_S$. We can associate such a differential form with a vertex operator of the form (9). Unfortunately, the resulting vertex operator need not be BRST-closed, as the corresponding differential form is not $\overline{\partial}$-invariant, and because of the boundary conditions on some of the θ_i.

Let us now work out the action of the BRST operator on this vertex operator. So, act on the vertex operator with the BRST operator, or equivalently, act on the zero form with $\overline{\partial}$, to generate a closed one-form. We now have a closed one-form valued in $TX|_S$. Even better – exactly as in the discussion of the coboundary map in section 2.3, our closed $TX|_S$-valued one-form is mathematically the image of a closed TS-valued one-form.

In other words, after applying $\overline{\partial}$, we can now apply the boundary conditions $\theta_i = (\mathrm{Tr}\, F_{i\overline{\jmath}})\,\eta^{\overline{\jmath}}$ in a fashion that makes sense globally on S. After applying this contraction, and comparing to the explicit description of d_2 of the spectral sequence from the previous section, we see that the action of the BRST operator is the same as the action of d_2; demanding that the vertex operator be BRST-closed is equivalent to demanding that it lie in the kernel of d_2.

Thus, in this fashion we see that the spectral sequence relating sheaf cohomology to Ext groups is encoded physically in the BRST cohomology.

In this section we have described how the spectral sequence can be realized physically, in the special case that the Chan-Paton gauge fields on either side of the open string are the same. In principle, of course, one would also like to check that the spectral sequence is realized physically more generally. We hope to address this in future work, as this seems extremely plausible.

3 Parallel branes on submanifolds of different dimension

3.1 Basic analysis

For another class of examples, consider a set of branes wrapped on $i : S \hookrightarrow X$, with gauge fields defined by holomorphic bundle $\mathcal{E}$, and another set of branes wrapped on $j : T \hookrightarrow S \hookrightarrow X$, with gauge fields determined by bundle $\mathcal{F}$. Following the same analysis as above, and ignoring the twisting of [13], we find boundary vertex operators given by

$$b_{\overline{\imath}_1 \cdots \overline{\imath}_n}^{\alpha\beta j_1 \cdots j_m}(\phi_0)\eta^{\overline{\imath}_1} \cdots \eta^{\overline{\imath}_n} \theta_{j_1} \cdots \theta_{j_m}$$

where the η indices are tangent to T, and the θ indices are normal to $S \hookrightarrow X$. Note that since the fields must respect the boundary conditions on either side of the operator, there can be no fields with indices in $N_{T/S}$, as any such η fermions would be killed by boundary conditions on one side, and any such θ fermions would be killed by boundary conditions on the other side. Equivalently, if we think about fermions on an infinite strip, fermions with mixed Dirichlet, Neumann boundary conditions are half-integrally moded, and so cannot contribute to massless modes in the Ramond sector (where the zero-point energy is already zero). The only possible factors can come from fermions with only Neumann or only Dirichlet boundary conditions; hence, the vertex operators above couple to the tangent bundle of T and to $N_{S/X}$, but not $N_{T/S}$. These vertex operators are in one-to-one correspondence with bundle-valued differential forms, counted by the (sheaf cohomology) groups

$$H^n\left(T, (\mathcal{E}|_T)^\vee \otimes \mathcal{F} \otimes \Lambda^m N_{S/X}|_T\right). \tag{10}$$

Again, we are here ignoring the boundary condition twisting described in [13].

We should mention that our expressions for sheaf cohomology groups describing modes of open strings connecting parallel branes of different dimensions do not assume that the difference in dimensions is a multiple of four. If the difference in dimensions is not a multiple of four, then in a physical theory, supersymmetry is[3] broken – although the Ramond sector ground state will always have vanishing zero-point energy, it is well-known that only when the difference in dimensions is a multiple of four will it be possible to find corresponding massless modes in the Neveu-Schwarz sector. However, there are massless modes in the Ramond sector for any difference in dimensions, and also corresponding BRST-invariant TFT states for any difference in dimensions.

As in the last section, one would hope that these open string states should be related to global Ext groups of the form

$$\mathrm{Ext}^p_X\left(i_* \mathcal{E}, j_* \mathcal{F}\right)$$

where $i : S \hookrightarrow X$ and $j : T \hookrightarrow X$ are inclusion maps. As before, we have a minor puzzle, in that the open string vertex operators are not counted by such Ext groups, but rather by the sheaf cohomology groups (10). As before, the resolution of this puzzle is that there is a spectral sequence relating the sheaf cohomology groups (10) counting the open string vertex operators to the desired Ext groups. Specifically, there is a spectral sequence generalizing (4) as follows:

$$E_2^{p,q} \;=\; H^p\left(T, \mathcal{E}^\vee|_T \otimes \mathcal{F} \otimes \Lambda^q N_{S/X}|_T\right) \;\Longrightarrow\; \mathrm{Ext}^{p+q}_X\left(i_* \mathcal{E}, j_* \mathcal{F}\right). \qquad (11)$$

(See appendix A for a derivation.)

It is plausible to assume that, as in the last section, when the Chan-Paton-induced boundary condition twisting described in [13] is properly taken into account, the effect will be to realize the spectral sequence above physically in the BRST cohomology, so that the states of the massless Ramond sector spectrum will be in one-to-one correspondence with Ext group elements. We would like to check this explicitly in future work. For the rest of this section, we shall describe the vertex operators in terms of sheaf cohomology groups, and leave explicit checks of the physical realization of the spectral sequence above to future work.

Let us now look for an example where the spectral sequence (11) is nontrivial and $T \neq S$. If we are to have a nontrivial d_r with $r \geq 2$ then

[3]Technically speaking we are discussing 'undissolved' branes, not 'dissolved' branes. If the second brane is not really a second boundary condition on the open string, but only merely curvature in the Chan-Paton bundle, then of course the difference in 'dimensions' need not be a multiple of four.

clearly some

$$E_2^{p,q} = H^p \left(T, \mathcal{E}^\vee|_T \otimes \mathcal{F} \otimes \Lambda^q N_{S/X}|_T \right)$$

with $p \geq 2$ must be nonzero. In particular, it must be the case that $\dim(T) \geq 2$. Since T is a proper subset of S, it must be the case that $\dim(S) \geq 3$. But if $\dim(S) = \dim X = 3$, then $N_{S/X} = 0$, hence $E_2^{p,q} = 0$ for all $q \geq 1$, and again the spectral sequence must degenerate. The conclusion is that nontrivial spectral sequences (11) can occur only if $\dim(X) \geq 4$.

3.2 Example: ADHM construction

As a quick check of our spectrum computation, let us check that our description in terms of sheaf cohomology groups correctly reproduces details of the ADHM construction. For k $D5$-branes on N $D9$-branes, say, one should expect to recover a single six-dimensional hypermultiplet valued in (k, N) of $U(k) \times U(N)$, and a single hypermultiplet valued in the adjoint of $U(k)$. Assume T is a point on a $K3$ surface $S = X$ (so $N_{S/X}$ is trivial), and $\mathcal{E}$, $\mathcal{F}$ are both trivial, then the only nonzero sheaf cohomology is

$$H^0 \left(\mathrm{pt}, (\mathcal{E}|_T)^\vee \otimes \mathcal{F} \right) = \mathbf{C}^{kN} \tag{12}$$

from open strings of one orientation between the $D5$ and $D9$ branes, determining

$$\mathrm{Ext}^n_{K3} \left(i_* \mathcal{E}, j_* \mathcal{F} \right) = \begin{cases} \mathbf{C}^{kN} & n = 0, \\ 0 & n \neq 0, \end{cases}$$

a well-known mathematical result, and also

$$H^0 \left(\mathrm{pt}, \mathcal{F}^\vee \otimes \mathcal{F} \right), \; H^0 \left(\mathrm{pt}, \mathcal{F}^\vee \otimes \mathcal{F} \otimes N_{T/X} \right), \; H^0 \left(\mathrm{pt}, \mathcal{F}^\vee \otimes \mathcal{F} \otimes \Lambda^2 N_{T/X} \right) \tag{13}$$

from our previous analysis applied to strings connecting $D5$ branes to $D5$ branes.

Now, in a physical theory these sheaf cohomology groups are counting massless fermions, so from (12), we see we get a single (k, N)-valued fermion in six dimensions. This is precisely the fermionic content[4] of a six-dimensional hypermultiplet valued in (k, N) of $U(k) \times U(N)$. The set of

[4]Recall that a single six-dimensional Weyl fermion is equivalent to a pair of "symplectic-Majorana" Weyl fermions, which allow us to write the supersymmetry transformations in a form some readers might find more familiar. Put another way, in order to get a pair of four-dimensional Weyl fermions after compactification on T^2, one must have started with a single six-dimensional Weyl fermion.

states (13) from open strings connecting $D5$ branes to $D5$ branes, precisely describe the fermion content of the six-dimensional gauge multiplet, a $U(k)$-adjoint-valued hypermultiplet, and their antiparticles. (The antiparticles of the $D5 - D9$ string states are given by strings with opposite orientation, as we will be able to check later after doing the general case which includes in particular the situation $S \subset T$ arising when the opposite orientation is chosen.)

Now, in closed string theories, the GSO projection uniquely determines the type of low-energy field (*e.g.*, chiral multiplet or vector multiplet) from the $U(1)$ charge of the vertex operator. In particular, vertex operators of $U(1)$ charge one correspond to low-energy chiral multiplets in compactifications to four dimensions. With that in mind, it would be natural to assume that the degree of the Ext group determines the type of low-energy field, in the same way. In other words, it would be natural to assume that a field associated to a vertex operator corresponding to an Ext group element of degree one (or whose Serre dual has degree one) should correspond to a chiral multiplet, and so forth.

Unfortunately that naive assumption is not true in general, as we see in this example. Specifically, in this ADHM example the hypermultiplets are coming from Ext groups of degree zero, not one. (We are describing Ext groups on K3's, but the problem persists even after performing the obvious further compactification on T^2, as we shall check shortly.) Thus, we see explicitly in this ADHM example that such a hypothetical correspondence between degrees of Ext groups and type of matter content simply does not hold in general. Here, our hypermultiplets are coming from Ext groups of degree zero, not one.

Note that we have not used any results from this paper in making this observation. Also note we are not claiming that scalar fields are never associated with Ext groups of degree one in nontrivial cases. For example, in the next section, we shall see a nontrivial example in which the scalar fields are associated with Ext groups of degree one. Note furthermore that this ADHM example is not the only example in which this naive mismatch occurs. For example, in work to be published shortly we shall see that the same problem arises when describing configurations of D5 branes and D9 branes on orbifolds, as relevant to, for example,the ADHM/ALE construction.

There are several possible resolutions of this discrepancy. One possibility is that the $U(1)$ charge of a state and the degree of the Ext group do not match, perhaps via (fractionally) charged vacua. (This has also been suggested by others; see *e.g.* [3].) Perhaps the states have $U(1)$ charge one, and non-matching Ext degree. Of course, it would be absurd to then claim that the corresponding Ext groups are actually of degree one just because of their $U(1)$ charges, as those degrees are uniquely determined mathematically and, in fact, are well-known. In this paper we have specifically avoided talking about $U(1)$ charges of states, so as to avoid having to sort out such issues. We do not intend to try to give a definitive account of the resolution of this puzzle in this paper.

Instead, we merely wish to observe, based on this very clean example, that in general the type of matter content is obviously not determined solely by the degree of the Ext group; just because an Ext group element is not of degree one does not mean it cannot describe scalar states. The correct statement is obviously more complicated.

Also note that if we compactify the $D5$ branes on a T^2, then (12) is replaced by the sheaf cohomology groups

$$\begin{aligned} H^0\left(T^2, (\mathcal{E}|_T)^\vee \otimes \mathcal{F}\right) &= \mathbf{C}^{kN}, \\ H^1\left(T^2, (\mathcal{E}|_T)^\vee \otimes \mathcal{F}\right) &= \mathbf{C}^{kN} \end{aligned}$$

in one open string orientation, corresponding to

$$\mathrm{Ext}^n_{K3\times T^2}\left(i_*\mathcal{E}, j_*\mathcal{F}\right) = \begin{cases} \mathbf{C}^{kN} & n = 0, 1, \\ 0 & n > 1, \end{cases}$$

respectively, which give us two fermions in the (k, N) of $U(k) \times U(N)$, again precisely correct to match the fermionic content of four-dimensional hypermultiplets. Note in this case, when X has complex dimension three instead of two, one of the matter fields does come from an Ext group of degree one, though neither the other, nor its Serre dual, come from an Ext group of degree one.

Again, we shall not attempt in this paper to give a definitive account of the relationship between degrees of Ext groups and type of matter field; rather, we merely wish to point out that the relationship is obviously rather more complicated than seems to be often assumed.

3.3 Serre duality invariance of the spectrum

In this section we shall point out a puzzle involving Serre duality. Ordinarily spectra are Serre duality invariant, but in the present case, we shall see that Serre duality invariance is naively lost in certain cases. We shall explore this naive loss of Serre duality invariance in this section, and in a later section we shall point out how Serre duality invariance is restored by an interesting physical effect.

How does Serre duality act on our boundary vertex operators? In general, we can use the relation[5] $K_T \cong K_S|_T \otimes \Lambda^{top} N_{T/S}$. As a result, under Serre duality,

$$\begin{aligned} & H^n\left(T, (\mathcal{E}|_T)^\vee \otimes \mathcal{F} \otimes \Lambda^m N_{S/X}|_T\right) \\ \cong\ & H^{t-n}\left(T, \mathcal{F}^\vee \otimes \mathcal{E}|_T \otimes \Lambda^m N_{S/X}^\vee|_T \otimes K_T\right)^* \\ \cong\ & H^{t-n}\left(T, \mathcal{F}^\vee \otimes \mathcal{E}|_T \otimes \Lambda^{r-m} N_{S/X}|_T \otimes K_S^\vee|_T \otimes K_T\right)^* \\ \cong\ & H^{t-n}\left(T, \mathcal{F}^\vee \otimes \mathcal{E}|_T \otimes \Lambda^{r-m} N_{S/X}|_T \otimes \Lambda^{top} N_{T/S}\right)^* \end{aligned}$$

where t is the dimension of T and r is the codimension of S in X. In the second isomorphism we have used $\Lambda^r N_{S/X} \simeq K_S$.

This result is rather interesting, and somewhat unexpected. Ordinarily, the spectrum of string states is invariant under Serre duality – not only for the open string boundary states for parallel coincident branes that we discussed in the last section, but also in other contexts, such as large-radius heterotic compactifications [11]. By contrast, we seem to see here that the open string spectrum connecting parallel branes of different dimensions is not invariant under Serre duality in general.

To shed a little more light on this subject, let us try to find a maximal-charge boundary vertex operator corresponding to the holomorphic top form on the Calabi-Yau. Such operators are deeply intertwined with Serre duality invariance of spectra, and they play important roles in $\mathcal{N}=2$ supersymmetry algebras. For example, these operators are typically identified with spectral flow by one unit; recall spectral flow by half a unit is part of the spacetime supercharge.

As one might have expected by now, we find that such maximal-charge boundary vertex operators do not always exist. We can write[6]

$$\Lambda^{top} T^* X \ \cong\ \Lambda^{top} T^* T \otimes \Lambda^{top} N_{T/S}^\vee \otimes \Lambda^{top} N_{S/X}^\vee|_T.$$

[5]This can be derived by taking determinants in the exact sequence $0 \to T(T) \to T(S)|_T \to N_{T/S} \to 0$, i.e. $(K_S)^\vee|_T \simeq K_T^\vee \otimes \Lambda^{\rm top} N_{S/T}$. It is also theorem III.7.11 in [21], and when T is complex codimension one in S, this reduces to the adjunction formula [18, p. 147]. We shall use analogous formulas repeatedly in the rest of this paper, but will not give such detailed justification in future.

[6]As mentioned earlier, in general, the restriction of the tangent bundle of X to a submanifold is merely an extension of the normal bundle by the tangent bundle and need not split, so in general it need not be true that $TX|_T \cong T^*T \oplus N_{T/S} \oplus N_{S/X}|_T$.

If $\Lambda^{top} N_{T/S}$ is trivial, then the holomorphic top form on the Calabi-Yau determines a section h of $\Lambda^{top} T^* T \otimes \Lambda^{top} N_{S/X}^{\vee}|_T$, and so if $\mathcal{E} = \mathcal{O}_S$, $\mathcal{F} = \mathcal{O}_T$ we have a maximal-charge boundary vertex operator given by

$$\overline{h}_{\bar{\imath}_1 \cdots \bar{\imath}_t}^{\; j_{t+1} \cdots j_n} \, \eta^{\bar{\imath}_1} \cdots \eta^{\bar{\imath}_t} \theta_{j_{t+1}} \cdots \theta_{j_n}$$

where $t = \dim T$ and, in this one example, $n = \dim X$. On the other hand, if $\Lambda^{top} N_{T/S}$ is not trivial, then it is not clear that the holomorphic top-form on the Calabi-Yau determines any boundary vertex operator.

Thus, whenever the line bundle $\Lambda^{top} N_{T/S}$ is nontrivial, the spectrum of boundary vertex operators appears to lose Serre duality invariance, and there is a corresponding lack of a maximal-charge boundary vertex operator induced by the holomorphic top form of the Calabi-Yau. In the next section, when we discuss general brane intersections, we shall return to this issue. Specifically, we shall find that in order to be able to relate boundary vertex operators to Ext groups for general intersections, the vacua are necessarily forced to be sections of line bundles over the intersection of the brane worldvolumes. This shall not only make it possible to relate boundary vertex operators to Ext groups in the general intersection case, but will also restore Serre duality invariance of the spectra, and will also appear to have other interesting physical implications for such brane configurations.

4 General intersecting branes

4.1 Basic analysis

Next, consider two branes, both wrapped on complex submanifolds of a Calabi-Yau, intersecting nontrivially. As before, we shall work out boundary vertex operators corresponding to Ramond sector ground states. Also as before, since the Ramond sector vacuum has vanishing zero-point energy, such ground states are guaranteed to exist, regardless of whether the corresponding brane configuration is supersymmetric.

In the general intersecting brane case we have the additional complication that we must now treat branes intersecting at general angles. Previously we have only considered parallel branes, so all worldsheet fermions were either integrally or half-integrally moded, depending upon boundary conditions. For branes at general angles, fermions can be fractionally[7] moded [20], which

[7]The actual calculation in [20] is more nearly appropriate to branes wrapped on special Lagrangian submanifolds; however, it is easy to repeat the analysis for branes on complex manifolds at angles, and one recovers the same result that the moding is shifted.

naively would appear to greatly complicate our calculations.

A moment's thought reveals that no great complication is introduced. We are calculating Ramond sector ground states, and the Ramond sector vacuum has vanishing zero-point energy, so there can be no contribution from any fermions whose moding is non-integral. Fractionally moded fermions are therefore irrelevant.

Following the same analysis as before, if S and T denote two intersecting complex submanifolds of the Calabi-Yau X, with inclusions i, j respectively, and holomorphic bundles $\mathcal{E}$, $\mathcal{F}$, respectively, such that their intersection $S \cap T$ is another submanifold, then as before we would naively find boundary vertex operators given by

$$b^{\alpha\beta j_1 \cdots j_m}_{\bar{\imath}_1 \cdots \bar{\imath}_n}(\phi_0) \eta^{\bar{\imath}_1} \cdots \eta^{\bar{\imath}_n} \theta_{j_1} \cdots \theta_{j_m}$$

where the ϕ zero modes describe sheaf cohomology on the intersection $S \cap T$, the η indices are tangent to the intersection $S \cap T$, and the θ indices are normal to both S and T. More formally, the θ's are sections of the bundle

$$\tilde{N} \;=\; TX|_{S\cap T} / \left(TS|_{S\cap T} + TT|_{S\cap T} \right) \qquad (14)$$

defined on $S \cap T$, so the would-be boundary vertex operators above are in one-to-one correspondence with elements of the sheaf cohomology groups

$$H^n \left(S \cap T, \mathcal{E}^\vee|_{S\cap T} \otimes \mathcal{F}|_{S\cap T} \otimes \Lambda^m \tilde{N} \right). \qquad (15)$$

Proceeding as before, it would be natural to conjecture the existence of a spectral sequence

$$E_2^{p,q} \;=\; H^p \left(S \cap T, \mathcal{E}^\vee|_{S\cap T} \otimes \mathcal{F}|_{S\cap T} \otimes \Lambda^q \tilde{N} \right) \;\Longrightarrow\; \mathrm{Ext}^{p+q}_X \left(i_* \mathcal{E}, j_* \mathcal{F} \right). \qquad (16)$$

Unfortunately, we have a problem – no such spectral sequence exists in general, as we shall argue in the next section. After we have demonstrated where our analysis has been slightly too naive, we shall describe the physics subtlety that we have glossed over, and describe how to correctly count both physical states, as well as the relation between the correctly-counted physical states and Ext groups.

As before, we are also ignoring the twisting of boundary conditions, induced by Chan-Paton curvature, described in [13]. Judging from the case of parallel coincident branes, it is extremely plausible that all spectral sequences are realized physically in BRST cohomology, but we are not at present able to explicitly perform that check. For the remainder of this section, we shall ignore the Chan-Paton-induced boundary condition twisting, and leave the study of its effects to future work.

4.2 Failure of the naive analysis

The proposed spectral sequence (16) that would be needed to relate the proposed sheaf cohomology groups in (15) to the desired Ext groups does not exist in general, as we shall now demonstrate. Our counterexample consists of two complex submanifolds S and T, intersecting transversely in a point, such that S is a divisor in the ambient Calabi-Yau X. Since these are transverse submanifolds intersecting in a point, from the analysis above the only possible boundary vertex operators are charge zero operators of the form $b^{\alpha\beta}(\phi_0)$, corresponding to elements of $H^0\left(S \cap T, \mathcal{E}^\vee|_{S\cap T} \otimes \mathcal{F}|_{S\cap T}\right)$, and hence if the desired spectral sequence existed in general, the only nonzero Ext group would be $\mathrm{Ext}^0_X\left(i_*\mathcal{E}, j_*\mathcal{F}\right)$.

Since S is a divisor in X, we can calculate the Ext groups directly. For simplicity, assume that $\mathcal{E} = \mathcal{O}_S$ and $\mathcal{F} = \mathcal{O}_T$. Without loss of generality assume S is the zero locus of a section of a line bundle $\mathcal{O}_X(S)$, then we have a projective resolution of $\mathcal{O}_S$ given by

$$0 \longrightarrow \mathcal{O}_X(-S) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_S \longrightarrow 0.$$

Local $\underline{\mathrm{Ext}}^*_X\left(\mathcal{O}_S, \mathcal{O}_T\right)$ sheaves are given by the cohomology sheaves of the complex

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}\left(\mathcal{O}_X, \mathcal{O}_T\right) \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{O}_X}\left(\mathcal{O}_X(-S), \mathcal{O}_T\right)$$

which we can rewrite as

$$\mathcal{O}_T \longrightarrow \mathcal{O}_T(S|_T).$$

The map above is injective, with cokernel $\mathcal{O}_{S\cap T}\left(S|_{S\cap T}\right) \cong N_{S/X}|_{S\cap T}$. Thus,

$$\underline{\mathrm{Ext}}^n_{\mathcal{O}_X}\left(\mathcal{O}_S, \mathcal{O}_T\right) = \begin{cases} N_{S/X}|_{S\cap T} & n = 1 \\ 0 & n \neq 1 \end{cases}$$

so from the local-global spectral sequence we immediately compute that

$$\mathrm{Ext}^n_X\left(\mathcal{O}_S, \mathcal{O}_T\right) = \begin{cases} H^0\left(S \cap T, N_{S/X}|_{S\cap T}\right) & n = 1 \\ 0 & n \neq 1 \end{cases}$$

(using the fact that $S \cap T$ is a point) but this contradicts the claim above, as here we see in this example that the nonzero Ext groups all have degree one or greater, whereas in order for our conjectured spectral sequence (16) to hold, the only nonzero Ext group must be at degree zero.

4.3 Corrected analysis

We have a puzzle. Previously, in discussions of parallel branes, we were able to relate boundary vertex operators to Ext groups in a reasonably straightforward fashion that worked for all parallel brane configurations, both BPS and non-BPS. In the general case, however, after repeating the same analysis as before, we find it is not possible to relate our boundary vertex operators to Ext groups in the general case. Given our previous success, we must surely have made an error in our analysis. But where? In (15) we gave the most general possible BRST-invariant boundary states. However, in order to correctly count physical states, we must also take into account the vacuum $|0>$, something we previously neglected. In order to be able to relate boundary vertex operators to Ext groups, we shall propose that the boundary vacuum should be understood as a holomorphic section of either $\Lambda^{top} N_{S\cap T/T}$ or $\Lambda^{top} N_{S\cap T/S}$ over $S \cap T$, depending upon the orientation of the open string.

Now, in Calabi-Yau compactifications, the idea that the vacuum can be a section of a line bundle over the moduli space of CFT's is an old and standard notion. (See for example [22, 23] for two references where this matter is discussed in great detail.) Our proposal is slightly different – instead of thinking about vacua as sections of line bundles over a moduli space, we are proposing that a boundary vacuum should be understood as a section of a line bundle over part of the ambient Calabi-Yau. Indeed, specifying a vacuum (classically) involves specifying field configurations. In the present case, that means specifying zero modes for the worldsheet bosons, *i.e.*, specifying a point on $S \cap T$. Thus, in principle $|0>$ could be a function or line bundle section over $S \cap T$.

For now, we shall check the consistency of this proposal. We shall also see that this proposal fixes the naive breakdown in Serre duality invariance observed in section (3). We shall also give a more rigorous physical justification later, that will be independent of any notions of relating boundary vertex operators to Ext groups. Before trying to give such a first-principles derivation, however, let us begin by first checking the consistency of this proposal.

Boundary states should be written as

$$b^{\alpha\beta j_1 \cdots j_m}_{\bar{\imath}_1 \cdots \bar{\imath}_n}(\phi_0) \eta^{\bar{\imath}_1} \cdots \eta^{\bar{\imath}_n} \theta_{j_1} \cdots \theta_{j_m} |0>$$

(the same expression as before, except that we are now making the vacuum explicit), and taking into account our proposal for the vacuum we can read off that these states are in one-to-one correspondence with elements of the sheaf cohomology groups

$$\begin{array}{l} H^p\left(S\cap T, \mathcal{E}^\vee|_{S\cap T}\otimes\mathcal{F}|_{S\cap T}\otimes\Lambda^{q-m}\tilde{N}\otimes\Lambda^{top}N_{S\cap T/T}\right) \\ H^p\left(S\cap T, \mathcal{E}|_{S\cap T}\otimes\mathcal{F}^\vee|_{S\cap T}\otimes\Lambda^{q-n}\tilde{N}\otimes\Lambda^{top}N_{S\cap T/S}\right) \end{array} \tag{17}$$

(depending upon open string orientation) where m is the rank of $N_{S\cap T/T}$, and n is the rank of $N_{S\cap T/S}$.

Unlike the attempt described above in (15) to associate sheaf cohomology groups with physical states, our new sheaf cohomology groups above in (17) *are* related to Ext groups, via the spectral sequences below:

$$\begin{aligned} E_2^{p,q} &= H^p\left(S\cap T, \mathcal{E}^\vee|_{S\cap T}\otimes\mathcal{F}|_{S\cap T}\otimes\Lambda^{q-m}\tilde{N}\otimes\Lambda^{top}N_{S\cap T/T}\right) \\ &\qquad\qquad \Longrightarrow \ \mathrm{Ext}^{p+q}_X\left(i_*\mathcal{E}, j_*\mathcal{F}\right) \end{aligned}$$

$$\begin{aligned} E_2^{p,q} &= H^p\left(S\cap T, \mathcal{E}|_{S\cap T}\otimes\mathcal{F}^\vee|_{S\cap T}\otimes\Lambda^{q-n}\tilde{N}\otimes\Lambda^{top}N_{S\cap T/S}\right) \\ &\qquad\qquad \Longrightarrow \ \mathrm{Ext}^{p+q}_X\left(j_*\mathcal{F}, i_*\mathcal{E}\right) \end{aligned}$$

where m is the rank of $N_{S\cap T/T}$ and n is the rank of $N_{S\cap T/S}$. Mathematical proofs of these spectral sequences can be found in appendix A. The form of these spectral sequences is of course the motivation for the specific line bundles that we are proposing the vacua should be sections of; however, we shall also later give other independent checks that our mathematical motivation is actually physically correct.

Note that our example in subsection 4.2 is fixed by our new proposal. Recall there we considered two branes with trivial bundles wrapped on two transverse submanifolds S and T, intersecting in a point, such that S is a divisor in the ambient Calabi-Yau. From our new analysis (17), the possible boundary vertex operators are classified by the single sheaf cohomology group

$$H^0\left(S\cap T, N_{S\cap T/T}\right)$$

(for one open string orientation). Using the fact that

$$\tilde{N} = N_{S/X}|_{S\cap T}/(N_{S\cap T/T})$$

and that $\tilde{N}=0$ to see that $N_{S\cap T/T} = N_{S/X}|_{S\cap T}$, we find that the only physical states (in one orientation) are naively counted by the sheaf cohomology group

$$H^0\left(S\cap T, N_{S/X}|_{S\cap T}\right)$$

and so the corresponding Ext groups are

$$\mathrm{Ext}^n_X\left(\mathcal{O}_S, \mathcal{O}_T\right) \; = \; \begin{cases} H^0\left(S \cap T, N_{S/X}|_{S\cap T}\right) & n = 1 \\ 0 & n \neq 1 \end{cases}$$

completely agreeing with the computations described in section 4.2.

Let us check our claims in another example. Consider (following [24]) a pair of sets of orthogonal D-branes on $\mathbf{C}^3$, which we shall describe with complex coordinates x, y, z. Put N branes on the divisor $y = z = 0$ in $\mathbf{C}^3$ and k branes on the divisor $x = y = 0$ in $\mathbf{C}^3$. In [24], it was claimed that open strings stretching between these D-branes should form a hypermultiplet valued in the (k, N) of $U(k) \times U(N)$. So, in order to agree, the sheaf cohomology groups for one orientation must be two copies of $\mathbf{C}^{kN}$ (as four-dimensional hypermultiplets contain a pair of Weyl fermions). If we take S to be the worldvolume of the first set of branes, and T the worldvolume of the second set, with $\mathcal{E}$ a trivial rank N bundle on S and $\mathcal{F}$ a trivial rank k bundle on T, then we find that $TX|_{S\cap T}/\left(TS|_{S\cap T} + TT|_{S\cap T}\right)$ is the trivial rank 1 complex vector bundle over $S \cap T$ (*i.e.*, the origin of $\mathbf{C}^3$), corresponding to the directions y, $\overline{y}$, along which the open string has Dirichlet boundary conditions on both sides. Also, $N_{S\cap T/S}$ and $N_{S\cap T/T}$ are both rank one trivial bundles over the point $S \cap T$ (the origin of $\mathbf{C}^3$), and so we get two sheaf cohomology groups in each orientation, namely

$$\begin{aligned} H^0\left(S \cap T, \mathcal{E}^\vee|_{S\cap T} \otimes \mathcal{F}|_{S\cap T} \otimes N_{S\cap T/T}\right) \; &= \; \mathbf{C}^{kN}, \\ H^0\left(S \cap T, \mathcal{E}^\vee|_{S\cap T} \otimes \mathcal{F}|_{S\cap T} \otimes \tilde{N} \otimes N_{S\cap T/T}\right) \; &= \; \mathbf{C}^{kN} \end{aligned}$$

for one orientation, determining

$$\mathrm{Ext}^n_{\mathbf{C}^3}\left(i_* \mathcal{E}, j_* \mathcal{F}\right) \; = \; \begin{cases} \mathbf{C}^{kN} & n = 1, 2, \\ 0 & n \neq 1, 2. \end{cases}$$

This is the correct number of states to give a four-dimensional hypermultiplet valued in the (k, N) of $U(k) \times U(N)$, precisely reproducing the result in [24]. Note that in this case, each Ext group (or its Serre dual) corresponding to a matter field has degree one, agreeing with current lore, unlike the ADHM example discussed previously.

As another check, we shall rederive from our (corrected) general analysis our results for the case of parallel branes of different dimension. Suppose that T is a submanifold of S. Then, in the expressions above, $\tilde{N} = TX|_{S\cap T}/\left(TS|_{S\cap T} + TT|_{S\cap T}\right) = N_{S/X}|_T$, $N_{S\cap T/T} = 0$, and $N_{S\cap T/S} = N_{T/S}$ in this case. Thus, the two spectral sequences (17) reduce to

$$E_2^{p,q} \; = \; H^p\left(T, \mathcal{E}^\vee|_T \otimes \mathcal{F} \otimes \Lambda^q N_{S/X}|_T\right) \; \Longrightarrow \; \mathrm{Ext}^{p+q}_X\left(i_* \mathcal{E}, j_* \mathcal{F}\right)$$

and

$$E_2^{p,q} = H^p\left(T, \mathcal{E}|_T \otimes \mathcal{F}^\vee \otimes \Lambda^{q-n} N_{S/X}|_T \otimes \Lambda^n N_{T/S}\right) \Longrightarrow \mathrm{Ext}_X^{p+q}\left(j_* \mathcal{F}, i_* \mathcal{E}\right)$$

for $n = \mathrm{rk}\ N_{T/S}$. The first of these expressions is the first spectral sequence we discussed in describing how to generate Ext groups from boundary vertex operators for parallel branes of different dimension, and the second we discussed later in that section in connection with Serre duality in non-supersymmetric cases.

Note in passing that there are two families of cases in which the spectral sequences below (17) completely degenerate, and Ext groups can be identified canonically with single sheaf cohomology groups:

1. Suppose S and T intersect transversely. In this case, $\tilde{N} = 0$ as[8] $TS + TT = TX$ over $S \cap T$, so $E_2^{p,q} = 0$ if $q \neq \mathrm{rk}\ N_{S \cap T/T}$ in the first spectral sequence, and $E_2^{p,q} = 0$ if $q \neq \mathrm{rk}\ N_{S \cap T/S}$ in the second. Hence, the spectral sequences completely degenerate, and the Ext groups are straightforward to compute. $\mathrm{Ext}_X^p\left(i_* \mathcal{E}, j_* \mathcal{F}\right)$ is given by
$$H^{p-m}\left(S \cap T, \mathcal{E}^\vee|_{S \cap T} \otimes \mathcal{F}|_{S \cap T} \otimes \Lambda^{top} N_{S \cap T/T}\right)$$
for $p \geq \mathrm{rk}\ N_{S \cap T/T}$, and 0 otherwise. $\mathrm{Ext}_X^p\left(j_* \mathcal{F}, i_* \mathcal{E}\right)$ is given by
$$H^{p-n}\left(S \cap T, \mathcal{E}|_{S \cap T} \otimes \mathcal{F}^\vee|_{S \cap T} \otimes \Lambda^{top} N_{S \cap T/S}\right)$$
for $p \geq \mathrm{rk}\ N_{S \cap T/S}$, and 0 otherwise.

2. Suppose $S \cap T$ is zero-dimensional. In this case, $E_2^{p,q} = 0$ for $p \neq 0$ in both spectral sequences, and again the Ext groups are straightforward to calculate. $\mathrm{Ext}_X^p\left(i_* \mathcal{E}, j_* \mathcal{F}\right)$ is given by
$$H^0\left(S \cap T, \mathcal{E}^\vee|_{S \cap T} \otimes \mathcal{F}|_{S \cap T} \otimes \Lambda^{p-m} \tilde{N} \otimes \Lambda^{top} N_{S \cap T/T}\right)$$
for $p \geq \mathrm{rk}\ N_{S \cap T/T}$, and 0 otherwise. $\mathrm{Ext}_X^p\left(j_* \mathcal{F}, i_* \mathcal{E}\right)$ is given by
$$H^0\left(S \cap T, \mathcal{E}|_{S \cap T} \otimes \mathcal{F}^\vee|_{S \cap T} \otimes \Lambda^{p-n} \tilde{N} \otimes \Lambda^{top} N_{S \cap T/S}\right)$$
for $p \geq \mathrm{rk}\ N_{S \cap T/S}$, and 0 otherwise.

[8] For transversely intersecting submanifolds, this is usually stated for the tangent bundles as C^∞ real vector bundles; however, it is also true for the associated holomorphic vector bundles we have here.

4.4 Restoration of Serre duality invariance

In section 3 we saw a breakdown in Serre duality invariance of the open string spectra. However, at the time we did not take into account the possibility that the boundary vacua could be sections of bundles over part of the Calabi-Yau. In this section, we shall see explicitly that with the assumption that the vacua are sections of $\Lambda^{top} N_{S \cap T/T}$ and $\Lambda^{top} N_{S \cap T/S}$ (depending upon orientation), Serre duality invariance of the spectrum is restored.

How does Serre duality act on our states? The sheaf cohomology groups

$$H^p \left(S \cap T, \mathcal{E}^\vee|_{S \cap T} \otimes \mathcal{F}|_{S \cap T} \otimes \Lambda^{q-m} \tilde{N} \otimes \Lambda^m N_{S \cap T/T} \right) \tag{18}$$

are isomorphic to

$$H^{s-p} \left(S \cap T, \mathcal{E}|_{S \cap T} \otimes \mathcal{F}^\vee|_{S \cap T} \otimes \Lambda^{b-q+m} \tilde{N} \otimes \mathcal{M} \right)^*$$

where

$$\mathcal{M} \;=\; \Lambda^{top} \tilde{N}^\vee \otimes \Lambda^m N^\vee_{S \cap T/T} \otimes K_{S \cap T}$$

and b is the rank of $\tilde{N}$ and s is the dimension of the intersection $S \cap T$. Next, use the fact (to be demonstrated below) that

$$\Lambda^{top} \tilde{N}^\vee \otimes \Lambda^{top} N^\vee_{S \cap T/T} \;\cong\; \Lambda^{top} N_{S \cap T/S} \otimes \Lambda^{top} N^\vee_{S \cap T/X}$$

so that the sheaf cohomology groups (18) are isomorphic to

$$H^{s-p} \left(S \cap T, \mathcal{E}|_{S \cap T} \otimes \mathcal{F}^\vee|_{S \cap T} \otimes \Lambda^{b-q+m} \tilde{N} \otimes \tilde{\mathcal{M}} \right)^*$$

where

$$\tilde{\mathcal{M}} \;=\; \Lambda^{top} N_{S \cap T/S} \otimes \Lambda^{top} N^\vee_{S \cap T/X} \otimes K_{S \cap T}$$

but $K_{S \cap T} \;\cong\; \Lambda^{top} N_{S \cap T/X}$, so we finally see that the sheaf cohomology groups (18) are isomorphic to

$$H^{s-p} \left(S \cap T, \mathcal{E}|_{S \cap T} \otimes \mathcal{F}^\vee|_{S \cap T} \otimes \Lambda^{b-q+m} \tilde{N} \otimes \Lambda^{top} N_{S \cap T/S} \right)^* .$$

Thus, Serre duality acts to exchange the sheaf cohomology groups appearing in our two spectral sequences. In other words, taking into account our proposal that the open string boundary vacua should be interpreted as sections of line bundles over $S \cap T$, we find that the physical spectrum is Serre duality invariant.

Let us also determine under what circumstances the holomorphic top form on the ambient Calabi-Yau induces a maximal-charge boundary vertex operator. Proceeding as before, we have that

$$\Lambda^{top} T^* X|_{S\cap T} \;=\; \Lambda^{top} T^*(S\cap T) \otimes \Lambda^{top} N^{\vee}_{S\cap T/X}.$$

Next, use the fact that

$$\begin{aligned} \Lambda^{top} N_{S\cap T/X} &= \Lambda^{top} N_{S\cap T/T} \otimes \Lambda^{top} N_{T/X}|_{S\cap T} \\ &= \Lambda^{top} N_{S\cap T/S} \otimes \Lambda^{top} N_{S/X}|_{S\cap T} \end{aligned}$$

and as

$$\tilde{N} \;=\; \frac{TX|_{S\cap T}}{TS|_{S\cap T} + TT|_{S\cap T}} \;=\; \frac{N_{S/X}|_{S\cap T}}{N_{S\cap T/T}} \;=\; \frac{N_{T/X}|_{S\cap T}}{N_{S\cap T/S}}$$

we see that

$$\Lambda^{top} N_{S\cap T/X} \;=\; \Lambda^{top} N_{S\cap T/T} \otimes \Lambda^{top} \tilde{N} \otimes \Lambda^{top} N_{S\cap T/S}$$

so finally

$$\Lambda^{top} T^* X|_{S\cap T} \;=\; \Lambda^{top} T^*(S\cap T) \otimes \Lambda^{top} N^{\vee}_{S\cap T/T} \otimes \Lambda^{top} N^{\vee}_{S\cap T/S} \otimes \Lambda^{top} \tilde{N}^{\vee}.$$

Thus, if both $\Lambda^{top} N_{S\cap T/S}$ and $\Lambda^{top} N_{S\cap T/T}$ are trivial, then the holomorphic top form on the ambient Calabi-Yau is equivalent to a section of $\Lambda^{top} T^*(S\cap T) \otimes \Lambda^{top} \tilde{N}^{\vee}$, which is equivalent to a maximal-charge boundary vertex operator

$$h_{\bar{\imath}_1 \cdots \bar{\imath}_s}^{j_{s+1} \cdots j_n} \eta^{\bar{\imath}_1} \cdots \eta^{\bar{\imath}_s} \theta_{j_{s+1}} \cdots \theta_{j_n}$$

Of course, by including the vacua in the discussion, we find that if at least one of $\Lambda^{top} N_{S\cap T/T}$, $\Lambda^{top} N_{S\cap T/S}$ is trivial, then one could still get a maximal-charge boundary vertex operator induced by the holomorphic top form on the Calabi-Yau.

Recall in section 3 we ran into an apparent problem with Serre duality. At the time, we did not consider that the boundary vacuum might be a section of a line bundle over part of the Calabi-Yau. Let us take a moment to work through the details. First, if $T \subseteq S$, then $N_{S\cap T/T} = \emptyset$, so for one string orientation we were consistent in section 3 to assume that $|0>$ is a scalar, and so the boundary vertex operator analysis in section 3 need not be redone. At the same time, $N_{S\cap T/S} = N_{T/S}$, so we see in our present language that if $\Lambda^{top} N_{T/S}$ is nontrivial, then we would naively run into problems with Serre duality, as indeed we saw in section 3. By assuming that the open string boundary vacuum of the opposite orientation is a section of $\Lambda^{top} N_{T/S}$, we are able to restore Serre duality invariance of the open string spectrum. Thus, we have solved the puzzle presented in section 3.

4.5 Vacua as sections of line bundles

So far we have proposed that the boundary Fock vacua $|0>$ should be understood as sections of line bundles over $S \cap T$. We have seen how this proposal was necessary in order to be able to relate boundary vertex operators to Ext groups. We have performed self-consistency checks on the resulting spectrum of boundary vertex operators, and we have seen how this proposal has the happy effect of also resolving an earlier puzzle involving an apparent breakdown in Serre duality invariance of certain spectra. However, these are all merely self-consistency checks; we have yet to give any sort of first-principles argument in favor of such a description of the boundary vacua.

In this section we shall provide a nearly rigorous physical argument in favor of this proposal. (Note, however, that our analysis ignores the Chan-Paton-induced twisting of boundary conditions described in [13]. Properly taking it into account will almost certainly modify the claims in this section. However, elsewhere in the discussion of general intersections we have also approximated the boundary conditions by ignoring that twisting; we are consistent in also omitting it here.)

Why might one believe that the boundary vacua should be interpreted as sections of line bundles? To justify this assertion, consider the zero modes of the partition function for the B-twisted CFT on a disk, describing open strings stretched between D-branes on S and on T. There are fermion zero modes coupling to $T(S \cap T)$ and to $\tilde{N} = TX|_{S\cap T}/(TS|_{S\cap T} + TT|_{S\cap T})$, hence because of those zero modes the partition function is a section of[9]

$$\Lambda^{top} T^*(S \cap T) \otimes \Lambda^{top} \tilde{N}^{\vee}$$

but as demonstrated earlier this bundle is isomorphic to

$$\Lambda^{top} N_{S\cap T/S} \otimes \Lambda^{top} N_{S\cap T/T}.$$

Thus, schematically, the disk partition function $< 0|0 >$ of the B-twisted open string sigma model is a section of $\Lambda^{top} N_{S\cap T/S} \otimes \Lambda^{top} N_{S\cap T/T}$.

We proposed earlier, in order to be able to form any relationship between boundary vertex operators and Ext groups, that the boundary vacua should be sections of $\Lambda^{top} N_{S\cap T/T}$ and $\Lambda^{top} N_{S\cap T/S}$ (one for each open string

[9]Note that since we want a correlation function containing just enough fermions to cancel out the zero modes to have a chance of being a nonzero constant, the zero modes must couple to duals of the bundles. Otherwise, a correlation function with fermions matching zero modes would be a section of a nontrivial bundle over T, and so could only be a constant if that constant were zero.

orientation). Here we see that that proposal, which was motivated mathematically, is completely consistent with the anomaly structure of the open string theory on a disk.

As a check, let us apply the same analysis to the case $S = T$ (*i.e.*, parallel coincident branes). In this case, the fermion zero modes (in the case the worldsheet is a strip) couple to TS and $N_{S/X}$, but $\Lambda^{top} T^* S \otimes \Lambda^{top} N_{S/X}^{\vee}$ is trivial, so the same analysis tells us that for parallel coincident branes, the Fock vacuum should not be interpreted as a section of a nontrivial bundle, which is consistent with our earlier results.

Now, arguing that $< 0|0 >$ formally behaves as if it were a section of $\Lambda^{top} N_{S \cap T/T} \otimes \Lambda^{top} N_{S \cap T/S}$ is not sufficient to argue that the open string vacua $|0>$ are individually sections of $\Lambda^{top} N_{S \cap T/T}$ and $\Lambda^{top} N_{S \cap T/S}$. Perhaps the vacua are sections of some other bundles, such that the tensor product of those other bundles is isomorphic to $\Lambda^{top} N_{S \cap T/T} \otimes \Lambda^{top} N_{S \cap T/S}$. So we have not given an unambiguous first-principles description of why the vacua must necessarily be the sections of the particular line bundles proposed earlier; our formal first-principles argument only applies to the product $< 0|0 >$. However, from self-consistency, it is easy to see that the stronger statement about the vacua should be true. For example, in section 3 we discovered a relationship between boundary vertex operators and Ext groups without having to worry about vacua being sections of bundles. This is all consistent so long as in that case, $|0>$ is a section of a trivial bundle, and indeed, our claim is that in that case, $|0>$ is a section of $\Lambda^{top} N_{S \cap T/T} = \Lambda^{top} N_{T/T}$, which is clearly trivial. If $|0>$ were not a section of $\Lambda^{top} N_{S \cap T/T}$, if only the weaker statement that $< 0|0 >$ is a section of $\Lambda^{top} N_{S \cap T/T} \otimes \Lambda^{top} N_{S \cap T/S}$ were true, then in section 3 we should not have been able to relate boundary vertex operators to Ext groups, and yet, we were able to do so.

4.6 Proposal for new selection rule

In this section, we shall make a proposal for a new selection rule for BPS brane configurations. Specifically, we propose that whenever either of the line bundles

$$\Lambda^{top} N_{S \cap T/T}$$
$$\Lambda^{top} N_{S \cap T/S}$$

is nontrivial, the corresponding brane configuration is non-BPS, when working near large radius, and when the B field vanishes identically.

Formally this proposal is motivated by our earlier claims that the boundary vacua are sections of nontrivial line bundles over $S \cap T$. Identify the

BRST operator with $\overline{\partial}$, and the other supersymmetry generator with ∂, in the standard fashion; then for the vacuum to be invariant under both supersymmetry generators of the $\mathcal{N} = 2$ algebra means that it is annihilated by both ∂ and $\overline{\partial}$, hence is a constant, or equivalently a section of a trivial line bundle. By contrast, if the vacuum is a holomorphic section of a nontrivial line bundle, then although it is annihilated by $\overline{\partial}$ (by holomorphicity), it is not in general annihilated by ∂, and hence it must spontaneously break supersymmetry, as it is not annihilated by both of the supersymmetry generators.

Thus, if $< 0|$ were a holomorphic section of $\Lambda^{top} N_{T/S}$ over T, one might expect nontriviality of the bundle $\Lambda^{top} N_{T/S}$ to imply spontaneous supersymmetry breaking, and that is precisely what we described empirically in the last subsection.

Of course, the formal analysis we have just described is extremely naive; our remarks in this section are not meant to be considered definitive, but rather merely suggestive.

Let us check this proposal empirically. Suppose $S = X$, which is taken to be a Calabi-Yau threefold with holonomy precisely equal to $SU(3)$ (so in particular X is not T^6 or $K3 \times T^2$). Let T be a curve in X, other than an elliptic curve. In this case, $\Lambda^{top} N_{S \cap T/S} = \Lambda^{top} N_{T/S}$ is nontrivial. Now, the difference in dimensions between S and T is a multiple of four, so naively this brane configuration appears[10] to be BPS. However, the ambient Calabi-Yau breaks too much supersymmetry. After all, in a type II compactification, the ambient Calabi-Yau leaves one with only $\mathcal{N} = 2$ supersymmetry in four dimensions, which is broken to $\mathcal{N} = 1$ by the first brane. However, $\mathcal{N} = 1$ has no BPS states, so a second non-coincident brane cannot be a BPS configuration.

Similarly, if $S = X$, a Calabi-Yau threefold as above, and T is a divisor in X, then $\Lambda^{top} N_{S \cap T/S} = K_T$ is nontrivial (unless T is itself Calabi-Yau) and again the brane configuration appears to be generically non-BPS, although this time the reason is much more basic, namely the difference in dimensions is not a multiple of four.

In every example we have checked in which the brane configuration is BPS, both of the bundles $\Lambda^{top} N_{S \cap T/T}$ and $\Lambda^{top} N_{S \cap T/S}$ are trivial. For ex-

[10]Precisely at the large radius limit point, this brane configuration *is* BPS. Our remarks involving curvature of the ambient space are irrelevant at the limit point, as the space has become infinitely large and curvature has spread infinitely thin. However, if one is interested in results merely near large radius, not actually at large radius, then our curvature considerations become important.

ample, consider parallel coincident branes *i.e.*, $S = T$, then both $N_{S \cap T/S} = 0$ and $N_{S \cap T/T} = 0$. Similarly, if T is a point on $S = X = K3$, then again both $\Lambda^{top} N_{S \cap T/T}$ and $\Lambda^{top} N_{S \cap T/S}$ are trivial, consistent with the fact that the corresponding branes are mutually supersymmetric.

Note also that it is possible to have non-BPS configurations such that both $\Lambda^{top} N_{S \cap T/T}$ and $\Lambda^{top} N_{S \cap T/S}$ are trivial. For example, if $S = X$ and T is a divisor that is also itself a Calabi-Yau manifold, then both of those line bundles are trivial. So we are not conjecturing that a brane configuration is supersymmetric if and only if both of those line bundles are trivial. Rather we are only making the weaker conjecture that if either of those line bundles is nontrivial, then close to large radius, with zero B field, the brane configuration will not be BPS.

5 Nonintersecting branes

In this section, for completeness we shall very briefly discuss boundary spectra describing open strings between D-branes on two completely disjoint complex submanifolds of a Calabi-Yau manifold X. Let the two submanifolds be denoted S_1, S_2, say, with inclusion maps i_1, i_2, respectively.

In this case, if the two complex submanifolds are completely disjoint, then there are no massless open string states connecting them. Happily, it is also easy to check that in such circumstances, all the groups

$$\mathrm{Ext}^n_X \left(i_{1*}\mathcal{E}_1, i_{2*}\mathcal{E}_2 \right)$$

must vanish. We can check this by noting that if there are no points on X where at least one of $i_{1*}\mathcal{E}_1$, $i_{2*}\mathcal{E}_2$ are zero, then all the corresponding local $\underline{\mathrm{Ext}}$ sheaves must vanish, and so by the local-global spectral sequence, the global Ext groups must all vanish as well.

6 Ext groups of complexes

Another claim often made concerning the relationship between D-branes and derived categories is that if open strings with boundaries corresponding to two complexes should have open string modes counted by Ext groups. In other words, for an open string strip diagram, if $\mathcal{E}.$ is a complex describing one boundary, and $\mathcal{F}.$ is a complex describing the other boundary, then open

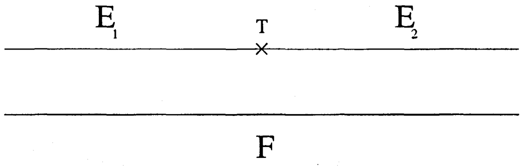

Figure 1: Open string realizing map between simple complexes.

string modes should be counted by elements of

$$\mathrm{Ext}^n_{D(X)}\left(\mathcal{E}.,\mathcal{F}.\right) \;=\; H^n\mathbf{R}\mathrm{Hom}\;\left(\mathcal{E}.,\mathcal{F}.\right)$$

One can ask how these groups are realized physically, just as earlier in this paper we asked how Ext groups between coherent sheaves could be realized physically. We saw how Ext groups between coherent sheaves are realized by vertex operators. What is the analogous procedure for Ext groups of complexes?

We shall consider simple configurations involving only branes, no antibranes. We will find that boundary vertex operators can be used to determine countably many possible Ext groups between complexes. It is tempting to conjecture that this ambiguity is closely related to possible reinterpretations of this calculation in terms of brane/antibrane configurations; however, we shall not say anything further here.

Let us consider the simplest possible nontrivial case, in which

$$\mathcal{E}.: \cdots \longrightarrow 0 \longrightarrow \mathcal{E}_1 \stackrel{T}{\longrightarrow} \mathcal{E}_2 \longrightarrow 0 \longrightarrow \cdots$$

and

$$\mathcal{F}.: \cdots \longrightarrow 0 \longrightarrow \mathcal{F}_1 \longrightarrow 0 \longrightarrow \cdots$$

so $\mathcal{E}.$ has only two nonzero elements, and $\mathcal{F}.$ has only a single nonzero element. The corresponding open string diagram is shown in figure (1).

Now, how can we see the states counted by $\mathrm{Ext}(\mathcal{E}.,\mathcal{F}.)$? We shall loosely follow the analysis of [25]. The only boundary degrees of freedom are the asymptotic incoming and asymptotic outgoing states. A state coming in asymptotically from the left of figure (1) would only see boundaries $\mathcal{E}_1$ and $\mathcal{F}$; both $\mathcal{E}_2$ and the boundary operator T would be effectively invisible. Hence, assuming for simplicity that $\mathcal{E}_1$ and $\mathcal{F}$ are both bundles on the same submanifold S of the Calabi-Yau X, our earlier analysis tells us that the asymptotic incoming states are naively counted by the sheaf cohomology groups

$$H^n\left(S, \mathcal{E}_1^\vee \otimes \mathcal{F} \otimes \Lambda^m N_{S/X}\right)$$

which determine (via a spectral sequence) elements of

$$\mathrm{Ext}_X^{n+m}\left(i_* \mathcal{E}_1, i_* \mathcal{F}\right).$$

Similarly, the asymptotic outgoing states (on the far right) only see $\mathcal{E}_2$ and $\mathcal{F}$, and so are naively counted by the sheaf cohomology groups

$$H^n\left(S, \mathcal{E}_2^\vee \otimes \mathcal{F} \otimes \Lambda^m N_{S/X}\right)$$

which determine a corresponding Ext group.

Now, these asymptotic states determine an element of the desired group

$$\mathrm{Ext}_{D(X)}^n\left(\mathcal{E}., \mathcal{F}.\right)$$

as follows. First, note that there is a short exact sequence of complexes

$$0 \longrightarrow \mathcal{E}_2 \longrightarrow \mathcal{E}. \longrightarrow \mathcal{E}_1[1] \longrightarrow 0 \tag{19}$$

an immediate consequence of the following trivial commuting diagram:

$$\begin{array}{lccccccccccc}
0: & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathcal{E}_2: & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{E}_2 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & \downarrow & & \downarrow & & \| & & \downarrow & & \\
\mathcal{E}.: & \cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{E}_1 & \xrightarrow{T} & \mathcal{E}_2 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & \downarrow & & \| & & \downarrow & & \downarrow & & \\
\mathcal{E}_1[1]: & \cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\
 & & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0: & \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots
\end{array}$$

As a result of the short exact sequence (19), we have a long exact sequence of Ext groups given by

$$\begin{aligned}
&\longrightarrow \mathrm{Ext}_{D(X)}^n\left(\mathcal{E}_1[1], \mathcal{F}.\right) \longrightarrow \mathrm{Ext}_{D(X)}^n\left(\mathcal{E}., \mathcal{F}.\right) \longrightarrow \mathrm{Ext}_{D(X)}^n\left(\mathcal{E}_2, \mathcal{F}.\right) \longrightarrow \\
&\longrightarrow \mathrm{Ext}_{D(X)}^{n+1}\left(\mathcal{E}_1[1], \mathcal{F}.\right) \longrightarrow \cdots
\end{aligned}$$

For the complex $\mathcal{F}.$ described above, we can simplify these expressions. The simplification depends upon the relative grading of $\mathcal{E}.$ and $\mathcal{F}.$. In the special case that $\mathcal{F}_1$ and $\mathcal{E}_2$ have the same grading

$$\begin{aligned} \mathrm{Ext}^n_{D(X)}\left(\mathcal{E}_2, \mathcal{F}.\right) &= \mathrm{Ext}^n_X\left(\mathcal{E}_2, \mathcal{F}_1\right) \\ \mathrm{Ext}^n_{D(X)}\left(\mathcal{E}_1[1], \mathcal{F}.\right) &= \mathrm{Ext}^{n-1}_X\left(\mathcal{E}_1, \mathcal{F}_1\right) \end{aligned}$$

so we can rewrite the long exact sequence above more usefully as follows:

$$\begin{aligned} \cdots &\longrightarrow \mathrm{Ext}^{n-1}_X\left(\mathcal{E}_1, \mathcal{F}_1\right) \longrightarrow \mathrm{Ext}^n_{D(X)}\left(\mathcal{E}., \mathcal{F}_1\right) \longrightarrow \mathrm{Ext}^n_X\left(\mathcal{E}_2, \mathcal{F}_1\right) \longrightarrow \\ &\longrightarrow \mathrm{Ext}^n_X\left(\mathcal{E}_1, \mathcal{F}_1\right) \longrightarrow \cdots \end{aligned}$$

More generally, if the grading of $\mathcal{F}_1$ is shifted j units to the left of $\mathcal{E}_2$, then

$$\begin{aligned} \mathrm{Ext}^n_{D(X)}\left(\mathcal{E}_2, \mathcal{F}.\right) &= \mathrm{Ext}^{n+j}_X\left(\mathcal{E}_2, \mathcal{F}_1\right) \\ \mathrm{Ext}^n_{D(X)}\left(\mathcal{E}_1[1], \mathcal{F}.\right) &= \mathrm{Ext}^{n-1+j}_X\left(\mathcal{E}_1, \mathcal{F}_1\right) \end{aligned}$$

in which case we can rewrite the long exact sequence as

$$\begin{aligned} &\longrightarrow \mathrm{Ext}^{n-1+j}_X\left(\mathcal{E}_1, \mathcal{F}_1\right) \longrightarrow \mathrm{Ext}^n_{D(X)}\left(\mathcal{E}., \mathcal{F}.\right) \longrightarrow \mathrm{Ext}^{n+j}_X\left(\mathcal{E}_2, \mathcal{F}_1\right) \longrightarrow \\ &\longrightarrow \mathrm{Ext}^{n+j}_X\left(\mathcal{E}_1, \mathcal{F}_1\right) \longrightarrow \cdots \end{aligned}$$

Thus, we see that boundary vertex operators can be used to determine Ext groups between complexes, but there is an ambiguity in the grading.

7 Conclusions

In this paper we have explored recent claims that, for D-branes wrapped on complex submanifolds of Calabi-Yau's, open string states between D-branes are counted by Ext groups. We have given much more detailed checks of this claim than have appeared previously, and have worked out vertex operators corresponding to Ext group elements in some generality.

In general terms, we have found that naively massless states in the Ramond sector of open strings between intersecting D-branes (wrapped on complex submanifolds, near large radius, with zero B field) are in one-to-one correspondence with sheaf cohomology groups, which are related to the desired Ext groups via spectral sequences. We have checked in a subclass of cases that those spectral sequences are realized physically via BRST cohomology, ultimately because of a Chan-Paton-induced modification of the open string boundary conditions [13]. We conjecture (though have not been able to explicitly check) that the same is true in general, that in all cases, the spectral sequences are realized physically in BRST cohomology, so that in general, massless Ramond sector states are in one-to-one correspondence with Ext group elements. These spectral sequences are nontrivial in general, in the sense that the unsigned sum of the dimensions of the sheaf cohomology groups is not the same as the unsigned sum of the dimensions of the corresponding Ext groups, so understanding their physical realization is an important issue.

For parallel (but not necessarily coincident) branes, relating boundary vertex operators to Ext group elements is straightforward physically. However, for more general brane intersections, we find some interesting new physics. Specifically, we found that in order to be able to relate boundary vertex operators to Ext groups, and in the context of ignoring the Chan-Paton-induced boundary condition twisting, the boundary vacua are necessarily forced to be sections of holomorphic line bundles over the brane intersections. This proposal not only allows us to find a relationship with Ext groups, but also fixes a naive breakdown in Serre duality invariance of the spectrum. We also gave a purely physics-based motivation for this proposal, by considering anomalies in the B-twisted sigma model on the disk. Finally, we speculated that this phenomenon might be an indication of a new (and very obscure) selection rule for BPS states.

In future work, we hope to return to the issue of the physical realization of the spectral sequences in the remaining cases. We conjecture that those spectral sequences are realized physically via BRST cohomology, so that the massless Ramond sector states are in one-to-one correspondence with Ext group elements, but we have only checked this explicitly in the case of parallel coincident branes.

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A Derivation of spectral sequences

In this appendix, we give rigorous derivations of the spectral sequences that are used in the paper.

A.1 Parallel coincident branes

Let X be a complex manifold. In our applications, X will be Calabi-Yau but this is not necessary so we do this more general situation which could conceivably be of interest for more general topological string theories than have been considered here.

Let S be a smooth complex submanifold of X, and let $i : S \hookrightarrow X$ be the inclusion. Finally, let $\mathcal{E}$ and $\mathcal{F}$ be bundles on S. The goal of this section is to compute $\mathrm{Ext}_X^{p+q}(i_*\mathcal{E}, i_*\mathcal{F})$, verifying the spectral sequence (4) which we reproduce here for convenience:

$$E_2^{p,q} : H^p\left(S, \mathcal{E}^\vee \otimes \mathcal{F} \otimes \Lambda^q N_{S/X}\right) \Longrightarrow \mathrm{Ext}_X^{p+q}\left(i_*\mathcal{E}, i_*\mathcal{F}\right)$$

The method is to compute the local Ext sheaves $\underline{\mathrm{Ext}}^*(i_*\mathcal{E}, i_*\mathcal{F})$ (which are supported on S), and then use the local to global spectral sequence

$$H^p(X, \underline{\mathrm{Ext}}^q(i_*\mathcal{E}, i_*\mathcal{F})) \Longrightarrow \mathrm{Ext}_X^{p+q}(i_*\mathcal{E}, i_*\mathcal{F}). \tag{20}$$

Note that

$$H^p(X, \underline{\mathrm{Ext}}^q(i_*\mathcal{E}, i_*\mathcal{F})) = H^p(S, \underline{\mathrm{Ext}}^q(i_*\mathcal{E}, i_*\mathcal{F}))$$

when $\underline{\mathrm{Ext}}^q(i_*\mathcal{E}, i_*\mathcal{F})$ is viewed as a sheaf on S.

Since

$$\underline{\mathrm{Ext}}^q(i_*\mathcal{E}, i_*\mathcal{F}) = \underline{\mathrm{Ext}}^q(i_*\mathcal{O}_S, i_*\mathcal{O}_S) \otimes \mathcal{E}^\vee \otimes \mathcal{F}, \tag{21}$$

we can and will assume temporarily that $\mathcal{E}$ and $\mathcal{F}$ are both $\mathcal{O}_S$. Since S is smooth, it is a local complete intersection [18, p. 20], so we can work locally and assume that S is the zero locus of a regular section $s \in H^0(E)$ where E is a bundle on X. We will eliminate dependence on E in the results by noting $E|_S \simeq N_{S/X}$, as will be verified shortly.

We have the Koszul resolution

$$0 \to \cdots \to \wedge^2 E^\vee \to E^\vee \to \mathcal{O}_X \to i_*\mathcal{O}_S \to 0 \tag{22}$$

where all maps except the last restriction map are defined as contraction by s. Explicitly, the map $\wedge^q E^\vee \to \wedge^{q-1} E^\vee$ is given by

$$(\omega_1 \wedge \ldots \wedge \omega_q) \mapsto \sum (-1)^{j-1} \omega_j(s) \omega_1 \wedge \ldots \wedge \omega_{j-1} \wedge \omega_{j+1} \wedge \ldots \wedge \omega_q.$$

The Koszul complex (22) is exact if and only if s is a regular section, i.e. if and only if the rank of E is equal to the codimension of S in X. See [18, 26].

For later use, note that (22) omitting $\mathcal{O}_X$ is self-dual up to a twist: if E has rank r so that $\wedge^r E$ is a line bundle, then $\wedge^k E \simeq \wedge^{r-k} E^\vee \otimes (\wedge^r E)$. See [26, Proposition 17.15].

We can truncate (22) to obtain the surjection

$$E^\vee \to \mathcal{I}_{S/X} \to 0 \tag{23}$$

where $\mathcal{I}_{S/X}$ denotes the ideal sheaf of S in X. Tensoring (23) with $i_* \mathcal{O}_S = \mathcal{O}_X / (\mathcal{I}_{S/X})$ we get the surjection $E^\vee|_S \to \mathcal{I}_{S/X} / (\mathcal{I}_{S/X})^2$ which can be seen to be an isomorphism by using local equations for S in X. Since

$$\mathcal{I}_{S/X} / (\mathcal{I}_{S/X})^2 \simeq N_{S/X}^\vee,$$

we see that $E|_S \simeq N_{S/X}$ as claimed.

We use (21) and (22) to calculate $\underline{\mathrm{Ext}}^*_{\mathcal{O}_X}(i_* \mathcal{E}, i_* \mathcal{F})$ as the cohomology sheaves of the complex

$$\left(\wedge^* E \otimes \mathcal{E}^\vee \otimes \mathcal{F}\right)|_S. \tag{24}$$

Note that $s|_S = 0$ by construction, so all maps in (24) are 0. Combining with $E|_S \simeq N_{S/X}$, we get

$$\underline{\mathrm{Ext}}^q_{\mathcal{O}_X}(i_* \mathcal{E}, i_* \mathcal{F}) \simeq \mathcal{E}^\vee \otimes \mathcal{F} \otimes \Lambda^q N_{S/X}. \tag{25}$$

Then the claimed spectral sequence (4) comes from substituting (25) into the local to global spectral sequence (20).

This spectral sequence has previously appeared in the string theory literature, e.g. [27].

A.2 Parallel branes of different dimension

In this section we shall derive the spectral sequence (11) which we reproduce here for convenience:

$$E_2^{p,q} = H^p\left(T, \mathcal{E}^\vee|_T \otimes \mathcal{F} \otimes \Lambda^q N_{S/X}|_T\right) \Longrightarrow \mathrm{Ext}^{p+q}_X(i_* \mathcal{E}, j_* \mathcal{F}).$$

Here T is a complex submanifold of S, which is a complex submanifold of X, $\mathcal{E}$ is a holomorphic bundle on $\mathcal{S}$, $\mathcal{F}$ is a holomorphic bundle on $\mathcal{T}$, and $i : S \hookrightarrow X$, $j : T \hookrightarrow X$ are inclusions.

Now, recall that we can relate local Ext sheaves to global Ext groups by the local to global spectral sequence generalizing (20)

$$E_2^{p,q} = H^p\left(S, \underline{\mathrm{Ext}}^q_{\mathcal{O}_S}(\mathcal{S}_1, \mathcal{S}_2)\right) \Longrightarrow \mathrm{Ext}^{p+q}_S(\mathcal{S}_1, \mathcal{S}_2) \qquad (26)$$

which is valid for any coherent sheaves $\mathcal{S}_1, \mathcal{S}_2$ on S.

To derive the result formally, we shall first show how to compute local Ext sheaves in terms of analogous data, then apply the local-global spectral sequence (26).

As in the previous section, we can work locally and assume that S is the zero locus of a regular section s of a bundle E on X. Then we have the Koszul resolution of $\mathcal{O}_S$:

$$\cdots \longrightarrow \Lambda^2 E^\vee \longrightarrow E^\vee \longrightarrow \mathcal{O}_X \longrightarrow i_* \mathcal{O}_S \to 0,$$

where E is a holomorphic bundle on X of rank equal to the complex codimension of S in X, with a section s whose zero set is S. The bundle E also has the property that $E|_S = N_{S/X}$.

To compute the sheaf $\underline{\mathrm{Ext}}^q_{\mathcal{O}_X}(i_*\mathcal{O}_S, j_*\mathcal{F})$, we use the Koszul resolution above to provide a projective resolution of $\mathcal{O}_S$. Thus, the local Ext sheaf desired is the degree q cohomology sheaf of the complex

$$\underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{O}_X, j_*\mathcal{F}) \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{O}_X}(E^\vee, j_*\mathcal{F}) \longrightarrow \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\Lambda^2 E^\vee, j_*\mathcal{F}) \longrightarrow .$$

Since $j_*\mathcal{F}$ is supported on $T \subset S$ and $s|_S = 0$, again we have that all maps are 0 and so

$$\underline{\mathrm{Ext}}^q_{\mathcal{O}_X}(i_*\mathcal{O}_S, j_*\mathcal{F}) = j_*\underline{\mathrm{Hom}}_{\mathcal{O}_T}\left(\Lambda^q N^\vee_{S/X}|_T, \mathcal{F}\right) \simeq \Lambda^q N_{S/X}|_T \otimes \mathcal{F}.$$

Locally on X we can form a bundle $\overline{\mathcal{E}}$ such that $\overline{\mathcal{E}}|_S = \mathcal{E}$, and by tensoring the projective resolution of $\mathcal{O}_S$ with $\overline{\mathcal{E}}$ and repeating the analysis above we immediately get the result

$$\begin{aligned}\underline{\mathrm{Ext}}^q_{\mathcal{O}_X}(i_*\mathcal{E}, j_*\mathcal{F}) &= j_*\underline{\mathrm{Hom}}_{\mathcal{O}_T}\left(\mathcal{E}|_T \otimes \Lambda^q N^\vee_{S/X}|_T, \mathcal{F}\right)\\ &= j_*\left(\left(\mathcal{E}^\vee \otimes \Lambda^q N_{S/X}\right)|_T \otimes \mathcal{F}\right).\end{aligned}$$

Finally, using the result [21, Lemma III.2.10] that

$$H^*(X, j_*\mathcal{F}) = H^*(T, \mathcal{F})$$

we see that

$$H^p\left(X, \underline{\mathrm{Ext}}^q\left(i_*\mathcal{E}, j_*\mathcal{F}\right)\right) = H^p\left(T, \left(\mathcal{E}^\vee \otimes \Lambda^m N_{S/X}\right)|_T \otimes \mathcal{F}\right)$$

which together with the local-global spectral sequence tells us that we have the desired level two spectral sequence

$$E_2^{p,q} : H^p\left(T, \left(\mathcal{E}^\vee \otimes \Lambda^m N_{S/X}\right)|_T \otimes \mathcal{F}\right) \Longrightarrow \mathrm{Ext}_X^{p+q}\left(i_*\mathcal{E}, j_*\mathcal{F}\right).$$

A.3 General brane intersections

Let's now turn to the general case. We have to interpret (24), which up to tensoring with bundles is the dual of a Koszul complex on a (not necessarily regular) section s of $E \otimes j_*\mathcal{O}_T = E|_T$. Koszul complexes are exact over the locus where the section is regular, so in particular (24) is exact on the complement of $S \cap T$. In other words, the cohomology sheaves of (24) are supported on $S \cap T$ as was already clear geometrically since these compute $\underline{\mathrm{Ext}}^*(i_*E, j_*F)$.

If we restrict (23) to T we again get a surjection

$$E^\vee|_T \to \mathcal{I}_{S\cap T,T} \to 0 \tag{27}$$

but the restriction of (27) to $S \cap T$, i.e. $E^\vee|_{S\cap T} \to N^\vee_{(S\cap T)/T}$, while certainly a surjection, need not be an isomorphism. Since $E|_S \simeq N_{S/X}$ this further restriction of (27) leads to a surjection $N^\vee_{S/X}|_{S\cap T} \to N^\vee_{(S\cap T),T}$. Letting $N = (N_{S/X})|_{S\cap T}$ and $N' = N_{S\cap T/T}$, this can be rewritten as a surjection $N^\vee \to (N')^\vee$. Dualizing, we see that N' is a subbundle of N. [11]

Denote the codimension of $S \cap T$ in T by k. Since considerations are local we can and will assume that $s|_T$ is a section of a rank k subbundle $E' \subset E|_T$ whose restriction to $S \cap T$ is the subbundle $N' \subset N$. Note that $s|_T$ is immediately seen to be a regular section of E' since its zero locus $S \cap T$ has codimension k.

So we see that (24) is up to tensoring with bundles the dual of a Koszul complex on a section of $E|_T$ which is regular as a section of the subbundle E'. Let $\tilde{N} = N/N'$ be the bundle on $S \cap T$ introduced in (14). We claim

[11] The inclusion $N' \subset N$ can also be seen directly from geometry. Consider the natural composition $\psi : T(T)|_{S\cap T} \to T(X)|_{S\cap T} \to N$ of the natural inclusion and quotient. The kernel of ψ at $p \in S \cap T$ consists of $T_pT \cap T_pS$; but this is $T_p(S \cap T)$ since $S \cap T$ is a submanifold. So $T(T)|_{S\cap T}/\ker(\psi) \simeq N'$ and we have the claimed inclusion $N' \subset N$.

that the q^{th} cohomology of this Koszul complex is $\Lambda^k(N') \otimes \Lambda^{q-k}\tilde{N}$, so that $\underline{\text{Ext}}^q(i_* \mathcal{O}_S, j_* \mathcal{O}_T) = \wedge^k(N') \otimes \Lambda^{q-k}(\tilde{N})$. Thus

$$\underline{\text{Ext}}^q(i_* \mathcal{E}, j_* \mathcal{F}) = \Lambda^k(N') \otimes \Lambda^{q-k}(\tilde{N}) \otimes (\mathcal{E}|_{S \cap T})^\vee \otimes \mathcal{F}|_{S \cap T}. \tag{28}$$

Then (28) immediately leads to the spectral sequence claimed in Section 4.3 by considerations of vertex operators.

It remains to explain our claim. This is justified by linear algebra and local coordinates.

Rather than give a careful proof, we content ourselves with explaining the idea. We can do this most easily if we assume that $E|_T$ splits holomorphically into a direct sum $E' \oplus E''$ with $E''|_{S \cap T} \simeq \tilde{N}$.

The only cohomology of the Koszul complex $\Lambda^\bullet(E')^\vee$ is on the far right giving $\mathcal{O}_{S \cap T}$. So the only cohomology of the dual complex $\Lambda^\bullet E'$ is on the far right; by the self-duality we have mentioned earlier, we use $\Lambda^\bullet E' \simeq \Lambda^\bullet(E')^\vee \otimes \Lambda^k E'$ to compute the cohomology of the dual complex as $\Lambda^k E'|_{S \cap T} = \Lambda^k N'$.

Now using the full bundle E rather than E', we note that

$$\Lambda^q E|_T = \bigoplus_i \Lambda^i E' \otimes \Lambda^{q-i} E''. \tag{29}$$

The dualized Koszul complex then decomposes into a direct sum of the dualized Koszul complex on E' tensored with various $\Lambda^* E''$. Computing cohomology and using $E''|_{S \cap T} \simeq \tilde{N}$, we get $\Lambda^k(N') \otimes \Lambda^{q-k}(\tilde{N})$. Then we tensor with $(\mathcal{E}|_{S \cap T})^\vee \otimes \mathcal{F}|_{S \cap T}$ to arrive at (28) as claimed.

For the general case, we can show that $\Lambda^p E|_T$ has a natural filtration with graded quotients $\Lambda^i E' \otimes \Lambda^{p-i}(E|_T / E')$. Its restriction to $S \cap T$ is again $\Lambda^i(N') \otimes \Lambda^{p-i}(\tilde{N})$. This filtration can be used to modify the argument that we gave above.

As an interesting aside, note that this spectral sequence is closely related to a standard adjunction calculation in algebraic geometry. For any complex manifold Y, if Z is a complex submanifold of complex codimension r, then it is straightforward to show [18, section 5.3] that

$$\underline{\text{Ext}}^q_{\mathcal{O}_Y}(\mathcal{O}_Z, K_Y) = \left\{ \begin{array}{cc} 0 & q < r \\ K_Z & q = r \end{array} \right. \tag{30}$$

This also follows readily from our computations above. By taking determinants in the exact sequence

$$0 \to T(Z) \to T(Y)|_Z \to N_{Z/Y} \to 0$$

we see that $K_Z \simeq (K_Y)|_Z \otimes \Lambda^{\text{top}} N_{Z/Y}$. We now can compare (30) to (28) with $S = Z$, $T = X = Y$, $\mathcal{E} = \mathcal{O}_Z$ and $\mathcal{F} = K_Y$. Then $S \cap T = Z$, $N' = N_{Z/Y}$, and $\tilde{N} = 0$. Then (28) becomes precisely (30) for $q \leq r$.

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