

# An analytical computation of asymptotic Schwarzschild quasinormal frequencies

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## Abstract

Recently it has been proposed that a strange logarithmic expression for the so-called Barbero-Immirzi parameter, which is one of the ingredients that are necessary for loop quantum gravity (LQG) to predict the correct black hole entropy, is not a sign of an inconsistency of this approach to quantization of general relativity, but is a meaningful number that can be independently justified in classical GR. The alternative justification involves the knowledge of the real part of the frequencies of black hole quasinormal modes whose imaginary part blows up. In this paper we present an analytical derivation of the states with frequencies approaching a large imaginary number plus  $\ln 3/8\pi G_N M$ ; this constant has been only known numerically so far. We discuss the structure of the quasinormal modes for perturbations of various spin. Possible implications of these states for thermal physics of black holes and quantum gravity are mentioned and interpreted in a new way. A general conjecture about the asymptotic states is stated.

# 1 Introduction

While there are several open issues about the way quantum dynamics is currently handled in loop quantum gravity, the pretty kinematics—or perhaps the mathematically elegant attempts to understand de Sitter space in this framework (see [1])—can be separated and could well serve as a useful piece of our future understanding of quantum gravity without recourse to a background.

Under some assumptions, loop quantum gravity implies that the black hole entropy is proportional to the area of the horizon  $A$  [2, 3] as well as a numerical constant called the Barbero-Immirzi parameter [4, 5] whose value must be chosen appropriately so that the result agrees with the Bekenstein-Hawking entropy. The frustrating role of the Barbero-Immirzi parameter recently changed a bit because of an interesting paper by Dreyer [6]. Building on a proposal by Bekenstein and Mukhanov [7] and extending a numerical observation by Hod [8], Dreyer proposed that the gauge group of LQG should be  $SO(3)$  rather than  $SU(2)$ . Such a minor modification changes the Barbero-Immirzi parameter required to reproduce the Bekenstein-Hawking entropy from  $\gamma_{SU(2)} = \ln(2)/(\pi\sqrt{3})$  to  $\gamma_{SO(3)} = \ln(3)/(2\pi\sqrt{2})$ . More importantly, this new constant leads to a new value of the minimal positive area  $A_0 = 4G_N \ln 3$ . The frequency emitted by a black hole which is changing its area by this amount turns out to coincide with another frequency that has been only known numerically: the asymptotic real part of the frequencies of black hole's quasinormal modes.<sup>1</sup> They will be the focus of our paper.

Although an immediate reaction is that this agreement is a coincidence, we will present some evidence that it does not have to be an accident. For example, a concern is that the real part agrees with the conjectured value  $\ln(3)T_{\text{Hawking}}$  only in the first few decimal places and the following ones will show a discrepancy. We will show that this is not the case. The most important result of this paper is an analytical proof that the asymptotic frequency is indeed equal to  $\ln(3)T_{\text{Hawking}}$  for perturbations of even spin. We will also present several other arguments that the agreement might indicate something important. We will use methods of classical general relativity. Every good theory of gravity should probably reproduce these results (for

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<sup>1</sup>We were informed that this idea of Dreyer was actually first proposed by Kirill Krasnov in an unpublished email correspondence between K. Krasnov, J. Baez, and A. Ashtekar in 1999, soon after Hod's paper [8]. The physicists stopped thinking about this proposal because of two reasons. The continuous character of the Hawking radiation did not agree with their attempts to interpret the frequency of the quasinormal modes. We will mention this problem in subsection 2.2. The main reason was puzzling numerical results by Kokkotas et al. [9] that we will discuss in subsection 3.3.

large black holes, the frequencies can be still taken to be much smaller than  $m_{\text{Planck}}$ , for example). The thermodynamic nature of our answer suggests that the result might tell us something nontrivial about quantum gravity.

## 2 The quantum mechanical problem

The possible radial spin- $j$  perturbations of the four-dimensional Schwarzschild background [10] are governed by the following master differential equation [11]:

$$\left(-\frac{\partial^2}{\partial x^2} + V(x) - \omega^2\right)\phi = 0. \quad (1)$$

We will treat this equation as a Schrödinger equation with the (Regge-Wheeler) potential  $V(x)$ :

$$V(x) = V[x(r)] = \left(1 - \frac{1}{r}\right) \left(\frac{l(l+1)}{r^2} + \frac{1-j^2}{r^3}\right) \quad (2)$$

Morally speaking, the perturbation  $\phi$  contains a factor of the spherical harmonic  $Y_{lm}$  with the orbital angular momentum equal to  $l$  that appears in the “centrifugal” term of (2). The numerator  $(1-j^2)$  of the other term is usually called  $\sigma$  and is characteristic for perturbations of various spins. It might be useful to summarize its values for the most important examples of  $j$ :

$$\sigma \equiv 1 - j^2 = \begin{cases} 1 : & \text{scalar perturbation} & j = 0 \\ 0 : & \text{electromagnetic perturbation} & j = 1 \\ -3 : & \text{gravitational perturbation} & j = 2 \end{cases} \quad (3)$$

The “tortoise” coordinate  $x$  is related to  $r$  by

$$x = \ln(r-1) + r, \quad \frac{\partial}{\partial x} = \left(1 - \frac{1}{r}\right) \frac{\partial}{\partial r}. \quad (4)$$

We chose units with unit radius of the black hole’s horizon:  $2G_N M = 1$ . Therefore the difference  $(1 - 1/r)$  is simply the usual Schwarzschild warp factor. Also, in the equivalent quantum mechanical problem (1) the doubled mass equals  $2m = 1$ .

We want to study the equation (1) on the interval

$$x \in (-\infty, +\infty) \quad \text{i.e.} \quad r \in (1, \infty). \quad (5)$$

Because the potential is mostly positive (for  $j \leq 1$  it is strictly positive), there are no discrete normalizable bound states. Nevertheless it makes mathematical sense to look for discrete *quasinormal* states, analogous to quasistationary states in quantum mechanics whose frequency is allowed to be complex. These states are required to have purely outgoing boundary conditions both at the horizon ( $r = 1$ ) and in the asymptotic region ( $r = \infty$ ):

$$\phi(x) \sim c_{\pm} e^{\mp i\omega x} \quad \text{for} \quad x \rightarrow \pm\infty. \quad (6)$$

Such a constraint is as strong as the requirement of normalizability and singles out discrete solutions  $\omega$ . It is however very hard to look for the right wave functions numerically because (6) requires the exponentially decreasing component of the wave function to be absent asymptotically. Separating the exponentially small contribution from the rest is an extremely difficult task, especially for large decay rates.

The time dependence of the wave function in our quantum mechanical model is then  $e^{i\omega t}$ , and because we want to study exponentially decaying<sup>2</sup> modes, the imaginary part of  $\omega$  will always be taken positive. This also implies that while the solutions can oscillate, they must also exponentially increase with  $|x|$ . There is a reflection symmetry

$$\omega \leftrightarrow -\omega^* \quad (7)$$

which changes the sign of  $\text{Re}(\omega)$ . Although the usual conventions describe the wave  $e^{-i\omega x}$  as outgoing at  $x \sim +\infty$  for  $\text{Re}(\omega) < 0$ , we will use the reflection symmetry and assume, without loss of generality,  $\text{Re}(\omega) \geq 0$ . On the other hand, the assumption  $\text{Im}(\omega) > 0$  is physical and important because our final result will originate, in a sense, from positive integers  $n$  such that  $(n + 2i\omega) \approx 1$ .

While the quasinormal modes with small enough  $\text{Im}(\omega)$  represent the actual damped vibrations of the black hole—those that bring it quickly to the perfect spherical shape, the physical interpretation of the very unstable, high-overtone states with  $|\text{Im}(\omega)| \geq \text{Re}(\omega)$  is more problematic. Despite these conceptual problems, the existence of such formal solutions implies the presence of poles in the transmission amplitude (or equivalently, the one-particle Green's function). Such poles can be important just like in many other situations and we will investigate some possible consequences at the end of this paper.

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<sup>2</sup>The relation between the energy in the quantum mechanical model and the spacetime energy is indirect: note that the former equals  $\omega^2$  while the latter is equal to  $\omega$ . The spacetime equations of motion are second order equations. Therefore one should not worry about our nonstandard sign in  $e^{i\omega t}$  that was chosen to agree with literature.

The following two subsections review the existing literature on the quasinormal modes and basics of LQG. The reader interested in our analytical computation is advised to continue with section 3.

## 2.1 Existing calculations

Numerically it has been found by Leaver [12], Nollert [13], Andersson [14] and others that the  $n^{\text{th}}$  quasinormal mode of a gravitational  $j = 2$  (or scalar  $j = 0$ ) perturbation has frequency

$$\omega_n = \frac{i(n - 1/2)}{2} + \frac{\ln(3)}{4\pi} + O(n^{-1/2}), \quad n \rightarrow \infty. \quad (8)$$

This asymptotic behavior is independent of the orbital angular momentum  $l$  although the detailed values are irregular and  $l$ -dependent for finite  $\text{Im}(\omega)$ . The asymptotic value (8) is valid for  $j = 2$  as well as  $j = 0$ , for example; we will argue that  $\text{Re}(\omega)$  must asymptotically vanish for odd and half-integer values of  $j$ . The number of quasinormal modes is infinite. This fact was proved ten years ago [15] by methods that did not allow the authors to say anything about the frequencies. The real part in (8) equals

$$\text{Re}(\omega_n) \rightarrow \frac{\ln(3)}{4\pi} \approx 0.087424. \quad (9)$$

This constant has been determined numerically with the indicated accuracy. Such a frequency is related to a natural value of the minimal area quantum  $4\ln(3)G_N$ , one of the values  $4\ln(k)G_N$  favored by Bekenstein and Mukhanov [7] because of their ability to describe the black hole entropy from area-degenerate states.<sup>3</sup> Four years ago, Hod [8] became the first one to notice that the constant (9) can be written in terms of  $\ln(3)$  and proposed a heuristic picture trying to explain this fact. As far as we know, our paper contains the first analytical proof that this number is indeed  $\ln(3)/(4\pi)$ .

Guinn et al. [18] presented two similar and very accurate higher-order WKB calculations. However, an undetermined subtlety makes their results invalid because they obtain 0 instead of  $\ln(3)/(4\pi)$  for the real part. The very recent paper [19] can be viewed as a refined version of [18]; it leads to results that are compatible with the present paper.

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<sup>3</sup>The idea of an equally spaced black hole spectrum goes back to 1974 when it was first proposed by Bekenstein. While Bekenstein and Mukhanov prefer the value  $k = 2$  [16] which also agrees with Wheeler's idea *It from Bit*, Hod found the value  $k = 3$  more reasonable. Also the authors of [17] propose some arguments in favor of  $k = 3$ . Although we disagree with the proposal of equally spaced black hole spectrum, we are grateful to J. Bekenstein for correspondence about the early history of black hole thermodynamics.

## 2.2 Loop quantum gravity and Dreyer's observation

Ashtekar proposed new variables for canonical quantization of Einstein's theory. See for example [20] for an efficient review of Loop Quantum Gravity or [21] for a more extensive one. The spatial three-dimensional components  $g^{ab}$  of the metric at  $t = 0$  are expressed in terms of the inverse densitized dreibeins  $E_i^a$

$$(\det g_{3 \times 3})g^{ab} = \sum_{i=1}^3 E_i^a E_i^b \quad (10)$$

The index  $i = 1, 2, 3$  is an adjoint index of some  $SO(3)$  (or formerly  $SU(2)$ ) gauge theory and the components  $E_i^a$  themselves are interpreted as the dual variables to the gauge field:

$$E_i^a(x) = -8\pi i \gamma G_N \frac{\delta}{\delta A_a^i(x)} \quad (11)$$

Note that the case of three spatial dimensions is special because the adjoint and vector representations of  $SO(3)$  have the same dimension. The Hilbert space is *a priori* the space of functionals of  $A_a^i(x)$ . However it is possible to consider the subspace of *functions* of  $SO(3)$ -valued open Wilson lines defined on links of a “spin network”; this is known as *loop transform*. The union of such subspaces for all possible spin networks is dense in the original Hilbert space. The functions on the Cartesian product of many copies of the  $SO(3)$  group manifold can be expanded into spherical harmonics. The open Wilson lines, i.e. the links, then carry a spin  $J_i$  associated with the spherical harmonic, “coloring” the spin network. Colored spin networks then form an orthogonal basis of the Hilbert space.

The most interesting result of this approach to quantization of gravity is the following area quantization law [22, 23] that results directly from (10) and (11):

$$A = 8\pi G_N \gamma \sum_i \sqrt{J_i(J_i + 1)} \quad (12)$$

Here  $J_i = 0, 1, 2, \dots$  are spins defined on the links  $i$  of a spin network (not to be confused with  $j$  introduced in (2)) that intersect a given two-dimensional sheet in the coordinate space. The physical area  $A$  of the sheet depends on the dynamical metric variables. According to LQG, the physical area is “concentrated” in the intersections with the spin network. Recall that colored spin networks form a basis of the Hilbert space of an  $SO(3)$  gauge

theory.<sup>4</sup> The constant  $\gamma$  denotes the so-called Barbero-Immirzi parameter [4, 5], a pure number that can be interpreted as the finite renormalization of  $G_N$  between the Planck scale and low energies. A specific value of  $\gamma$  is required for LQG to reproduce the Bekenstein-Hawking entropy as we will recall soon.

The area (12) becomes a quasicontinuous variable if it is large because of the contributions with  $J > 1$ : the ratios between numbers  $\sqrt{J_i(J_i + 1)}$  for different  $J_i$  are irrational. Nevertheless the typical microstates of a horizon are dominated by links with the minimal possible value of  $J$ . If the links with higher values of  $J$  were absent altogether, the spectrum of the area operator (12) would become equally spaced. The step would be equal to  $4G_N \ln(2J^{(\min)} + 1)$  for the entropy to come out correctly [7]. If the relation between the area and the mass is preserved (which is however hardly the case microscopically), the links with  $J > 1$  are *essential* to keep the energy spectrum as well as the spectrum of Hawking radiation continuous. Furthermore the emission of a typical Hawking quantum should rearrange a large number of links—including links with  $J > J^{(\min)}$ —if LQG is to be compatible with Hawking’s semiclassical results, otherwise the spectrum would be peaked around discrete frequencies. This is one of the problems that was mentioned in footnote in the introduction. We believe that the assumption that the energy spectrum is discrete with spacing  $\ln(3)/(8\pi G_N M)$  was incompatible with the spectrum of Hawking radiation whose energies are of the same order but continuous. The correct interpretation is probably more subtle. If we reject Hawking’s prediction of highly thermal radiation, then we also have no more reasons to believe the calculation of black hole entropy that is an implication of the temperature calculation; at least the numerical coefficient would be unjustified. We find it hardly acceptable to reject Hawking’s semiclassical calculations, in part because they have been confirmed by many developments in String Theory.

The need for a macroscopic rearrangement of many links (including those with  $J > J^{(\min)}$ ) in LQG to explain a single Hawking particle should not be too surprising because LQG needs a similar miraculous interplay between a large number of links to generate the flat space itself. One should also note that the problem is equally serious regardless of the value of  $J^{(\min)}$  and

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<sup>4</sup>Originally, Ashtekar and Baez worked with an  $SU(2)$  theory instead of  $SO(3)$ . They were motivated by the possibility of incorporating fermions into a future version of loop quantum gravity. We believe that there is no reason why a theory that is meant to rewrite gravitational physics should “know” about our desire to incorporate fermions. Therefore,  $SO(3)$  is more natural than  $SU(2)$ . The  $SU(2)$  gauge theory could lead to similar physics but  $J_i$  could also be half-integers in this case. The resulting  $SU(2)$  Barbero-Immirzi parameter  $\ln(2)/(\pi\sqrt{3})$  has not been supported by any numerical coincidences.

therefore we should not discard Dreyer's proposal to switch from  $SU(2)$  to  $SO(3)$  on these grounds.

While loop quantum gravity does not yet offer a well-defined framework to explain the key question why black hole entropy has no volume extensive contributions coming from the interior, there exist arguments that LQG implies the correct entropy of the horizons of all possible black holes i.e. the entropy proportional to their area (up to the universal constant mentioned a moment ago). See [2] and [3] as well as references therein for more information. Instead of the sophisticated Chern-Simons computations initiated by Krasnov, let us review a simpler version of the argument. The observables  $J_i^{(z)}$  living on the links intersecting a given area become undetermined and physical if the area happens to be a horizon. The mixed state of the black hole is dominated by microstates with the minimal allowed value of  $J^{(\min)} = 1$  occupying most links (for the  $SU(2)$  gauge group, we would have  $J^{(\min)} = 1/2$ ). Because of (12), the number of such links is

$$N_{\text{links}} = \frac{A}{A_0} = \frac{A}{8\pi G_N \gamma \sqrt{J^{(\min)}(J^{(\min)} + 1)}}. \quad (13)$$

Each link carries  $(2J^{(\min)} + 1)$  possible  $z$ -components of the spin and therefore the number of microstates equals the corresponding power of  $(2J^{(\min)} + 1)$ , namely

$$N_{\text{microstates}} = \exp \left[ \ln(2J^{(\min)} + 1) \frac{A}{8\pi G_N \gamma \sqrt{J^{(\min)}(J^{(\min)} + 1)}} \right]. \quad (14)$$

This is equal to the exponential of the black hole entropy  $A/4G_N$  for

$$\gamma = \frac{\ln(2J^{(\min)} + 1)}{2\pi \sqrt{J^{(\min)}(J^{(\min)} + 1)}} \bigg|_{J^{(\min)}=1} = \frac{\ln(3)}{2\pi\sqrt{2}}. \quad (15)$$

Such a value of  $\gamma$  may seem very unnatural. However Dreyer [6] has recently suggested an independent argument for this particular value of  $\gamma$ . Because we disagree with several details of his argument, we will present a modified version of it.

Assume that a new link with  $J^{(\min)} = 1$  is *absorbed* by a black hole horizon (or it is *created* there). Its area therefore increases by  $A_0$ , which is according to (12) and (15) equal to

$$\Delta A = A_0 = 4\ln(3)G_N. \quad (16)$$



Because  $A = 4\pi R^2 = 16\pi(G_N M)^2$ , this variation can be written as

$$16\pi\Delta(M^2) = \frac{4\ln(3)}{G_N} \Rightarrow \Delta M \approx \frac{\ln(3)}{8\pi G_N M}. \quad (17)$$

It follows there should be a preferred frequency (energy)

$$\omega_{\text{link}} = \ln(3)/(8\pi G_N M) \quad (18)$$

associated with the emission of the most generic link. While black holes emit Hawking radiation with continuous frequencies, the special frequency  $\omega_{\text{link}}$  turns out to be exactly equal to the asymptotic real part of the quasinormal frequencies in (8) (divided by  $2G_N M = 1$ ). We believe that the special frequency should *not* be interpreted as the energy of a physical Hawking quantum which is continuous.

Because Dreyer was forced to switch from  $SU(2)$  to  $SO(3)$  and because the justification of  $\ln(3)$  on one side was only numerical, it is clear that his argument is nontrivial. Although we do not think that the imaginary part can be ignored in Dreyer's mechanistic arguments involving the Bohr's frequency and one can criticize other issues, such a coincidence nevertheless deserves an explanation. The first step is to extend the numerical evidence for the  $\ln(3)$  form of this asymptotic frequency into an exact analytical derivation. We will realize this step in the next section.

### 3 The asymptotic expansion for large frequencies

The solution of (1) can be expanded in the following way (see for example [13]):

$$\phi(r) = \left(\frac{r-1}{r^2}\right)^{i\omega} e^{-i\omega(r-1)} \sum_{n=0}^{\infty} a_n \left(\frac{r-1}{r}\right)^n. \quad (19)$$

The prefactor is meant to satisfy the boundary conditions (6):

- $e^{-i\omega(r-1)}$  is necessary for the correct leading behavior at  $x \rightarrow \infty$  i.e.  $r \rightarrow \infty$ ;
- $(r-1)^{i\omega}$  fixes the behavior at  $x \rightarrow -\infty$  i.e.  $r \rightarrow 1^+$ ;
- $r^{-2i\omega}$  repairs the subleading behavior at  $r \rightarrow \infty$  coming from the logarithmic term in (4).

The power series (19) converges for  $1/2 < r < \infty$  and also for  $r = \infty$  if the boundary conditions at  $r = \infty$  are preserved. The equation (1) is equivalent to the recursion relation

$$c_0(n, \omega)a_n + c_1(n, \omega)a_{n-1} + c_2(n, \omega)a_{n-2} = 0 \quad (20)$$

where the coefficients  $c_k(n, \omega)$  can be extracted from the equation and rewritten in the following convenient way:

$$c_0(n, \omega) = +n(n + 2i\omega) \quad (21)$$

$$c_1(n, \omega) = -2(n + 2i\omega - 1/2)^2 - l(l + 1) + j^2 - 1/2 \quad (22)$$

$$c_2(n, \omega) = +(n + 2i\omega - 1)^2 - j^2 \quad (23)$$

Note that except for the factor  $n$  in  $c_0(n)$ , the coefficients  $c_k(n, \omega)$  depend on  $n, \omega$  only through their combination  $(n + 2i\omega)$ . The initial conditions for the recursion relation guarantee that the boundary conditions at  $r = 1$  will be preserved. They are  $a_0 = 1$  (or any nonzero constant) and  $a_{-1} = 0$  (which also implies  $a_{-n} = 0$  for all positive integers  $n$  as a result of (20)). We also define

$$R_n = -\frac{a_n}{a_{n-1}}. \quad (24)$$

The minus sign was chosen to agree with [13]. The relation (20) then implies

$$c_1(n, \omega) - c_0(n, \omega)R_n = \frac{c_2(n, \omega)}{R_{n-1}} \quad (25)$$

or, equivalently,

$$R_{n-1} = \frac{c_2(n, \omega)}{c_1(n, \omega) - c_0(n, \omega)R_n}. \quad (26)$$

$R_n$  can be consequently written as an infinite continued fraction. The boundary condition at  $r = 1$ , i.e.  $a_{-1} = 0$ , take the form

$$R_0 = \infty \quad \text{i.e.} \quad c_1(1, \omega) - c_0(1, \omega)R_1 = 0. \quad (27)$$

The convergence of (19) at  $r = \infty$  implies a particular asymptotic form of  $R_n$  for large  $n$  (and  $|R_n| < 1$  for  $n \rightarrow \infty$ ) and the boundary conditions require (27). The equation (27), written in terms of continued fractions, is then a condition that  $\omega$  must satisfy for a quasinormal mode to exist:

$$0 = c_1(1, \omega) - c_0(1, \omega) \frac{c_2(2, \omega)}{c_1(2, \omega) - c_0(2, \omega) \frac{c_2(3, \omega)}{c_1(3, \omega) - c_0(3, \omega) R_3^{\text{full in } \dots}}} \quad (28)$$

### 3.1 Going to the limit

Let us now consider  $\omega$  with a huge positive imaginary part. Because  $R_n \rightarrow -1$  as  $n \rightarrow \infty$  [13] (in other words,  $a_n/a_{n-1} \rightarrow 1$ ) and because for large  $|\omega|$ , the changes of  $R_n$  slow down, it is an extremely good approximation to assume that  $R_n$  changes adiabatically as long as  $n$ ,  $|\omega|$  and  $|n + 2i\omega| \gg 1$ :

$$\frac{R_n}{R_{n-1}} = 1 + O(|\omega|^{-1/2}). \quad (29)$$

Such an approximation is good for  $\text{Re}(n + 2i\omega) > 0$  (where we calculate  $R_n$  recursively from  $R_\infty = -1$ ) as well as  $\text{Re}(n + 2i\omega) < 0$  (where we start the computation with  $R_0 = \infty$ ). Note that a similar fact holds for the coefficients of the Taylor expansion of the exponential. By inserting  $R_{n-1} = R_n$  into (25) we obtain a quadratic equation<sup>5</sup> with the following solutions (we keep only the leading terms for  $n \sim |2\omega| \rightarrow \infty$ ):

$$R_n^\pm = \frac{-(n + 2i\omega) \pm \sqrt{2i\omega(n + 2i\omega)}}{n} + O(|\omega|^{-1/2}). \quad (30)$$

This approximation for the continued fractions defined in (26) has been successfully checked with *Mathematica*. Irregularities in  $R_n$  would imply violations of the boundary conditions although we do not claim to have a totally rigorous proof.  $R_n$  should satisfy (30) for one of the signs; it is a *necessary* condition and a more detailed discussion whether the condition is also *sufficient* is needed. For  $\text{Re}(n + 2i\omega) < 0$  the sign is determined by the condition  $R_1$  is small; two terms in (30) must approximately cancel. The sign for  $\text{Re}(n + 2i\omega) > 0$  follows from the requirement  $|R_n| < 1$  for  $n \rightarrow \infty$  which is necessary for the convergence of (19) for  $r = \infty$ .

For very large  $|\omega|$ , we can choose integer  $M$  such that

$$1 \ll M \ll |\omega|. \quad (31)$$

For such intermediate values of  $M$  the equation (30) can still be trusted when we want<sup>6</sup> to determine  $R_{[-2i\omega] \pm M}$ ; and only the second term of (30) is relevant in the limit (31). This term makes the numbers  $R_{[-2i\omega] + x}$  scale like  $\pm i\sqrt{x}/\sqrt{-2i\omega}$  for moderately small  $x$  such that  $1 \ll x \ll |\omega|$  while the first

<sup>5</sup>Matthew Schwartz independently realized that this quadratic equation gives a good estimate of the remainders. In fact, he even showed that the solution (30) of this quadratic equation coincides with the fully resummed expression by Nollert [13] in the scaling limit  $n \sim |2i\omega| \rightarrow \infty$ .

<sup>6</sup>The symbol for the integer part  $[-2i\omega]$  should be actually understood as an arbitrary integer that differs from  $-2i\omega$  by a number much smaller than  $M$ ; we will assume that it is even.

term in (30) scales like  $x/\omega$  and is subleading. Note that neglecting this first term is equivalent to neglecting  $c_1(n, \omega)$  in the original quadratic equation (25); this term will turn out to be irrelevant for *all* values of  $n$  in the large  $|\omega|$  limit, with the possible exception of some purely imaginary frequencies where  $c_0(n)$  or  $c_2(n)$  vanish. The ratio of  $R_{[-2i\omega]\pm M}$  can be calculated from (30) as the ratio of  $\sqrt{x}$  for  $x = \pm M$ :

$$\frac{R_{[-2i\omega]+M}}{R_{[-2i\omega]-M}} = \pm i + O(|\omega|^{-1/2}) \quad (32)$$

Because the assumptions that led to (30) *break down* for  $|n + 2i\omega| \sim 1$  where the coefficients  $c_0(n), c_1(n), c_2(n)$  strongly depend on  $n$  and furthermore contain terms of order 1 (or  $j^2$ ) that now cannot be neglected,<sup>7</sup> the quantities  $R_{[-2i\omega]+M}$  and  $R_{[-2i\omega]-M}$  must be related through the original continued fraction (instead of the adiabatic approximation which breaks down in this region). In the next paragraph we will be able to calculate the continued fraction exactly in the  $|\omega| \rightarrow \infty$  limit. The continued fraction must predict the same ratio (32) as the adiabatic argument because these two solutions must be eventually “connected.” This agreement will impose a non-trivial constraint on  $\omega$ .

The scaling of  $R_{[-2i\omega]+x}$  as  $1/\sqrt{|\omega|}$  (see (30)) also implies that  $c_1(n)$  in the denominator of (26) (which is of order one) is negligible if compared to the other term (which scales like  $1/R_{[-2i\omega]+x}$  or  $\sqrt{|\omega|}$ ). Keeping  $M$  fixed and scaling  $|\omega| \rightarrow \infty$ , it is clear that the effect of  $c_1(n)$  in (26) goes to zero. The only possible exception is the case when  $c_0(n)$  and/or  $c_2(n)$  become almost zero for some  $n$ ; this exception cannot occur for nonzero  $\text{Re}(\omega)$  but we will study it in the next subsection.

Because  $c_1(n)$  does not affect the asymptotic frequencies, the orbital angular momentum  $l$  is irrelevant, too. Therefore the asymptotic frequencies will depend on  $j$  only (which is contained in the definition (23) of  $c_2(n)$ ). The factor  $n$  in (21) can be replaced by the constant  $[-2i\omega]$  as the frequency goes to infinity if we want to study a relatively small neighborhood of  $n \sim [-2i\omega]$  only. Moreover, the continued fraction simplifies into an ordinary fraction.

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<sup>7</sup>Except for the discrete character of  $n$  in the present case, the breakdown of the generic result (30) is analogous to the way how the simple *Ansatz* for the wave function in the WKB approximation *breaks down* near the classical turning points.

Inserting (26) recursively into itself  $2M$  times tells us that

$$R_{[-2i\omega]-M} = \prod_{k=1}^M \left( \frac{c_2([-2i\omega] - M + 2k - 1)c_0([-2i\omega] - M + 2k)}{c_0([-2i\omega] - M + 2k - 1)c_2([-2i\omega] - M + 2k)} \right) R_{[-2i\omega]+M} \quad (33)$$

The dependence of all coefficients  $c_k(n)$  on  $\omega$  was suppressed. Now we can combine (32) and (33) to eliminate the remainders  $R_n$ ; we will formulate the requirement that the generic adiabatic solution (30) that is valid almost everywhere can be “patched” with the exact solution for  $n \sim [-2i\omega]$  (33) that has also been simplified by the large  $|\omega|$  limit.

Fortunately, the products of  $c_0(n)$  and  $c_2(n)$  can be expressed in the language of the  $\Gamma$  function (partially because  $x^2 - j^2 = (x+j)(x-j)$ ; this is a deeper reason why we write  $\sigma$  as  $1 - j^2$ ). Since  $c_2(n)$  is bilinear, the four factors inside the product (33) actually lead to a product of  $6 (= 2+1+1+2)$  factors, each of which is equal to a ratio of two  $\Gamma$  functions. Summary: we obtain a ratio of twelve  $\Gamma$  functions.

The resulting condition conveniently written in terms of a shifted frequency  $f \sim O(1)$  where  $-2if = -2i\omega - [-2i\omega]$  becomes

$$\pm i = \frac{\Gamma(\frac{+M+2if+j}{2})\Gamma(\frac{+M+2if-j}{2})\Gamma(\frac{-M+2if+1}{2})}{\Gamma(\frac{-M+2if+j}{2})\Gamma(\frac{-M+2if-j}{2})\Gamma(\frac{+M+2if+1}{2})} \times \quad (34)$$

$$\times \frac{\Gamma(\frac{-M+2if+j+1}{2})\Gamma(\frac{-M+2if-j+1}{2})\Gamma(\frac{+M+2if+2}{2})}{\Gamma(\frac{+M+2if+j+1}{2})\Gamma(\frac{+M+2if-j+1}{2})\Gamma(\frac{-M+2if+2}{2})} \quad (35)$$

The factors  $n$  from (23) cancelled. Among these twelve  $\Gamma$  functions, six have argument with a huge negative real part and can be converted into  $\Gamma$  of positive numbers due to the well-known formula

$$\Gamma(x) = \frac{\pi}{\sin(\pi x)\Gamma(1-x)}. \quad (36)$$

The factors of  $\pi$  cancel, just like the Stirling approximations for the  $\Gamma$  functions with a huge positive argument:

$$\Gamma(n+1) \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n. \quad (37)$$

Only the  $\sin(x)$  factors survive. The necessary condition for the *regular* asymptotic frequencies (i.e. frequencies where our analysis is valid; we will explain this point later) becomes

$$\pm i = \frac{\sin \pi(if+1)}{\sin \pi(if+\frac{1}{2})} \frac{\sin \pi(if+\frac{j}{2})}{\sin \pi(if+\frac{1+j}{2})} \frac{\sin \pi(if-\frac{j}{2})}{\sin \pi(if+\frac{1-j}{2})} \quad (38)$$

We chose  $M \in 4\mathbb{Z}$  so that we could erase  $M$  from the arguments of the trigonometric functions. We are also allowed to replace  $f$  by  $\omega$  again because the functions in (38) are periodic with the right periodicity and we could have chosen the number  $[-2i\omega]$  even. The terms  $(\pi/2)$  in the denominator can be used to convert the sin functions into cos. The three arguments happen to be the same like in the numerator and we can also rewrite (38) as

$$\pm i = \tan(\pi i \omega) \tan(\pi i \omega + \pi j/2) \tan(\pi i \omega - \pi j/2) \quad (39)$$

For generic complex values of  $j$ , (39) has formally six generic nontrivial solutions in every strip  $L \leq \text{Im}(\omega) < L + 1$  of the complex plane: three roots for each sign on the left hand side of (39). However two of them are infinite,  $|\omega| \rightarrow \infty$ ; the remaining four can be rewritten as  $\exp(\pm 4\pi\omega) = -1 - 2\cos(\pi j)$ .<sup>8</sup> This form of our result can be easily derived if we multiply (38) by the denominator of the right hand side and expand the sin functions in terms of the exponentials. One can be more careful about the signs to see that the correct result must satisfy

$$\exp[\epsilon(\text{Re } \omega) \cdot 4\pi\omega] = -1 - 2\cos(\pi j) \quad (40)$$

where  $\epsilon(y)$  is the sign function. Note that the modulus of the left hand side is never smaller than one, and therefore the asymptotic quasinormal modes do not exist for  $\cos(\pi j) < (-1/2)$ .

Let us now study (40) for various physical values of  $j$ . If  $j$  is a half-integer, the right-hand side of (40) equals  $(-1)$ . The allowed frequencies are therefore<sup>9</sup>

$$j \in \mathbb{Z} + \frac{1}{2} : \quad \omega = \frac{i(n - 1/2)}{2} + O(n^{-1/2}), \quad n \in \mathbb{Z} \quad (41)$$

If  $j$  is an odd integer, the right-hand side of (40) equals  $(+1)$  and we obtain

$$j \in 2\mathbb{Z} + 1 : \quad \omega = \frac{in}{2} + O(n^{-1/2}), \quad n \in \mathbb{Z} \quad (42)$$

Because the frequencies in (41) and (42) lead to an indeterminate form in (38)—in other words,  $c_0(n)$  and/or  $c_2(n)$  vanishes for certain values of  $n$ —our analysis, strictly speaking, breaks down for these frequencies and more effort is needed to decide whether these solutions are physical. We will discuss these issues in the next subsection. At any rate, all these possible frequencies have a vanishing real part.

<sup>8</sup>This form of the result was obtained in collaboration with Andrew Neitzke. [19]

<sup>9</sup>I am indebted to C. Herzog for pointing out an oversimplification in the first version of this paper.

On the other hand, the result for even  $j$  (scalar or gravitational perturbations) is more interesting. There is no cancellation and (40) simplifies to

$$j \in 2\mathbb{Z} : \quad \omega = \frac{i(n - 1/2)}{2} \pm \frac{\ln(3)}{4\pi} + O(n^{-1/2}), \quad n \in \mathbb{Z} \quad (43)$$

This is the solution that has been used to support loop quantum gravity.

**A partial summary:** The only “regular” quasinormal frequencies that follow from our equations—without encountering indeterminate forms—are those with the real part equal to  $\pm \ln(3)$  (43) for  $j \in 2\mathbb{Z}$ . It is also conceivable that there are solutions with  $\omega = in/2$  for odd  $j$  and  $\omega = i(n - 1/2)/2$  for half-integer  $j$  (with vanishing real part) where our analysis breaks down. In the next subsection we will show that these states indeed *do* exist. At any rate,  $\pm \ln(3)$  is the only allowed nonzero asymptotic real part of the frequency.

We cannot resist the temptation to say that the equation (40) simplifies not only for integer and half-integer values of  $j$ , but also for  $j \in 2\mathbb{Z} \pm 2/3$ . In this (most likely unphysical) case we again obtain no solutions (even if one allows the sign function in (40) to be replaced by an arbitrary sign).

### 3.2 Subtleties for purely imaginary frequencies

The purpose of this subsection is to resolve the question marks regarding the special frequencies where our approximation, neglecting  $c_1(n)$ , breaks down.<sup>10</sup>

First, we want to argue that the “regular” solutions are guaranteed to exist. Once we are able to relate the remainders  $R_{[-2i\omega \pm M]}$  through the continued fraction where we are allowed to neglect  $c_1(n)$  as we said, we can also extrapolate them to (30). The boundary conditions  $R_0 = \infty$ ,  $R_\infty = -1^+$  imply that a specific sign of the square root in (30) must be separately chosen for  $n < [-2i\omega]$  and for  $n > [-2i\omega]$ . But the signs turn out to agree with the signs of  $\pm i$  that lead to our solutions automatically.

The *necessary* condition for  $\omega$  is also *sufficient*. The only solutions of this kind are the  $\ln(3)$  solutions from (43). Their existence is guaranteed.

In order to find all irregular solutions, we start with two useful observations.

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<sup>10</sup>This subsection did not appear in the original version of the article.

- The continued fraction (28) depends on the coefficients  $c_0(n)$  and  $c_2(n')$ , but only through their products  $c_0(n)c_2(n+1)$ .
- If zeroes appear in (33) exclusively in the numerator or the denominator, the ratio  $R_{[-2i\omega]-M}/R_{[-2i\omega]+M}$  will be effectively either zero, or infinite. After we take  $c_1(n)$  into the account, “zero” or “infinity” is replaced by a negative or positive power of  $\sqrt{|\omega|}$ , respectively.

The second point implies that the only possibility to obtain irregular solutions is to find  $\omega$  such that (33) is an indeterminate form of type 0/0. Because  $c_0(n)$  can vanish at most for one value of  $n$ , there must be at least one value of  $n$  where  $c_2(n)$  vanishes. Equation (23) then implies that  $2i\omega + j$  or  $2i\omega - j$  (or both) must be integer.

For an integer or half-integer spin,  $j$ , these two conditions are equivalent and therefore both numbers  $2i\omega \pm j$  must be integers for the quasinormal mode to have a chance to exist. In the half-integer case this condition precisely agrees with the solutions proposed in (41):

$$j \in \mathbb{Z} + \frac{1}{2} : \quad \omega = \frac{i(n - 1/2)}{2}, \quad n \in \mathbb{Z} \quad (44)$$

Because the numbers  $2i\omega \pm j$  differ by an odd integer for a half-integer  $j$ , one of the corresponding  $c_2(n)$  factors appears in the numerator and the other appears in the denominator of (33) and allows for an indeterminate form. The coefficients  $c_0(n)$  are then guaranteed to be nonzero for all  $n$ . The deviation of  $\omega$  from (44) affects the ratio in a generic fashion. For a suitable choice of this deviation, the indeterminate form is regularized to  $\pm i$  once we include the effect of two relevant coefficients  $c_1(n)$ . We have checked the existence of solutions (44) with *Mathematica*. In the case of  $j = 1/2$ , for each allowed value of  $\text{Im}(\omega)$  there are two solutions with nonzero  $\pm \text{Re}(\omega)$  that converge to zero.

Now consider the case of integer  $j$ . As stated above (44), both numbers  $2i\omega \pm j$  must be integers. But if  $j$  is integer, then these two numbers differ by an *even* number. It follows that both vanishing factors of  $c_2(n)$  appear in the numerator of (33), or both appear in the denominator of this fraction. Without loss of generality, let us assume that they appear in the denominator. Then the indeterminate form is obtained only if the vanishing  $c_0(n)$  appears in the numerator. A closer look at (21), (23), and (33) implies that  $2i\omega \pm j$  and  $2i\omega$  must be different modulo two, and therefore  $j$  must be odd. We can again find such a deviation from (42) that the effect of  $c_1(n)$  gives us any desired result. For  $j \in 2\mathbb{Z} + 1$  we therefore confirmed the



existence of the states (42) while for  $j \in 2\mathbb{Z}$  the *regular* states from (43) are the *only* solutions. Our numerical calculations done using *Mathematica* furthermore indicate that for each allowed value of  $\text{Im}(\omega)$  there are two states with nonzero values of  $\pm \text{Re}(\omega)$  that approach zero as  $\text{Im}(\omega) \rightarrow \infty$ .

To amuse the reader, we would like to mention that for the special unphysical choice of the spin  $j \in 2\mathbb{Z} \pm 2/3$  discussed previously, at most one of the numbers  $2i\omega$ ,  $2i\omega + j$ ,  $2i\omega - j$  can be integer and therefore we obtain only one zero in (33) either in the numerator *or* in the denominator. This is not enough to end up with the indeterminate form of type  $0/0$ . Therefore for  $j \in 2\mathbb{Z} \pm 2/3$  there are *no* asymptotic quasinormal modes: neither *regular* nor *irregular*. This contrasts with the behavior for general  $j$  where we find four finite *regular* asymptotic quasinormal modes in every strip  $L \leq \text{Im}(\omega) < L + 1$ .

We showed that all irregular solutions that we have found are precisely the solutions where (39) led to a cancellation of zero against infinity. All solutions can be summarized by the equation (40) that we mentioned previously.

### 3.3 Generalizations and alternative computations

It would be interesting to generalize our general procedures to related cases and apply the methods of answer analysis to other calculational frameworks. We have the following points in mind:

- **Reissner-Nordström charged black holes.** It is possible to write down the wave function as an expansion in  $(r - r_+)/ (r - r_-)$  similar to (19) [12, 24]. The analysis of the critical behavior could be analogous to ours. One might be forced to add one or two terms to (20) because the Reissner-Nordström warp factor contains an extra term  $(Q^2/r^2)$ . It has been recently shown by functional [19] and numerical [24] methods that the universal Hod-like result [8]

$$\text{Re}(\omega) \rightarrow \ln(3)T_{\text{Hawking}} \quad \text{as} \quad \text{Im}(\omega) \rightarrow \infty. \quad (45)$$

is replaced by a more complicated equation:

$$e^{\beta_{\text{Hawking}}\omega} + 2 + 3e^{k^2\beta_{\text{Hawking}}\omega} = 0, \quad k \equiv \frac{r_-}{r_+} \quad (46)$$

- **Kerr and Kerr-Newman rotating (and charged) black holes.** In the case of Kerr black holes (with  $Q = 0$ ), it is still possible to separate the differential equations into radial and angular equations; see

for example [12, 25]. Investigation of the quasinormal modes by the methods of this paper as well as the methods of [19] is in progress [26]. But some numerical calculations have already been done [24]. They indicate that the answer  $\ln(3)T_{\text{Hawking}}$  is not universal; in fact, their calculation seem to imply that the quasinormal modes do not exist for  $a > a_{\text{crit}}$  where the critical angular momentum is  $a_{\text{crit}} \sim c/|\omega|$ . This result would mean that the connection between the area quantum and quasinormal modes of the Schwarzschild black holes [6] was a coincidence that does not generalize to other black holes. Therefore it is desirable to confirm or reject the results of [24] by other means [26].

- **Black holes in AdS spaces.** A more detailed analysis of black hole quasinormal modes in Anti de Sitter space might be interesting especially because the Maldacena's AdS/CFT correspondence might shed a new light on their origin.<sup>11</sup> Papers [27, 28] show that the facts about the black hole quasinormal modes in the AdS space can be reinterpreted in terms of thermal physics of the dual Conformal Field Theory. In [29], the continued fractions play an important role. However, the asymptotic structure of the quasinormal modes of the AdS black holes differs from the flat space in many important respects. These differences follow from a different asymptotic geometry of the spacetime. For example, even the asymptotic frequencies depend on  $l$ . Not only the imaginary part, but also the real part of the frequencies diverge. There is no  $\ln(3)$  limit. The recurrence relation describing the quasinormal modes of AdS black holes has also been studied [30, 31, 32].
- **De Sitter space and Rindler space.** The usual perturbations of de Sitter space form an integrable system that does not exhibit the  $\ln(3)$  effect, at least in the simplest models that we checked (which however corresponded to  $j = 1$  where our prediction for  $\text{Re}(\omega)$  vanishes anyway). The perturbations of Rindler space can be rewritten as a quantum mechanical model with the potential  $\exp(x)$ . This is the asymptotic behavior of (2) for  $x \rightarrow -\infty$  and a general property associated with the horizons. The solutions are simply Bessel functions; see [33] for another application of the same Bessel equation. We do not know a clear argument determining what the boundary condition at  $x \rightarrow +\infty$  should be. Therefore we cannot derive a similar  $\ln(3)$  effect in the Rindler space; it is even impossible to associate a unique temperature scale with the whole Rindler space and therefore it might be unreasonable to expect any definite answer. Another contradiction is that the result should be  $j$ -dependent as we have seen in

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<sup>11</sup>I am grateful to V. Cardoso for reminding me of the Anti de Sitter space.

the Schwarzschild case, but the Regge-Wheeler-like potential for the Rindler space seems to be  $j$ -independent.

- **Higher-dimensional black holes.** The calculation of the higher-dimensional Schwarzschild black hole's quasinormal modes could lead to much more important and interesting results. This is not just our belief; see the recent speculative attempt [34] to predict the higher-dimensional behavior. Even if the general structure of the modes remains valid, the factor of  $\ln(3)$  in (45) could be replaced by a different constant—for example  $\ln(d-1)$ . The recent paper [35] showed that it was possible to calculate quasinormal modes in higher dimension; their asymptotic behavior was finally calculated in a very recent paper [19] that is based on functional methods. The quasinormal modes of scalar perturbations of Schwarzschild black holes seem to approach  $\ln(3)T_{\text{Hawking}}$  in any dimension. This result seems puzzling from the viewpoint of loop quantum gravity [36]. It is desirable to reproduce or reject this result by numerical calculations and/or by the methods of this paper. However, the five-dimensional Schwarzschild black hole would probably lead to a recursion relation of the fifth order that is much more difficult to handle.

Let us mention a very unlikely (but not quite impossible) speculative scenario. Imagine that for 11-dimensional black holes or black branes of some kind,  $\ln(3)$  in (45) is replaced by  $\ln(248)$  in the case of *gravitational* perturbations. Such a coincidence would probably encourage many to search for a spin network description of (the bosonic part of) the 11-dimensional supergravity based on the  $E_8$  gauge theory. Such a description might be related to the usefulness of  $E_8$  gauge theory in the bulk for the path integral quantization of the three-form potential, studied by Diaconescu, Moore, and Witten [37]. The  $E_8$  gauge field has a sufficient number of components to include the three-potential (expressed as the Chern-Simons three-form) as well as the metric that could be perhaps written in the (almost) usual LQG fashion:

$$(\det g_{10 \times 10})g^{ab} = - \sum_{i=1}^{248} \frac{\delta}{\delta A_a^i} \frac{\delta}{\delta A_b^i} \quad a, b = 1, 2, \dots, 10. \quad (47)$$

The field  $A_a^i$  has many components,  $10 \times 248$ . While it might sound like a very redundant choice, we think that the meaningful proposals of LQG can be generalized to any spacetime dimension, as long as we allow the gauge theory configuration space to be bigger than the configuration space of pure gravity. For example, the quantization of two-dimensional areas in four spacetime dimension *must* generalize to the

quantization of  $(d-2)$ -dimensional areas in  $d$  spacetime dimensions—which is directly implied e.g. by (47). The reason is simply that the  $(d-2)$ -dimensional areas determine the entropy.

- **The WKB approximation.** The article [18] tried to solve the problem of the highly damped quasinormal modes in the WKB approximation. Although the authors could calculate the subleading corrections at very high orders, it was later realized that a subtlety invalidates this WKB approximation even at zeroth order because their asymptotic  $\text{Re}(\omega)_{\text{Guinn}} = 0$  for the highly damped modes is incorrect—certainly for general real values of  $j$ . Subsequently Andersson and Linnæus [38, 14] improved the method of [18] and looked for the highly damped quasinormal modes again but the asymptotic behavior was not understood analytically. We should note that the WKB prescription of [18] seems to break down in a controllable way. Furthermore, the resulting frequencies should be a periodic function of  $j$  that appears, through the combination  $(1-j^2)$ , in the coefficient of the  $(1/r^4)$  term of the potential in [18]. The correct result was finally reproduced by functional methods in a very recent paper [19].

## 4 Speculations on implications for gravity

We start this section by rewriting the equation (8) for the quasinormal modes in usual units where  $2M \neq 1$  and  $G_N \neq 1$ :

$$G_N \omega_n = \frac{i(n-1/2)}{4M} + \frac{\ln(3)}{8\pi M} + O(n^{-1/2}), \quad n \rightarrow \infty. \quad (48)$$

Note that the ratio between the real part and the spacing of the imaginary part equals  $\ln(3)/2\pi$ , regardless of the choice of units. It is useful to recall that the Hawking temperature of the Schwarzschild black hole of mass  $M$  equals  $T_{\text{Hawking}} = 1/(8\pi G_N M)$ . Consequently, the equation (48) can be also written as

$$\frac{\omega_n}{T_{\text{Hawking}}} = 2\pi i(n-1/2) + \ln 3 + O(n^{-1/2}) \quad (49)$$

We can remove the term proportional to  $n$  simply by exponentiating ( $\pi i$  in (49) gives the minus sign below):

$$\exp\left(\frac{\omega_n}{T_{\text{Hawking}}}\right) = -3 + O(n^{-1/2}) \quad (50)$$

Because the quasinormal modes show the position of poles of the transmission amplitude in the corresponding quantum mechanical model—and these

poles will probably also have some interpretation in the spacetime terms—the asymptotic structure of this transmission amplitude (for large  $\text{Im}(\omega)$ , and perhaps for large  $|\omega|$  in general) will be proportional to the factor

$$T(\omega) \sim \frac{1}{e^{\beta_{\text{Hawking}}\omega} + 3}, \quad \beta_{\text{Hawking}} \equiv \frac{1}{T_{\text{Hawking}}}. \quad (51)$$

The roots obtained by the reflection symmetry have a negative imaginary part  $-\ln(3)$  and lead to a similar factor  $1/(\exp(\beta_{\text{Hawking}}\omega) + 1/3)$ . These denominators are reminiscent of thermal physics. Actually, had we started with our results for the odd spin (42) and the half-integer spin (44), the same procedure would have given us a more familiar factor in the transmission amplitude proportional to

$$T(\omega) \sim \frac{1}{e^{\beta_{\text{Hawking}}\omega} \mp 1}. \quad (52)$$

This is exactly the average occupation number in Bose-Einstein statistics (for  $j$  odd the denominator has a minus sign) or Fermi-Dirac statistics (for  $j$  half-integer the denominator has a plus sign). In both cases the statistics agrees with the spin  $j$  of the perturbation. We can imagine that the transmission amplitude results from an interaction of the given particle with a thermal bath of temperature  $T_{\text{Hawking}}$ , containing particles of the same statistics. In other words, the amplitude is related to the thermal Green's function in the Schwarzschild background; this explains the general form of (51) and (52).

This agreement makes the result (51) for the even values of  $j$  even more puzzling. Why do we fail to obtain the same Bose-Einstein factor as we did for odd  $j$ ? Instead, we calculated a result more similar to the half-integer case, i.e. Fermi-Dirac statistics with the number 3 replacing the usual number 1; let us call it *Tripled Pauli statistics*. Such an occupation number (51) can be derived for objects that satisfy the Pauli's principle, but if such an object does appear (only one of them can be present in a given state), it can appear in three different forms. Does it mean that scalar quanta and gravitons near the black hole become (or interact with)  $J = 1$  links (triplets) in a spin network that happen to follow the Pauli's principle? Our puzzling results are summarized in the table below.

Spin	Asymptotic frequencies	Corresponding poles	Naively implied statistics
$j \in \mathbb{Z} + 1/2$	$\frac{i(n-1/2)}{2}$	$\frac{1}{\exp(\beta_{\text{Hawking}}\omega)+1}$	Fermi-Dirac
$j \in 2\mathbb{Z} + 1$	$\frac{in}{2}$	$\frac{1}{\exp(\beta_{\text{Hawking}}\omega)-1}$	Bose-Einstein
$j \in 2\mathbb{Z}$	$\frac{i(n-1/2)}{2} \pm \ln(3)$	$\frac{1}{\exp(\beta_{\text{Hawking}}\omega)+3}$	Tripled Pauli?

(53)

#### 4.1 Path integrals and speculations on the black hole chemical potential and spin network resonances

It is a well-known fact that the asymptotic periodicity of the time coordinate in the Euclidean black hole solutions equals the inverse Hawking temperature  $\beta_{\text{Hawking}}$  exactly if there is no deficit (or excess) angle at the horizon (which is a “tip” of the solution). In this sense, the Euclidean solution “knows” about the Hawking temperature. Black holes can only be in equilibrium with a heat bath of the Hawking temperature because if we want to consider black hole configurations contributing to the path integral, we are forced to take the right periodicity of the time direction.

If we glance at (52), we see that the quasinormal modes also “know” about the right temperature. The asymptotic quasinormal frequencies in the case of spin  $j \notin 2\mathbb{Z}$  satisfy  $\exp(\beta_{\text{Hawking}}\omega) = \pm 1$ . But  $\exp(\beta_{\text{Hawking}}\omega)$  is the evolution operator by  $\beta_{\text{Hawking}}$  in the Euclidean time. So the allowed asymptotic frequencies are exactly the frequencies that respect the time periodicity. Alternatively, the transmission amplitude (52) contains the information about the thermal ensemble. Both Hawking radiation as well as quasinormal modes deal with general relativity linearized around the black hole solution, so it is perhaps not too surprising that they contain a similar piece of information.

We understand that (52) is a natural expression related to the thermal properties of the black hole. But what about (51), which has +3 instead of  $-1$  in the denominator? These quasinormal frequencies appear for  $j \in 2\mathbb{Z}$ . They satisfy  $\exp(\beta_{\text{Hawking}}\omega) = -3$  (or  $-1/3$ , if we consider the solutions with the negative real part). Just like the solutions (42) and (44) lead to (52) which correspond to the path integral with periodic or antiperiodic boundary conditions on the time coordinate of periodicity  $\beta_{\text{Hawking}}$ , the extra frequencies (43) seem to lead to the path integral where the fluctuations get multiplied by  $(-3)$  (or  $(-1/3)$ , which is related to  $(-3)$  by the time reversal symmetry) if we perform a rotation around the Euclidean time. Such boundary conditions correspond to the evaluation of the thermal expectation value of the operator

$$\langle (-3)^{\hat{N}} \rangle = \text{Tr} [\exp(-\beta_{\text{Hawking}} \hat{H}) (-3)^{\hat{N}}] \quad (54)$$

where  $\hat{N}$  is the number of gravitons (or scalar quanta) in the state. Just as the frequencies (41) and (42) inform us that the thermal expectation values of various ordinary operators contain the denominator (52), the extra quasinormal modes (43) might perhaps hide a similar insight. Note that the

insertion of  $(-3)^{\hat{N}}$  in (54) might be interpreted as a chemical potential<sup>12</sup> for gravitons (or scalar quanta) in a grand canonical ensemble  $\exp(-\beta\hat{H} + \mu\hat{N})$ . Something special should happen when the chemical potential  $\mu$  formally approaches the (complex) value  $2\pi i(n + 1/2) + \ln(3)$ .

In loop quantum gravity,  $\ln(3)$  arises as the entropy of a single link with the minimal possible spin  $J^{(\min)} = 1$  (we assume the  $SO(3)$  version of LQG). An advocate of loop quantum gravity could argue in the following way:  $(-3)^{\hat{N}}$  is an operator that receives some “resonant” contributions from the nontrivial quasinormal modes. The reason could be that  $(-3)^{\hat{N}}$  might actually be the operator that creates (or destroys) a single link of the spin network. The number 3 (or  $(-3)$ , we have no understanding of the minus sign at this point; it might arise from Fermi-like statistics of some objects such as the  $J = 1$  links) would be related to the number of possible  $J_z$  polarizations of the new link. The number of gravitons  $\hat{N}$  would be correlated with the number of the links in some unknown way. Finally, there could also be an explanation why the creation operator of a link has a large resonant expectation value in the canonical ensemble describing a black hole. Only time can show whether these speculations can be supported by a consistent formalism.

## 5 Conclusions and outlook

In this paper we calculated the asymptotic form of the frequencies of the quasinormal modes on the four-dimensional Schwarzschild background. We used the method of Taylor expansion and continued fractions that can be approximated very well for large frequencies by imposing the constraint that the quantities  $R_n$ , describing the continued fractions, change slowly. For the values of index  $n$  near  $|n + 2i\omega| \approx 1$ , this estimate breaks down for general values of  $\omega$ . But for large  $|\omega|$  the recursion relations near  $|n + 2i\omega| \approx 1$  can be approximated by an exactly solvable system. The continued fractions become ordinary fractions. The expressions can be rewritten in terms of the  $\Gamma$  function and then even in terms of trigonometric functions. The result for odd and half-integer spins  $j$  of the perturbation are the frequencies

$$\omega = T_{\text{Hawking}} \cdot 2\pi i(k + j), \quad k \in \mathbb{Z} \quad (55)$$

but for even integer spins  $j$  we proved the existence of modes with

$$\omega = T_{\text{Hawking}}[2\pi i(k + 1/2) \pm \ln 3], \quad k \in \mathbb{Z}. \quad (56)$$

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<sup>12</sup>We are grateful to Alex Maloney for suggesting the possible relevance of the term “chemical potential” in this context.

We conjectured that these asymptotic frequencies of quasinormal modes can be found for other black holes, too, as long as the frequency is written as a multiple of the Hawking temperature; the result (56) was recently confirmed in the case of higher-dimensional Schwarzschild black holes [19], but ruled out in the Reissner-Nordström case [19] as well as the case of Kerr black holes [24]. While (55) carries some information on the thermal structure of the black hole, the significance of the nontrivial frequencies (56) remains clouded in mystery. We presented speculations how (56) might be understood in the language related to loop quantum gravity.

It would be interesting to

- check more carefully whether all *a priori* allowed solutions in (55) and (56) exist; look for the “irregular” solutions (55) numerically; it is however true that some very difficult numerical calculations become less interesting because their computational goals can now be achieved analytically
- generalize the calculation to higher-dimensional black holes and derive what  $\ln(3)T_{\text{Hawking}}$  is replaced by; a very recent paper [19] confirmed that  $\ln(3)T_{\text{Hawking}}$  appears for scalar perturbations of Schwarzschild black holes in an arbitrary dimension
- try to apply similar methods to the case of charged black holes and black holes in the Anti de Sitter space; functional methods have been used to derive the aperiodic structure of the Reissner-Nordström black holes in four dimensions [19] and this structure has been confirmed in a very recent paper [24]
- investigate the asymptotic structure of the quasinormal modes of rotating Kerr black holes in four dimensions; preliminary results suggest that  $\ln(3)T_{\text{Hawking}}$  is *not* the universal answer in this case [24]
- calculate the subleading contributions in  $1/(\text{Im}(\omega))^{1/2}$  to  $\text{Re}(\omega)$  using our methods or the methods of [19] as the zeroth order approximation
- find an interpretation of the real part that does not involve loop quantum gravity
- or even more ambitiously, find a solid interpretation in terms of loop quantum gravity; our analysis never led to the number  $\ln(2)$ ; does it show that loop quantum gravity is inconsistent with the existence of fermions?

Obviously, there are still many tasks to be solved.



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