

Exact chiral ring of $\text{AdS}_3/\text{CFT}_2$

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Abstract

We carry out an exact worldsheet computation of tree level three-point correlators of chiral operators in type IIB string theory on $\text{AdS}_3 \times S^3 \times T^4$ with NS–NS flux. We present a simple representation for the string chiral operators in the coordinate basis of the dual boundary CFT. Striking cancelations occur between the three-point functions of the H_3^+ and the $\text{SU}(2)$ WZW models which result in a simple factorized form for the final correlators. We show, by fixing a single free parameter in the H_3^+ WZW model, that the fusion rules and the structure constants of the $N = 2$ chiral ring in the bulk are in precise agreement with earlier computations in the boundary CFT of the symmetric product of T^4 at the orbifold point in the large N limit.

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1 Introduction

Within the family of $\text{AdS}_{n+1}/\text{CFT}_n$ holographically dual theories [1–4], the $\text{AdS}_3/\text{CFT}_2$ case has a concrete realization in the form of a duality between string theory in $\text{AdS}_3 \times S^3 \times M^4$, where M^4 is either a torus T^4 or a $K3$ surface, and a two-dimensional CFT in the moduli space of a non-linear sigma model whose target space is the symmetric product of M^4 .

In the heuristic derivation of this $\text{AdS}_3/\text{CFT}_2$ correspondence via the near horizon of the D1/D5 system in type IIB 10D supergravity [1] (see [5, 6] for reviews), there is RR flux through the S^3 factor of the $\text{AdS}_3 \times S^3 \times M^4$ geometry, which makes the theory more difficult to quantize [7, 8]. It is convenient to use an S-dual description [9], where the D1/D5 system becomes a system of Q_1 fundamental-strings and Q_5 NS5 branes. The near horizon geometry is still $\text{AdS}_3 \times S^3 \times M^4$, but the RR flux is traded for $k = Q_5$ units of NS flux through AdS_3 and through S^3 , and the common radius of AdS_3 and S^3 is $\sqrt{\alpha' k}$. The resulting model has an exact worldsheet description through level k supersymmetric $\text{SL}(2, R)$ and $\text{SU}(2)$ WZW models. This allows a much more detailed treatment of the bulk theory that goes beyond the leading supergravity approximation. In this controlled setting, the emergence of the two-dimensional superconformal symmetry of the dual theory from the worldsheet has been studied in [10–13].

String propagation in this background is of interest because it is related to the Strominger–Vafa black hole [14] and one can construct black hole solutions by taking quotients [5, 15].

Checks of the AdS_3/CFT_2 duality in this background focused so far in comparing the moduli space [16], and the spectrum of both theories [9, 17–19], and to our knowledge no successful comparison of dynamical quantities was performed yet, of the kind done in [20, 21] for the $N = 4$ SYM/ $AdS_5 \times S^5$ duality. In this paper, we make a step in that direction. We show that the fusion rules and the structure constants of the entire $N = 2$ chiral ring of the symmetric product, computed at the orbifold point, agree precisely with computations in the $AdS_3 \times S^3 \times M^4$ worldsheet. The correlators for the AdS_3 factor are better studied in its Euclidean version, the $H_3^+ = SL(2, C)/SU(2)$ WZW model, and to find complete agreement with the boundary we should fix a free parameter in the H_3^+ WZW model to a specific value. In this work, we consider only “unflowed” $SL(2, R)$ representations [22]. We also consider only $M^4 = T^4$ for simplicity, but the results can be easily extended to $M^4 = K3$.

This precise agreement is quite surprising because our computations in the bulk are carried out at a point in the moduli space which does not correspond in the boundary SCFT to the orbifold point of the symmetric product. One expects that the latter has a boundary B-field turned on as theta angles, whereas in the bulk theory the corresponding field is switched off since there is no background RR field. To go from one to other would require for example to turn on RR field in the bulk or twist fields in the boundary. Our results suggest a non-renormalization of these correlators analogous to what is found in the context of AdS_5/CFT_4 even though we have only half as much supersymmetry in our case. Clearly, it would be very interesting to understand this agreement from general principles.

In the bulk, we express the chiral spectrum in the isospin variables (x, \bar{x}) and (y, \bar{y}) of the $SL(2, R)$ and $SU(2)$ current algebras, respectively. This basis makes for an easy comparison between the boundary and bulk correlators and is also well suited for computing the tensor products required in the construction of the chiral operators. A crucial point of the computation is that all the factors in the three-point function of the H_3^+ WZW model which mix the quantum numbers of the three vertices, cancel against similar factors from the $SU(2)$ WZW model.¹

¹This cancellation was first observed in [23] in the related background $SL(2, R)/U(1) \times SU(2)/U(1)$. A similar cancellation occurs in the product of the Liouville and minimal models three-point functions [24, 25].

The outline of the paper is as follows. In Section 2 we review the spectrum, operators and correlators of the chiral sector of the symmetric product of T^4 . In Section 3, we review the relevant aspects of the $\text{SL}(2, R)$ and $\text{SU}(2)$ WZW models and the chiral spectrum in the $\text{AdS}_3 \times S^3 \times T^4$ worldsheet. In Section 4 we show how the fusion rules and structure constants of the symmetric product are obtained from the string worldsheet. Finally, in Section 5 we discuss possible directions for further explorations. Appendix A contains a detailed derivation of the chiral states in the bulk in the $(x, \bar{x}) - (y, \bar{y})$ basis. In Appendix B we elaborate on the relationship between the three-point functions of the H_3^+ and $\text{SU}(2)$ WZW models in order to better understand the cancelations between them. In particular, we show how the conformal bootstrap method used in [26] for H_3^+ can also be applied to $\text{SU}(2)$.

2 The chiral ring of the symmetric product

The boundary theory is a $(4, 4)$ SCFT on the moduli space of the symmetrized product $\text{Sym}^N(M^4)$ of N copies of M^4 . Here $N = Q_1 Q_5$ for $M^4 = T^4$ and $N = Q_1 Q_5 + 1$ for $M^4 = K3$. For simplicity we consider $M^4 = T^4$, but these considerations generalize easily to $M^4 = K3$.

We work at a point in the moduli space of the theory where we can think of $\text{Sym}^N(T^4)$ as an orbifold $(T^4)^N/S_N$, where S_N is the symmetric group action on N objects. Before orbifolding, the $(T^4)^N$ theory has $4N$ free bosons ϕ_I^i that coordinatize the space, with $i = 1, 2, 3, 4$ and $I = 1, \dots, N$. Their superpartners are $4N$ Majorana–Weyl fermions ξ_I^i . We will define the complex combinations

$$X_I^1 = \frac{\phi_I^1 + i\phi_I^2}{\sqrt{2}}, \quad X_I^2 = \frac{\phi_I^3 + i\phi_I^4}{\sqrt{2}}, \quad (2.1)$$

$$\lambda_I^1 = \frac{\xi_I^1 + i\xi_I^2}{\sqrt{2}}, \quad \lambda_I^2 = \frac{\xi_I^3 + i\xi_I^4}{\sqrt{2}}. \quad (2.2)$$

The fields are normalized as

$$X_I^a(z)X_J^{\dagger b}(w) \sim -\delta^{ab}\delta_{IJ}\log(z-w), \quad (2.3)$$

$$\lambda_I^a(z)\lambda_J^{\dagger b}(w) \sim \frac{\delta^{ab}\delta_{IJ}}{z-w}, \quad a, b = 1, 2. \quad (2.4)$$

These fields form a representation of the $N = 4$ superconformal algebra with $c = 6N$, with generators

$$\begin{aligned}
 T(z) &= -\partial X_I^{\dagger i} \partial X_I^i - \frac{1}{2} \lambda_I^{\dagger a} \partial \lambda_I^a + \frac{1}{2} \partial \lambda_I^{\dagger a} \lambda_I^a, \\
 J^1 &= -\frac{i}{2} (\lambda_I^1 \lambda_I^2 + \lambda_I^{\dagger 1} \lambda_I^{\dagger 2}), \\
 J^2 &= \frac{1}{2} (\lambda_I^{\dagger 1} \lambda_I^{\dagger 2} - \lambda_I^1 \lambda_I^2), \\
 J^3 &= \frac{1}{2} (\lambda_I^1 \lambda_I^{\dagger 1} + \lambda_I^2 \lambda_I^{\dagger 2}), \\
 G^a &= \sqrt{2} \begin{bmatrix} i\lambda_I^1 \\ -\lambda_I^{\dagger 2} \end{bmatrix} \partial X_I^{\dagger 1} + \sqrt{2} \begin{bmatrix} i\lambda_I^2 \\ \lambda_I^{\dagger 1} \end{bmatrix} \partial X_I^{\dagger 2}, \\
 G^{\bar{a}} &= \sqrt{2} \begin{bmatrix} i\lambda_I^{\dagger 1} \\ \lambda_I^2 \end{bmatrix} \partial X_I^1 + \sqrt{2} \begin{bmatrix} i\lambda_I^{\dagger 1} \\ -\lambda_I^1 \end{bmatrix} \partial X_I^2.
 \end{aligned}
 \tag{2.5}$$

There is a similar antiholomorphic copy of all the fields and the algebra. The global part of the $SU(2)$ R-symmetry algebra J^i along with the antiholomorphic $SU(2)$ correspond to the $SO(4) \sim SU(2) \times SU(2)$ isometries of the S^3 factor in the bulk.

We will be interested in (c, c) and (a, a) fields under an $N = 2$ subalgebra, satisfying [27]

$$\Delta = Q, \quad \bar{\Delta} = \bar{Q}
 \tag{2.6}$$

and

$$\Delta = -Q, \quad \bar{\Delta} = -\bar{Q},
 \tag{2.7}$$

respectively, where $\Delta, \bar{\Delta}$ are the conformal dimensions and Q, \bar{Q} are the charges under J^3, \bar{J}^3 .²

2.1 Chiral spectrum

The Hilbert space of the symmetric orbifold is the direct sum of twisted sectors, each sector corresponding to a conjugacy class of S_N . The latter can be represented by disjoint cyclic permutations of various lengths n_i ,

$$(n_1)^{N_1} (n_2)^{N_2} \dots (n_r)^{N_r}
 \tag{2.8}$$

such that

$$\sum_i n_i N_i = N.
 \tag{2.9}$$

Twisted sectors are thus classified by the various ways of partitioning the integer N in terms of smaller integers. The full twist operator of a given

²In the standard normalization of the $N = 2$ subalgebra, the $U(1)$ R-current is $2J^3$.

conjugacy class can then be built as S_N invariant combinations of the Z_{n_i} twist operators that generate cycles of lengths n_i .

Chiral states in a manifold are in correspondence with the elements of its Dolbeault cohomology [28]. Under this correspondence, an element of $H^{p,q}$ corresponds to a chiral operator with chiral charge p on the left and q on the right. For a symmetric product of a manifold M^4 , the cohomology can be expressed as a Fox space of free particles [29]. In the AdS/CFT context, this acquires a physical meaning in terms of the number of particles in the gravity side [9, 17]. The first quantized spectrum of the symmetric product corresponds to the second quantized spectrum of the string dual. Chiral vertex operators representing BPS single particle states in first-quantized string theory correspond in the boundary to chiral primaries in the conjugacy class of a *single* Z_n cycle.

For a Z_n cycle, the generator of the orbifold group acts cyclically on the fields $\{X_I^a\}$ for $I = 1, \dots, n$ taking $X_I^a \rightarrow X_{I+1}^a$ for $I < n$ and $X_n^a \rightarrow X_1^a$, and similarly on the λ_I^a 's. To analyze the twist fields, we can diagonalize this cyclic action by defining the fields

$$Y_l^a(z) = \frac{1}{\sqrt{n}} \sum_{I=1}^n e^{-2\pi i l I/n} X_I^a, \quad a = 1, 2 \tag{2.10}$$

$$y_l^a(z) = \frac{1}{\sqrt{n}} \sum_{I=1}^n e^{-2\pi i l I/n} \lambda_I^a(z) \tag{2.11}$$

for $l = 0, 1, 2, \dots, (n - 1)$. These fields are orthogonal,

$$Y_l^a(z) Y_m^{\dagger b}(w) \sim -\delta^{ab} \delta_{lm} \log(z - w), \tag{2.12}$$

$$y_l^a(z) y_m^{\dagger b}(w) \sim \frac{\delta^{ab} \delta_{lm}}{z - w}. \tag{2.13}$$

Therefore we have $2n$ independent complex bosons and fermions with boundary conditions

$$Y_l^a(z e^{2\pi i}) = e^{2\pi i l/n} Y_l^a(z), \tag{2.14}$$

$$y_l^a(z e^{2\pi i}) = e^{2\pi i l/n} y_l^a(z). \tag{2.15}$$

For each field Y_l^a , with $l > 1$, this sector is created by the action of a twist field σ_l^a , with conformal dimension $\Delta = \frac{l}{2n}(1 - \frac{l}{n})$ [30]. For the fermions,

we can bosonize them as

$$y_l^1 = e^{iF_l^1}, \quad y_l^{\dagger 1} = e^{-iF_l^1}, \tag{2.16}$$

$$y_l^2 = e^{iF_l^2}, \quad y_l^{\dagger 2} = e^{-iF_l^2}, \tag{2.17}$$

with $F_l^a(z)F_m^b(w) \sim -\delta_{lm}\delta^{ab}\log(z-w)$, and their twist fields are $e^{(il/n)F_l^1}$ $e^{(il/n)F_l^2}$, with conformal dimension $\frac{l^2}{n^2}$. Collecting the factors for all the fields, the n -cycle twist operator is

$$\Sigma_{(12\dots n)} = \prod_{l=1}^{n-1} \sigma_l^1(z, \bar{z})\sigma_l^2(z, \bar{z})e^{(il/n)(F_l^1+\bar{F}_l^1)}e^{(il/n)(F_l^2+\bar{F}_l^2)}, \tag{2.18}$$

where we have included also the antiholomorphic dependence. For each l in the above product, the conformal dimension of the twist fields is

$$\Delta_l = \bar{\Delta}_l = 2 \times \frac{l}{2n} \left(1 - \frac{l}{n}\right) + 2 \times \frac{l^2}{2n^2} = \frac{l}{n}, \tag{2.19}$$

and therefore the conformal dimension of $\Sigma_{(12\dots n)}$ is

$$\Delta = \bar{\Delta} = \sum_{l=1}^{n-1} \Delta_l = \frac{n-1}{2}. \tag{2.20}$$

The charge of each factor, measured with

$$J^3 = \frac{i}{2} \sum_{l=0}^{n-1} (\partial F_l^1 + \partial F_l^2) + \frac{1}{2} \sum_{I=n+1}^N (\lambda_I^1 \lambda_I^{\dagger 1} + \lambda_I^2 \lambda_I^{\dagger 2}), \tag{2.21}$$

is

$$Q_l = \frac{l}{n} = \Delta_l, \tag{2.22}$$

so the total charge is $Q = \Delta = \frac{n-1}{2}$, and also $\bar{Q} = \bar{\Delta} = \frac{n-1}{2}$, and therefore $\Sigma_{(12\dots n)}$ is chiral–chiral. More generally, every chiral operator in T^4 will give a chiral operator in every twisted sector [29], and $\Sigma_{(12\dots n)}$ corresponds to the identity field in T^4 . The other (c, c) fields in each T^4 are, for the scalar sector (no sum on I),

$$\lambda_I^a \bar{\lambda}_I^{\bar{a}}, \quad a, \bar{a} = 1, 2, \tag{2.23}$$

which have $\Delta = Q = \bar{\Delta} = \bar{Q} = 1/2$ and correspond to the four $(1, 1)$ forms in T^4 , and

$$\lambda_I^1 \lambda_I^2 \bar{\lambda}_I^{\bar{1}} \bar{\lambda}_I^{\bar{2}} \tag{2.24}$$

which has $\Delta = Q = \bar{\Delta} = \bar{Q} = 1$ and corresponds to the $(2, 2)$ form in T^4 . In each case, we should multiply $\Sigma_{(12\dots n)}$ by a combination of the above chiral

fields invariant under the cyclic permutation. But this last requirement is not restrictive enough. It turns out that there is a 2-fold ambiguity in the construction of the chiral operators. The first ambiguity is related to the fact that both [5, 31]

$$\Sigma_{(12\dots n)} \sum_I \lambda_I^a \bar{\lambda}_I^{\bar{a}} \tag{2.25}$$

and [32]

$$\Sigma_{(12\dots n)} \left(\sum_I \lambda_I^a \right) \left(\sum_I \bar{\lambda}_I^{\bar{a}} \right) \tag{2.26}$$

are such that the fermions are invariant under the cyclic permutation. The second ambiguity is the range of I in the above sums. In order to obtain an operator invariant under $\Sigma_{(12\dots n)}$, one can sum over $I = 1, \dots, N$ [5] or $I = 1, \dots, n$ [32, 33] (more generally one could sum over $I = 1, \dots, m$ for $m \geq n$). Similar ambiguities exist for operators built from the chiral fields (2.24).

Clearly each option leads to different correlation functions. If we sum as in (2.26), then the fermionic contributions will be factorized into holomorphic and antiholomorphic contributions, which will not be the case in (2.25). Regarding the range of the sum, if it runs over $I = 1, \dots, N$, the fermions will not only commute with $\Sigma_{(12\dots n)}$, but also with the spin field associated with any other cycle. As a consequence, correlators will factorize into a trivial part involving the fermions, and a part involving $\Sigma_{(12\dots n)}$ which will be universal for a given choice of Z_n cycles. On the other hand, for $I = 1, \dots, n$, the fermions will generally not commute with the spin fields corresponding to other cycles, and correlators will depend on the operators multiplying each twist field.

The comparison we perform in the next sections with the string theory correlators shows that the correlators in the gravity dual are factorized into holomorphic and antiholomorphic contributions, and they do depend on the type of fermionic dressing of the twist fields. Therefore, the boundary operators we should consider are

$$\Sigma_{(12\dots n)}^{(0,0)} = \Sigma_{(12\dots n)}, \tag{2.27}$$

$$\Sigma_{(12\dots n)}^{(a,\bar{a})} = \Sigma_{(12\dots n)} y_0^a \bar{y}_0^{\bar{a}}, \quad a, \bar{a} = 1, 2, \tag{2.28}$$

$$\Sigma_{(12\dots n)}^{(2,2)} = \Sigma_{(12\dots n)} y_0^1 y_0^2 \bar{y}_0^{\bar{1}} \bar{y}_0^{\bar{2}}, \tag{2.29}$$

$$\Sigma_{(12\dots n)}^{(0,2)} = \Sigma_{(12\dots n)} \bar{y}_0^{\bar{1}} \bar{y}_0^{\bar{2}}, \quad \text{etc.}, \tag{2.30}$$

where y_0^a was defined in (2.11) and $\bar{y}_0^{\bar{a}}$ is its antiholomorphic counterpart. The fact that these operators have the form (2.26) is natural from the point

of view of the orbifold twisting, since the orbifold action on the fermions is diagonalized in the y_l (2.15), and the fields y_0 are neutral under the orbifold action. In each twisted sector therefore the chiral fields can be constructed from the twist fields and the y_0, \bar{y}_0 fields which do not suffer any twisting.

Using these operators in the Z_n sectors, the chiral primaries in the full S_N orbifold theory can be constructed by symmetrization. Summing over all permutations, we obtain, for $n > 1$, the final expressions for the scalar chiral primaries

$$O_n^{(0,0)}(z, \bar{z}) = \frac{1}{(N!(N-n)!n)^{1/2}} \sum_{h \in S_N} \Sigma_{h(1, \dots, n)h^{-1}}^{(0,0)}(z, \bar{z}), \tag{2.31}$$

$$O_n^{(a,\bar{a})}(z, \bar{z}) = \frac{1}{(N!(N-n)!n)^{1/2}} \sum_{h \in S_N} \Sigma_{h(1, \dots, n)h^{-1}}^{(a,\bar{a})}(z, \bar{z}), \tag{2.32}$$

$$O_n^{(2,2)}(z, \bar{z}) = \frac{1}{(N!(N-n)!n)^{1/2}} \sum_{h \in S_N} \Sigma_{h(1, \dots, n)h^{-1}}^{(2,2)}(z, \bar{z}), \tag{2.33}$$

and similar expressions for non-scalar operators. The corresponding antichiral fields are obtained by conjugation and the prefactors are fixed by normalizing as [32]

$$\langle O_n^{(0,0)\dagger}(\infty) O_n^{(0,0)}(0) \rangle = \langle O_n^{(2,2)\dagger}(\infty) O_n^{(2,2)}(0) \rangle = 1, \tag{2.34}$$

$$\langle O_n^{(a,\bar{a})\dagger}(\infty) O_n^{(b,\bar{b})}(0) \rangle = \delta^{ab} \delta^{\bar{a}\bar{b}}, \tag{2.35}$$

and similarly for non-scalar operators. For $n = 1$, the expressions (2.27) to (2.30) are already normalized. To compare with the string theory computations, it will be useful to express n in terms of h defined as

$$n = 2h - 1. \tag{2.36}$$

In terms of the variable h , the quantum numbers of the three families of chiral operators can be summarized in the following table:

Field	$\Delta = Q$	Range of Δ
$O_h^{(0,0)}$	$h - 1$	$0, \frac{1}{2}, \dots, \frac{N-1}{2}$
$O_h^{(a,a)}$	$h - 1/2$	$\frac{1}{2}, 1, \dots, \frac{N}{2}$
$O_h^{(2,2)}$	h	$1, \frac{3}{2}, \dots, \frac{N+1}{2}$

The form (2.18) for $\Sigma_{(12\dots n)}$ was useful to find its quantum numbers, but it is not useful to compute correlation functions of twist fields corresponding to cycles which have a partial overlap.

Correlators of twist fields in symmetric products were studied by different authors along complementary lines . The works [34, 35], using the path integral formalism, computed correlators of fields $O_n^{(\alpha, \bar{\alpha})}$ with only $\alpha, \bar{\alpha} = 0, 2$, but considered all the elements of the $SU(2)$ multiplet of the $N = 4$ algebra, of which our $N = 2$ chiral and antichiral fields correspond to the highest and lowest J_0^3 eigenvalue.

On the other hand, the work [32], whose results we will mostly use in this paper, applied the conformal bootstrap to the results of [36, 37] and computed the structure constants of all the $N = 2$ chiral ring, i.e., all the operators $O_n^{(\alpha, \bar{\alpha})}$ with $\alpha, \bar{\alpha} = 0, a, 2$.³

2.2 Fusion rules and structure constants

The chiral ring is defined by its fusion rules and its structure constants. The scalar sector of the (c, c) ring can be shown to be closed, and its fusion rules are [32]

$$\begin{aligned} (0, 0) \times (0, 0) &= (0, 0) + (2, 2), \\ (0, 0) \times (2, 2) &= (2, 2), \\ (0, 0) \times (a, a) &= (a, a), \\ (a, a) \times (a, a) &= (2, 2), \end{aligned} \tag{2.37}$$

where the length n of the cycle of each operator should be such that the chiral charge is conserved. Therefore, there are five non-zero structure constants, given by [32]⁴

$$\begin{aligned} \langle O_{n+k-1}^{(0,0)\dagger} O_k^{(0,0)} O_n^{(0,0)} \rangle &= F(N, n, k) \left[\frac{(n+k-1)^3}{nk} \right]^{1/2}, \\ \langle O_{n+k-3}^{(2,2)\dagger} O_k^{(0,0)} O_n^{(0,0)} \rangle &= F(N, n, k) \left[\frac{(N - (n+k) + 3)}{(N - (n+k) + 2)nk(n+k-3)} \right]^{1/2}, \end{aligned}$$

³It would be interesting to study the chiral correlators of the symmetric product with the topological field theory techniques used in [38] for abelian orbifolds.

⁴The expression in [32] for the second correlator in (2.38) has an additional factor of 2 which was corrected in [35].

$$\begin{aligned}
 \langle O_{n+k-1}^{(2,2)\dagger} O_k^{(0,0)} O_n^{(2,2)} \rangle &= F(N, n, k) \left[\frac{n^3}{k(n+k-1)} \right]^{1/2}, \\
 \langle O_{n+k-1}^{(a,\bar{a})\dagger} O_k^{(0,0)} O_n^{(b,\bar{b})} \rangle &= F(N, n, k) \left[\frac{n(n+k-1)}{k} \right]^{1/2} \delta^{ab} \delta^{\bar{a}\bar{b}}, \\
 \langle O_{n+k-1}^{(2,2)\dagger} O_k^{(a,\bar{a})} O_n^{(b,\bar{b})} \rangle &= F(N, n, k) \left[\frac{nk}{(n+k-1)} \right]^{1/2} \xi^{ab} \xi^{\bar{a}\bar{b}}, \tag{2.38}
 \end{aligned}$$

where

$$F(N, n, k) = \left[\frac{(N-n)!(N-k)!}{(N-(n+k-1))! N!} \right]^{1/2}, \tag{2.39}$$

$$\xi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{2.40}$$

and the operators were set at $z = \bar{z} = 0, 1, \infty$. All these correlators assume $n, k > 1$, and in each case it can be checked that the chiral charge is conserved.

Both the fusion rules and the correlators are easily extended to the non-scalar sector. From the analysis in [32], it follows that the process $(0, 0) \times (0, 0) \rightarrow (2, 2)$ can only occur simultaneously in the holomorphic and antiholomorphic sector, but the other four processes in (2.37) can be combined independently among themselves. Moreover, for these cases, the structure constants are completely factorized into holomorphic and antiholomorphic contributions and are given by multiplying pairwise the square roots of the structure constants of (2.38). Thus the chiral ring is fully characterized by $1 + 4 \times 4 = 17$ structure constants. It can also be shown that the (c, a) ring has the same fusion rules and structure constants of the (c, c) ring.

To compare with the bulk tree level computations, we fix the charges and take $N \rightarrow \infty$. We also change the labels n, k and express them in terms of $h_{1,2}$ as

$$n = 2h_1 - 1, \tag{2.41}$$

$$k = 2h_2 - 1. \tag{2.42}$$

Using

$$\lim_{N \rightarrow \infty} F(N, n, k) = \left(\frac{1}{N} \right)^{1/2}, \tag{2.43}$$

the tree level chiral structure constants become

$$\langle O_{h_3}^{(0,0)\dagger} O_{h_2}^{(0,0)} O_{h_1}^{(0,0)} \rangle = \left(\frac{1}{N} \right)^{1/2} \left[\frac{(2h_3 - 1)^3}{(2h_1 - 1)(2h_2 - 1)} \right]^{1/2}, \quad (2.44)$$

$$\langle O_{h_3}^{(2,2)\dagger} O_{h_2}^{(0,0)} O_{h_1}^{(0,0)} \rangle = \left(\frac{1}{N} \right)^{1/2} \left[\frac{1}{(2h_1 - 1)(2h_2 - 1)(2h_3 - 1)} \right]^{1/2}, \quad (2.45)$$

$$\langle O_{h_3}^{(2,2)\dagger} O_{h_2}^{(0,0)} O_{h_1}^{(2,2)} \rangle = \left(\frac{1}{N} \right)^{1/2} \left[\frac{(2h_1 - 1)^3}{(2h_2 - 1)(2h_3 - 1)} \right]^{1/2}, \quad (2.46)$$

$$\langle O_{h_3}^{(a,\bar{a})\dagger} O_{h_2}^{(0,0)} O_{h_1}^{(b,\bar{b})} \rangle = \left(\frac{1}{N} \right)^{1/2} \left[\frac{(2h_1 - 1)(2h_3 - 1)}{(2h_2 - 1)} \right]^{1/2} \delta^{ab} \delta^{\bar{a}\bar{b}}, \quad (2.47)$$

$$\langle O_{h_3}^{(2,2)\dagger} O_{h_2}^{(a,\bar{a})} O_{h_1}^{(b,\bar{b})} \rangle = \left(\frac{1}{N} \right)^{1/2} \left[\frac{(2h_1 - 1)(2h_2 - 1)}{(2h_3 - 1)} \right]^{1/2} \xi^{ab} \xi^{\bar{a}\bar{b}}, \quad (2.48)$$

where h_3 is fixed from the conservation of $U(1)$ R-charge, and is given by $h_3 = h_1 + h_2 - 2$ for (2.45) and by $h_3 = h_1 + h_2 - 1$ for the other cases. The fusion rules and these simple factorized formulae for the structure constants can be reproduced by a completely different worldsheet calculation in the string theory dual as we show below.

3 The $\text{AdS}_3 \times S^3 \times T^4$ worldsheet theory

The supersymmetric $\text{SL}(2, R)_k$ model has symmetries generated by the supercurrents $\psi^A + \theta J^A$, $A = 1, 2, 3$. Their OPEs are

$$J^A(z) J^B(w) \sim \frac{\frac{k}{2} \eta^{AB}}{(z-w)^2} + \frac{i \epsilon^{AB}{}_C J^C(w)}{z-w}, \quad (3.1)$$

$$J^A(z) \psi^B(w) \sim \frac{i \epsilon^{AB}{}_C \psi^C(w)}{z-w}, \quad (3.2)$$

$$\psi^A(z) \psi^B(w) \sim \frac{\frac{k}{2} \eta^{AB}}{z-w}, \quad (3.3)$$

where $\epsilon^{123} = 1$ and capital letter indices are raised and lowered with $\eta^{AB} = \eta_{AB} = (+ + -)$. Similarly, the supersymmetric $\text{SU}(2)_k$ model has

supercurrents $\chi^a + \theta K^a$, $a = 1, 2, 3$, with OPEs

$$K^a(z)K^b(w) \sim \frac{\frac{k}{2}\delta^{ab}}{(z-w)^2} + \frac{i\epsilon^{ab}{}_c K^c(w)}{z-w}, \quad (3.4)$$

$$K^a(z)\chi^b(w) \sim \frac{i\epsilon^{ab}{}_c \chi^c(w)}{z-w}, \quad (3.5)$$

$$\chi^a(z)\chi^b(w) \sim \frac{\frac{k}{2}\delta^{ab}}{z-w}, \quad (3.6)$$

and lower case indices are raised and lowered with $\delta^{ab} = \delta_{ab} = (+, +, +)$. We will often use the linear combinations

$$J^\pm \equiv J^1 \pm iJ^2, \quad \psi^\pm \equiv \psi^1 \pm i\psi^2, \quad (3.7)$$

$$K^\pm \equiv K^1 \pm iK^2, \quad \chi^\pm \equiv \chi^1 \pm i\chi^2. \quad (3.8)$$

As usual in supersymmetric WZW models, it is convenient to split the J^A, K^a currents into

$$J^A = j^A + \hat{j}^A, \quad (3.9)$$

$$K^a = k^a + \hat{k}^a, \quad (3.10)$$

where

$$\hat{j}^A = -\frac{i}{k}\epsilon^A{}_{BC}\psi^B\psi^C, \quad (3.11)$$

$$\hat{k}^a = -\frac{i}{k}\epsilon^a{}_{bc}\chi^b\chi^c. \quad (3.12)$$

The currents j^A and k^a generate *bosonic* $SL(2, R)$ and $SU(2)$ affine algebra at levels $k+2$ and $k-2$, respectively, and commute with the free fermions ψ^A, χ^a . The latter in turn form a pair of supersymmetric $SL(2, R)$ and $SU(2)$ models at levels -2 and $+2$, whose bosonic currents are \hat{j}^A and \hat{k}^a . The spectrum and the interactions of the original level k supersymmetric WZW models are factorized into the bosonic WZW models and the free fermions. In terms of the split currents the stress tensor and supercurrent of the worldsheet theory are

$$T = \frac{1}{k}j^A j_A - \frac{1}{k}\psi^A \partial\psi_A + \frac{1}{k}k^a k_a - \frac{1}{k}\chi^a \partial\chi_a + T(T^4), \quad (3.13)$$

$$T_F = \frac{2}{k}(\psi^A j_A + \frac{2i}{k}\psi^1\psi^2\psi^3) + \frac{2}{k}(\chi^A k_A - \frac{2i}{k}\chi^1\chi^2\chi^3) + T_F(T^4), \quad (3.14)$$

and one can check that the central charge adds up to $c = 15$. Let us see now some specifics of the $SL(2, R)$ and $SU(2)$ models separately.

3.1 The $\mathrm{SL}(2, R)$ model: currents and observables

A primary field of spin h in the $\mathrm{SL}(2, R)_{k+2}$ WZW model satisfies

$$j^A(z)\Phi_h(x, \bar{x}; w, \bar{w}) \sim -\frac{D_x^A \Phi_h(x, \bar{x}; w, \bar{w})}{z-w}, \quad (3.15)$$

where the operators D_x^A are

$$D_x^- = \partial_x, \quad (3.16)$$

$$D_x^3 = x\partial_x + h, \quad (3.17)$$

$$D_x^+ = x^2\partial_x + 2hx, \quad (3.18)$$

and there is a similar antiholomorphic copy. We will sometimes omit writing explicitly the antiholomorphic dependence of the operators. The conformal dimension of $\Phi_h(x, \bar{x}; z, \bar{z})$ is

$$\Delta_h = \bar{\Delta}_h = -\frac{h(h-1)}{k}, \quad (3.19)$$

and it can be expanded in modes as

$$\Phi_h(x, \bar{x}) = \sum_{m, \bar{m}} \Phi_{h, m, \bar{m}} x^{-h-m} \bar{x}^{-h-\bar{m}}, \quad (3.20)$$

but the range of the summation is not always well defined [12]. Yet, the action of the zero modes of the currents on $\Phi_{h, m, \bar{m}}$ is well defined and can be read from (3.15) to be

$$j_0^3 \Phi_{h, m, \bar{m}} = m \Phi_{h, m, \bar{m}}, \quad (3.21)$$

$$j_0^\pm \Phi_{h, m, \bar{m}} = (m \mp (h-1)) \Phi_{h, m \pm 1, \bar{m}}, \quad (3.22)$$

and similarly for the antiholomorphic currents. In this work, we will mostly use the (x, \bar{x}) basis. These variables are interpreted as the local coordinates of the two-dimensional conformal field theory living in the boundary of AdS_3 .

The spectrum of the bosonic $\mathrm{SL}(2, R)_{k+2}$ was obtained in [22] and consists of delta-normalizable *continuous* representations, with $h = \frac{1}{2} + i\mathbb{R}$ and $m = \alpha + \mathbb{Z}$ ($\alpha \in [0, 1)$), and non-normalizable *discrete* highest/lowest weight

representations, with $h \in \mathbb{R}$ obeying

$$\frac{1}{2} < h < \frac{k+1}{2}, \tag{3.23}$$

and $m = h, h+1, \dots$ (lowest weight) or $m = -h, -h-1, \dots$ (highest weight). Along with these *unflowed* representations, one should include the states generated from them by spectral flow [22], which will be treated in [39].

The bound (3.23) on h is slightly stricter than the bound $0 < h < k/2 + 1$ needed for the no-ghost theorem to hold [40–47]. The stricter bound is required for the normalizability of the primary operators [48], and its two ends are consistent with the spectral flow symmetry, which relates a highest weight representation with spin h to a lowest weight representation with spin $k/2 + 1 - h$ [22].

Expressions similar to (3.15) hold also for the total currents J^A and the fermionic currents \hat{j}^A of the decomposition (3.9). We will use the letters \hat{h} , h and H to denote the $SL(2, R)$ spins associated to the currents \hat{j}^A , j^A and J^A . In particular, H is the conformal dimension of a field $\Phi_H(x, \bar{x})$ in the dual CFT [10].

The OPEs like (3.15) between the currents J^A and a field $\Phi_H(x)$ can be expressed in a compact way by means of the current

$$J(x; z) = -J^+(z) + 2xJ^3(z) - x^2J^-(z) \tag{3.24}$$

as

$$J(x_1; z)\Phi_H(x_2; w) \sim \frac{1}{z-w} [(x_1-x_2)^2\partial_{x_2} - 2H(x_1-x_2)] \Phi_h(x_2; w). \tag{3.25}$$

Similarly, the OPEs (3.1) between the currents J^A can be also expressed through $J(x, z)$ as

$$\begin{aligned} J(x_1; z)J(x_2; w) &\sim k \frac{(x_1-x_2)^2}{(z-w)^2} \\ &+ \frac{1}{z-w} [(x_1-x_2)^2\partial_{x_2} + 2(x_1-x_2)] J(x_2; w), \end{aligned} \tag{3.26}$$

and from here we see that $J(z; x)$ is not an $SL(2, R)$ primary due to the first term. On the other hand, its superpartner,

$$\psi(x; z) = -\psi^+(z) + 2x\psi^3(z) - x^2\psi^-(z), \tag{3.27}$$

satisfies

$$J(x_1; z)\psi(x_2; w) \sim \frac{1}{z-w} [(x_1 - x_2)^2 \partial_{x_2} + 2(x_1 - x_2)] \psi(x_2; w), \quad (3.28)$$

which follows from (3.2). Comparing with (3.25), we see that the field $\psi(x; z)$ is an $\text{SL}(2, R)$ primary with $H = \hat{h} = -1$. It will appear below in the construction of the chiral operators.

As in (3.9), the current $J(x; z)$ can be split into purely bosonic and fermionic terms as

$$J(x; z) = j(x; z) + \hat{j}(x; z), \quad (3.29)$$

where

$$\hat{j}(x; z) = -\hat{j}^+(z) + 2x\hat{j}^3(z) - x^2\hat{j}^-(z), \quad (3.30)$$

and $j(x; z)$ is similarly expressed in terms of j^A .

3.2 The $\text{SU}(2)$ model: currents and observables

The bosonic $\text{SU}(2)_{k-2}$ WZW model has primaries $V_{j,m,\bar{m}}$ with $m, \bar{m} = -j, \dots, +j$, and the spin j is bounded by [49, 50]

$$0 \leq j \leq \frac{k-2}{2}. \quad (3.31)$$

The conformal dimension of $V_{j,m,\bar{m}}$ is

$$\Delta = \bar{\Delta} = \frac{j(j+1)}{k}. \quad (3.32)$$

Similarly to the x, \bar{x} variables of the $\text{SL}(2, R)$ model, isospin coordinates y, \bar{y} can be introduced for $\text{SU}(2)$ [49], and the primaries are summed into

$$V_j(y, \bar{y}) = \sum_{m=-j}^j V_{j,m,\bar{m}} y^{-m+j} \bar{y}^{-\bar{m}+j}. \quad (3.33)$$

The action of the k^a currents on $V_j(y; z)$ is

$$k^a(z)V_j(y; w) \sim -\frac{P_y^a V_j(y; w)}{z-w}, \quad (3.34)$$

where the differential operators

$$P_y^- = -\partial_y, \tag{3.35}$$

$$P_y^3 = y\partial_y - j, \tag{3.36}$$

$$P_y^+ = y^2\partial_y - 2jy, \tag{3.37}$$

are the $SU(2)$ counterparts of D_x^A , and there is a similar antiholomorphic copy. The action of the zero modes of k^a on $V_{j,m,\bar{m}}$ can be read from (3.34) to be

$$k_0^3 V_{j,m,\bar{m}} = m V_{j,m,\bar{m}}, \tag{3.38}$$

$$k_0^\pm V_{j,m,\bar{m}} = (\pm m + 1 + j) V_{j,m\pm 1,\bar{m}} \quad (m \neq \pm j) \tag{3.39}$$

$$k_0^+ V_{j,j,\bar{m}} = k_0^- V_{j,-j,\bar{m}} = 0, \tag{3.40}$$

and similarly for \bar{k}_0^a . There are similar expressions for \hat{k}^a and K^a , and we will denote by \hat{j} , j and J the spins associated to \hat{k}^a , k^a and K^a . Defining now the current

$$K(y; z) = -K^+(z) + 2yK^3(z) + y^2K^-(z), \tag{3.41}$$

the OPEs (3.4) and the K^a version of (3.34) can be expressed as

$$K(y_1; z)K(y_2; w) \sim -k \frac{(y_1 - y_2)^2}{(z - w)^2} + \frac{1}{z - w} [(y_1 - y_2)^2 \partial_{y_2} + 2(y_1 - y_2)] K(y_2; w), \tag{3.42}$$

$$K(y_1; z)V_J(y_2; w) \sim \frac{1}{z - w} [(y_1 - y_2)^2 \partial_{y_2} + 2J(y_1 - y_2)] V_J(y_2; w). \tag{3.43}$$

The superpartner of $K(y)$,

$$\chi(y; z) = -\chi^+(z) + 2y\chi^3(z) + y^2\chi^-(z), \tag{3.44}$$

is an $SU(2)$ primary field of spin $J = \hat{j} = 1$, which satisfies

$$K(y_1; z)\chi(y_2; w) \sim \frac{1}{z - w} [(y_1 - y_2)^2 \partial_{y_2} + 2(y_1 - y_2)] \chi(y_2; w) \tag{3.45}$$

and will appear in the chiral operators below.

Finally, the current $K(y)$ can be split as

$$K(y; z) = k(y; z) + \hat{k}(y; z), \quad (3.46)$$

where

$$\hat{k}(y; z) = -\hat{k}^+(z) + 2yk^3(z) + y^2\hat{k}^-(z), \quad (3.47)$$

and $k(y; z)$ is similarly expressed in terms of k^a .

3.3 Ramond sector

It is convenient to consider the Ramond sector of the $SL(2, R)$ and $SU(2)$ models together. For this, let us bosonize the ψ^A, χ^a fermions as

$$\partial H_1 = \frac{2}{k} \psi^2 \psi^1, \quad (3.48)$$

$$\partial H_2 = \frac{2}{k} \chi^2 \chi^1, \quad (3.49)$$

$$\partial H_3 = \frac{2}{k} i \psi^3 \chi^3. \quad (3.50)$$

We normalize the four fermions of T^4 , η^i , $i = 1, \dots, 4$, as

$$\eta^i(z) \eta^j(w) \sim \frac{\delta^{ij}}{z - w}, \quad (3.51)$$

and they can be bosonized as

$$\partial H_4 = \eta^2 \eta^1, \quad (3.52)$$

$$\partial H_5 = \eta^4 \eta^3. \quad (3.53)$$

All bosons are normalized as

$$H_i(z) H_j(w) \sim -\delta_{ij} \log(z - w). \quad (3.54)$$

In order to get the correct anticommutation among the fermions in their bosonized form, we should also introduce proper cocycles [51]. For that, we first define the number operators

$$N_i = i \oint \partial H_i, \quad (3.55)$$

and then work in terms of bosons redefined as

$$\hat{H}_i = H_i + \pi \sum_{j < i} N_j. \quad (3.56)$$

The fermions are expressed in terms of \hat{H}_i as

$$e^{\pm i\hat{H}_1} = \frac{\psi^1 \pm i\psi^2}{\sqrt{k}}, \quad e^{\pm i\hat{H}_2} = \frac{\chi^1 \pm i\chi^2}{\sqrt{k}}, \quad e^{\pm i\hat{H}_3} = \frac{\chi^3 \mp \psi^3}{\sqrt{k}}, \quad (3.57)$$

$$e^{\pm i\hat{H}_4} = \frac{\eta^1 \pm i\eta^2}{\sqrt{2}}, \quad e^{\pm i\hat{H}_5} = \frac{\eta^3 \pm i\eta^4}{\sqrt{2}}, \quad (3.58)$$

and the cocycles pick the right signs using the relation

$$e^{iaN_j} e^{ibH_j} = e^{ibH_j} e^{iaN_j} e^{iab}, \quad j = 1, \dots, 4. \quad (3.59)$$

The Ramond ground state is created by acting on the vacuum with the spin fields

$$S(z) = e^{i/2 \sum_I \epsilon_I \hat{H}_I}, \quad (3.60)$$

where $\epsilon_I = \pm 1$, and the GSO projection imposes the mutual locality condition

$$\prod_{I=1}^5 \epsilon_I = +1. \quad (3.61)$$

In particular, the spin fields are used to build the spacetime supercharges as

$$Q = \oint dze^{-\phi/2} S(z). \quad (3.62)$$

BRST invariance imposes on the supercharges a constraint which is not present in flat space [10]. Commutation of Q with the BRST charge requires that no $(z-w)^{-3/2}$ singularities appear in the OPE between the supercurrent T_F and S . Let us express T_F in (3.14) as

$$T_F = T_F^\alpha + T_F^\beta + T_F(T^4), \quad (3.63)$$

where

$$T_F^\alpha = \frac{1}{\sqrt{k}} \left[e^{+i\hat{H}_1} j^- + e^{-i\hat{H}_1} j^+ + \left(e^{+i\hat{H}_3} - e^{-i\hat{H}_3} \right) j^3 \right. \\ \left. + e^{+i\hat{H}_2} k^- + e^{-i\hat{H}_2} k^+ + \left(e^{+i\hat{H}_3} + e^{-i\hat{H}_3} \right) k^3 \right] \quad (3.64)$$

and

$$T_F^\beta = \frac{1}{\sqrt{k}} \left[(i\partial\hat{H}_2 - i\partial\hat{H}_1) e^{-i\hat{H}_3} + (i\partial\hat{H}_2 + i\partial\hat{H}_1) e^{+i\hat{H}_3} \right]. \quad (3.65)$$

Using these expressions, it is easy to check that to avoid $(z - w)^{-3/2}$ singularities in the OPE between T_F^β and S , we must impose the constraint

$$\prod_{I=1}^3 \epsilon_I = +1. \tag{3.66}$$

Equations (3.61) and (3.66) imply that $\epsilon_4\epsilon_5 = +1$ in $S(z)$, and leave a total of eight supercharges, which correspond to the eight supercharges of the global $N = 4$ superconformal algebra in the boundary theory [10].

The currents are expressed in terms of the \hat{H}_i bosons as

$$\hat{j}^3 = i\partial\hat{H}_1, \tag{3.67}$$

$$\hat{j}^\pm = \pm e^{\pm i\hat{H}_1} \left(e^{-i\hat{H}_3} - e^{+i\hat{H}_3} \right), \tag{3.68}$$

$$\hat{k}^3 = i\partial\hat{H}_2, \tag{3.69}$$

$$\hat{k}^\pm = \mp e^{\pm i\hat{H}_2} \left(e^{-i\hat{H}_3} + e^{+i\hat{H}_3} \right). \tag{3.70}$$

The spin fields provide two $(\frac{1}{2}, \frac{1}{2})$ representations of the \hat{j}^A, \hat{k}^a currents, with opposite six-dimensional chirality. Defining

$$S_{[\epsilon_1, \epsilon_2, \epsilon_3]} = e^{i(\epsilon_1/2)\hat{H}_1 + i(\epsilon_2/2)\hat{H}_2 + i(\epsilon_3/2)\hat{H}_3}, \tag{3.71}$$

then the $(\frac{1}{2}, \frac{1}{2})$ representation with $\epsilon_1\epsilon_2\epsilon_3 = +1$ is given by

$$|\epsilon_1, \epsilon_2\rangle_+ = (-1)^{(1-\epsilon_1)/2} (i)^{(1-\epsilon_2)/2} S_{[\epsilon_1, \epsilon_2, \epsilon_1\epsilon_2]}|0\rangle \tag{3.72}$$

and that with $\epsilon_1\epsilon_2\epsilon_3 = -1$ is

$$|\epsilon_1, \epsilon_2\rangle_- = (i)^{(1-\epsilon_2)/2} S_{[\epsilon_1, \epsilon_2, -\epsilon_1\epsilon_2]}|0\rangle. \tag{3.73}$$

In both cases the zero modes of the currents act as

$$\hat{j}_0^3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad \hat{j}_0^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{j}_0^- = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \tag{3.74}$$

$$\hat{k}_0^3 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad \hat{k}_0^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \hat{k}_0^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{3.75}$$

and the phases that come from the cocycles in $S_{[\epsilon_1, \epsilon_2, \epsilon_3]}$ are crucial to obtain these results. Given a $(\frac{1}{2}, \frac{1}{2})$ representation, the linear combination

$$S(x, y)|0\rangle = xy| - - \rangle + x| - + \rangle + y| + - \rangle + | + + \rangle \tag{3.76}$$

is a well defined primary in the (x, y) basis for $SL(2, R)$ and $SU(2)$, with $H = \hat{h} = -1/2$ and $J = \hat{j} = 1/2$. For each chirality, the explicit expressions of $S(x, y)$ are

$$S^+(x, y) = -xyiS_{[-\ -+]} - xS_{[-\ +\ -]} + yiS_{[+\ -\ -]} + S_{[+\ +\ +]}, \quad (3.77)$$

$$S^-(x, y) = +xyiS_{[-\ -\ -]} + xS_{[-\ +\ +]} + yiS_{[+\ -\ +]} + S_{[+\ +\ -]}. \quad (3.78)$$

In the table below we summarize the properties of the fields $\psi(x)$, $\chi(y)$ and $S^\pm(x, y)$, defined in (3.27), (3.44) and (3.77) and (3.78). They all belong to the Hilbert space of the free fermions and will play an important role below in the construction of the chiral operators.

Field	\hat{h}	\hat{j}	Sector	Expansion
$\psi(x)$	-1	—	NS	$-\psi^+ + 2x\psi^3 - x^2\psi^-$
$\chi(y)$	—	1	NS	$-\chi^+ + 2y\chi^3 + y^2\chi^-$
$S^\pm(x, y)$	-1/2	1/2	R	$\mp xyiS_{[-\ -\ \pm]} \mp xS_{[-\ +\ \mp]} + yiS_{[+\ -\ \mp]} + S_{[+\ +\ \pm]}$

3.4 Spectrum of chiral operators

Chiral operators belong to $SU(2)$ multiplets which satisfy

$$H = J. \quad (3.79)$$

A chiral (antichiral) operator corresponds to the state with K_0^3 eigenvalue $M = J$ ($M = -J$), but it will be convenient to keep the whole $SU(2)$ multiplet to compute the correlators. The spectrum of physical chiral operators in the worldsheet of the bulk theory was obtained in [18] in the m, \bar{m} basis. In Appendix A, we rederive it in the x, \bar{x} basis, which is more appropriate for the computation of correlation functions.

The result is that all the chiral operators are built from the basic $k - 1$ operators

$$\mathcal{O}_h(x, y) \equiv \Phi_h(x)V_{h-1}(y), \quad h = 1, \frac{3}{2}, \dots, \frac{k}{2}, \quad (3.80)$$

where $\Phi_h(x)$ and $V_{h-1}(y)$ are primaries of the bosonic $SL(2, R)_{k+2}$ and $SU(2)_{k-2}$ models. Note that $\Delta(\mathcal{O}_h(x, y)) = 0$ and that the operators cover the whole range of $k - 1$ values for $j = h - 1$ allowed by the $SU(2)$ bound (3.31).

In the holomorphic sector, there are three families of chiral operators. In the -1 ($-1/2$) picture of the NS (R) sector, they are obtained by multiplying

$\mathcal{O}_h(x, y)$ by any of the operators $e^{-\phi}\psi(x)$, $e^{-\phi}\chi(y)$ or $e^{-\phi/2}s_-^a(x, y)$ ($a = 1, 2$), where

$$s_{\pm}^1(x, y) = S^{\pm}(x, y)e^{+(i/2)(\hat{H}_4 - \hat{H}_5)}, \quad (3.81)$$

$$s_{\pm}^2(x, y) = S^{\pm}(x, y)e^{-(i/2)(\hat{H}_4 - \hat{H}_5)}, \quad (3.82)$$

and ϕ comes from the bosonization of the $\beta - \gamma$ ghosts [52]. We will use a tilde ($\tilde{}$) to denote the representation of the operators in the 0 and $-3/2$ pictures. We summarize the holomorphic spectrum in the following table:⁵

Op.	Pic.	Expansion	$H = J$	NSR	Range of H
\mathcal{O}_h^0	-1	$e^{-\phi}\mathcal{O}_h(x, y)\psi(x)$	$h - 1 = j$	NS	$0, \frac{1}{2}, \dots, \frac{k-2}{2}$
$\tilde{\mathcal{O}}_h^0$	0	$((1-h)\hat{j}(x) + j(x) + \frac{2}{k}\psi(x)\chi_a P_y^a)\mathcal{O}_h(x, y)$			
\mathcal{O}_h^a	$-\frac{1}{2}$	$e^{-\phi/2}\mathcal{O}_h(x, y)s_-^a(x, y)$	$h - \frac{1}{2} = j + \frac{1}{2}$	R	$\frac{1}{2}, 1, \dots, \frac{k-1}{2}$
$\tilde{\mathcal{O}}_h^a$	$-\frac{3}{2}$	$-\sqrt{k}(2h-1)^{-1}e^{-3\phi/2}\mathcal{O}_h(x, y)s_+^a(x, y)$			
\mathcal{O}_h^2	-1	$e^{-\phi}\mathcal{O}_h(x, y)\chi(y)$	$h = j + 1$	NS	$1, \frac{3}{2}, \dots, \frac{k}{2}$
$\tilde{\mathcal{O}}_h^2$	0	$(h\hat{k}(y) + k(y) + \frac{2}{k}\chi(y)\psi_A D_x^A)\mathcal{O}_h(x, y)$			

The antiholomorphic part of the operators is fixed by multiplying by an antiholomorphic field $e^{-\bar{\phi}}\bar{\psi}(\bar{x})$, $e^{-\bar{\phi}}\bar{\chi}(\bar{y})$ or $e^{-\bar{\phi}/2}\bar{s}_{\pm}^a(\bar{x}, \bar{y})$. The full chiral operators have then the form

$$\mathcal{O}_h^{(0,2)} = e^{-\phi-\bar{\phi}}\mathcal{O}_h(x, \bar{x}, y, \bar{y})\psi(x)\bar{\chi}(\bar{y}) \quad (3.83)$$

and so on, giving a total of nine families, whose spectrum and degeneracies can be compared [18] with the KK modes of supergravity computed in [17, 53]. The scalar sector is composed by $\mathcal{O}_h^{(0,0)}$, $\mathcal{O}_h^{(a,\bar{a})}$ and $\mathcal{O}_h^{(2,2)}$.

As was studied in [54], the chiral operators are also chiral with respect to the $N = 2$ superconformal symmetry of the worldsheet.

The labeling of the operators makes explicit the bulk-boundary dictionary we propose in this work. This dictionary is based on matching the lowest conformal dimension in each family, and in the degeneracy in the

⁵The correspondence with the notation in [18] is $\mathcal{W}_{h-1}^- \leftrightarrow \mathcal{O}_h^0$, $\mathcal{X}_{h-1}^+ \leftrightarrow \mathcal{O}_h^2$, $\mathcal{Y}_{h-1}^{\pm} \leftrightarrow \mathcal{O}_h^a$.

indices (a, \bar{a}) , which correspond both in the bulk and the boundary to the elements of $H^{(1,1)}(T^4)$.

Special mention deserves the $h = 1$ operator in the $\mathcal{O}_h^{(0,0)}$ family, which does not seem to have a counterpart in the boundary. In the zero picture it is

$$\tilde{\mathcal{O}}_{h=1}^{(0,0)} = j(x)\bar{j}(\bar{x})\Phi_{h=1}(x, \bar{x}). \quad (3.84)$$

It has conformal dimension zero in the boundary, and appears in the central extension of the boundary symmetries built from the string worldsheet [12]. But it fails to behave as the identity in a correlator, since its insertion in an n -point function does not give the $(n - 1)$ -point function, as we will see below for $n = 3$. Several properties of this operator were studied in [55].

Assuming $Q_5 = k$, the number of operators in each family in the bulk is $Q_5 - 1$, less than the $N = Q_1 Q_5$ operators in the boundary. Even though this will not prevent us from performing a successful comparison of the correlators for those operators present both in the bulk and in the boundary, a few words on this point are in order.

A complete treatment of the $SL(2, R)$ WZW model must include the spectrally flowed representations of $SL(2, R)$ [22], to be considered in [39].⁶ But including them leads to an infinite number of chiral operators in the bulk. A resolution to this problem was proposed in [19, 62]. The idea is that the spectral flow parameter w , which in perturbation theory spans all the integers, should be restricted to $0 \leq w \leq Q_1 - 1$. This “stringy exclusion principle” [9] is not seen in the worldsheet because the six-dimensional string coupling is [10]

$$g_6^2 = \frac{Q_5}{Q_1}, \quad (3.85)$$

so string perturbation theory needs $Q_1 \gg 1$. With this prescription, there are $Q_1(k - 1)$ operators in the bulk and the missing Q_1 operators can be explained away along the lines of [61].⁷

Given this split between Q_1 and Q_5 for the quantum numbers in the string side, it was further suggested in [19] that the boundary theory is more

⁶These representations are necessary in order to obtain a modular invariant partition function [56, 57] (see also [58, 59]), to ensure that the spacetime energy does not have an unphysical upper bound, and to properly account for the states corresponding to the long string excitations [60, 61].

⁷It was argued in [63] that in the plane wave limit that the missing chiral operators appear in the continuous representations of $SL(2, R)$. Notice that if we would identify $Q_5 = k - 1$ there would be no operators missing. This shift is allowed in the large Q_5 limit needed for supergravity to be valid.

naturally identified as a deformation of the iterated symmetric product

$$\text{Sym}^{Q_1}(\text{Sym}^{Q_5}(M^4)). \tag{3.86}$$

In this setting, the computations of this paper for the $w = 0$ sector in the bulk would correspond to the identity sector in Sym^{Q_1} .

4 Three-point correlation functions

4.1 The basic cancelation

Since the basic building block of all the chiral operators is the field $\mathcal{O}_h(x, y) = \Phi_h(x)V_{h-1}(y)$, any three-point correlator among them will involve the value of

$$\langle \mathcal{O}_{h_1}(x_1, y_1)\mathcal{O}_{h_2}(x_2, y_2)\mathcal{O}_{h_3}(x_3, y_3) \rangle, \tag{4.1}$$

which is the product of

$$\begin{aligned} &\langle \Phi_{h_1}(x_1, \bar{x}_1)\Phi_{h_2}(x_2, \bar{x}_2)\Phi_{h_3}(x_3, \bar{x}_3) \rangle \\ &= \frac{C_H(h_1, h_2, h_3)}{|x_{12}|^{2h_1+2h_2-2h_3}|x_{23}|^{2h_2+2h_3-2h_1}|x_{31}|^{2h_3+2h_1-2h_2}} \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} \langle V_{j_1}(y_1, \bar{y}_1)V_{j_2}(y_2, \bar{y}_2)V_{j_3}(y_3, \bar{y}_3) \rangle &= C_S(j_1, j_2, j_3)|y_{12}|^{2j_1+2j_2-2j_3} \\ &\times |y_{23}|^{2j_2+2j_3-2j_1}|y_{31}|^{2j_3+2j_1-2j_2} \end{aligned} \tag{4.3}$$

evaluated at

$$j_i = h_i - 1 \quad (i = 1, 2, 3). \tag{4.4}$$

Equation (4.1) has no dependence on the z_i 's because $\Delta(\mathcal{O}_h) = 0$, and for (4.2) and (4.3) the dependence on the z_i 's is standard and we have omitted it.

The expressions $C_H(h_1, h_2, h_3)$ and $C_S(j_1, j_2, j_3)$ are the three-point functions of the H_3^+ and $SU(2)$ WZW models at levels $k + 2$ and $k - 2$, respectively. For the $SU(2)$ case, they are [49]

$$C_S(j_1, j_2, j_3) = N_{j_1, j_2, j_3} \sqrt{\gamma(b^2)} P(j + 1) \prod_{i=1}^3 \frac{P(j - 2j_i)}{P(2j_i) \sqrt{\gamma((2j_i + 1)b^2)}}, \tag{4.5}$$

where

$$j = j_1 + j_2 + j_3, \tag{4.6}$$

$$b = 1/\sqrt{k}, \tag{4.7}$$

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}. \tag{4.8}$$

The function $P(s)$ is defined for s a non-negative integer as

$$P(s) = \prod_{n=1}^s \gamma(nb^2), \quad P(0) = 1, \tag{4.9}$$

and the coefficients N_{j_1, j_2, j_3} are the $SU(2)_{k-2}$ fusion rules:

$$N_{j_1, j_2, j_3} = \begin{cases} 1 & \text{for } k-2 \geq j_1 + j_2 + j_3 \geq \max(2j_1, 2j_2, 2j_3), \\ & \text{and } j_1 + j_2 + j_3 = 0 \pmod{2}, \\ 0 & \text{otherwise.} \end{cases} \tag{4.10}$$

For the H_3^+ model, the three-point functions are [26, 64]⁸ (see also [65–71])

$$C_H(h_1, h_2, h_3) = -\frac{b^{1+2b^2} \Upsilon(b)}{2\pi^2 \gamma(1+b^2)} [\nu b^{2b^2}]^{-h+1} \frac{1}{\Upsilon(b(h-1))} \prod_{i=1}^3 \frac{\Upsilon(2bh_i - b)}{\Upsilon(b(h-2h_i))}, \tag{4.11}$$

where

$$h = h_1 + h_2 + h_3, \tag{4.12}$$

and the function Υ , introduced in [72], is related to the Barnes double gamma function and can be defined by

$$\log \Upsilon(x) = \int_0^\infty \frac{dt}{t} \left[\left(\frac{Q}{2} - x \right)^2 e^{-t} - \frac{\sinh^2\left(\left(\frac{Q}{2} - x\right)\frac{t}{2}\right)}{\sinh \frac{bt}{2} \sinh \frac{t}{2b}} \right]. \tag{4.13}$$

The integral converges in the strip $0 < \text{Re}(x) < Q$. Outside this range it is defined by the relations

$$\Upsilon(x+b) = b^{1-2bx} \gamma(bx) \Upsilon(x), \quad \Upsilon(x+1/b) = b^{-1+2x/b} \gamma(x/b) \Upsilon(x). \tag{4.14}$$

⁸We use the normalization of [64].

The two-point functions for delta-normalizable states (with $h = \frac{1}{2} + i\rho, \rho \in \mathbb{R}$) can be obtained by taking one of the operators to be the identity and gives

$$\begin{aligned} & \langle \Phi_{h=1/2+i\rho}(x_1) \Phi_{h'=1/2+i\rho'}(x_2) \rangle \\ &= \lim_{\epsilon \rightarrow 0} \langle \Phi_{h=1/2+i\rho}(x_1) \Phi_{h'=1/2+i\rho'}(x_2) \Phi_{\epsilon}(x_3) \rangle \end{aligned} \quad (4.15)$$

$$= \frac{1}{|z_{12}|^{4\Delta_h}} \left(\delta^{(2)}(x_1 - x_2) \delta(\rho + \rho') + \frac{B(h)}{|x_{12}|^{4h}} \delta(\rho - \rho') \right), \quad (4.16)$$

where

$$B(h) = -\frac{\nu^{-2h+1}}{\pi b^2} \gamma(1 - b^2(2h - 1)). \quad (4.17)$$

To obtain (4.16) from (4.11) one should use that

$$\Upsilon(\epsilon) \sim \epsilon \Upsilon(b), \quad (4.18)$$

which follows from (4.14), and the distributional limits

$$\lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\rho^2 + \epsilon^2} = \pi \delta(\rho), \quad (4.19)$$

$$\lim_{\epsilon \rightarrow 0} |x|^{-2+2\epsilon} = \pi \frac{\delta^{(2)}(x)}{\epsilon}. \quad (4.20)$$

After taking the limit (4.15), the resulting expression can be analytically continued to non-normalizable states. The overall constant in C_H (4.11) is not determined by the functional equations of the conformal bootstrap (see Appendix B), and is fixed by requiring the coefficient of the first term in (4.16) to be 1.

The parameter ν is free in the H_3^+ WZW model. It is not a parameter of the action, which only depends on the level k , but rather of the vertex operators, and in the conformal bootstrap is left undetermined by the crossing symmetry (see Appendix B). In the interacting theory, vertex operators are the sum of an incoming and an outgoing wave, and ν appears in the reflection coefficient between these two terms.⁹ In [64] it was proposed to

⁹In the free field realization of the theory, ν does appear in the action multiplying the interaction term [10], because the latter is used to transform the vertex operators of the free theory into those of the interacting theory, via insertion of screening charges in the correlators. The value of ν is also related to the dilaton in AdS₃ [55].

fix ν by demanding that the constant

$$c_\nu = \frac{\pi\Gamma(1 - b^2)}{\nu\Gamma(1 + b^2)}, \tag{4.21}$$

which appears in the OPE

$$\Phi_1(x_1)\Phi_h(x_2) \sim c_\nu\delta^{(2)}(x_1 - x_2)\Phi_h(x_2), \tag{4.22}$$

be set to $c_\nu = 1$. We leave ν undetermined for the moment, and it will be fixed below holographically by comparing the bulk and the boundary correlators.

We will evaluate the expression (4.11) for C_H at values of h_i such that $2h_i$ and h are non-negative integers. For these values, equation (4.11) can be expressed in terms of $P(s)$, defined in (4.9), by means of the identity

$$P(s) = \frac{\Upsilon(sb + b)}{\Upsilon(b)} b^{s((s+1)b^2-1)}, \tag{4.23}$$

which is easily verified by iterating s times the first equation in (4.14). We get thus

$$C_H(h_1, h_2, h_3) = -\frac{\nu^{-h+1}}{2\pi^2\gamma(1 + b^2)} \frac{1}{P(h - 2)} \prod_{i=1}^3 \frac{P(2h_i - 2)}{P(h - 2h_i - 1)}. \tag{4.24}$$

We are interested in the product

$$C(h_1, h_2, h_3) \equiv C_H(h_1, h_2, h_3)C_S(h_1 - 1, h_2 - 1, h_3 - 1), \tag{4.25}$$

which, from (4.5) and (4.24) is equal to

$$C(h_1, h_2, h_3) = N_{h_1-1, h_2-1, h_3-1} \frac{\nu^{-h+1}}{2\pi^2 b^4 \sqrt{\gamma(b^2)}} \prod_{i=1}^3 \frac{1}{\sqrt{\gamma(b^2(2h_i - 1))}}. \tag{4.26}$$

This expression has the remarkable property that the four $P(s)$'s, in C_S and C_H that depend on more than one of the h_i 's have canceled against each other. In particular, the poles of C_H that appear at particular linear combinations of several h_i 's, whose physical meaning was analyzed in [73], have disappeared. This cancelation of the structure constants follows from the close relationship between $SU(2)$ and H_3^+ structure constants, which we explore further in Appendix B. Note that the remaining h_i -dependent

factors can be absorbed by rescaling the \mathcal{O}_{h_i} operators, as we will do below when normalizing the chiral operators.

Finally, note that $\langle \mathcal{O}_{h_1} \mathcal{O}_{h_2} \mathcal{O}_{h_3} \rangle$ is independent of z_i, \bar{z}_i , again due to a cancellation between the dependence on z_i, \bar{z}_i of the two factors (4.2) and (4.3).

4.2 Three-point correlators of chiral multiplets

Now that we have the correlation function of three $\mathcal{O}_h(x, y)$'s, in order to compute the three-point function of the chiral $SU(2)$ multiplet we only need to add factors involving the fermions, spin fields and current algebra descendants. To illustrate the steps involved, let us compute in detail the following correlator

$$\langle c\bar{c}\mathcal{O}_{h_1}^{(0,0)} c\bar{c}\mathcal{O}_{h_2}^{(0,0)} c\bar{c}\tilde{\mathcal{O}}_{h_3}^{(0,0)} \rangle, \quad (4.27)$$

where the pictures have been chosen so that the total picture number adds up to -2 . This correlator will be proportional to $C(h_1, h_2, h_3)$, but we are interested in the precise prefactor. The last term in (A.15) of the zero picture vertex $\tilde{\mathcal{O}}_{h_3}^{(0,0)}$ can be discarded in this particular correlator since $\langle \chi_a \rangle = \langle \bar{\chi}_a \rangle = 0$. For the computation we need to use

$$\langle \psi(x_1; z_1) \psi(x_2; z_2) \rangle = k \frac{(x_{12})^2}{z_{12}}, \quad (4.28)$$

and

$$\langle \psi(x_1; z_1) \psi(x_2; z_2) \hat{j}(x_3; z_3) \rangle = -2k \frac{x_{12} x_{23} x_{31}}{z_{13} z_{23}}, \quad (4.29)$$

which follows from

$$\langle \psi^A(z_1) \psi^B(z_2) \hat{j}^C(z_3) \rangle = \frac{i \frac{k}{2} \epsilon^{ABC}}{z_{13} z_{23}}. \quad (4.30)$$

We also need the value of $\langle \mathcal{O}_{h_1} \mathcal{O}_{h_2} j(x_3) \mathcal{O}_{h_3} \rangle$, where normal order implies

$$j(x_3) \mathcal{O}_{h_3}(x_3) = (-j_{-1}^+ + 2x_3 j_{-1}^3 - x_3^2 j_{-1}^-) \mathcal{O}_{h_3}(x_3). \quad (4.31)$$

The correlation functions of these current algebra descendants can be expressed in terms of correlators of the primaries by combining the Ward

identity

$$\langle j^A(w)\Phi_{h_1}(x_1; z_1) \cdots \Phi_{h_3}(x_3; z_3) \rangle = - \sum_{i=1}^3 \frac{D_{x_i}^A}{w - z_i} \langle \Phi_{h_1}(x_1; z_1) \cdots \Phi_{h_3}(x_3; z_3) \rangle \tag{4.32}$$

with the OPE

$$j^A(w)\Phi_{h_3}(x_3; z_3) \sim - \frac{D_{x_3}^a}{w - z_3} \Phi_{h_3}(x_3; z_3) + j_{-1}^A \Phi_{h_3}(x_3; z_3) + j_{-2}^A \Phi_{h_3}(x_3; z_3)(w - z_3) + \cdots \tag{4.33}$$

Expanding the $i = 1, 2$ denominators in (4.32) as $(w - z_i)^{-1} = (z_3 - z_i)^{-1} \sum_{n=0}^{\infty} \left(\frac{w-z_3}{z_i-z_3}\right)^n$, equations (4.32) and (4.33) give

$$\langle \mathcal{O}_{h_1}(x_1)\mathcal{O}_{h_2}(x_2)j_{-1}^A\mathcal{O}_{h_3}(x_3) \rangle = \left(\frac{D_{x_1}^A}{z_{13}} + \frac{D_{x_2}^A}{z_{23}} \right) \langle \mathcal{O}_{h_1}(x_1)\mathcal{O}_{h_2}(x_2)\mathcal{O}_{h_3}(x_3) \rangle, \tag{4.34}$$

and using now the dependence of $\langle \mathcal{O}_{h_1}\mathcal{O}_{h_2}\mathcal{O}_{h_3} \rangle$ on x_i given by (4.2), we obtain

$$\langle \mathcal{O}_{h_1}\mathcal{O}_{h_2}j(x_3)\mathcal{O}_{h_3} \rangle = (-h_1 - h_2 + h_3) \frac{z_{12}}{z_{31}z_{23}} \frac{x_{23}x_{31}}{x_{12}} \langle \mathcal{O}_{h_1}\mathcal{O}_{h_2}\mathcal{O}_{h_3} \rangle. \tag{4.35}$$

Note that although $\langle \mathcal{O}_{h_1}\mathcal{O}_{h_2}\mathcal{O}_{h_3} \rangle$ is independent of z_i, \bar{z}_i , the above expression does depend on z_i . Using

$$\langle c(z_1)c(z_2)c(z_3) \rangle = z_{12}z_{23}z_{31}, \tag{4.36}$$

collecting all the terms, and including the antiholomorphic factors, we get finally

$$\begin{aligned} \langle c\bar{c}\mathcal{O}_{h_1}^{(0,0)}c\bar{c}\mathcal{O}_{h_2}^{(0,0)}c\bar{c}\tilde{\mathcal{O}}_{h_3}^{(0,0)} \rangle &= g_s v_4 k^2 (h_1 + h_2 + h_3 - 2)^2 C(h_1, h_2, h_3) \\ &\times \left| \frac{y_{12}}{x_{12}} \right|^{2H_1+2H_2-2H_3} \left| \frac{y_{23}}{x_{23}} \right|^{2H_2+2H_3-2H_1} \\ &\times \left| \frac{y_{31}}{x_{31}} \right|^{2H_3+2H_1-2H_2} \end{aligned} \tag{4.37}$$

where we have included the correct power of $g_s = g_s^{-2+3}$ and the volume v_4 of the T^4 . The other correlators are computed similarly, and in all the cases holomorphic and antiholomorphic contributions factorize. Labeling

the operators with $\alpha, \bar{\alpha} = 0, a, 2$, all the correlators have the form (we omit the dependence on x_i, y_i , which is the same as above)

$$\begin{aligned} & \langle c\bar{c}\mathcal{O}_{h_1}^{(\alpha_1, \bar{\alpha}_1)} c\bar{c}\mathcal{O}_{h_2}^{(\alpha_2, \bar{\alpha}_2)} c\bar{c}\tilde{\mathcal{O}}_{h_3}^{(\alpha_3, \bar{\alpha}_3)} \rangle \\ & = g_s v_4 C(h_1, h_2, h_3) g(h_i; \alpha_1, \alpha_2, \alpha_3) g(h_i; \bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) \end{aligned} \quad (4.38)$$

with

$$g(h_i; 0, 0, 0) = k(-h_1 - h_2 - h_3 + 2), \quad (4.39)$$

$$g(h_i; 0, 0, 2) = k(-h_1 - h_2 + h_3 + 1), \quad (4.40)$$

$$g(h_i; 0, 2, 2) = k(-h_1 + h_2 + h_3), \quad (4.41)$$

$$g(h_i; 2, 2, 2) = k(h_1 + h_2 + h_3 - 1), \quad (4.42)$$

$$g(h_i; 0, a, b) = \sqrt{k} \xi^{ab}, \quad (4.43)$$

$$g(h_i; 2, a, b) = \sqrt{k} \xi^{ab}, \quad (4.44)$$

and the results are independent of the picture chosen for the operators, as long as the total picture number is -2 .

4.3 Two-point functions

In order to compare the three-point functions of the bulk to those of the symmetric product orbifold, operators of both sides should be equally normalized. In the symmetric product, the normalization is given by (2.34) and (2.35). To compute a two-point function in the string theory side, when fixing two vertex operators in the sphere there is a zero coming from dividing by the volume of the conformal group. This zero is canceled against the divergence of the delta $\delta(h_1 - h_2)$ in the two-point function in the H_3^+ WZW model (4.16), which can be interpreted as the volume of the Killing group in the target space which leaves invariant the positions x_1, x_2 of the two operators [12]. As we review now, the finite result of this cancellation is h -dependent.

The string two-point function can be obtained, following [73] (see also [23]), by exploiting the Ward identity for affine currents in the boundary CFT. Given an holomorphic affine current $K^a(z)$ in the inner CFT of an AdS_3 compactification, it was shown in [12] that the vertex operator

$$K^a(x; z) = -\frac{1}{k c_\nu} K^a(z) \bar{j}(\bar{x}) \Phi_1(x, \bar{x}) \quad (4.45)$$

is the corresponding holomorphic affine current in the dual CFT, with c_ν defined in (4.21) and (4.22). Note that it has the correct conformal dimensions $(\Delta, \bar{\Delta}) = (1, 1)$ in the bulk, and $(H, \bar{H}) = (1, 0)$ in the boundary.

The Ward identity for the above current in the boundary CFT is

$$\langle c\bar{c}K^a(x)\Phi_h(x_1)P_1\Phi_h(x_2)P_2 \rangle = \left(\frac{q_1}{x-x_1} + \frac{q_2}{x-x_2} \right) \langle \Phi_h(x_1)P_1\Phi_h(x_2)P_2 \rangle, \quad (4.46)$$

where the $P_{1,2}$ stand for the ghosts, fermions and operators of the internal theory, and $q_1 = -q_2$ are the charges of $P_{1,2}$ under $K^a(z)$. Note that the expression for $K^a(x; z)$ in (4.45) is in the zero picture, so that on both sides of (4.46) the operators $\Phi_h(x_i)P_i (i = 1, 2)$ are in the same picture. The lhs of (4.46) can be computed as we did in the previous subsection (see equation (4.35)). Comparing the resulting expression with the rhs yields the string theory two-point function

$$\langle \Phi_h(x_1)P_1\Phi_h(x_2)P_2 \rangle = -\frac{1}{k c_\nu} \frac{(2h-1)C_H(1, h, h)p_{12}}{|x_{12}|^{4h}}, \quad (4.47)$$

where p_{12} is

$$p_{12} = \langle c(\infty)\bar{c}(\infty)P_1(1)P_2(0) \rangle. \quad (4.48)$$

There are no powers of g_s for the two-point functions. Note that

$$C(1, h, h) = \frac{b^2\gamma(-b^2)}{2\pi\nu} B(h), \quad (4.49)$$

so the difference between the string theory two-point function (4.47) and that of the H_3^+ WZW model (4.16) is the factor $(2h-1)$, plus h -independent factors.

Choosing the chiral operators so that the total picture is -2 , we get, for the scalar sector,

$$\langle c\bar{c}\mathcal{O}_h^{(0,0)}c\bar{c}\mathcal{O}_h^{(0,0)} \rangle = \langle c\bar{c}\mathcal{O}_h^{(2,2)}c\bar{c}\mathcal{O}_h^{(2,2)} \rangle = -c_\nu^{-1}k(2h-1)C(1, h, h)v_4 \left| \frac{y_{12}}{x_{12}} \right|^{4H}, \quad (4.50)$$

$$\langle c\bar{c}\tilde{\mathcal{O}}_h^{(a,\bar{a})}c\bar{c}\mathcal{O}_h^{(b,\bar{b})} \rangle = -c_\nu^{-1}(2h-1)^{-1}C(1, h, h)v_4 \xi^{ab}\xi^{\bar{a}\bar{b}} \left| \frac{y_{12}}{x_{12}} \right|^{4H}, \quad (4.51)$$

where we used $C_H(1, h, h) = C(1, h, h)$. In the last line we have taken into account the prefactor carried by the operators in the $-3/2$ picture (see (A.28)).

The (c, c) elements of the $N = 2$ chiral ring are those operators with $M = \bar{M} = J$, where M, \bar{M} are the eigenvalues of K_0^3, \bar{K}_0^3 . Similarly, the (a, a) operators correspond to $M = \bar{M} = -J$. Therefore, we define the normalized (c, c) operators as (see the expansion (3.33))

$$\mathbb{O}_h^{(0,0)} = c\bar{c}\mathcal{O}_h^{(0,0)}(y = \bar{y} = 0) [-c_\nu^{-1}k(2h - 1)C(1, h, h)v_4]^{-1/2}, \tag{4.52}$$

$$\mathbb{O}_h^{(2,2)} = c\bar{c}\mathcal{O}_h^{(2,2)}(y = \bar{y} = 0) [-c_\nu^{-1}k(2h - 1)C(1, h, h)v_4]^{-1/2}, \tag{4.53}$$

$$\mathbb{O}_h^{(a,a)} = c\bar{c}\mathcal{O}_h^{(a,a)}(y = \bar{y} = 0) [-c_\nu^{-1}(2h - 1)^{-1}C(1, h, h)v_4]^{-1/2}, \tag{4.54}$$

and the (a, a) operators as

$$\mathbb{O}_h^{(0,0)\dagger} = \lim_{y, \bar{y} \rightarrow \infty} |y|^{-4H} c\bar{c}\mathcal{O}_h^{(0,0)}(y, \bar{y}) [-c_\nu^{-1}k(2h - 1)C(1, h, h)v_4]^{-1/2}, \tag{4.55}$$

and similarly for $\mathbb{O}_h^{(2,2)\dagger}$ and $\mathbb{O}_h^{(a,a)\dagger}$. Note that we have included the ghosts $c\bar{c}$ in the definition. These operators are thus normalized as

$$\langle \mathbb{O}_h^{(0,0)\dagger} \mathbb{O}_h^{(0,0)} \rangle = \langle \mathbb{O}_h^{(2,2)\dagger} \mathbb{O}_h^{(2,2)} \rangle = \frac{1}{|x_{12}|^{4H}} \tag{4.56}$$

$$\langle \tilde{\mathbb{O}}_h^{(a,\bar{a})\dagger} \mathbb{O}_h^{(b,\bar{b})} \rangle = \frac{\delta^{a\bar{b}}\delta^{\bar{a}b}}{|x_{12}|^{4H}} \tag{4.57}$$

One can define similarly normalized operators for the non-scalar sector.

4.4 Fusion rules

Before computing the structure constants, let us see how the boundary fusion rules (2.37) are obtained in the bulk. The chiral (antichiral) operator in each $SU(2)$ multiplet corresponds to $M = J$ ($M = -J$). Therefore, conservation of the $U(1)$ R-charge, measured with K_0^3 , implies that the fusion

$$\mathcal{O}_{h_1}^* \times \mathcal{O}_{h_2}^* = [\mathcal{O}_{h_3}^*] \tag{4.58}$$

is possible only if

$$J_3 = J_1 + J_2. \tag{4.59}$$

On the other hand,

$$J_i = j_i + \hat{j}_i, \tag{4.60}$$

where $\hat{j}_i = 0, 1/2, 1$ for $\mathcal{O}^0, \mathcal{O}^a, \mathcal{O}^2$, respectively. Now, from the fusion rules (4.10) it follows that

$$\hat{j}_3 \leq \hat{j}_1 + \hat{j}_2, \tag{4.61}$$

and therefore (4.59) implies¹⁰

$$\hat{j}_3 \geq \hat{j}_1 + \hat{j}_2. \quad (4.62)$$

The equality corresponds to the four cases

$$\begin{aligned} (0) \times (0) &= (0), \\ (0) \times (2) &= (2), \\ (0) \times (a) &= (a), \\ (a) \times (a) &= (2), \end{aligned} \quad (4.63)$$

and the inequality to the case

$$(0) \times (0) = (2). \quad (4.64)$$

Note that operators of the Ramond sector appear in pairs. For the antiholomorphic quantum numbers we get similarly

$$\bar{j}_3 \geq \bar{j}_1 + \bar{j}_2. \quad (4.65)$$

For chiral operators we should combine both rules (4.62) and (4.65), taking into account that the j_i quantum numbers are the same for holomorphic and antiholomorphic part. This implies that the four cases (4.63) can be freely combined between holomorphic and antiholomorphic, and the case (4.64) should be the same in the holomorphic and antiholomorphic sectors. These are precisely the same fusion rules as in the boundary theory, associated to the same $1 + 4 \times 4 = 17$ processes.

4.5 Structure constants

We have already all the elements to compute the structure constants of the chiral ring. Consider first $\langle \mathbb{O}_{h_3}^{(0,0)\dagger} \mathbb{O}_{h_2}^{(0,0)} \tilde{\mathbb{O}}_{h_1}^{(0,0)} \rangle$. From $H_3 = H_1 + H_2$ we get $h_3 = h_1 + h_2 - 1$. Plugging this into (4.37) and using (4.26), (4.52) and (4.55) give

$$\langle \mathbb{O}_{h_3}^{(0,0)\dagger} \mathbb{O}_{h_2}^{(0,0)} \tilde{\mathbb{O}}_{h_1}^{(0,0)} \rangle = \frac{g_s}{\sqrt{v_4}} \left(\frac{2\pi}{\nu\gamma(1+b^2)} \right)^{1/2} \left[\frac{(2h_3-1)^3}{(2h_1-1)(2h_2-1)} \right]^{1/2} \quad (4.66)$$

¹⁰The inequality (4.62) does not mean that there is a violation of the rules of $SU(2)$ tensor product for the \hat{k}^a algebra. Since the operators appear in different pictures, the label \hat{j}_i here only denotes the family to which an operator belongs, but not necessarily its spin under \hat{k}^a .

where we have fixed $x_1 = \bar{x}_1 = 0, x_2 = \bar{x}_2 = 1, x_3 = \bar{x}_3 = \infty$. In order to get the prefactor $N^{-1/2}$ to agree with (2.44), we use that

$$\frac{1}{\sqrt{N}} = \frac{1}{\sqrt{Q_1 Q_5}} = \sqrt{\frac{Q_5}{Q_1} \frac{1}{Q_5}} = g_6 \frac{1}{Q_5} = \frac{g_s}{\sqrt{v_4}} \frac{1}{Q_5}, \tag{4.67}$$

and this fixes the value of ν , which was the only free parameter in the H_3^+ WZW model, as

$$\nu = \frac{2\pi}{b^{4\gamma}(1+b^2)}. \tag{4.68}$$

We are considering fixed Q_5 but large Q_1 so that the string coupling is small and N is large. With this value for ν , the other four scalar correlators are computed similarly and are

$$\langle \mathbb{O}_{h_3}^{(2,2)\dagger} \mathbb{O}_{h_2}^{(0,0)} \tilde{\mathbb{O}}_{h_1}^{(0,0)} \rangle = \left(\frac{1}{N}\right)^{1/2} \left[\frac{1}{(2h_1-1)(2h_2-1)(2h_3-1)} \right]^{1/2}, \tag{4.69}$$

$$\langle \mathbb{O}_{h_3}^{(2,2)\dagger} \mathbb{O}_{h_2}^{(0,0)} \tilde{\mathbb{O}}_{h_1}^{(2,2)} \rangle = \left(\frac{1}{N}\right)^{1/2} \left[\frac{(2h_1-1)^3}{(2h_2-1)(2h_3-1)} \right]^{1/2}, \tag{4.70}$$

$$\langle \mathbb{O}_{h_3}^{(a,\bar{a})\dagger} \mathbb{O}_{h_2}^{(0,0)} \mathbb{O}_{h_1}^{(b,\bar{b})} \rangle = \left(\frac{1}{N}\right)^{1/2} \left[\frac{(2h_1-1)(2h_3-1)}{(2h_2-1)} \right]^{1/2} \delta^{ab} \delta^{\bar{a}\bar{b}}, \tag{4.71}$$

$$\langle \mathbb{O}_{h_3}^{(2,2)\dagger} \mathbb{O}_{h_2}^{(a,\bar{a})} \mathbb{O}_{h_1}^{(b,\bar{b})} \rangle = \left(\frac{1}{N}\right)^{1/2} \left[\frac{(2h_1-1)(2h_2-1)}{(2h_3-1)} \right]^{1/2} \xi^{ab} \xi^{\bar{a}\bar{b}}, \tag{4.72}$$

where in all the cases h_3 is obtained from $h_{1,2}$ using $H_3 = H_1 + H_2$. All correlators coincide precisely with the boundary results (2.44) to (2.48). It is immediate to extend the agreement to the 12 correlators of the non-scalar sector, using the factorization of the structure constants into holomorphic and antiholomorphic contributions, which holds both in the bulk and in the boundary. Much like in the boundary, it is easy to see that the (c, a) ring in the bulk has the same fusion rules and structure constants as the (c, c) ring.

Contrary to the boundary correlators (2.44) to (2.47), the above bulk correlators are defined for $\mathbb{O}_{h_2=1}^{(0,0)}$, but, as we mentioned above, the three-point functions do not reduce to the two-point functions as would be for an identity operator.

Given the different normalizations of the \mathbb{O}_h operators and the different powers of k in (4.39) to (4.44), it is remarkable that the definition (4.68) gives the correct prefactor in all the cases.

Note that the volume v_4 of the inner T^4 disappears from the correlators and from ν , and we could have used g_6 from the beginning. This is consistent

with the supergravity result that in the frame with NS–NS flux, the value of v_4 is an arbitrary number not related to the value of T^4 in the boundary or to other parameters of the theory.

5 Discussion

The remarkable agreement between bulk and boundary quantities computed at different points in moduli space begs for an explanation. A similar agreement between the three-point correlators of $N = 4$ super Yang–Mills theory and the dual $Ad\mathcal{S}_5 \times S^5$ case [20, 21] was explained by showing that two- and three-point functions do not receive g_{YM}^2 corrections [74–80]. We conjecture that a similar non-renormalization exists in our case, which should be further investigated.

Another possibility, if we accept the model of the iterated structure (3.86), is that the Z_2 twists which deform the orbifold [81] to the point where the bulk string theory has NS–NS flux only are such that they only mix different Q_1 copies but do not mix the Q_5 copies, and therefore the deformation is not seen when considering unflowed $SL(2, R)$ representations in the bulk. To settle this question definitively more work is needed. In particular, it would be useful to compute the correlators of spectrally flowed operators in the bulk as well as the correlators in the iterated symmetric product in the boundary. These computations would hopefully provide enough additional information to arrive at the correct interpretation. We plan to return to these questions in [39].

In other $Ad\mathcal{S}_{n+1}/CFT_n$ backgrounds with RR fields, bulk computations are mostly limited to supergravity and it is not clear how to even begin the computation of loop amplitudes. An important advantage of the $Ad\mathcal{S}_3/CFT_2$ background with NS–NS flux considered here is that one has an exact worldsheet description available for the bulk string theory. It is natural to ask if the striking agreement found at tree level extends to higher loops. Fortunately, exact answers for finite N are available in the boundary. In the bulk, quantum corrections to three-point correlators can in principle be computed systematically by evaluating higher genus string amplitudes. It would be very interesting to see if the technical tools can be developed sufficiently to carry out such a comparison. We hope our results and further investigations will lead to a better understanding of the chiral sector of the theory for finite N and also to more stringent tests of the gauge-string duality.

The cubic couplings of chiral primaries in this background have been studied in the supergravity limit in [82–85], but no agreement was found

with the boundary results. We believe our results might help to better understand those computations.

More generally, it has been pointed out in the past that some aspects of holography in this background follow a paradigm close to the matrix models duals of non-critical strings [31]. The cancelations between the three-point functions, similar to the minimal strings case, strengthen this idea. Note that in our case the holographic correspondence does not involve legpole factors. Moreover, a ground ring exists also in our background [54],¹¹ and much like for non-critical strings, it might lead to an integrable structure shared by the two holographic descriptions.

Note: Upon completion of this work, we learnt of the preprint [86], where one of the five families of correlators discussed here has been computed independently.

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A Derivation of the chiral spectrum in the bulk

In this appendix we will compute the spectrum of chiral operators of the bulk theory [18] in the x, \bar{x} basis. Chiral operators belong to $SU(2)$ multiplets

¹¹The existence of a ground ring in AdS_3 backgrounds is not in contradiction with the vanishing theorem shown to hold in [46, 47], which states that the cohomology of strings in AdS_3 backgrounds is concentrated at ghost number one (except for a few states at ghost number zero). The reason is that the vanishing theorem assumes that the $SL(2, R)$ spin satisfies the bound (3.23), and the ground ring elements violate this bound.

satisfying

$$H = J. \tag{A.1}$$

From (3.9) and (3.10), we see that the representations of J^A, K^a arise as tensor products of those of the bosonic currents j^A, k^a and the fermionic currents \hat{j}^A, \hat{k}^a . The latter have representations of spins $\hat{h}, \hat{j} = 0, 1$ in the NS sector, and $\hat{h} = \hat{j} = 1/2$ in the R sector.

Since $2J \in \mathbb{Z}^+$, it follows from (A.1) that the $SL(2, R)$ factor of a chiral operator will belong to a discrete representation. States in the Hilbert space of the bosonic $SL(2, R)$ and $SU(2)$ WZW models should then satisfy the bounds (3.23) and (3.31),

$$\frac{1}{2} < h < \frac{k+1}{2}, \tag{A.2}$$

$$0 \leq j \leq \frac{k-2}{2}. \tag{A.3}$$

A physical operator should also be BRST invariant and survive the GSO projection. We will consider first the holomorphic spectrum, and afterwards we will discuss its tensoring with the antiholomorphic sector.

A.1 Neveu–Schwarz sector

In this sector, commutation with the BRST charge implies two conditions. Firstly, the vertex operator must be a Virasoro primary satisfying the mass shell condition. In the -1 picture this is

$$\Delta = -\frac{h(h-1)}{k} + \frac{j(j+1)}{k} + \frac{\hat{h}(\hat{h}+1)}{4} + \frac{\hat{j}(\hat{j}+1)}{4} + \Delta_T + N = \frac{1}{2}, \tag{A.4}$$

where $\Delta_T \geq 0$ corresponds to a primary of T^4 or $K3$ that may appear in the vertex operator, and N is the level of possible excited states. Secondly, there should be no double poles in the OPE between the vertex operator and the supercurrent T_F . Let us consider the four different $\hat{h}, \hat{j} = 0, 1$ cases:

1. $\hat{h} = \hat{j} = 0$

In this case $H = h = j = J$, and the mass shell condition is

$$\Delta = \frac{2h}{k} + \Delta_T + N = \frac{1}{2}. \tag{A.5}$$

Since there are no fermions from $SL(2, R)$ or $SU(2)$, in order to survive the GSO projection a fermion from the M^4 factor should be excited,

with $\Delta_T = 1/2$. This implies $h = 0$, which is forbidden by the bound (A.2) on h . Thus there are no physical chiral operators in this sector.

2. $\hat{h} = 1, \hat{j} = 0$

In this case we have $J = j$, and the tensor product of $\hat{h} = 1$ with h gives the representations $H = h - 1, h, h + 1$. Let us consider each one of the cases.

(a) $H = h + 1 = j = J$

In this case the mass shell condition is

$$\Delta = \frac{4h + 2}{k} + \frac{1}{2} + \Delta_T + N = \frac{1}{2} \quad (\text{A.6})$$

and would require $h \leq -1/2$, which violates the bound (A.2) on h , so there are no physical chiral operators from this sector.

(b) $H = h = j = J$

The mass shell condition is

$$\Delta = \frac{2h}{k} + \frac{1}{2} + \Delta_T + N = \frac{1}{2} \quad (\text{A.7})$$

and would require $h \leq 0$, which violates the bound (A.2) on h , so there are no physical chiral operators from this sector either.

(c) $H = h - 1 = j = J$

The mass shell condition is

$$\Delta = 0 + \frac{1}{2} + \Delta_T + N = \frac{1}{2} \quad (\text{A.8})$$

so it is satisfied by $\Delta_T = N = 0$. In Section A.3 we work out the details of the $h - 1$ representation coming from the tensor product of the $\text{SL}(2, R)$ bosonic representation of spin h and the fermionic representation of spin 1. The result in the x -basis is just

$$\Phi_h(x)\psi(x), \quad (\text{A.9})$$

where $\psi(x)$ is the fermionic $\text{SL}(2, R)$ primary with $h = -1$ defined in (3.27). The product of $\psi(x)$ with $\Phi_h(x)$ has no singularities, since $\Phi_h(x)$ is a primary of the purely bosonic currents j^A . Let us define the operator

$$\mathcal{O}_h(x, y) \equiv \Phi_h(x)V_{h-1}(y), \quad (\text{A.10})$$

and note that

$$\Delta(\mathcal{O}_h(x, y)) = 0. \quad (\text{A.11})$$

Then the chiral physical vertex operator is

$$\mathcal{O}_h^0 = e^{-\phi} \mathcal{O}_h(x, y) \psi(x). \tag{A.12}$$

By requiring the bounds (A.2) and (A.3) to be satisfied, we find that there are $k - 1$ operators \mathcal{O}_h , with $h = 1, \frac{3}{2}, \dots, \frac{k}{2}$. Finally, we verify that in the OPE

$$\begin{aligned} T_F(z) \mathcal{O}_h(x, y; w) \psi(x; w) &\sim (z - w)^{-2} (D_x^+ - 2xD_x^3 + x^2D_x^-) \mathcal{O}_h(x, y; w) \\ &+ O\left(\frac{1}{z - w}\right) \\ &\sim O\left(\frac{1}{z - w}\right) \end{aligned} \tag{A.13}$$

all the double poles cancel. Note that in flat space, this last condition imposes on a vertex like $\xi \cdot \psi e^{ik \cdot X}$ the polarization constraint $\xi \cdot k = 0$. Here the polarization is already fixed in (A.9) by the $SL(2, R)$ symmetry, in a way which is automatically BRST invariant.

For the computation of the three-point functions we will need the representation of this vertex operator in the 0 picture. Acting on \mathcal{O}_h^0 with the picture-changing operator $e^\phi T_F$ we get

$$\tilde{\mathcal{O}}_h^0 = \left(J(x) + \frac{2}{k} \psi(x) \psi_A D_x^A + \frac{2}{k} \psi(x) \chi_a P_y^a \right) \mathcal{O}_h(x, y) \tag{A.14}$$

$$= \left((1 - h) \hat{j}(x) + j(x) + \frac{2}{k} \psi(x) \chi_a P_y^a \right) \mathcal{O}_h(x, y), \tag{A.15}$$

where all the terms are normal ordered and in the second line we have used equation (3.29) and the identity

$$\psi(x) \psi_A D_x^A = -\frac{k}{2} h \hat{j}(x). \tag{A.16}$$

3. $\hat{h} = 0, \hat{j} = 1$

In this case we have $H = h$ and $J = j - 1, j, j + 1$. The analysis for the three cases is similar to the $\hat{h} = 1, \hat{j} = 0$ cases. The only physical chiral operators correspond to $H = h = j + 1 = J$. Using the results

of Section A.3 on the tensor product of spin j and spin 1 $SU(2)$ representations, the physical chiral vertex is given by

$$\mathcal{O}_h^2 = e^{-\phi} \mathcal{O}_h(x, y) \chi(y), \tag{A.17}$$

where $\chi(y)$ is the fermionic $SU(2)$ primary with $j = 1$ defined in (3.44). The absence of double poles in the OPE of T_F with $\mathcal{O}_h(x, y) \chi(y)$ is verified similarly to (A.13), and the number of \mathcal{O}_h^2 operators is again $k - 1$, as the number of $\mathcal{O}_h(x, y)$ operators.

In the 0 picture, the operator is

$$\tilde{\mathcal{O}}_h^2 = \left(K(y) + \frac{2}{k} \chi(y) \chi_a P_y^a + \frac{2}{k} \chi(y) \psi_A D_x^A \right) \mathcal{O}_h(x, y) \tag{A.18}$$

$$= \left(h \hat{k}(y) + k(y) + \frac{2}{k} \chi(y) \psi_A D_x^A \right) \mathcal{O}_h(x, y), \tag{A.19}$$

where in the second line we have used (3.46) and the identity

$$\chi(y) \chi_a P_y^a = j \frac{k}{2} \hat{k}(y), \tag{A.20}$$

with $j = h - 1$.

4. $\hat{h} = \hat{j} = 1$

For this case we have $H = h - 1, h, h + 1$ and $J = j - 1, j, j + 1$, so there are nine sectors. One can check that in all the cases, the mass shell condition

$$\Delta = -\frac{h(h - 1)}{k} + \frac{j(j + 1)}{k} + 1 + \Delta_T + N = \frac{1}{2} \tag{A.21}$$

cannot be satisfied without violating the bound (A.2) on h or the condition $\Delta_T \geq 0$. So there are no chiral physical operators in these sectors.

A.2 Ramond sector

The mass shell condition is now, in the $-1/2$ picture,

$$\Delta = \frac{5}{8} - \frac{h(h - 1)}{k} + \frac{j(j + 1)}{k} + \Delta_T + N = \frac{5}{8}, \tag{A.22}$$

and in the Ramond sector $\hat{j} = \hat{h} = 1/2$, so we have $H = h \pm 1/2$ and $J = j \pm 1/2$. One can check that the mass shell condition is satisfied without

violating the bound (A.2) on h , only by $H = h - 1/2 = j + 1/2 = J$. As shown in Section A.3, in the (x, y) basis, the tensor product corresponding to this case is given simply by

$$\Phi_h(x)V_{h-1}(y)S(x, y), \quad (\text{A.23})$$

where $S(x, y)$ is the field defined in (3.76), and which can be realized in our background in the two forms $S^\pm(x, y)$ defined in (3.77) and (3.78). Including the spin field for the T^4 factor, our candidates for the Ramond vertex operators are

$$\mathcal{O}_h(x, y)S^\pm(x, y)e^{(i/2)(\epsilon_4\hat{H}_4+\epsilon_5\hat{H}_5)}. \quad (\text{A.24})$$

We should now check the BRST invariance of these operators, which in the $-1/2$ picture implies the absence of $(z-w)^{-3/2}$ singularities in their OPE with T_F . Using the expressions (3.63) to (3.65) for T_F , it is easy to check that the combination $\mathcal{O}_h(x, y)S^-(x, y)$ is BRST invariant, due to a precise cancelation between the coefficients of $(z-w)^{-3/2}$ in its OPEs with T_F^α and T_F^β . On the other hand, $\mathcal{O}_h(x, y)S^+(x, y)$ is not BRST invariant, since its OPE with T_F^β has no $(z-w)^{-3/2}$ singularities to cancel those arising in its OPE with T_F^α .¹²

The GSO projection (3.61) imposes the further constraint $\epsilon_4\epsilon_5 = -1$, so the physical chiral operators in the R sector are finally

$$\mathcal{O}_h^a = e^{-\phi/2}\mathcal{O}_h(x, y)s_-^a(x, y), \quad a = 1, 2, \quad (\text{A.25})$$

where

$$s_\pm^1(x, y) = S^\pm(x, y)e^{+(i/2)(\hat{H}_4-\hat{H}_5)}, \quad (\text{A.26})$$

$$s_\pm^2(x, y) = S^\pm(x, y)e^{-(i/2)(\hat{H}_4-\hat{H}_5)}. \quad (\text{A.27})$$

In order to compute the two-point functions of \mathcal{O}_h^a , we will need their expressions in the $-3/2$ picture, which are

$$\tilde{\mathcal{O}}_h^a = -\frac{\sqrt{k}}{(2h-1)}e^{-3\phi/2}\mathcal{O}_h(x, y)s_+^a(x, y). \quad (\text{A.28})$$

This expression can be checked to be correct by acting on it with the picture raising operator $e^\phi T_F$, which yields \mathcal{O}_h^a (only the term T_F^α in (3.63) has a non-trivial action).

¹²One can check that the coefficient of the $(z-w)^{-3/2}$ singularity in the OPE between T_F^α and $\mathcal{O}_h(x, y)S^+(x, y)$ is zero only at $h = 1/2$. This lies at the boundary in the allowed range (A.2) for h , where a discrete representation becomes a continuous one, but violates the range (A.3) for j , since $j = h - 1 = -1/2$.

In summary, all the chiral operators are obtained, in the canonical $-1/2$, -1 pictures, by multiplying the basic field $\mathcal{O}_h(x, y)$ defined in (A.10) by any of the operators $e^{-\phi}\psi(x)$, $e^{-\phi}\chi(y)$ or $e^{-\phi/2}s^a_-(x, y)$. The antiholomorphic part of the operators is fixed by multiplying also by an antiholomorphic field $e^{-\bar{\phi}}\bar{\psi}(\bar{x})$, $e^{-\bar{\phi}}\bar{\chi}(\bar{y})$ or $e^{-\bar{\phi}/2}\bar{s}^a_-(\bar{x}, \bar{y})$.

A.3 Tensoring bosonic and fermionic representations of $SL(2, R)$ and $SU(2)$

In this section we will work out the tensor products between the bosonic and fermionic representations of $SL(2, R)$ and $SU(2)$ that appear in the chiral operators.

Let us first obtain the $h - 1$ representation appearing in the tensor product of a bosonic representation of $SL(2, R)$ with quantum number h , and the spin 1 representation provided by the free fermions ψ^A . We work in a normalization for the modes $\Phi_{h,m}$ such that

$$J_0^3 \Phi_{h,m} = m \Phi_{h,m}, \tag{A.29}$$

$$J_0^\pm \Phi_{h,m} = (m \mp (h - 1)) \Phi_{h,m \pm 1}. \tag{A.30}$$

The operators also depend on an antiholomorphic index \bar{m} which we omit. We wish to determine the Clebsch–Gordon coefficients in the expansion

$$(\psi\Phi)_{h-1,m} = a_m \psi^3 \Phi_{h,m} + b_m \psi^+ \Phi_{h,m-1} + c_m \psi^- \Phi_{h,m+1}. \tag{A.31}$$

Acting on this operator with both sides of $J_0^- = j_0^- + \hat{j}_0^-$, we get

$$\begin{aligned} (m + h - 2) (a_{m-1} \psi^3 \Phi_{h,m-1} + b_{m-1} \psi^+ \Phi_{h,m-2} + c_{m-1} \psi^- \Phi_{h,m}) = \\ = (a_m(m + h - 1) + 2b_m) \psi_3 \Phi_{h,m-1} + b_m(m + h - 2) \psi^+ \Phi_{h,m-2} \\ + (a_m + c_m(m + h)) \psi^- \Phi_{h,m}. \end{aligned} \tag{A.32}$$

A second equation is obtained by acting on $(\psi\Phi)_{h-1,m-1}$ with both sides of $J_0^+ = j_0^+ + \hat{j}_0^+$,

$$\begin{aligned} (m - h + 1) (a_m \psi^3 \Phi_{h,m} + b_m \psi^+ \Phi_{h,m-1} + c_m \psi^- \Phi_{h,m+1}) = \\ = (a_{m-1}(m - h) - 2c_m) \psi_3 \Phi_{h,m} + (b_{m-1}(m - 1 - h) - a_{m-1}) \psi^+ \Phi_{h,m-1} \\ + (c_{m-1}(m - h + 1)) \psi^- \Phi_{h,m+1}. \end{aligned} \tag{A.33}$$

Equating the coefficients of both sides of (A.32) and (A.33), we get six homogeneous equations for the six coefficients $a_m, b_m, c_m, a_{m-1}, b_{m-1}, c_{m-1}$. Inserting the resulting values in (A.31), gives, up to an overall rescaling

$$(\psi\Phi)_{h-1,m} = 2\psi^3\Phi_{h,m} - \psi^+\Phi_{h,m-1} - \psi^-\Phi_{h,m+1}. \tag{A.34}$$

This expression can be recast in the x basis as

$$(\psi\Phi)_{h-1}(x) = \sum_m (\psi\Phi)_{h-1,m} x^{-h+1-m} \tag{A.35}$$

$$= (-\psi^+ + 2x\psi^3x - x^2\psi^-) \times \sum_m \Phi_{h,m} x^{-h-m} \tag{A.36}$$

$$= \psi(x)\Phi_h(x). \tag{A.37}$$

One can check that this result holds both for discrete representations, where the sum runs over a semi-infinite range ($m = h, h + 1, \dots$ or $m = -h, -h - 1, \dots$), and for continuous representations, where the sum runs over an infinite range ($m = \alpha + \mathbb{Z}, \alpha \in [0, 1)$).

The spin $j + 1$ $SU(2)$ representation in the tensor product between a bosonic representation of spin j and the spin 1 representation provided by the fermions is χ^α is similarly obtained. By exploiting the action of $K_0^\pm = k_0^\pm + \hat{k}_0^\pm$, and using the normalization (3.38) and (3.39), we get

$$(\chi V)_{j+1,m} = -\chi^+ V_{j,m-1} + 2\chi^3 V_{j,m} + \chi^- V_{j,m+1}. \tag{A.38}$$

In the isospin y basis, this becomes

$$(\chi V)_{j+1}(y) = \sum_{m=-j-1}^{j+1} (\chi V)_{j+1,m} y^{-m+j+1}, \tag{A.39}$$

$$= (-\chi^+ + 2y\chi^3 + y^2\chi^-) \times \sum_{m=-j}^j V_{j,m} y^{-m+j}, \tag{A.40}$$

$$= \chi(y)V_j(y). \tag{A.41}$$

Finally, the representation with $SL(2, R)$ and $SU(2)$ spins $(h - 1/2, j + 1/2)$ in the tensor product of representations with spins (h, j) and $(\frac{1}{2}, \frac{1}{2})$ is obtained by acting with both $J_0^\pm = j_0^\pm + \hat{j}_0^\pm$ and $K_0^\pm = k_0^\pm + \hat{k}_0^\pm$, and is

given by

$$\begin{aligned}
 (S\Phi V)_{\substack{(h-1/2, m+1/2) \\ (j+1/2, n+1/2)}} &= |+\rangle\langle +| \Phi_{h,m} V_{j,n} + |+\rangle\langle -| \Phi_{h,m} V_{j,n+1} + |-\rangle\langle +| \\
 &\times \Phi_{h,m+1} V_{j,n} + |-\rangle\langle -| \Phi_{h,m+1} V_{j,n+1}. \tag{A.42}
 \end{aligned}$$

In the (x, y) basis this becomes

$$\begin{aligned}
 (S\Phi V)_{h-1/2, j+1/2}(x, y) &= \sum_m \sum_{n=-j-1}^j (S\Phi V)_{\substack{(h-1/2, m+1/2) \\ (j+1/2, n+1/2)}} x^{-m-h} y^{-n+j} \tag{A.43}
 \end{aligned}$$

$$= (|+\rangle\langle +| + y|+\rangle\langle -| + x|-\rangle\langle +| + xy|-\rangle\langle -|) \times \tag{A.44}$$

$$\begin{aligned}
 &\times \sum_m \Phi_{h,m} x^{-m-h} \sum_{n=-j}^j V_{j,n} y^{-n+j} \\
 &= S(x, y) \Phi_h(x) V_j(y). \tag{A.45}
 \end{aligned}$$

B Interactions in generalized SU(2) WZW models

In order to better understand the cancelation between the factors in the three-point functions C_S and C_H of the SU(2) and H_3^+ WZW models at levels $k - 2$ and $k + 2$, respectively, we will see in this appendix that these two quantities are solutions of functional equations that are related by a sort of “Wick rotation”.

For convenience let us define ($b = 1/\sqrt{k}$)

$$\alpha_i \equiv bh_i, \quad \alpha \equiv bh = \alpha_1 + \alpha_2 + \alpha_3, \tag{B.1}$$

$$a_i \equiv bj_i, \quad a \equiv bj = a_1 + a_2 + a_3, \tag{B.2}$$

and

$$c_H(\alpha_1, \alpha_2, \alpha_3) \equiv C_H(h_1, h_2, h_3), \tag{B.3}$$

$$c_S(a_1, a_2, a_3) \equiv C_S(j_1, j_2, j_3). \tag{B.4}$$

The three-point function $c_H(\alpha_1, \alpha_2, \alpha_3)$ is determined by requiring it to be a solution of the functional equations

$$\frac{c_H(\alpha_1 + b/2, \alpha_2, \alpha_3) c_H^-(\alpha_1)}{c_H(\alpha_1 - b/2, \alpha_2, \alpha_3) c_H^+(\alpha_1)} = \frac{\gamma^2(b(2\alpha_1 - b))\gamma(b(\alpha - 2\alpha_1 - b/2))\gamma(1 - b\alpha + 3b^2/2)}{\gamma(b(\alpha - 2\alpha_3 - b/2))\gamma(b(\alpha - 2\alpha_2 - b/2))}, \quad (\text{B.5})$$

$$\frac{c_H(\alpha_1 + b^{-1}/2, \alpha_2, \alpha_3) \tilde{c}_H^-(\alpha_1)}{c_H(\alpha_1 - b^{-1}/2, \alpha_2, \alpha_3) \tilde{c}_H^+(\alpha_1)} = \frac{\gamma(b^{-1}2\alpha_1 - b^{-2})\gamma(2b^{-1}\alpha_1 - 1)\gamma(b^{-1}(\alpha - 2\alpha_1 - b^{-1}/2))}{\gamma(b^{-1}(\alpha - 2\alpha_3 - b^{-1}/2))\gamma(b^{-1}(\alpha - 2\alpha_2 - b^{-1}/2))\gamma(b^{-1}(\alpha - b^{-1}/2 - b))}, \quad (\text{B.6})$$

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. These equations were obtained in [26] by imposing crossing symmetry on a four-point function, with one of the fields corresponding to the degenerate primaries $h = -1/2$ or $h = -k/2$ of the $SL(2, R)$ current algebra (see also [55]). The functions $c_H^\pm(\alpha_1), \tilde{c}_H^\pm(\alpha_1)$ are special structure constants that appear in the fusion of these degenerate fields with a generic primary

$$\Phi_{-1/2}\Phi_h = c_H^+(\alpha)[\Phi_{h-1/2}] + c_H^-(\alpha)[\Phi_{h+1/2}], \quad (\text{B.7})$$

$$\Phi_{-k/2}\Phi_h = \tilde{c}_H^+(\alpha)[\Phi_{h-k/2}] + \tilde{c}_H^-(\alpha)[\Phi_{h+k/2}] + \tilde{c}_H^\times(\alpha)[\Phi_{1-h+k/2}], \quad (\text{B.8})$$

where $[\Phi_h]$ denotes the primary field and all its current algebra descendants. The special structure constants can be obtained by a perturbative calculation [55, 87, 88], and are given by¹³

$$c_H^+(\alpha_1) = \tilde{c}_H^+(\alpha_1) = 1, \quad (\text{B.9})$$

$$c_H^-(\alpha_1) = \nu \frac{\gamma(b(2\alpha_1 - b))}{\gamma(2b\alpha_1)}, \quad (\text{B.10})$$

$$\tilde{c}_H^-(\alpha_1) = \tilde{\nu} \frac{\gamma(b^{-1}(2\alpha_1 - b))}{\gamma(2b^{-1}\alpha_1)}. \quad (\text{B.11})$$

¹³To obtain the special structure constants self-consistently without any perturbative computation, one should apply to H_3^+ the method used for Liouville theory in [89]. A functional equation that constrains the special structure constants was obtained in [26].

Plunging these values into (B.5) and (B.6) yields

$$\frac{c_H(\alpha_1 + b, \alpha_2, \alpha_3)}{c_H(\alpha_1, \alpha_2, \alpha_3)} = \frac{(\nu)^{-1} \gamma(2b\alpha_1) \gamma(b(2\alpha_1 + b)) \gamma(b(\alpha - 2\alpha_1 - b))}{\gamma(b(\alpha - 2\alpha_3)) \gamma(b(\alpha - 2\alpha_2)) \gamma(b(\alpha - b))}, \tag{B.12}$$

$$\frac{c_H(\alpha_1 + b^{-1}, \alpha_2, \alpha_3)}{c_H(\alpha_1, \alpha_2, \alpha_3)} = \frac{(\tilde{\nu})^{-1} \gamma(2b^{-1}\alpha_1) \gamma(b^{-1}(2\alpha_1 + b^{-1})) \gamma(b^{-1}(\alpha - 2\alpha_1 - b^{-1}))}{\gamma(b^{-1}(\alpha - 2\alpha_3)) \gamma(b^{-1}(\alpha - 2\alpha_2)) \gamma(b^{-1}(\alpha - b))}. \tag{B.13}$$

The solution of these functional equations is, up to a multiplicative constant,

$$c_H(\alpha_1, \alpha_2, \alpha_3) = \left[\nu b^{-2b^2} \right]^{-h+1} \frac{1}{\Upsilon(\alpha - b)} \prod_{i=1}^3 \frac{\Upsilon(2\alpha_i)}{\Upsilon(\alpha - 2\alpha_i)} \tag{B.14}$$

with $\tilde{\nu} = \nu b^2 b^{-4}$. After rescaling the operators as

$$\Phi_h \rightarrow \frac{\Phi_h}{\gamma(b^2(2h - 1))} \tag{B.15}$$

we get the three-point function given in (4.11), up to a multiplicative constant. Note that c_H is not an analytic function of b , since Υ has a branch cut for positive imaginary values of b [24].

The conformal bootstrap method used for the H_3^+ WZW model can also be applied in the $SU(2)$ case.¹⁴ Indeed, the steps in [26] that lead to the level $k + 2$ H_3^+ functional equations, have level $k - 2$ $SU(2)$ counterparts which are simply obtained by the replacements

$$h \rightarrow -j, \tag{B.16}$$

$$b \rightarrow -ib, \tag{B.17}$$

$$b^{-1} \rightarrow ib^{-1}. \tag{B.18}$$

¹⁴The technique used in [49] to compute the $SU(2)$ WZW three-point functions (4.5) was different, and consisted in exploiting the relation between generic $SU(2)$ four-point function and a five-point function of the minimal models.

Applying this to (B.5) and (B.6) we get the functional equations

$$\frac{c_S(a_1 - b/2, a_2, a_3) c_S^-(a_1)}{c_S(a_1 + b/2, a_2, a_3) c_S^+(a_1)} = \frac{\gamma^2(b(2a_1 + b))\gamma(b(a - 2a_1 + b/2))}{\gamma(b(a - 2a_3 + b/2))\gamma(b(a - 2a_2 + b/2))\gamma(b(a + 3b/2))}, \tag{B.19}$$

$$\frac{c_S(a_1 + b^{-1}/2, a_2, a_3) \tilde{c}_S^-(a_1)}{c_S(a_1 - b^{-1}/2, a_2, a_3) \tilde{c}_S^+(a_1)} = \frac{\gamma(b^{-1}(-2a_1 + b^{-1}))\gamma(-2b^{-1}a_1 - 1)\gamma(b^{-1}(-a + 2a_1 + b^{-1}/2))}{\gamma(b^{-1}(-a + 2a_3 + b^{-1}/2))\gamma(b^{-1}(-a + 2a_2 + b^{-1}/2))\gamma(b^{-1}(-a + b^{-1}/2 - b))}. \tag{B.20}$$

Under (B.16) to (B.18), the degenerate primaries go to $j = 1/2$ and $j = -k/2$. The latter does not belong to the standard spectrum of the $SU(2)$ WZW model, but it is a degenerate vector of the $SU(2)$ affine algebra [90]. The fusion rules are now [91]

$$V_{1/2}V_j = c_S^+(a)[V_{j+1/2}] + c_S^-(a)[V_{j-1/2}], \tag{B.21}$$

$$V_{-k/2}V_j = \tilde{c}_S^+(a)[V_{j-k/2}] + \tilde{c}_S^-(a)[V_{j+k/2}] + \tilde{c}_S^\times(a)[V_{-1-j+k/2}]. \tag{B.22}$$

Since in the case of the $SU(2)$ WZW model, the primaries are normalized as

$$\langle V_{j_1}(y_1)V_{j_2}(y_2) \rangle = \delta_{j_1, j_2} |y_{12}|^{2j_1}, \tag{B.23}$$

we can identify the special structure constants with particular three-point functions,

$$c_S^\pm(a_1) = c_S(a_1, b/2, a_1 \pm b/2), \tag{B.24}$$

$$\tilde{c}_S^\pm(a_1) = c_S(a_1, -b^{-1}/2, a_1 \mp b^{-1}/2). \tag{B.25}$$

This in turn allows to determine them from (B.19) and (B.20). Specializing (B.19) to $a_3 = a_1, a_2 = b/2$, and (B.20) to $a_3 = a_1, a_2 = -b^{-1}/2$ we get

$$\left(\frac{c_S^-(a_1)}{c_S^+(a_1)}\right)^2 = \frac{\gamma^2(b(2a_1 + b))}{\gamma(2ba_1)\gamma(2b(a_1 + b))} \tag{B.26}$$

and

$$\left(\frac{\tilde{c}_S^-(a_1)}{\tilde{c}_S^+(a_1)}\right)^2 = \frac{\gamma(-2a_1b^{-1} + b^{-2})\gamma(-2a_1b^{-1} - 1)}{\gamma(-2a_1b^{-1})\gamma(b^{-1}(-2a_1 + b^{-1} - b))}. \tag{B.27}$$

Plugging these values back into (B.19) and (B.20) gives the functional equations

$$\frac{c_S(a_1 + b, a_2, a_3)}{c_S(a_1, a_2, a_3)} = \frac{\gamma(b(a - 2a_2 + b))\gamma(b(a - 2a_3 + b))\gamma(b(a + 2b))}{\gamma(b(2a_1 + 2b))[\gamma(b(2a_1 + b))\gamma(b(2a_1 + 3b))]^{1/2}\gamma(b(a - 2a_1))}, \quad (\text{B.28})$$

$$\frac{c_S(a_1 + b^{-1}, a_2, a_3)}{c_S(a_1, a_2, a_3)} = \frac{\gamma(-b^{-1}(a - 2a_1 - b^{-1}))}{\gamma(-b^{-1}(a - 2a_2))\gamma(-b^{-1}(a - 2a_3))\gamma(-b^{-1}(a + b))} \times \frac{1}{[\gamma(2 + 2a_1b^{-1})\gamma(1 + 2a_1b^{-1} + b^{-2})\gamma(1 + 2a_1b^{-1})\gamma(2 + 2a_1b^{-1} + b^{-2})]^{1/2}}. \quad (\text{B.29})$$

Using (4.14), it follows that the above two functional equations are solved by

$$c_S(a_1, a_2, a_3) = \frac{\sqrt{\gamma(b^2)}b^{\frac{1}{2}-b^2}}{\Upsilon(b)} \Upsilon(a + 2b) \prod_{i=1}^3 \frac{\Upsilon(a - 2a_i + b)}{[\Upsilon(2a_i + b)\Upsilon(2a_i + 2b)]^{1/2}}, \quad (\text{B.30})$$

where we have fixed the arbitrary constant by requiring $c_S(a_1, a_1, 0) = 1$. Using (4.23) one can express $c_S(a_1, a_2, a_3)$ in terms of $P(s)$, and it is immediate to check that the resulting expression for (B.30) precisely coincides with $C_S(j_1, j_2, j_3)$ in (4.5) — up to the factor N_{j_1, j_2, j_3} which should be added to (B.30).

The above form of the $SU(2)$ three-point functions is defined for any value of $2j_i$, not only for positive integers.¹⁵ Thus $c_S(a_1, a_2, a_3)$ are the three-point functions of a *generalized* $SU(2)$ WZW model, similar to the generalized minimal models studied in [24, 25].¹⁶ In those works, similar relations and cancelations between Liouville theory and the minimal models three-point functions were observed.

Finally, note that to obtain $c(\alpha_1, \alpha_2, \alpha_3) \equiv c_H(\alpha_1, \alpha_2, \alpha_3)c_S(\alpha_1 - b, \alpha_2 - b, \alpha_3 - b)$ in (4.26), instead of multiplying the two expressions, we could have combined equations (B.12), (B.13), (B.28) and (B.29). This gives the

¹⁵Indeed, we have used this freedom in (B.25), since we have evaluated c_S at a value corresponding to $j_2 = -k/2 < 0$.

¹⁶V. Petkova has pointed out to us that it should be possible to obtain the structure constants for these generalized $SU(2)$ from those of the generalized minimal models by using the relation between the structure constants in both models found in [92].

functional equations,

$$\frac{c(\alpha_1 + b, \alpha_2, \alpha_3)}{c(\alpha_1, \alpha_2, \alpha_3)} = (\nu)^{-1} \left[\frac{\gamma(b(2\alpha_1 + b))}{\gamma(b(2\alpha_1 - b))} \right]^{1/2}, \quad (\text{B.31})$$

$$\frac{c(\alpha_1 + b^{-1}, \alpha_2, \alpha_3)}{c(\alpha_1, \alpha_2, \alpha_3)} = (\tilde{\nu})^{-1} [(2\alpha_1 b^{-1} - 1)^2 (2\alpha_1 b^{-1} + b^{-2} - 1)^2]^{1/2}, \quad (\text{B.32})$$

whose solution, after the rescaling (B.15), is given by (4.26), up to a constant.

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