String orientations on simplicial homology manifolds

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Abstract

Simplicial homology manifolds are proposed as an interesting class of geometric objects, more general than topological manifolds but still quite tractable, in which questions about the microstructure of space-time can be naturally formulated. Their string orientations are classified by $H^3$ with coefficients in an extension of the usual group of $D$-brane charges, by cobordism classes of homology three-spheres with trivial Rokhlin invariant.

1 Notation and background

1.1 In homotopy-theoretic terms, a string structure on a smooth manifold $M$ is a lift

\[
\begin{array}{ccc}
BO \langle 8 \rangle & \rightarrow & BO \\
\downarrow & & \\
M & \rightarrow & BO
\end{array}
\]
of the map classifying the stable tangent bundle of $M$; where the seven-connected cover $BO(8)$ of $BO$ is the fiber

$$BO(8) \to BO \to (BO)_{(7)}$$

of the map to its Postnikov approximation [28] having homotopy groups concentrated in degrees seven and below. Alternately:

$$(BO)(8) = B(O(7))$$

is the classifying space for the topological group

$$O(7) \to O \to O_{(6)}$$

whose homotopy groups agree with those of $O$ in degrees greater than or equal to seven.

A string structure can thus be interpreted as a “reduction” of the structure group of the stable tangent bundle of $M$, from $O$ to $O_{(7)}$, just as a spin structure is similarly a “reduction” of that structure group to

$$Spin = O(2) \to O \to \mathbb{Z}_2 \times H(\mathbb{Z}_2, 1).$$

An oriented manifold $M$ admits a spin structure iff the Stiefel–Whitney class $w_2 = 0$; similarly, the existence of a string structure requires the vanishing of a version $p_1/2$ of the Pontrjagin class defined for spin manifolds. The set of spin structures on $M$ admits a transitive free action of $H^1(M, \mathbb{Z}_2)$, and by essentially the same argument the set of string structures is an $H^3(X, \mathbb{Z})$-torsor. The associated twisted $K$-groups of $X$ are natural repositories [7] for Ramond–Ramond $D$-brane charges.

The interest in string orientations comes from the quantum field theory, where they were recognized as necessary to define an analog of the Dirac operator on the space $LM$ of free loops on $M$, but mathematical interest [1] in their properties goes back to the early 1970s [17]. There is a parallel interest in the representation of three-dimensional (3D) cohomology classes in local geometric terms, analogous [6] to the description of 2D classes by complex line bundles.

1.2 If $G$ is a connected, simply connected simple Lie group, then

$$\pi_3(G) = \mathbb{Z}$$

by a classical theorem of Bott. A theorem of Kuiper says that the group $\text{Gl}(\mathbb{H})$ of invertible bounded operators on an (infinite-dimensional) Hilbert
space is contractible; the projective general linear group

$$\text{PGL}(\mathbb{H}) = \text{GL}(\mathbb{H})/\mathbb{C}^\times \sim B\mathbb{T}$$

is therefore an Eilenberg–MacLane space of type $H(\mathbb{Z}, 2)$, from which it follows that the classifying space for $\text{PGL}(\mathbb{H})$-bundles is an EM space of type $H(\mathbb{Z}, 3)$. The associated bundle

$$1 \to \text{PGL}(\mathbb{H}) \to G\langle 4 \rangle \to G \to 1$$

is an extension of topological groups [29]. The following construction is due to Kitchloo [19, appendix]:

A level one projective representation of the loop group of $G$ on $\mathbb{H}$ defines a homomorphism to $\text{PGL}(\mathbb{H})$, which pulls $\text{GL}(\mathbb{H})$ back to a version of the universal central extension of $LG$. Let $\mathcal{A}$ denote the topological Tits building of $LG$, modeled by the contractible space of connections on a trivial $G$-bundle over a circle; the pointed loops act freely on it, and the holonomy map makes it a principal $\Omega G$ bundle. The diagonal action of $LG$ on

$$\mathcal{A} \times_{\Omega G} \text{PGL}(\mathbb{H}) := G\langle 4 \rangle$$

factors through an action of $G$ on $G\langle 4 \rangle$ lifting the action of $G$ on itself by conjugation.

Note that because $\text{PGL}(\mathbb{H})$ is the group of automorphisms of $\text{GL}(\mathbb{H})$, the cohomology group

$$H^1(X, \text{PGL}(\mathbb{H})) \cong H^1(X, H(\mathbb{Z}, 2)) \cong H^3(X, \mathbb{Z})$$

can be interpreted as a Brauer group of bundles of $C^*$-algebras (up to Morita equivalence) over $X$. A refinement of this argument [22] represents the Brauer group of bundles of graded $C^*$-algebras over $X$ by the generalized Eilenberg–MacLane space

$$H(\mathbb{Z}_2, 1) \times H(\mathbb{Z}, 3),$$

which suggests interpreting $O\langle 7 \rangle = O\langle 4 \rangle$ in terms of a bundle of such algebras over $(S)O$. 
2 Simplicial homology manifolds

2.1 I also want to thank Kitchloo for observing that in low dimensions, the map

$$\text{PL}/\text{O} \to \text{BO} \to \text{BPL}$$

is almost an equivalence: the homotopy groups of the fiber are the Kervaire–Milnor groups of differentiable structures on spheres, which (aside from the smooth 4D Poincaré conjecture . . . ) are trivial below dimension seven. This implies that a string structure on a smooth manifold is the same as a smoothing of a topological manifold endowed with a lift

$$\begin{align*}
\text{BTop}(8) \\
\downarrow \\
M \\
\downarrow \\
\text{BTop}
\end{align*}$$

of the map classifying its tangent topological block bundle. The theorem

$$\text{Top}/\text{PL} \sim H(\mathbb{Z}_2, 3)$$

of Kirby and Siebenman [25] seems also to point in this direction.

2.2 The classification of string structures on geometric objects more general than smooth manifolds is accessible nowadays, thanks to many researcher-years of deep work related to the Hauptvermutung, suggesting that questions like “Who ordered the differentiable structure” may not be out of reach. I will summarize some background from Ranicki’s elegant account [23], but in some cases I’ll use terminology from [16]:

**Definition 2.1.** A space $X$ is a $d$-dimensional homology manifold iff for any $x \in X$,

$$H_*(X, X - \{x\}; \mathbb{Z}) \cong H_*(S^{d-1}; \mathbb{Z});$$

but a *simplicial* homology manifold is a simplicial complex $K$ such that for any $k$-simplex $\sigma \in K$,

$$H_*(\text{link}_K(\sigma); \mathbb{Z}) \cong H_*(S^{d-k-1}; \mathbb{Z}).$$

The polyhedron $|K|$ of $K$ is a homology manifold iff $K$ is a simplicial homology manifold. A manifold homology resolution $f : M \to X$ of a space $X$ is a compact topological manifold $M$ together with a surjective map $f$ with acyclic point inverses.
The element \( \kappa_k(K) \in H^k(|K|, \Theta_{k-1}) \) Poincaré dual to the cycle

\[
\sum_{|\sigma|=d-k} [\text{link}_K(\sigma)] \cdot \sigma \in H_{d-k}(|K|, \Theta_{k-1})
\]

with coefficients in the group of simplicial homology spheres (up to cobordism through PL homology cylinders) is trivial unless \( k = 4 \): for \( \Theta := \Theta_3 \) is the only nontrivial coefficient group. (It is not finitely generated \([13, 14, 26]\).) There is a block bundle theory \([15, 16]\) for simplicial homology manifolds, resulting in a fibration

\[
B_{\text{PL}} \to B_{\text{HL}} \to H(\Theta, 4)
\]

of classifying spaces.

**Theorem 2.2** (cf. \([8, 23 \S 5]\)). A simplicial homology manifold \( K \) of dimension \( \geq 5 \) admits a PL manifold homology resolution iff

\( \kappa_4(K) = 0 \).

The resolutions themselves are classified by maps to \( H(\Theta, 3) \).

**2.3** Using this technology, the question motivating this note can be formulated as the problem of understanding the map

\[
B(\text{Top}\langle 7 \rangle) = B(\text{PL}\langle 7 \rangle) \to B_{\text{HL}}.
\]

Its fiber can be decomposed as

\[
\text{PL}/\text{Top}\langle 7 \rangle \to \text{HL}/\text{PL}\langle 7 \rangle \to \text{HL}/\text{PL} = H(\Theta, 3);
\]

the fibration

\[
\text{PL}/\text{Top}\langle 7 \rangle \to \text{Top}/\text{Top}\langle 7 \rangle = H(\mathbb{Z}_2, 1) \times H(\mathbb{Z}, 3) \to \text{Top}/\text{PL} = H(\mathbb{Z}_2, 3)
\]

shows that \( \text{PL}/\text{Top}\langle 7 \rangle \) is a three-stage Postnikov system, with homotopy group \( \mathbb{Z} \) in degree three and \( \mathbb{Z}_2 \) in degrees one and two. The group \( \pi_*(\text{HL}/\text{PL}\langle 7 \rangle) \) is therefore \( \mathbb{Z}_2 \) in degree one and zero in degree two, while in
degree three there is an exact sequence
\[ 0 \to \mathbb{Z} \to \pi_3(\text{HL}/\text{Top}(7)) \to \Theta \to \mathbb{Z}_2 \to 0. \]

The map on the right can be identified with Rokhlin’s invariant
\[ \Sigma \mapsto \rho(\Sigma) := \frac{1}{8} \text{signature} (W) \mod (2) : \Theta \to \mathbb{Z}_2 \]
of a homology three-sphere \( \Sigma \) (where \( W \) is a spin four-manifold with \( \partial W = \Sigma \)). Let \( \Theta_0 \) be the kernel of \( \rho \).

**Corollary 2.3.** When \( K \) is a smoothable (\( \varkappa(K) = 0 \)) simplicial homology string manifold of dimension \( \geq 5 \), PL manifold structures on a homology resolution are classified by elements of \( H^3(|K|, \tilde{\Theta}) \), where
\[ \tilde{\Theta} := \mathbb{Z} \oplus \Theta_0 \cong \pi_3(\text{HL}/\text{Top}(7)). \]

**Proof.** The exact sequence above splits, because the infinite cyclic group maps isomorphically to the third homotopy group of Top. This suggests, among other things, the existence of a combinatorial formula [3] for its Pontrjagin class. When that class vanishes, lifts of the classifying map from \( |K| \) to \( \text{BHL} \) to a map from a homology resolution \( X \) are classified by maps to \( \text{HL}/\text{Top}(7) \). \( \square \)

### 2.4 The inclusion of the fiber in
\[ H(\Theta, 3) \to \text{BPL} \to \text{BHL} \]
is trivial at odd primes: \( \text{BPL} \cong B \otimes [31] \), but \( K \)-theory is blind to Eilenberg–MacLane spaces \( H(A, n) \) for \( n > 2 \). At the prime two, there are still open questions. In particular, it is not known if the Rokhlin homomorphism \( \rho \) splits: this is equivalent to the conjecture that all topological manifolds of \( \dim > 4 \) are simplicial complexes.

Freed [11] identifies the classifying space of the Picard category of \( \mathbb{Z}_2 \)-graded complex lines as a two-stage Postnikov system. Its associated infinite-loop spectrum
\[ \Sigma^2 \tilde{I}(0) := \mathbb{L}_{\pm} \xrightarrow{\beta \text{Sq}^2} H\mathbb{Z}_2 \xrightarrow{} \Sigma^3 H\mathbb{Z} \]
of the double suspension of the Anderson dual \( \tilde{I} \) [20, Appendix B, 20] of the sphere spectrum, which is characterized by a short exact sequence
\[ 0 \to \text{Ext}(E_{* - 1}, \mathbb{Z}) \to \tilde{I}^*(E) \to \text{Hom}(E_{*}, \mathbb{Z}) \to 0, \]
associated to a spectrum $E$. This same small Postnikov system appears as the base

$$F/\text{PL} \cong \Sigma^4*HZ \times \Sigma^4*HZ_2 \times \Sigma^2\mathbb{L}_\pm (* > 1),$$

of the (two-localization) of the infinite-loop space classifying piecewise-linear structures on a Poincaré-duality space [21, Chapter VII]. These observations can then be assembled into a diagram

\[
\begin{array}{ccc}
\Sigma^3H\Theta & \xrightarrow{=} & \text{HL}/\text{PL} \\
\rho & & \kappa \\
\beta & & \Sigma^2\tilde{l} \\
\end{array}
\]

of spectra.

On the other hand, Anderson’s exact sequence implies $\tilde{l}^2(H\Theta_0) = 0$, so (the nonzero invariant defined by Rokhlin’s invariant) $\rho \in \tilde{l}^1(H\mathbb{Z}_2)$ in the exact sequence

$$\cdots \rightarrow \tilde{l}^2(H\Theta_0) \rightarrow \tilde{l}^1(H\mathbb{Z}_2) \rightarrow \tilde{l}^1(H\Theta) \rightarrow \cdots;$$

maps to (the non-zero invariant defined by) $\kappa \in \tilde{l}^1(H\Theta)$: which defines a homomorphism from $B\Theta$ to $\mathbb{L}_\pm$, interpretable as a topological field theory mapping the category with one object, and 3D homology cylinders as morphisms, to the category of $\mathbb{Z}_2$-graded complex lines.

This can be regarded as a lift of Rokhlin’s invariant, regarded as a topological field theory taking values in the Picard category of real lines. It suggests the interest of super-Chern–Simons theories [10, 12, §9] defined on simplicial homology spin manifolds.

3 Über die Hypothesen, welche zu Grunde der Physik liegen

Following [6, §VII], it is tempting to interpret the elements of $\mathbb{Z}$ in $\tilde{\Theta}$ as topological twists “in the large” (or at infinity), and elements of $\Theta_0$ as twists “in the small”. Physics has a tradition of concern (cf. eg Wheeler) with the possibility that the microstructure of the Universe might in some way be non-Euclidean. This seems legitimate: experiments in very high-energy physics probe the topology of space-time at very short distance, and it is conceivable that at very fine scales physical space might be described by some kind of quantum foam model [24], possibly involving ensembles with varying topology.
These ideas have a big literature, but interested researchers seem not to be very aware of the long history of interest in analogous questions among topologists. In particular, the extended homology cobordism group $\tilde{\Theta}$ seems to capture rather precisely the idea that the space-time “bubbles” in which very-short-distance interactions occur—I’m thinking of the way Feynman diagrams are often drawn—might be bounded by non-standard spheres.

If physics starts by hypothesizing the existence of a simplicial homology manifold structure\(^1\) on some (say, 10D or 11D) space-time $K$, then the vanishing of $w_2$ decides the existence of a spin structure and the vanishing of $\chi(K)$ decides the existence of a PL resolution $X$. When $p_1/2 = 0$, resolutions of $|K|$ admit string structures; this is quite like the standard situation. However, the possible \textit{twists} of that string resolution (related to $B$-fields \cite{27}, Vafa’s discrete torsion \cite[§1.6]{2} and perhaps more generally to $D$-brane charges \cite[§4.4]{5}) lie in $H^3(|K|, \tilde{\Theta})$, which is much bigger than the usual group of gerbes over $|K|$. There may even be ‘experimental’ evidence for the physical relevance of twisting by homology three-spheres, in that deep results about the structure of such manifolds are derived by scattering Yang–Mills bosons off them: i.e., from Donaldson theory \cite{9}.

As for exotic homology manifolds \cite{4}, hypotheses non fingo: in part because they, unlike simplicial homology manifolds, seem to lack the clocks and measuring rods that play the role of rulers and compasses in classical geometry. This is probably a lack of imagination on my part; a deeper concern is that major questions in 4D geometric topology remain open. Here I want only to make the point that simplicial homology manifolds are understood well enough to test against the models of contemporary physics.

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\(^1\)conceptually similar to a causal structure
References


