On the vector bundles associated to the irreducible representations of cocompact lattices of $\text{SL}(2,\mathbb{C})$

Indranil Biswas$^1$ and Avijit Mukherjee$^2$

$^1$School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Bombay 400005, India
indranil@math.tifr.res.in
$^2$Department of Physics, Jadavpur University, Raja S. C. Mullick Road, Jadavpur, Kolkata 700032, India
avijit00@gmail.com

Abstract

We prove the following: let $\Gamma \subset \text{SL}(2,\mathbb{C})$ be a cocompact lattice and let $\rho : \Gamma \rightarrow \text{GL}(r,\mathbb{C})$ be an irreducible representation. Then the holomorphic vector bundle $E_\rho \rightarrow \text{SL}(2,\mathbb{C})/\Gamma$ associated to $\rho$ is polystable. The compact complex manifold $\text{SL}(2,\mathbb{C})/\Gamma$ has natural Hermitian structures; the polystability of $E_\rho$ is with respect to these natural Hermitian structures. We show that the polystable vector bundle $E_\rho$ is not stable in general.
1 Introduction

We first recall the set-up, and some results, of [1]. Let

$$\Gamma \subset \text{SL}(2, \mathbb{C})$$

be a discrete cocompact subgroup. Fixing a SU(2)-invariant Hermitian form on the Lie algebra $\text{sl}(2, \mathbb{C})$, we get a Hermitian structure $h$ on the compact complex manifold $M := \text{SL}(2, \mathbb{C})/\Gamma$. The $(1, 1)$-form $\omega_h$ on $M$ associated to $h$ satisfies the identity $d\omega^2_h = 0$. Take any homomorphism

$$\rho : \Gamma \rightarrow \text{GL}(r, \mathbb{C}).$$

This $\rho$ produces a holomorphic vector bundle $E_\rho$ of rank $r$ on $M$ equipped with a flat holomorphic connection $\nabla^\rho$. The homomorphism $\rho$ is called irreducible if $\rho(\Gamma)$ is not contained in some proper parabolic subgroup of $\text{GL}(r, \mathbb{C})$.

If $\rho(\Gamma) \subset U(r)$, then $E_\rho$ is equipped with a Hermitian structure $H^\rho$ such that the associated Chern connection is $\nabla^\rho$.

If

- $\rho(\Gamma) \subset U(r)$ and
- $\rho(\Gamma)$ is irreducible,

then the vector bundle $E_\rho$ is stable [1, Proposition 4.5].

Now assume that $\rho$ is irreducible, but do not assume that $\rho(\Gamma) \subset U(r)$. Our aim here is to prove the following (see Theorem 2.2):

The holomorphic vector bundle $E_\rho$ is polystable with respect to the Hermitian structure $h$ on $M$.

It is known that under some minor condition, the group $\Gamma$ admits some free groups of more than one generators as quotients [6, p. 3393, Theorem 2.1]. Therefore, there are many examples of pairs $(\Gamma, \rho)$ of the above type satisfying the irreducibility condition.

Since $E_\rho$ is polystable, the holomorphic vector bundle $E_\rho$ has an Hermitian–Yang–Mills structure $\mathcal{H}^\rho$ [7] (see also [3]). It may be worthwhile to investigate this Hermitian structure $\mathcal{H}^\rho$. We should clarify that $\mathcal{H}^\rho$ need not be flat. An Hermitian–Yang–Mills structure on a polystable vector bundle with vanishing Chern classes over a compact Kähler manifold is flat, but $M$ is not Kähler.
It is natural to ask whether the polystable vector bundle $E_{\rho}$ is stable. If we take $\rho$ to be the inclusion of $\Gamma$ in $SL(2, \mathbb{C})$, then $\rho$ is irreducible, but the associated holomorphic vector bundle $E_{\rho}$ is holomorphically trivial, in particular, $E_{\rho}$ is not stable (see Lemma 2.3 for the details).

Infinitesimal deformations of the complex structure of $M$ are investigated in [8].

### 2 Polystability of associated vector bundle

The Lie algebra of $SL(2, \mathbb{C})$, which will be denoted by $sl(2, \mathbb{C})$, is the space of complex $2 \times 2$ matrices of trace zero. Consider the adjoint action of $SU(2)$ on $sl(2, \mathbb{C})$. Fix an inner product $h_0$ on $sl(2, \mathbb{C})$ preserved by this action; for example, we may take the Hermitian form $(A, B) \mapsto -\text{trace}(AB^*)$ on $sl(2, \mathbb{C})$. Let $h_1$ be the Hermitian structure on $SL(2, \mathbb{C})$ obtained by right-translating the Hermitian form $h_0$ on $T_{\text{Id}} SL(2, \mathbb{C}) = sl(2, \mathbb{C})$.

Let $\Gamma$ be a cocompact lattice in $SL(2, \mathbb{C})$. So $\Gamma$ is a discrete subgroup of $SL(2, \mathbb{C})$ such that the quotient

$$M := SL(2, \mathbb{C})/\Gamma$$

is compact. This $M$ is a compact complex manifold of complex dimension three. The left-translation action of $SL(2, \mathbb{C})$ on itself descends to an action of $SL(2, \mathbb{C})$ on $M$. We will call this action of $SL(2, \mathbb{C})$ on $M$ the left-translation action. The Hermitian structure $h_1$ on $SL(2, \mathbb{C})$ descends to an Hermitian structure on $M$. This descended Hermitian structure on $M$ will be denoted by $h$. Let $\omega_h$ be the $C^\infty (1,1)$-form on $M$ associated to $h$. Then

$$d\omega_h^2 = 0$$

[1, Corollary 4.1].

For a torsionfree nonzero coherent analytic sheaf $F$ on $M$, define

$$\text{degree}(F) := \int_M c_1(F) \wedge \omega_h^2 \in \mathbb{R} \quad \text{and} \quad \mu(F) := \frac{\text{degree}(F)}{\text{rank}(F)} \in \mathbb{R}.$$ 

A torsionfree nonzero coherent analytic sheaf $F$ on $M$ is called stable (respectively, semistable) if for every coherent analytic subsheaf $V \subset F$
such the rank($V$) $\in [1, \text{rank}(F) - 1]$ and the quotient $F/V$ is torsionfree, the inequality

$$\mu(V) < \mu(F) \quad \text{(respectively, } \mu(V) \leq \mu(F))$$

holds (see [5, Chapter V, Section 7]). A torsionfree nonzero coherent analytic sheaf $F$ on $M$ is called *polystable* if it is semistable and is isomorphic to a direct sum of stable sheaves.

**Remark 2.1.** Since a polystable coherent analytic sheaf $F$ is semistable, if $F = \bigoplus_{i=1}^{r} F_i$, then $\mu(F_i) = \mu(F)$ for all $i$.

Take any homomorphism

$$\rho : \Gamma \longrightarrow \text{GL}(r, \mathbb{C}).$$

(2.2)

Let $(E_{\rho}, \nabla^{\rho})$ be the flat holomorphic vector bundle of rank $r$ over $M$ associated to the homomorphism $\rho$. We recall that the total space of $E_{\rho}$ is the quotient of $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^r$ where two points

$$(z_1, v_1), (z_2, v_2) \in \text{SL}(2, \mathbb{C}) \times \mathbb{C}^r$$

are identified if there is an element $\gamma \in \Gamma$ such that $z_2 = z_1 \gamma$ and $v_2 = \rho(\gamma^{-1})(v_1)$. The trivial connection on the trivial vector bundle $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^r \longrightarrow \text{SL}(2, \mathbb{C})$ of rank $r$ descends to the connection $\nabla^{\rho}$. The left-translation action of $\text{SL}(2, \mathbb{C})$ on $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^r$ together define an action of $\text{SL}(2, \mathbb{C})$ on $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^r$. This action of $\text{SL}(2, \mathbb{C})$ on $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^r$ descends to an action

$$\tau : \text{SL}(2, \mathbb{C}) \times E_{\rho} \longrightarrow E_{\rho}$$

(2.3)

of $\text{SL}(2, \mathbb{C})$ on the vector bundle $E_{\rho}$. The action $\tau$ in (2.3) is clearly a lift of the left-translation action of $\text{SL}(2, \mathbb{C})$ on $M$.

The homomorphism $\rho$ in (2.2) is called *reducible* if there is a nonzero linear subspace $S \subseteq \mathbb{C}^r$ such that $\rho(\Gamma)(S) = S$. The homomorphism $\rho$ is called *irreducible* if it is not reducible.

**Theorem 2.2.** Assume that the homomorphism $\rho$ in (2.2) is irreducible. Then the corresponding holomorphic vector bundle $E_{\rho}$ is polystable.

**Proof.** Since $E_{\rho}$ has a flat connection, the Chern class $c_1(\det E_{\rho}) = c_1(E_{\rho}) \in H^2(M, \mathbb{R})$ vanishes. Hence we have degree($E_{\rho}$) = 0 (see [1, Lemma 4.2]).
We will first show that $E_\rho$ is semistable. Assume that $E_\rho$ is not semistable. Let

$$0 < W_1 \subset \cdots \subset W_{\ell-1} \subset W_\ell = E_\rho$$

be the Harder–Narasimhan filtration $E_\rho$; see [2] for the construction of the Harder–Narasimhan filtration of vector bundles on compact complex manifolds. Since $E_\rho$ is not semistable, we have $\ell \geq 2$ and $W_1 \neq 0$.

Consider the action $\tau$ of $\text{SL}(2, \mathbb{C})$ on $E_\rho$ constructed in (2.3). From the uniqueness of the Harder–Narasimhan filtration it follows immediately that $\tau(\{g\} \times W_1) = W_1$ for every $g \in \text{SL}(2, \mathbb{C})$. Therefore, we have

$$\tau(\text{SL}(2, \mathbb{C}) \times W_1) = W_1.$$  (2.5)

Let $C(W_1) \subset M$ be the closed subset over which $W_1$ fails to be locally free. Since $\tau$ is a lift of the left-translation action of $\text{SL}(2, \mathbb{C})$ on $M$, from (2.5) we conclude that $C(W_1)$ is preserved by the left-translation action of $\text{SL}(2, \mathbb{C})$ on $M$. As the left-translation action of $\text{SL}(2, \mathbb{C})$ on $M$ is transitive, it follows that $C(W_1)$ is the empty set. Therefore, $W_1$ is a holomorphic vector bundle on $M$. Similarly, the closed proper subset of $M$ over which $W_1$ fails to be a subbundle of $E_\rho$ is preserved the left-translation action of $\text{SL}(2, \mathbb{C})$ on $M$. Hence this subset is empty, and $W_1$ is a holomorphic subbundle of $E_\rho$.

We will show that the flat connection $\nabla^\rho$ on $E_\rho$ preserves the subbundle $W_1$ in (2.4).

To show that $\nabla^\rho$ preserves $W_1$, first note that the flat sections of the trivial connection on the trivial vector bundle $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^r \longrightarrow \text{SL}(2, \mathbb{C})$ are of the form

$$\text{SL}(2, \mathbb{C}) \longrightarrow \text{SL}(2, \mathbb{C}) \times \mathbb{C}^r, \quad g \longmapsto (g, v_0),$$

where $v_0 \in \mathbb{C}^r$ is independent of $g$. On the other hand, the image of such a section is an orbit for the action of $\text{SL}(2, \mathbb{C})$ on $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^r$; recall that the action of $\text{SL}(2, \mathbb{C})$ on $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^r$ is the diagonal one for the left-translation action of $\text{SL}(2, \mathbb{C})$ on itself and the trivial action of $\text{SL}(2, \mathbb{C})$ on $\mathbb{C}^r$ (see the construction of $\tau$ in (2.3)). Also, recall that the connection $\nabla^\rho$ on $E_\rho$ is the descent of the trivial connection on the trivial vector bundle $\text{SL}(2, \mathbb{C}) \times \mathbb{C}^r \longrightarrow \text{SL}(2, \mathbb{C})$. Combining these, from (2.5) we conclude that $\nabla^\rho$ preserves $W_1$.

The homomorphism $\rho$ is given to be irreducible. Therefore, the only holomorphic subbundles of $E_\rho$ that are preserved by the associated connection
\( \nabla^\rho \) are 0 and \( E_\rho \) itself. But \( \ell \geq 2 \) and \( W_1 \neq 0 \) in (2.4). So \( W_1 \) neither 0 nor \( E_\rho \).

In view of the above contradiction, we conclude that the holomorphic vector bundle \( E_\rho \) is semistable.

We will now prove that \( E_\rho \) is polystable.

Consider all nonzero coherent analytic subsheaves \( V \) of \( E_\rho \) such that

- \( V \) is polystable and
- degree(\( V \)) = 0.

Let

\[ \mathcal{F} \subset E_\rho \]  

be the coherent analytic subsheaf generated by all \( V \) satisfying the above two conditions. It is known that \( \mathcal{F} \) is polystable with \( \mu(\mathcal{F}) = \mu(E_\rho) = 0 \) (see [4, p. 23, Lemma 1.5.5]). Therefore, the subsheaf \( \mathcal{F} \) is uniquely characterized as follows: the subsheaf \( \mathcal{F} \) is the unique maximal coherent analytic subsheaf of \( E_\rho \) such that

- \( \mathcal{F} \) is polystable and
- degree(\( \mathcal{F} \)) = 0.

Note that the quotient \( E_\rho/\mathcal{F} \) is torsionfree, because if \( T \subset E_\rho/\mathcal{F} \) is the torsion part, then \( \varphi^{-1}(T) \subset E_\rho \), where

\[ \varphi : E_\rho \rightarrow E_\rho/\mathcal{F} \]

is the quotient map, also satisfies the above two conditions, while \( \mathcal{F} \subset \varphi^{-1}(T) \) if \( T \neq 0 \).

Consider the action \( \tau \) of \( \text{SL}(2, \mathbb{C}) \) on \( E_\rho \) constructed in (2.3). From the above characterization of the subsheaf \( \mathcal{F} \) in (2.6) it follows immediately that

\[ \tau(\text{SL}(2, \mathbb{C}) \times \mathcal{F}) = \mathcal{F}. \]  

As it was done for \( W_1 \), from (2.7) we conclude that \( \mathcal{F} \) is a holomorphic subbundle of \( E_\rho \).

As it was done for \( W_1 \), from (2.7) it follows that the flat connection \( \nabla^\rho \) on \( E_\rho \) preserves the subbundle \( \mathcal{F} \) in (2.6). Since \( \rho \) is irreducible, either
\( \mathcal{F} = 0 \) or \( \mathcal{F} = E_\rho \). The rank of \( \mathcal{F} \) is at least one because the semistable vector bundle \( E_\rho \) of degree zero has a nonzero stable subsheaf of degree zero. Therefore, we conclude that \( \mathcal{F} = E_\rho \). Consequently, \( E_\rho \) is polystable. \( \square \)

We may now ask whether the polystable vector bundle \( E_\rho \) in Theorem 2.2 is stable. The following lemma shows that \( E_\rho \) is not stable in general.

Let
\[
\delta : \Gamma \hookrightarrow \text{SL}(2, \mathbb{C}) \tag{2.8}
\]
be the inclusion map. This homomorphism \( \delta \) is clearly irreducible. Let \((E_\delta, \nabla^\delta)\) be the corresponding flat holomorphic vector bundle on \( M \).

**Lemma 2.3.** The above holomorphic vector bundle \( E_\delta \) is holomorphically trivial.

**Proof.** Recall that the vector bundle \( E_\delta \) is a quotient of \( \text{SL}(2, \mathbb{C}) \times \mathbb{C}^2 \). Consider the holomorphic map
\[
\text{SL}(2, \mathbb{C}) \times \mathbb{C}^2 \longrightarrow \text{SL}(2, \mathbb{C}) \times \mathbb{C}^2
\]
defined by \((g, v) \longmapsto (g, g(v))\). This map descends to a holomorphic isomorphism of vector bundles
\[
E_\delta \longrightarrow M \times \mathbb{C}^2
\]
over \( M \). Therefore, this descended homomorphism provides a holomorphic trivialization of \( E_\delta \). \( \square \)

**References**


