CONVERGENCE OF CIRCLE PACKINGS OF FINITE VALENCE TO RIEMANN MAPPINGS

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ABSTRACT. In [R-S], the conjecture by W. Thurston [Th] that the hexagonal circle packings can be used to approximate the Riemann mapping (in the topology of uniform convergence in compact subsets) is proved; and in [He], the derivatives of these approximations are shown to be convergent.

We show in Section 1 that the methods used in [R-S] in the case of hexagonal packings can be easily extended to the case of non-hexagonal circle packing with bounded radii ratios. We note that Stephenson had taken the major steps toward such an extension in [Ste]. Although he follows the overall strategy of [R-S], he replaces certain key steps by parabolistic arguments which have an interesting interpretation in terms of the flow of electricity in a network.

In Section 2, we show that the method of [He] can be extended to a more general class of non-hexagonal packings. Specifically, the restriction in [Ste] that the radii ratios be bounded can be replaced by the much weaker condition that the circle packings have uniformly bounded valence.

INTRODUCTION

Let \( P \) be a circle packing which "almost" fills up some fixed simply connected domain \( \Omega \). Suppose that the nerve (or graph) of \( P \) is equivalent to the 1-skeleton of a triangulation of the closed unit disk \( \bar{D} = \{ z \in \mathbb{C}; |z| \leq 1 \} \). Then by the Circle Packing Theorem, there is a circle packing \( P' \) in \( \bar{D} \) which is combinatorially equivalent to \( P \), such that all the boundary circles of \( P' \) are tangent to the unit circle. Let \( \Omega_P \) be the union of all triangles of centers of mutually tangent triples of circles in \( P \). As in [R-S], there is a simplicial map \( f_P : \Omega_P \rightarrow D \) which maps the centers of circles of \( P \) into the centers of circles
of $P'$. We may normalize $f_P$ by a Möbius transformation so that
\begin{equation}
\tag{0.1}
f_P(z_0) = 0, \quad f_P(z_1) > 0,
\end{equation}
for some pre-assigned points $z_0, z_1 \in \Omega$. Note that if $\partial \Omega_P$ is sufficiently close to $\partial \Omega$, then $z_0, z_1 \in \Omega_P$.

Let $f: \Omega \to D$ be the Riemann mapping with
\begin{equation}
\tag{0.2}
f(z_0) = 0, \quad f(z_1) > 0.
\end{equation}
Suppose that both the Caratheodory distance
\[d(\partial \Omega_P, \partial \Omega) = \max\{ \sup_{w \in \partial \Omega} \inf_{z \in \partial \Omega_P} |z - w|, \sup_{w \in \partial \Omega_P} \inf_{z \in \partial \Omega} |z - w|\}\]
and the mesh,
\[m(P) = \max\{\text{radii of circles of } P\}\]
are small, say, $\leq \epsilon$. We are interested in the following problem: How close is $f_P$ to the Riemann mapping $f$ in terms of $\epsilon$?

In [R-S], it is shown that if the packings $P$ are subpackings of regular hexagonal circle packings, then in any compact subset $K \subseteq \Omega$,
\begin{equation}
\tag{0.3}
\|f_P - f\|_K \to 0 \text{ as } \epsilon \to 0;
\end{equation}
where $\|\cdot\|_K$ denotes $L^\infty(K)$ - norm. Then in [He], it is further shown that
\begin{equation}
\tag{0.4}
\left\| \frac{\partial f_P(z)}{\partial z} - \frac{\partial f(z)}{\partial z} \right\|_K \to 0 \text{ as } \epsilon \to 0.
\end{equation}
It follows from (0.4) that $\left\| \frac{\partial f_P(z)}{\partial z} \right\|_K \to 0$ and $\|r_P - |f'|\|_K \to 0$, where $r_P$ is the ratio of the radii of a target circle to its source circle nearest to $z$ (see [Ro]).

In this paper, we will show that the results of [R-S] and [He] can be generalized easily for more general circle packings. The generalized approach of [He] leads to the following result.
Theorem 2.1. Assume that the valences of $P$ are bounded by a positive integer $k_0$. Let $K \subseteq \Omega$ be a compact subset. Then

\[
\|f_P - f\|_K + \left\| \frac{\partial f_P(z)}{\partial z} - \frac{\partial f(z)}{\partial z} \right\|_K + \left\| \frac{\partial f_P(z)}{\partial \bar{z}} \right\|_K \to 0 \text{ as } \epsilon \to 0.
\]

The rate of convergence depends only on $k_0$, $K$ and $\Omega$.

The rest of the paper will be organized into two sections which can be read independently. In the first section, we will explain how to generalize the method of [R-S] to obtain uniform convergence for $f_P$; here we work with the additional hypothesis that the ratios of radii of circles of $P$ are uniformly bounded. Note that the hypothesis of uniformly bounded radii ratios implies that the valences of $P$ are also uniformly bounded. In the second section we show how to modify the argument of [He] to prove Theorem 2.1.

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1. Convergence in the Case of Bounded Radii Ratio

Let us first start with the following easy case; and see how the method of [R-S] can be extended.

Theorem 1.1. Assume that there is some fixed constant $M \geq 1$ so that the ratio of the radii of any two circles in $P$ is bounded by $M$. Then in any compact subset $K \subseteq \Omega$,

\[
\|f_P - f\|_K \to 0 \text{ as } \epsilon \to 0.
\]

Recall that the proof of [R-S] uses a compactness property of circle packings, a length-area inequality, and an approximate rigidity result for large pieces of the hexagonal packing. Let us start with the compactness property.

Lemma 1.2 (Compactness Property). Let $P_n$ be a sequence of circle packings with uniformly bounded valences, and let $c_{0,n}$ be a circle of $P_n$. Suppose that:
(1) there is a sequence $k_n$ of positive integers with \( \lim k_n = \infty \), such that for each \( n \), the first \( k_n \) generations of \( P_n \) around \( c_{0,n} \) is the 1-skeleton of a triangulation of the 2-disk, with circles in the first \( k_n - 1 \) generations corresponding to the interior vertices of the triangulation;

(2) the centers of \( c_{0,n} \) form a bounded subset of points and their radii are uniformly bounded both from above and from below.

Then a subsequence of \( P_n \) converges to a circle packing \( P_\infty \) in the plane whose carrier is a simply connected domain.

**Proof.** First choose a subsequence of \( c_{0,n} \) which converges to a circle, say \( c_{0,\infty} \). Now, since the valences of \( P_n \) are uniformly bounded, we may choose a further subsequence such that the valences of \( c_{0,n} \) in \( P_n \) are the same, say equal to \( j_1 \). Let \( c_{i,n} \), \( i = 1, 2, \ldots, j_1 \), be the circles of \( P_n \) which are tangent to \( c_{0,n} \). Then by the ring lemma [R-S], the radii of these circles are also uniformly bounded both from above and from below. So, we may select a further subsequence such that for each \( i = 1, 2, \ldots, j_1 \), the sequence \( c_{i,n} \) converges to some circle \( c_{i,\infty} \). Because of the bounded valence property, the number of circles in any finite number of generations of \( P_n \) around \( c_{0,n} \) is uniformly bounded, so we may continue this process for all generations to get a sequence of subsequences. Then the diagonal sequence will satisfy the requirement of the lemma. \( \square \)

In [R-S], a length-area argument was used to show that the border circles of \( P' \) shrink to points, and consequently the Caratheodory distance \( d(\partial \Omega_p, \partial \Omega) \) tends to zero, as \( \epsilon \to 0 \). Those arguments extend to nonhexagonal packings in the following way.

**Lemma 1.3 (Length-Area Result).** Let \( P \) be a circle packing in \( \Omega \) such that the mesh of \( P \) and Caratheodory distance \( d(\partial \Omega_p, \partial \Omega) \) are bounded above by \( \epsilon \), and the ratios of the radii of any two circles of \( P \) are bounded above by \( M \). Let \( P' \) be the isomorphic circle packing of \( \mathbb{D} \) such that a circle \( c_0 \) in \( P \) nearest to \( z_0 \) corresponds to a circle \( c'_0 \) in \( P' \) centered at the origin. Then every border circle of \( P' \) has diameter less than \( CM/\log(1/\epsilon) \), where \( C \) depends only on \( \text{dist}(z_0, \partial \Omega) \).
Proof. Consider a border circle \( c \) of \( P \) and crosscuts of \( \Omega \) centered at the center of \( c \) with radii \( 5k\epsilon, \) \((k = 1, 2, \ldots, k_{\text{max}}), \) where \( k_{\text{max}} = \lfloor \text{dist}(z_0, \partial \Omega)/5\epsilon \rfloor - 2. \) The crosscut of radius \( 5k\epsilon \) meets a number of circles of \( P; \) select from these a chain \( S_k \) of tangent circles which separates \( c \) from \( c_0 \) and which has a border circle at each end. The chains \( S_1, S_2, \ldots \) so obtained will be disjoint. Note that if \( m \) disks of radius \( \geq r \) with disjoint interiors intersect a circle of radius \( \rho \) then \( m \leq 4\rho/r \) (since, if the disks had radius exactly \( r \), the area of the annulus of radii \( \rho \pm r \) would be an upper bound for \( m\pi r^2 \)). Therefore, the combinatorial length of \( S_k \) is \( \leq 20k\epsilon M. \) By the Length-Area Lemma [R-S], the circle \( c' \) in \( \Omega' \) which corresponds to \( c \) has radius at most

\[
\left\{ \sum_{k=1}^{k_{\text{max}}} \left( \frac{1}{20k\epsilon M} \right) \right\}^{-1} \leq \frac{CM}{\log \frac{1}{\epsilon}}. \]

Corollary 3.3 in [Schwarz, Lemma II] makes use of a length-area argument incorporating the isoperimetric inequality, and arrives at a much stronger conclusion. Its proof also extends, \textit{mutatis mutandi}, to nonhexagonal packings and gives the following result.

Lemma 1.4 (Length-Area-Isoperimetric Inequality). With the notations and hypotheses of Lemma 1.3, every border circle of \( \Omega' \) has diameter less than \( CMe^{\pi^2/80M} \) where \( C \) depends only on \( d(z_0, \partial \Omega) \).

Proof of Theorem 1.1. Let \( P_n \) be a sequence of circle packings satisfying the hypothesis of Theorem 1.1 such that \( \epsilon_n = \max\{d(\partial \Omega_P, \partial \Omega), \text{mesh } m(P)\} \) converges to 0. Let \( f_{P_n} : \Omega_{P_n} \to D \) be the associated mappings constructed in the introduction. We need to show that \( f_{P_n} \) converges to the Riemann mapping \( f : \Omega \to D \) which satisfies (0.2).

By the ring lemma of [R-S], \( f_{P_n} \) are all uniformly quasiconformal mappings; and thus a subsequence (still denoted by the same notation) converges to some mapping, say \( f' : \Omega \to D \) which is either quasiconformal or a constant. But Lemma 1.3 implies the image of \( f' \) is \( D \) and thus \( f' \) is a quasiconformal homeomorphism of \( \Omega \) onto \( D. \) Clearly \( f' \) satisfies (0.2).

It remains to show that \( f' \) is conformal. Let \( \mu_n \) be the complex dilatation of \( f_{P_n}. \) We claim that \( \mu_n \) converges to zero at almost every point of \( \Omega. \) Suppose
by contradiction that there are a set of \( w \in \Omega \) of positive measure, a \( \delta > 0 \), and a subsequence (still denoted by \( \mu_n \)) of \( \mu_n \) such that

\[
|\mu_n(w)| \geq \delta.
\]

From the definition of the mappings \( f_{P_n} \), it follows that for each \( n \) there is a triple of mutually tangent circles \( c_{0,n}, c_{1,n} \) and \( c_{2,n} \) of \( P_n \) such that:

1. \( w \) is contained in the triangle of centers of \( c_{0,n}, c_{1,n} \) and \( c_{2,n} \);
2. if \( c_{0,n}', c_{1,n}' \) and \( c_{2,n}' \) are the corresponding circles of \( P_n' \) (constructed in the introduction), then the triangle of centers of \( c_{0,n}', c_{1,n}' \) and \( c_{2,n}' \) is not nearly similar to the triangle of centers of \( c_{0,n}, c_{1,n} \) and \( c_{2,n} \).

Clearly, the packings \( P_n \) and \( c_{0,n} \), and \( P_n' \) and \( c_{0,1} \) satisfy the conditions of Lemma 1.2 after transformations by affine similarities. It follows that (some subsequence) of the transformed packings \( P_n \) and \( P_n' \) converge to some infinite packings, say, \( P_\infty \) and \( P_\infty' \). Since the circles of \( P_n \) have uniformly bounded ratios, the carrier of \( P_\infty \) must fill up the whole plane. Then by the rigidity result of [R-S], \( P_\infty \) and \( P_\infty' \) are similar. This contradicts the above conclusion that the triangle of centers of \( c_{0,n}', c_{1,n}' \) and \( c_{2,n}' \) is not nearly similar to the triangle of centers of \( c_{0,n}, c_{1,n} \) and \( c_{2,n} \). Hence \( \mu_n(w) \) must converge to zero.

The rest of the proof is just the same as [R-S]. □

2. CONVERGENCE IN THE CASE OF BOUNDED VALENCE

In this section, we will prove the following stronger result under the weaker condition of bounded valence.

**Theorem 2.1.** Assume that the valences of \( P \) are bounded by a positive integer \( k_0 \). Let \( K \subseteq \Omega \) be a compact subset. Then

\[
\|f_P - f\|_K + \left\| \frac{\partial f_P(z)}{\partial z} - \frac{\partial f(z)}{\partial z} \right\|_K + \left\| \frac{\partial f_P(z)}{\partial \bar{z}} \right\|_K \to 0 \quad \text{as} \quad \epsilon \to 0.
\]

The rate of convergence depends only on \( k_0, K \) and \( \Omega \).

We will begin with the definition of some constants similar to the hexagonal circle packing constants \( s_n \) introduced in [R-S]. Let \( n \) be an integer \( \geq 2 \). Suppose that \( P_n \) is a circle packing in \( \mathbb{C} \) such that

1. The valence of \( P_n \) is bounded by \( k_0 \);
(2) The radii of the circles of $P_n$ are all bounded from above by some positive $r$; and there is some "center" circle $c_0$ of $P_n$ such that carrier of $P_n$ contains a closed disk of radius $(2n+1)r$ which is concentric with $c_0$.

Here the carrier of a circle packing is understood to be the union of all the closed disks and the triangular interstices bounded by the circles.

Let $P'_n$ be any other circle packing of $C$ combinatorially equivalent to $P_n$. Suppose that $c_0$ is surrounded by $c_1, c_2, \ldots, c_k$ in $P_n$; and $c'_0, c'_1, c'_2, \ldots, c'_k$ are the circles in $P'_n$ corresponding to $c_0, c_1, c_2, \ldots, c_k$ respectively. Let
\[
d_1(P_n, P'_n) = \max \left\{ \frac{\text{Radius } (c'_j)}{\text{Radius } (c'_l)} : 0 \leq j, l \leq k \right\},
\]
and let
\[
s(P_n) = \sup_{P'_n}[d_1(P_n, P'_n) - 1].
\]

The following theorem follows *mutatis mutandi* from [He]. Actually the proof under the new setting is more direct.

**Theorem 2.2.** There is a constant $C$ depending only on $k_0$ such that
\[
s(P_n) \leq \frac{C}{n}.
\]

**Proof.** Let us normalize $P_n$ so that $c_0$ becomes the unit circle $\partial D = \{z \in \mathbb{C}; |z| = 1\}$. Let $m$ be a positive integer with $m < n$. For $m = 1$, let $P_1$ be the first generation of $P_n$ around $c_0$; for $m > 1$, let $P_m$ be the subpacking of $P_n$ which consists of all circles which are contained in the closed disk of radius $(2m+1)r$ centered at the origin. Let $G_m$ be the Schottky group generated by inversions on the circles of $P_m$. Denote by $J_m$ the union of images by the transformations of $G_m$ of the interstices bounded by circles in $P_m$. We have the following area estimate.

**Lemma 2.3.** For any Möbius transformation $h$ preserving the unit disk $D$, we have
\[
|D \setminus h(J_m)| \leq C_3/m^2,
\]
where $C_3$ depends only on $k_0$. 

Proof. The proof follows by the inductive argument of Lemma 3.1 of [He] with the following modifications.

(a) Since $P_1$ is the first generation of $P_n$ around $c_0$ and the valence of $P_n$ is bounded by $k_0$, Lemma 3.2 of [He] also holds for $J_1$ defined here; where $\delta_2$ and $C_3$ denote some positive constants depending only on $k_0$.

(b) $H_1, H_2, \cdots, H_n$ in [He] should be replaced by $P_1, P_2, \cdots, P_n$. The other notations need no changes. For example, $U_1$ is the complement in $\hat{C}$ of the union of interstices and disks bounded by circles in $P_1$, etc.

(c) The last paragraph of page 405 should be replaced by:
Let $A$ be a disk bounded by some circle $c = \partial A$ of $P_k \setminus P_{k-1}$, $2 \leq k \leq l - 1$, and let $z_A$ be its center. Since the closed disk of radius $(2(l-k)+1)r$ centered at the center of $c$ is contained in the closed disk of radius $(2l+1)r$ centered at the center of $c_0$, we have by inductive hypothesis,

$$|\gamma(A \setminus J_1)| \leq \frac{C_3}{\pi(l-k)^2}|\gamma(\Delta)|.$$

Lines 4–11 of page 406 in [He] should be replaced by:
If $z \in \Delta$, $\partial \Delta \in P_k \setminus P_{k-1}$, $2 \leq k \leq l - 1$, then $|z| \geq rk \geq k$, since

$$r \geq \text{radius}(D) = 1.$$

Thus

$$\frac{C_3}{\pi(l-k)^2} \leq \frac{C_3}{\pi[(l-|z|)+1]^2}.$$

Let $\rho: U \rightarrow [0, 1]$ be the following function

$$(3.9') \quad \rho(z) = \rho(|z|) = \min \left( \frac{C_3}{\pi[(l-|z|)+1]^2}, 1 \right).$$

Then $\eta \leq \rho$ on $U_1$, and hence $\eta \gamma \leq \eta \circ \gamma^{-1} \leq \rho \circ \gamma^{-1}(z)$. By the Ring lemma, the sizes of the circles in $P_1$ are uniformly bounded, so there is some $R > 1$ depending only on $k_0$ such that $\{|z| > 3R\} \cup \{\infty\} \subseteq U_1$.

Let $V = \{|z| > 3R\} \cup \{\infty\}$. Then $\cdots$.

(d) In lines 1 and 2 of page 407 in [He], we should let $\sigma_1 = \frac{3R}{z}$; and then line 3 should be replaced by
Similarly, (3.12) and (3.7) in [He] should be replaced by

\[ (3.12') \quad \frac{1}{|\gamma(V)|} \int_{\gamma(V)} \rho \circ \gamma^{-1} dx dy \leq \frac{9R^2}{(1 - \sqrt{1 - \delta_2/2})^2 \ell^2} + \frac{C_3}{\pi(1 - \delta_2/2)\ell^2} \]

and

\[ (3.7') \quad \frac{|\gamma(U_1 \setminus J)\gamma(V)|}{|\gamma(V)|} \leq \frac{9R^2}{(1 - \sqrt{1 - \delta_2/2})^2 \ell^2} + \frac{C_3}{\pi(1 - \delta_2/2)\ell^2}. \]

(e) Line (3.14) of [He] should be replaced by

\[ (3.14') \quad C_3 = \max \left\{ 4\pi, \frac{18R^2 \pi(1 - \delta_2) (1 - \delta_2/2)}{(1 - \sqrt{1 - \delta_2/2})^2 \delta_2} \right\} \]

Proof of Theorem 2.2 (continued). The rest of the proof is identical to that of the estimate on \( s_n \) in [He, §2].

Proof of Theorem 2.1. The proof is the same as in the case of hexagonal packings (see also [Ro]) except the following lemma which replaces the length-area argument.
Lemma 2.4. Let $P$ be a circle packing in $\Omega$ such that the mesh of $P$ and Carathéodory distance $d(\partial \Omega, \partial \Omega)$ are bounded above by $\epsilon$, and the valence of $P$ is bounded above by $k_0$. Let $P'$ be the isomorphic circle packing of $D$ and let $f_D : \Omega \rightarrow D$ be the mapping constructed in the introduction such that $f_D(z_0) = 0$. Then the Carathéodory distance $d(\partial f_D(\Omega), \partial D)$ is less than or equal to $Ce^\alpha$, where $C$ and $\alpha$ are some positive constants which depend only on $k_0$ and $d(z_0, \partial \Omega)$.

Proof. We need only show that every border circle of $P'$ has diameter less than or equal to $Ce^\alpha$. By the Ring Lemma of [R-S], $f_D$ is $C_4$-quasiconformal onto its image where $C_4$ depends only on $k_0$. Let $c$ be a border circle of $P$ and let $c'$ be the corresponding border circle in $P'$. We assume that $c$ is chosen so that $c'$ has the biggest radius among the border circles of $P'$; and we want to show that the radius of $c'$ is bounded by $Ce^\alpha$. Consider the annulus $A$ bounded by the circle $c$ and the circle $c_1$ concentric with $c$ which passes through $z_0$. Because the radius of $c$ is $\leq \epsilon$ and the radius $c_1$ is at least $\text{dist}(z_0, \partial \Omega)$, the modulus of the annulus $A$ is at least $C_5|\log \epsilon|$. It follows that the extremal length between $c \cap \Omega_\epsilon$ and $c_1 \cap \Omega_\epsilon$ in the domain $A \cap \Omega_\epsilon$ is at least $C_5|\log \epsilon|$. Since $f_D$ is $C_4$-quasiconformal, the extremal length between $\gamma = f_D(c \cap \Omega_\epsilon) = c' \cap f_D(\Omega_\epsilon)$ and $\gamma_1 = f_0(c_1 \cap \Omega_\epsilon)$ in $f_D(A \cap \Omega_\epsilon)$ is at least $C_4C_5|\log \epsilon|$. Since $f_D(z_0) = 0$ and $z_0 \in c_1$, $\gamma_1$ is a curve which separates the domain $f_D(\Omega_\epsilon)$ and which passes the origin 0. On the other hand, because $c'$ is a biggest border circle, the domain $f_D(\Omega_\epsilon)$ contains a cone neighbourhood of the center of $c'$ of an angle bounded from below by some universal positive constant. It follows that the radius of $c'$ is bounded from above by $Ce^\alpha$, for some $\alpha > 0$ and $C > 0$. This completes the proof. □

References


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