RIGIDITY THEOREMS FOR PRIMITIVE FANO 3-FOLDS

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INTRODUCTION

A fundamental problem in the classification theory of algebraic manifolds is how many different projective structures can exist on a given manifold $X_0$. The answer may vary from only few structures to the existence of moduli spaces.

In case $X_0$ is the projective space $\mathbb{P}_n$, it is known by Hirzebruch-Kodaira [HK] and Yau [Y] that any projective manifold homeomorphic to $X_0$ is again $\mathbb{P}_n$. For $n$ even this requires the existence of a Kähler-Einstein metric on the potential candidate $X$ homeomorphic to $\mathbb{P}_n$. But already for the quadric $Q_n$ the analogous result is known only in case $n$ is odd (Brieskorn [Br]). Even the surface case is unsettled: there might be a surface of general type which is homeomorphic to $\mathbb{P}_1 \times \mathbb{P}_1$. The projective structures on $\mathbb{P}_1 \times \mathbb{P}_1$ of Kodaira dimension $\neq 2$ are just the ruled surfaces $\mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(-n)), n \in \mathbb{N}$ even.

Unknown are also the possible projective structures on $\mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(-1))$ different from $\mathbb{P}(\mathcal{O}_{\mathbb{P}_1} \oplus \mathcal{O}_{\mathbb{P}_1}(-n)), n \in \mathbb{N}$ odd, which again are suspected not to exist.

The next interesting surfaces to look at would be Fano surfaces $X_0$ (i.e. $-K_{X_0}$ is ample), which are classically called del Pezzo surfaces. It is well known that Barlow's surface (which is of general type) is homeomorphic to $\mathbb{P}_2$ blown up in 8 points. But for instance it is unknown whether there is a surface of general type homeomorphic to $\mathbb{P}_2$ blown up in, say, 2 points.

The aim of this paper is the study of projective structures on certain Fano 3-folds $X_0$. As we already saw in the surface case, difficulties arise to exclude possible $X$ with $K_X$ ample, or $K_X$ nef ($\langle K_X.C \rangle \geq 0$ for every curve $C$). In the 3-fold case this can be excluded if we know that $\chi(\mathcal{O}_X) > 0$ using a result
of Miyaoka. Of course, $\chi(O_{X_0}) = 1$, so we ask whether $\chi(O_X)$ is a topological invariant for projective 3-folds.

Clearly $\dim H^i(X, O_X)$ are topological invariants for $i = 1, 2$ if $b_2 \leq 2$ but whether $\dim H^3(X, O_X)$ is also invariant is a deep unsolved problem. We can force $H^3(X, O_X)$ to vanish by requiring $b_3(X_0) = 0$. So we deal only with Fano 3-folds with vanishing $b_3$. In case $b_2(X_0) = 1$ those $X_0$ are well understood and easy to deal with: $X_0$ is $\mathbb{P}_3, Q_3$, one 3-fold of index 2 and a family of index 1; any $X$ homeomorphic to $X_0$ is again of the same type.

So we turn to the case $b_2 \geq 2$; we will restrict ourselves here only to $b_2 = 2$, Fano 3-folds with $b_2 \geq 2$ are classified by Mori-Mukai [MM 1,2], the most interesting case being $b_2 = 2$ or 3. Such a $X_0$ is called primitive if it is not the blow-up of another 3-fold along a smooth curve. In order not to overload the paper we will also restrict ourselves to primitive $X_0$; but certainly similar results can be proved also in the imprimitive case using the same methods.

Our result is now:

**Theorem.** Let $X_0$ be a primitive Fano 3-fold with $b_2 = 2, b_3 = 0$. Let $X$ be a projective smooth 3-fold homeomorphic to $X$. Then either $X \simeq X_0$, or $X \simeq \mathbb{P}(E)$ with a rank 2-vector bundle $E$ on $\mathbb{P}_2$ whose Chern classes $(c_1, c_2)$ belong to the following set: $\{(0, 0), (-1, 1), (-1, 0), (0, -1), (0, 3)\}$ or $X = \mathbb{P}(O_{P_1}(a) \oplus O_{P_1}(b) \oplus O_{P_1}(c))$ with $a + b + c \equiv 0(3)$.

In fact, $X_0$ is by the Mori-Mukai classification of the form $\mathbb{P}(V)$ with $V$ a 2-bundle on $\mathbb{P}_2$ of the form:

$$O \oplus O(-n) \text{ with } 0 \leq n \leq 2, \quad T_{\mathbb{P}_2}, \quad \text{or } V \text{ is given by an extension}:$$

$$0 \to O_{\mathbb{P}_2}(-2) \to O_{\mathbb{P}_2}^3 \to V \to 0.$$

Now $E$ is just a bundle topologically isomorphic to $V$, i.e. with the same Chern classes.

Using analogous methods, we are able in § 7 to answer a question asked in [C2]: if $Z_0$ is a Moishezon non-projective twistor space, does there exist a projective threefold $Z$ which is homeomorphic to $Z_0$? The answer is no, at least when $b_2$ is odd. Let us recall that such a $Z_0$ is the first known example of a manifold of class $\mathcal{C}$ (i.e. : bimeromorphic to a compact Kähler one)
admitting arbitrarily small deformations which are not in the class $\mathcal{C}$. This exhibits another pathology of these $Z_0$. However, it would be interesting to have an example of a Moishezon manifold $Z_0$, diffeomorphic to some projective $Z$, but admitting arbitrarily small deformations which are not in $\mathcal{C}$.

The relationship with the other investigations of this paper is that $Z_0$ is nearly Fano in the sense that the Kodaira dimension of its anticanonical bundle is $3 = \dim_{\mathbb{C}}(Z_0)$.

1. Basic material on Fano 3-folds

Let $X$ be a projective manifold with canonical bundle $K_X$. $X$ is called Fano if $-K_X$ is ample. Fano manifolds are simply connected and satisfy

$$H^q(X, \mathcal{O}_X) = 0, q \geq 1$$

by Kodaira's vanishing theorem.

1.1. In case $b_2(X) = 1$ all Fano 3-folds are classified by Iskovskih, Shokurov and also Mukai [Is 1,2], [Mu]. Those with $b_3(X) = 0$ can be listed as follows:

(a) $X = \mathbb{P}^3$,
(b) $X = \mathbb{Q}_3$, the 3-dimensional smooth quadric,
(c) $X$ is of index 2, i.e. $-K_X = 2L$ with $L \in \text{Pic}(X)$ the ample generator of $\text{Pic}(X) \cong \mathbb{Z}$, and $L^3 = 5$. $X$ is unique by these properties and usually called $V_5$.
(d) $X$ is of index one, i.e. $-K_X = L$; $L^3 = 22$. These build up a family and we write $X = A_{22}$.

1.2. Fano 3-folds $X$ with $b_2 \geq 2$ are classified in [MM 1,2], we will only consider those with $b_2 = 2$. First recall that $X$ is called primitive if it is not the blow-up of a 3-fold $Y$ with $b_2 = 1$ along a smooth curve. It is obvious that this is equivalent to saying that $X$ is not the blow up of any 3-fold along a smooth curve. The classification heavily depends on Mori's theory of extremal rays, cone theorem etc. We will make freely use of this and refer e.g. to [KMM]. $X$ being Fano with $b_2 = 2$ we have exactly two extremal maps $R_i$ on $X$ giving rise to contractions

$$\varphi_i : X \to Y_i.$$
Then \( \text{Pic}(Y_i) \simeq \mathbb{Z} \), in fact \( Y_i \) are Fano with only terminal singularities with \( b_2 = 1 \), so fix ample generators \( L'_i \) on \( Y_i \) and put

\[ L_i = \varphi_i^*(L'_i). \]

**Lemma 1.3.** \( \text{Pic}(X) = \mathbb{Z}.L_1 \oplus \mathbb{Z}.L_2. \)

**Proof.** [MM 1] \( \Box \)

1.4. We now give a table of all primitive (five) Fano 3-folds \( X \) with \( b_2(X) = 2, \ b_3(X) = 0 \) and their relevant numerical properties needed in this paper, according to [MM 1,2]. As to notations, let \( D_{2,1} \) denote a smooth divisor of bidegree \((2,1)\) in \( \mathbb{P}_2 \times \mathbb{P}_2 \) and let \( W_4 \) be the Veronese cone in \( \mathbb{P}_6 \).

The last column means the following: \((a, b)\) is the pair determined by the equation (observe (1.3) !) \(-K_X = aL_1 + bL_2.\)

<table>
<thead>
<tr>
<th>( X )</th>
<th>( Y_1 )</th>
<th>( Y_2 )</th>
<th>(-K^3_X)</th>
<th>( L^3_1 )</th>
<th>( L^3_1L^2_2 )</th>
<th>( L^3_1L^3_2 )</th>
<th>( (a, b) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{P}_1 \times \mathbb{P}_2 )</td>
<td>( \mathbb{P}_1 )</td>
<td>( \mathbb{P}_2 )</td>
<td>54</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \mathbb{P}(T_{\mathbb{P}_2}) )</td>
<td>( \mathbb{P}_2 )</td>
<td>( \mathbb{P}_2 )</td>
<td>48</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( \mathbb{P} (O_{\mathbb{P}<em>2} \oplus O</em>{\mathbb{P}_2} (-1)) )</td>
<td>( \mathbb{P}_2 )</td>
<td>( \mathbb{P}_3 )</td>
<td>56</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \mathbb{P} (O_{\mathbb{P}<em>2} \oplus O</em>{\mathbb{P}_2} (-2)) )</td>
<td>( \mathbb{P}_2 )</td>
<td>( W_4 )</td>
<td>62</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>( D_{2,1} )</td>
<td>( \mathbb{P}_2 )</td>
<td>( \mathbb{P}_2 )</td>
<td>30</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

1.5. The structure of a Mori contraction \( \varphi : X \to Y \) of an extremal ray on a smooth 3-fold \( X \) is completely determined by [Mo] and given in the following list:

(a) \( \varphi \) is a modification. Then either \( \varphi \) is the blow-up of a smooth curve in the smooth 3-fold \( Y \). Or there is an unique irreducible divisor \( E \subset X \) contracted by \( \varphi \) to a point and either

(a1) \( E \simeq \mathbb{P}_2 \) with normal bundle \( N_E = \mathcal{O}(a), a = -1, -2 \)

(a2) \( E \simeq \mathbb{P}_1 \times \mathbb{P}_1 \) with \( N_E = \mathcal{O}(-1, -1) \)

(a3) \( E \) is a (singular) quadric cone with \( N_E = \mathcal{O}(-1) \).

(b) \( \dim Y = 2 \). Then \( \varphi \) is a \( \mathbb{P}_1 \)-bundle or a conic bundle.

(c) \( \dim Y = 1 \). Then \( \varphi \) is a \( \mathbb{P}_2 \) -bundle, a quadric bundle, or the general fibre \( F \) of \( \varphi \) is a del Pezzo surface with \( 1 \leq K^2_F \leq 6 \).

(d) \( \dim Y = 0 \) and \( X \) is Fano with \( b_2 = 1 \).
1.6. We now describe the structures of $\varphi_i$ in the table (1.4) according to (1.5); see again [MM 1,2].

In case $X = P_1 \times P_2$ this is obvious; for $X = P(T_{P_2})$ we have two $P_1$-bundle structures. $P(\mathcal{O} \oplus \mathcal{O}(-1))$ is a $P_1$-bundle over $P_2$ and the blow up of a point in $P_3$. $P(\mathcal{O} \oplus \mathcal{O}(-2))$ is a $P_1$-bundle over $P_2$ and also the blow-up of the unique singular (quadruple) point on $W_4$; the exceptional divisor $D$ is $P_2$ with normal bundle $\mathcal{O}(-2)$. Finally $D_{2,1}$ is a $P_1$-bundle over $Y_1 = P_2$ via $\varphi_1$ and a conic bundle over $Y_2 = P_2$ via $\varphi_2$ (by our choice of $(a, b)$) with $\varphi_i$ being the restriction of the projection $pr_i$ to $P_2$.

The $P_1$-bundle structure is given as $P(F)$ with $F$ a 2-bundle on $P_2$ defined by an extension

$$0 \to \mathcal{O}(-2) \to \mathcal{O}^3 \to F \to 0.$$ 

2. Topological invariants

Let $X_0$ be a smooth projective 3-fold with $b_1 = 0, b_2 \leq 2$ and assume $X$ to be another smooth projective 3-fold homeomorphic to $X_0$. By Hodge decomposition:

$$H^q(X, \mathcal{O}_X) = H^q(X_0, \mathcal{O}_{X_0}) = 0$$

for $q = 1, 2$.

Although $b_3(X) = b_3(X_0)$, the Hodge decomposition of $H^3$ might a priori be quite different, so let us formulate:

PROBLEM 2.1. Is $h^3(X, \mathcal{O}_X)$ a topological invariant for projective 3-folds? (Equivalently, we could ask for $h^{2,1}$, and the same can be asked also in general for $h^{2,0}$).

Because of the unsolved problem (2.1) we will always assume $b_3(X_0) = 0$. Then clearly

$$H^3(X, \mathcal{O}_X) = H^3(X_0, \mathcal{O}_{X_0}) = 0,$$

and hence:

$$\chi(\mathcal{O}_X) = \chi(\mathcal{O}_{X_0}) = 1.$$ 

This vanishing has far-reaching consequences by the following result of Miyaoka [Mi], which is an immediate consequence of his inequality $c_1^2 \leq 3c_2$. 


Theorem 2.2. Let $X$ be a projective 3-fold with $K_X$ nef. Then $\chi(O_X) \leq 0$.

Corollary 2.3. Let $X_0$ be a Fano 3-fold with $b_3 = 0$, $X$ a projective 3-fold homeomorphic to $X_0$. Then $K_X$ is not nef.

In particular, $X$ carries an extremal ray by [Mo] and we can use Mori theory to examine the structure of $X$ (if $b_2 \geq 2$). This will be done in §4. If we don’t assume $b_3 = 0$ in (2.3) then there is no apparent reason why $K_X$ could not be ample for instance.

We now come to important methods to determine $K_X$ going back to Hirzebruch-Kodaira [HK]. Here let us suppose $X_0$ to be a Fano 3-fold with $b_2 \leq 2$ for simplicity. In case $b_2 = 1$ we fix an ample generator $L_0$ on $X_0$. In case $b_2 = 2$ we let $L_1, L_2$ be as in (1.3). Then if $b_2 = 1$ we can write

$$c_1(X) = c_1(X_0) + 2sc_1(L), \quad s \in \mathbb{Z}$$

and for $b_2 = 2$:

$$c_1(X) = c_1(X_0) + 2(s_1c_1(L_1) + s_2c_1(L_2)).$$

Observe that the factor 2 comes from the invariance of the Stiefel-Whitney class $w_2(X)$ which is the residue class of $c_1(X)$ in $H^2(X, \mathbb{Z}_2)$. Then we have:

Proposition 2.4. Let $\mathcal{G}$ be a holomorphic line bundle on $X_0$, $\tilde{\mathcal{G}}$ the corresponding one on $X$. Then

- (a) $\chi(X, \tilde{\mathcal{G}}) = \chi(X_0, \mathcal{G} \otimes L^s)$ if $b_2(X_0) = 1$
- (b) $\chi(X, \tilde{\mathcal{G}}) = \chi(X_0, \mathcal{G} \otimes L_1^{s_1} \otimes L_2^{s_2})$ if $b_2(X_0) = 2$.

The line bundle $\tilde{\mathcal{G}}$ corresponding to $\mathcal{G}$ means the following: $\mathcal{G}$ can be viewed as a topological line bundle on $X$ and since $Pic(X) \simeq H^2(X, \mathbb{Z})$ by $H^q(X, O_X) = 0, q = 1, 2$, its carries a unique holomorphic structure, namely $\tilde{\mathcal{G}}$.

Proof. We prove only (a), (b) being completely the same. By Riemann-Roch (see e.g. [Hi])

$$\chi(X, \tilde{\mathcal{G}}) = \left[e^{\frac{1}{2}c_1(X) + c_1(\mathcal{G})} \sum_{i=0}^{\infty} \hat{A}_i(p_1, p_2, ...)\right]_3,$$
where \( p_i \) are the Pontrjagin classes of \( X \) and \( A_i \) certain universal functions. Since \( p_i(X) = p_i(X_0) \) (Novikov) and since \( c_1(X) = c_1(X_0) + 2sc_1(L) \) by assumption, we obtain:

\[
\chi(X, \mathcal{G}) = \left[ e^{\frac{1}{2} c_1(X_0) + c_1(L^s) + c_1(\mathcal{G})} \cdot \sum A_i(p_1(X_0), p_2(X_0), ...) \right]_3
\]

\[
= \chi(X_0, \mathcal{G} \otimes L^s),
\]
again by Riemann-Roch. □

**Remark 2.5.** Of course the arguments above are independent of dimension 3 and of the Fano property of \( X_0 \). The only requirements we need are that \( c_1(X) - c_1(X_0) \) contains a holomorphic line bundle on \( X \), that \( \mathcal{G} \) has a holomorphic structure \( \mathcal{G} \) on \( X \), and that, moreover: \( \text{Pic}(X_0) = \mathbb{Z} \) or \( \mathbb{Z}^2 \). We finish this section by stating for later use the following well-known result:

**Proposition 2.6.** Let \( S \) be an algebraic surface with \( \pi_1(S) \) finite and \( b_2(S) = 1 \). Then \( S \simeq \mathbb{P}_2 \).

A proof can be found in [BPV, p. 135].

3. **Fano 3-folds with \( b_2 = 1 \)**

We are going to study 3-folds homeomorphic to Fano 3-folds with \( b_2 = 1 \). From (2.3) we immediately obtain:

**Theorem 3.1.** If \( X \) is a projective 3-fold homeomorphic to the Fano 3-fold \( X_0 \) with \( b_2 = 1, b_3 = 0 \), then \( X \) is again Fano and in fact \( X \simeq X_0 \) resp. is of type \( A_{22} \) if \( X_0 \) is of type \( A_{22} \).

*Proof.* By (2.3) \( K_X \) is not nef. Since \( \text{Pic}(X) \simeq \mathbb{Z}, -K_X \) must be ample, so \( X \) is Fano. By the classification of Fano 3-folds it suffices now to prove \( c_1(X) = c_1(X_0) \). Writing \( c_1(X) = c_1(X_0) + 2sc_1(L) \) \((s \geq -\frac{1}{2} \text{ index } (X_0))\), \( L \) the ample generator, we obtain from (2.4):

\[
\chi(L^s) = \chi(O_X) = 1.
\]

Using Riemann-Roch for instance it is easy to solve this equation to obtain \( s = 0 \). □
Of course (3.1) is known by [HK] for $\mathbb{P}_3$, by [Br] for $Q_3$ and in the other cases by [LS]. We should mention that the use of (2.3) can be avoided by solving

$$\chi(L^s) = \chi(O_X) = 1$$

also for all $s < 0$. In fact $\chi(L^s) = -h^3(O_X)$ for $s < 0$, hence $\chi(L^s) \neq 1$.

This arguments works in all odd dimensions, on the other hand it is not known whether there is a projective $n$-fold $X$, $n$ even, homeomorphic to a quadric $Q_n$, with $K_X$ ample.

**Remark 3.2.** If we don't assume $b_3 = 0$ in (3.1) we cannot conclude $\chi(O_X) > 0$ and hence $K_X$ could be ample. If $K_X$ is known not to be ample or trivial, then clearly $X$ is Fano and one can apply Iskovshih's classification to $X$. We exclude the case $K_X = O_X$ as follows. Assume $K_X = O_X$. By the invariance of $w_2$, $X_0$ is a Fano 3-fold of index 2 or 4. Since $X_0 \neq \mathbb{P}_3$, $X_0$ has in fact index 2. Hence in the equation

$$0 = c_1(X) = c_1(X_0) + 2sc_1(L)$$

we have $s = -1$.

Let $\tilde{L} \in \text{Pic}(X)$ be the ample generator. By (2.4) we have

$$\chi(X, \tilde{L}^t) = \chi(X_0, L^{t-1}),$$

in particular

$$\chi(X, \tilde{L}) = \chi(O_{X_0}) = 1.$$

By Riemann-Roch we get

$$\chi(X, \tilde{L}) = \frac{c_1(\tilde{L})^3}{6} + \frac{1}{12}c_1(\tilde{L}) \cdot c_2(X).$$

Miyaoka's inequality $c_3(X) \leq 3c_2(X)$ ([Mi]) yields $c_1(\tilde{L}) \cdot c_2(X) \geq 0$. We even must have strict inequality; if $c_1(\tilde{L}) \cdot c_2(X) = 0$ we would get (by $b_2(X) = b_4(X) = 1$) $c_2(X) = 0$, so $X$ would be covered by a torus [Y], contradiction.

Thus it is possible, using (1) and (2), to compute the pair $(c_1(\tilde{L})^3, c_2(X))$, since by Iskovshih, $1 \leq c_2(\tilde{L})^3 = c_1(L)^3 \leq 4$ (observe $b_3(X_0) > 0$).
Identifying $H^2(X_0, \mathbb{Z})$ and $H^4(X_0, \mathbb{Z})$ with $\mathbb{Z}$, the intersection product is just multiplication, and we obtain: $(c_1(L)^3, c_2(X)) = (1, 10), (2, 8), (3, 6), (4, 4)$. Now consider the Pontrjagin class

$$p_1(X) = c_1^2(X) = c_2(X).$$

$p_1(X)$ is a topological invariant. We compute easily in the four cases: $p_1(X_0) = -8, -4, 0, 4$. On the other hand $p_1(X) = -c_2(X) = -10, -8, -6, -4$, contradiction.

We can try to determine the type of $K_X$ by (2.4). In fact, (2.4) gives, if we write $c_1(X) = c_1(X_0) + 2sc_1(L)$ as in (2.4),

$$\chi(X, O_X) = \chi(X_0, L^s).$$

Since $\chi(X, O_X) = 1 - h^3(O_X)$ and $h^3(O_X) \leq \frac{b_3(X_0)}{2} = \frac{b_3(X_0)}{2}$, we obtain:

$$\chi(X_0, L^s) \geq 1 - \frac{b_3(X_0)}{2}.$$  

Observe that we may assume $s < 0$, otherwise $X$ is already Fano. Now we can go to the list of Fano 3-folds $X_0$ with $b_2 = 1, b_3 > 0$ (of index 1 or 2); $b_3$ being known, we can try to solve the above inequality using Riemann-Roch on $X_0$. Then we obtain setting $c_1(X) = \mu c_1(L) = (2s + \tau), \tau$ the index of $X_0$:

<table>
<thead>
<tr>
<th>index</th>
<th>$L^3$</th>
<th>$\frac{b_3}{2}$</th>
<th>$s$</th>
<th>$\mu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>1</td>
<td>21</td>
<td>$-2 \geq s \geq -5$</td>
<td>$-2, -4, -6, -8$</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>10</td>
<td>$-2, -3$</td>
<td>$-2, -4$</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>5</td>
<td>$-2$</td>
<td>$-2$</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>2</td>
<td>$-2$</td>
<td>$-2$</td>
</tr>
</tbody>
</table>

(3.3)

| 1     | 2     | 52              | $-1 \geq s \geq -5$ | $-1, -3, \ldots, -9$ |
| 1     | 4     | 30              | $-1 \geq s \geq -3$ | $-1, -3, -5$         |
| 1     | 6     | 20              | $-1, -2$           | $-1, -3$           |
| 1     | 8     | 14              | $-1, -2$           | $-1, -3$           |
| 1     | $8 < L^3 \leq 18$ | $\cdots$ | $-1$ | $-1$ |

In any case there are only finitely many possibilities for $K_X$; in a lot of cases only the “dual” possibility $c_1(X) = -c_1(X_0)$. At least we can conclude that all the $X$ homeomorphic to a given Fano 3-fold $X_0$ with $b_2 = 1$ form a bounded family.
4. Structure of Mori contractions on topological primitive Fano 3-folds with $b_2 = 2, b_3 = 0$ and the main result

Let $X_0$ always denote a Fano 3-fold with $b_2 = 2, b_3 = 0$. We assume that $X_0$ is primitive, i.e. $X_0$ is not the blow-up of a (Fano) 3-fold along a smooth curve. Let $X$ be a projective smooth 3-fold homeomorphic to $X_0$. By (2.3) we know that $K_X$ is not nef, so there is a contraction $\varphi : X \to Y$ of an extremal ray on $X$.

We let $\varphi_i : X_0 \to Y_i$ be the two contractions on $X_0$ as on (1.2) and let $L_i$ be as in (1.2):

$$L_i = \varphi_i^*(L'_i)$$

for ample generators $L'_i$ on $Y_i$.

The list of all possible $X_0$ together with $\varphi_i : X_0 \to Y_i$ is given in (1.4) and (1.5). In order to determine $K_X$ we will make the following ansatz as in Section. 2:

$$c_1(X) = c_1(X_0) + 2s_1c_1(L_1) + 2s_2c_1(L_2)$$

and we know that for any line bundle $G$ on $X_0$, with corresponding bundle $\tilde{G}$ on $X$ (2.4 (b)):

$$\chi(X, G) = \chi(X_0, G \otimes L_1^{s_1} \otimes L_2^{s_2}),$$

(4.1.1) in particular

$$1 = \chi(X, O_X) = \chi(X_0, L_1^{s_1} \otimes L_2^{s_2});$$

(4.1.2) often we will abbreviate $L_1^a \otimes L_2^b$ by $O_{X_0}(a, b)$.

**Proposition 4.2.** Assume that $\varphi$ contracts a divisor $E$ to a point.

Then either $X \simeq X_0 = \mathbb{P}(O \oplus O(-1))$ or $X_0 = \mathbb{P}(O \oplus O(-2))$ and $E^3 = 4$.

**Proof.** According to (1.3) write:

$$E = a_1L_1 + a_2L_2, \quad a_i \in \mathbb{Z}.$$  

So $E^3 = a_1^3L_1^3 + 3a_1^2a_2L_1^2L_2 + 3a_1a_2^2L_1L_2^2 + a_2^3L_2^3$. On the other hand: $E^3 = 1, 2$ or 4 by (1.5).
If $X_0 = \mathbf{P}_1 \times \mathbf{P}_2$, $\mathbf{P}(T_{\mathbf{P}_2})$ or $D_{2,1}$ in (1.4), we conclude:

$$3(a_1^2a_2L_1^2L_2 + a_1a_1^2L_1L_2^2) = 1, 2, 4$$

which is impossible.

Hence $X_0 = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(\alpha)), \alpha = -1, -2$ (1.4).

(a) First assume $\alpha = -1$. Then we obtain:

$$3a_1^2a_2 + 3a_1a_2^2 + a_2^3 = 1, 2 \text{ or } 4.$$  
Trivial calculations show that $E^3 = 2$ or 4 are not possible, so $E^3 = 1$ and $\varphi$ is the blow-up of a simple point. In particular $Y$ is smooth with $Pic(Y) = \mathbb{Z}$,

$$K_X = \varphi^*(K_Y) + E,$$

and obviously $Y$ is Fano. In order to determine it, we solve:

(4.1.2)  \hspace{1cm} \chi(\mathcal{O}_{X_0}(s_1, s_2)) = 1;

it is an easy exercise to see e.g. via Riemann-Roch: $s_1 = s_2 = 0$. Hence $c_1(X) = c_1(X_0)$. So $K_X^3 = K_{X_0}^3 = -56$, hence $-56 = K_Y^3 + 8E^3$ yields $K_Y^3 = -64$ and by the classification we conclude $Y \simeq \mathbf{P}_3$; so $X \simeq X_0$.

(b) Finally let $\alpha = -2$.

Then our equation reads:

$$3a_1^2a_2 + 6a_1a_2^2 + 4a_2^3 = 1, 2 \text{ or } 4.$$  
The only solution for $E^3 = 1$ is $(a_1, a_2) = (-1, 1)$, $E^3 = 2$ being impossible. So it is sufficient to exclude $E^3 = 1$. (In this case $Y$ is Fano with $b_2 = 1$, $b_3 = 0$, so $Y = \mathbf{P}_3, Q_3, V_5$ or $A_{22}$).

Using $L = \varphi^*(\mathcal{O}_Y(1))$ we have $L^2E = 0$, on the other hand writing $c_1(L) = \alpha_1c_1(L_1) + \alpha_2c_1(L_2)$:

$$L^2E = (\alpha_1L_1 + \alpha_2L_2)^2(-L_1 + L_2)$$

$$= (\alpha_1 + \alpha_2)^2 + 2(\alpha_1 - 2^2 - \alpha_2).$$

Both equations imply $\alpha_1 = \alpha_2 = 0$, a contradiction. \(\Box\)

Remark 4.3. If $X_0 = \mathbf{P}(\mathcal{O} \oplus \mathcal{O}(-2))$ in (4.2) then we will show in (4.8) that in this case $X \simeq X_0$, too.
Proposition 4.4. \( \varphi \) is never the blow-up of a smooth curve in a smooth 3-fold \( Y \).

Proof. Assume that \( \varphi \) is the blow-up of the smooth curve \( C \) in \( Y \). Since \( b_3(X) = 0 \), we conclude \( b_3(Y) = 0 \) and \( C \cong \mathbb{P}_1 \). \( Y \) being Fano with \( b_2 = 1 \), we have \( Y = \mathbb{P}_3, Q_3, V_5 \) or \( A_{22} \).

Let \( \mathcal{O}_Y(1) \) be the ample generator and \( L = \varphi^*(\mathcal{O}_Y(1)) \). Then \( L^3 = 1, 2, 5 \) or \( 22 \), respectively. On the other hand, write again:

\[
c_1(L) = a_1c_1(L_1) + a_2c_1(L_2).
\]

Then we have the equation

\[
3a_1^2a_2L_1^2L_2 + 3a_1a_2^2L_1L_2^2 + a_2^3L_2^3 = 1, 2, 5 \text{ or } 22.
\]

From table (1.4) we conclude that necessarily \( X_0 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(\alpha)) \) with \( \alpha = -1, -2 \), because otherwise the left hand side would be divisible by 3.

(a) \( \alpha = -1 \).

Then the only solutions are \((0, 1)\) (with \( L^3 = 1 \)) and \((-1, 2)\) (with \( L^3 = 2 \)).

If \((a_1, a_2) = (0, 1)\) then \( c_1(L) = c_1(L_2) \).

Let \( F, F_2 \) be a general non-trivial fiber of \( \varphi \) resp. \( \varphi_2 \).

Then \((-K_X.F) = 1, (-K_{X_0}.F_2) = 2 \). Since \( c_1(X) = c_1(X_0) = 0 \) (proof of (4.2)), it follows via \( c_1(L) = c_1(L_2) \) that \([F]\) is an even multiple of \([F_4]\) in \( H^4(X_0, \mathbb{Z}) \), i.e. \([F]\) is divisible by 2 in \( H^4(X_0, \mathbb{Z}) \) which is clearly false. So assume now \((a_1, a_2) = (-1, 2)\).

Write \( E = \alpha_1c_1(L_1) + \alpha_2c_1(L_2) \).

Then the equation \( L^2E = 0 \) yields \( \alpha_2 = 0 \), so \( E^3 = 0 \). On the other hand \( E^3 = -c_1(N_{C|Y}) \) which is absurd since \( Y = Q_3 \).

(b) \( \alpha = -2 \).

Now the only solution is \((a_1, a_2) = (-1, 1)\) with \( L^3 = 1 \), so \( Y \cong \mathbb{P}_3 \). With \( E = \alpha_1c_1(L_1) + \alpha_2c_2(L_2) \) we obtain as in (a):

\[
0 = L^2E = -3\alpha_2, \text{ hence } E^3 = 0
\]

and we conclude \( c_1(N_{C|Y}) = 0 \), contradiction. \( \square \)

From now on we may assume that \( \varphi \) is not a modification, hence \( \dim Y = 1 \) or \( 2 \) and \( Y \) is smooth.

Proposition 4.5. Assume \( \dim Y = 2 \). Then \( Y \cong \mathbb{P}_2 \) and either:
(c1) φ is a $\mathbb{P}_1$-bundle, or

(c2) φ is a proper conic bundle over $\mathbb{P}_2$ and $X_0 = D_{2,1}$, a divisor of bidegree $(2,1)$ in $\mathbb{P}_2 \times \mathbb{P}_2$, moreover $c_1(X) = c_1(X_0)$ and $c_1(L) = c_1(L_2)$.

**Proof.** Since $\pi_1(Y) = 0$, $X$ being simply connected, and since $b_2(Y) = 1$, we conclude $Y \simeq \mathbb{P}_2$ by (2.6). So $X$ is a $\mathbb{P}_1$-bundle or a conic bundle over $\mathbb{P}_2$.

Let $L = \varphi^*(\mathcal{O}_{\mathbb{P}_2}(1))$ and write

\[ c_1(L) = a_1c_1(L_1) + a_2c_1(L_2). \]

We are going to solve the equation

\[(*) \quad 0 = L^3 = 3a_1^2a_2L_1^2L_2 + 3a_1a_2^2L_1L_2^2 + a_2^3L_2^3.\]

But first we claim:

(a) if $a_2 = 0$ in case of $X_0 \neq \mathbb{P}_1 \times \mathbb{P}_2$, $X$ is a $\mathbb{P}_1$-bundle over $Y$.

So assume for the proof : $a_2 = 0$. If $a_1 \neq \pm 1$, then $L$ would be divisible by some line bundle $L'$ which necessarily has to be of the form $\varphi^*(\mathcal{O}_{\mathbb{P}_2}(m))$, which is absurd. So $|a_1| = 1$.

Assume $\varphi$ is not a $\mathbb{P}_1$-bundle. Then let $F$ a component of a reducible fiber of $\varphi$. We have

\[ (-K_X.F) = 1. \]

Now let $F_1$ be a fiber of $\varphi$. Then $c_1(L) = \pm c_1(L_1)$ yields $[F] = \pm [F_1]$ in $H^4(X_0, \mathbb{Z})$.

Hence $(-K_{X_0}.F_1) = \pm (K_{X_0}.F) \equiv 1(2)$ by the invariance of the Stiefel-Whitney class $w_2$. On the other hand, $\varphi_1$ is a $\mathbb{P}_1$-bundle if $X_0 \neq \mathbb{P}_1 \times \mathbb{P}_2$, so

\[ (-K_{X_0}.F_1) = 2, \]

contradiction.

(b) Now let $X_0 = P(\mathcal{O} \oplus \mathcal{O}(+\alpha)), \alpha = -1, -2$. Then $(*)$ gives immediately $a_2 = 0$, so we are done by (a).

(c) If $X_0 = \mathbb{P}_1 \times \mathbb{P}_2$ then $(*)$ reads

\[ 3a_1a_2^2 = 0, \text{ so } a_1 = 0 \text{ or } a_2 = 0. \]

If $a_2 = 0$ we would have $L^2 = 0$ which is impossible. So we can apply (a).
(d) For $\mathbb{P}(T_{P_2})$, (*) gives:

$$3(a_1^2a_2 + a_1a_2^2) = 0$$

so $a_1 = 0$ or $a_2 = 0$ or $a_1 = -a_2$ and it is sufficient to exclude the latter possibility. But if $a_1 = -a_2$, then

$$L^2 = a_1^2(L_1 - L_2)^2.$$ 

Since $L^2 = F$, a fiber of $\varphi$, we obtain:

$$(-K_X \cdot F) = -2a_1^2$$

if we suppose $c_1(X) = c_1(X_0)$. Since $-(K_X \cdot F) > 0$, we have a contradiction. In order to verify: $c_1(X) = c_1(X_0)$, we write as usual:

$$c_1(X) = c_1(X_0) + 2s_1c_1(L_1) + 2s_2c_1(L_2)$$

and have (4.1.2) to solve the equation

$$\chi(\mathcal{O}_{X_0}(s_1, s_2)) = 1.$$ 

But $X_0 = \mathbb{P}(T_{P_2})$ can be viewed as divisor of bidegree $(1,1)$ in $\mathbb{P}_2 \times \mathbb{P}_2$, hence

$$\chi(\mathcal{O}_{X_0}(s_1, s_2)) = \chi(\mathcal{O}_{P_2 \times P_2}(s_1, s_2)) = \chi(\mathcal{O}_{P_2 \times P_2}(s_1 - 1, s_2 - 1)) = 1.$$ 

Now compute, using:

$$\chi(\mathcal{O}_{P_2}(t)) = \frac{(t + 1)(t + 2)}{2}$$

to get $s_1 = s_2 = 0$.

(e) It remains to treat $X_0 = D_{2,1}$.

In this case (*) reads

$$3a_1^2a_2 + 6a_1a_2^2 = 0.$$ 

If $a_2 = 0$, then $\varphi$ is a $\mathbb{P}_1$-bundle by (a) and we are done. So either $a_1 = 0$ or $a_1 = -2a_2$.

First we want to exclude the latter possibility. So assume $a_1 = -2a_2$. Using

$$c_1(X) = (1 + 2s_1)c_1(L_1) + (2 + 2s_2)c_1(L_2)$$

and

$$L^2.(-K_X) = F.(-K_X) = 2,$$
we obtain: $a_2^2 = 1$; moreover $s_1 = -2s_2 - 3$.

In order to determine $(s_1, s_2)$, we use:

$$\chi(\mathcal{O}_{X_0}(s_1, s_2)) = 1.$$ 

In fact,

$$\chi(\mathcal{O}_{X_0}(s_1, s_2)) = \chi(\mathcal{O}_{P_2 \times P_2}(s_1, s_2)) - \chi(\mathcal{O}_{P_2 \times P_2}(s_1 - 2, s_2 - 1))$$

is an explicit polynomial, and via the relation between $s_1$ and $s_2$, we easily obtain:

$$s_1 = 0, s_2 = -3.$$ 

Now consider the equation (4.1.1)

$$\chi(X, L') = \chi(\mathcal{O}_{X_0}(-2t, t - 3)).$$

Clearly $\chi(X, L') = \frac{(t+1)(t+2)}{2}$. The right hand side is also easily computed (go again to $P_2 \times P_2$), and it turns out that both polynomials are different, contradiction.

So we are left with the case $a_1 = 0$. Then we want to show that $\varphi$ is a conic bundle, that $c_1(X) = c_1(X_0)$ and $c_1(L) = c_1(L_2)$.

As before, by a divisibility argument we get $|a_2| = 1$, so $c_1(L) = \pm c_1(L_2)$ also it is easy to see that $\varphi_2$ cannot be a $P_1$-bundle, hence must be a proper conic bundle. We have

$$c_1(X) = (1 + s_1)L_1 + (2 + s_2)L_2.$$ 

Since (general) fiber of $\varphi$ and $\varphi_2$ have the same cohomology class, we obtain by intersecting $-K_X$ with a general fiber easily: $s_1 = 0$.

So by (4.1.1)

$$\chi(X, L') = \chi(X_0, \mathcal{O}_{X_0}(0, t + s_2)), \quad (\text{resp. } \chi(X_0, \mathcal{O}_{X_0}(-t + s_2)),$$

hence

$$\frac{(t+1)(t+2)}{2} = \frac{(t + s_2 + 1)(t + s_2 + 2)}{2} \quad (\text{resp. } \frac{(-t + s_2 + 1)(-t + s_2 + 2)}{2})$$

which gives $s_2 = 0$.

This ends the proof of (4.5) \(\square\)
Remark 4.6. We will see in sect. 5 that in fact if \( X_0 = D_{1,2} \) and \( \varphi \) is a conic bundle then \( X \simeq X_0 \).

**Proposition 4.7.** Assume \( \dim Y = 1 \). Then \( Y \simeq \mathbb{P}_1 \), \( X \) is a \( \mathbb{P}_2 \)-bundle over \( \mathbb{P}_1 \) and \( X_0 \simeq \mathbb{P}_1 \times \mathbb{P}_2 \). \( X \) is of the form \( \mathbb{P}(E) \) with \( E = \mathcal{O}(a) \oplus \mathcal{O}(b) \oplus \mathcal{O}(c) \) with \( a + b + c \equiv 0(3) \).

**Proof.** Obviously \( Y \) is rational. Write again:

\[
c_1(L) = a_1c_1(L_1) + a_2c_1(L_2).
\]

Then from \( L_1L_2 = 0 \) and \( L_3 = 0 \) we obtain \( a_2 = 0 \) and hence

\[
X_0 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1))
\]
and also easily: \( X \) is a \( \mathbb{P}_1 \)-bundle, or the two equations:

\[
\begin{align*}
2a_1L_1^2L_2 + a_2L_1L_2^2 &= 0 \\
3a_1^2L_1L_2 + 3a_1a_2L_1L_2^2 + a_2^2L_2^2 &= 0,
\end{align*}
\]

are satisfied.

Now using table (1.4) it is trivial to obtain a contradiction in all cases but \( a_2 = 0 \). If \( a_2 = 0 \) we proceed as above. So \( X = \mathbb{P}(E) \to \mathbb{P}_1 \), and the 3-bundle \( E \) has obviously the form as stated above. \( \square \)

We are coming now back to a special situation to be still treated (see (4.3)).

**Proposition 4.8.** Assume that \( \varphi \) contracts a divisor \( E \simeq \mathbb{P}_2 \) with \( E^3 = 4 \) to a point and assume \( X_0 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-2)) \). Then \( X \simeq X_0 \).

**Proof.** Write

\[
c_1(\mathcal{O}_X(E)) = \alpha_1c_1(L_1) + \alpha_2c_1(L_2).
\]

Then solve the equation

\[
4 = E^3 = \alpha_2(3\alpha_1^2 + 6\alpha_1\alpha_2 + 4\alpha_2^2)
\]
the solutions are \((\alpha_1,\alpha_2) = (0,1),(-1,4)\) and \((-2,1)\). Now put \( c_1(L) = \alpha_1c_1(L_1) + \alpha_2c_1(L_2) \) in to \( L^2.E = 0 \). Then this rules already \((\alpha_1,\alpha_2) = (0,1)\) resp. \((-1,4)\).

So \((\alpha_1,\alpha_2) = (-2,1)\). This gives by \( L^2.E = 0 \) : \((a_1,a_2) = (0,a_2)\), hence by divisibility as usual : \( a_2 = 1 \). Moreover we see that lines in \( E \) and lines in
the exceptional divisor of $\varphi_2$ have the same cohomology class. This implies by intersecting
\[ c_1(X) = (1 + s_1)c_1(L_1) + (2 + s_2)c_1(L_2) \]
with such a line:
\[ s_1 = 0, \text{ resp. } s_1 = 1 \text{ if } c_2 = -1. \]
The case $s_1 = -1, a_2 = -1$ is excluded as follows. From $1 = \chi(O_X) = \chi(O_{X_0}(-1, s_2))$ we first see $s_2 > 0$. By Serre duality we obtain $\chi(X_0, L_2^{-s_2-2}) = -1$. Computing on the Veronese cone $Y_2 = W_4$ we easily derive a contradiction. Now by (4.1.2):
\[ 1 = \chi(X, O_X) = \chi(X_0, O_{X_0}(0, s_2)) \]
and consequently $s_2 = 0$. So $c_1(X) = c_1(X_0)$. Let $L' \in \text{Pic}(Y)$ with $\varphi^*(L') = L$.

We want to compute Fujita's $\Delta$-invariant:
\[ \Delta(L') = 3 + L'^3 - h^\circ(L'). \]
First note: $L'^3 = L^3 = L_2^3 = 4$.

In order to compute $h^\circ(L') = h^\circ(L)$ we notice that because of $c_1(X) = c_1(X_0)$ and because of the invariance of $p_1(X) = c_1^2 - 2c_2$, we have $c_2(X) = c_2(X_0)$, too, and hence by Riemann-Roch:
\[ \chi(L) = \chi(L_2). \]
This $\chi(L') = \chi(L) = 7$.

Now $Y$ is 2-Gorenstein (see [Mo]), $\rho(Y) = 1$ and $L'$ is the ample generator of $\text{Pic}(Y) \simeq \mathbb{Z}$. Moreover we compute easily:
\[ -K_Y = \frac{3}{2}L'. \]
Hence we get
\[ H^q(Y, L') = 0 \]
by the vanishing theorem of Kawamata-Viehweg (see e.g. [KMM]), since $L' - K_Y$ is ample. Consequently $h^\circ(L') = 7$ and $\Delta(L') = 0$. By [Fj], the linear system $|L'|$ is base point free and in fact defines an embedding:
\[ Y \hookrightarrow \mathbb{P}_6. \]
Now the unique singular point $y_0 \in Y$ is a quadruple point by [Mo], hence if $l \subset P_5$ is a line through $y_0$, then either $l \cap Y = \{y_0\}$, or $l \subset Y$.

This $Y$ is the cone over the Veronese $P_2 \hookrightarrow P_5$ with vertex $y_0$. But this is also exactly the description of $Y_2 = W_4$, then $X \simeq X_0$. \hfill $\square$

Taking the results of sect. 5 for granted (see remark 4.6) we can rephrase the results of the section as follows.

**Theorem 4.9.** Let $X_0$ be a primitive Fano 3-fold with $b_2 = 2$, $b_3 = 0$. Let $X$ be a projective 3-fold homeomorphic to $X_0$. Then either $X \simeq X_0$ or $X = P(\mathcal{O}(a)\oplus\mathcal{O}(b)\oplus\mathcal{O}(c))$ with $a+b+c \equiv 0(3)$. $X \simeq P(E)$ with $E$ a rank 2-bundle on $P_2$ given in the following table (we normalise $E$ such that $c_1(E) = -1$ or 0).

In fact, every $X_0$ has the form $P(V)$ (unique up to $P(T_{P_2})$) over $P_2$ and $c_i(E) = c_i(V)$ (i.e. $E$ and $V$ are topologically the same).

<table>
<thead>
<tr>
<th>$X_0$</th>
<th>$c_1(E)$</th>
<th>$c_2(E)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P_1 \times P_2$</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$P(T_{P_2})$</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$P(\mathcal{O} \oplus \mathcal{O}(-1))$</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>$P(\mathcal{O} \oplus \mathcal{O}(-2))$</td>
<td>0</td>
<td>-1</td>
</tr>
<tr>
<td>$D_{2,1}$</td>
<td>0</td>
<td>3</td>
</tr>
</tbody>
</table>

(4.9)

**Proof.** We consider our extremal contraction $\varphi : X \rightarrow Y$.

(1) If $\varphi$ is a modification, then by (4.2), (4.4) and (4.8) : $X \simeq X_0$.

(2) If $\dim Y = 2$, then by (4.5) : $Y \simeq P_2$ and either $X$ is a $P_1$-bundle over $P_2$ or $X_0 \simeq D_{2,1}$ and $X$ is a conic bundle. In the latter case, $X \simeq X_0$ by (5.1).

So assume $X \simeq P(E) \rightarrow P_2$.

Now write $X_0 = P(V)$ with $V = \mathcal{O}_{P_2} \oplus \mathcal{O}_{P_2}(-n)$, $n = 0, 1, 2$ or $V = T_{P_2}$ or $c_1(V) = 0$, $c_2(V) = 3$ (in case $X_0 = D_{2,1}$).

Then $p_1(P(E)) = p_1(P(V))$.

Since $\varphi(p_1(P(E))) = (c_1^2(E) - 4c_2(E))$ for the projection $\varphi : X \rightarrow Y$ and since we know $\varphi_* = \varphi_{i*}$ for $i = 1$ or 2, we conclude

$$c_1^2(E) - 4c_2(E) = c_1^2(V) - 4c_2(V).$$
Since $E$ is normalized and $V$ is explicitly known we obtain our table.

(3) If $\dim Y = 1$, then apply (4.7). □

Remark 4.10. Of course if $X = \mathbf{P}(E)$ as in the table, then $X \simeq X_0$ topologically, since two rank 2-bundle on $\mathbf{P}_2$ with the same Chern classes are topologically equivalent (see [OSS]).

Some words to the existence of $E$ with $c_i(E)$ as given on the table. There are always a lot of instable 2-bundles $E$ which can be constructed by the Serre correspondence (see [OSS]). But a semi-stable $E$ (different from the original bundle) exists only in $X_0 = D_{2,1}$; they are described by a moduli space of dimension 9.

5. THE PROPER CONIC BUNDLE CASE

After 4.5 and 4.6, the last remaining case is the following:

Proposition 5.1. Let $X$ be a threefold homeomorphic to $X_0 = D_{2,1}$ (see 1.4 for notations).

Assume $\varphi : X \to \mathbf{P}_2$ is a proper conic bundle, that $c_1(X) = c_1(X_0) = L_1 + 2L_2$, and $L_2 = \varphi^*(O_{\mathbf{P}_2}(1))$, the identifications being obtained from the equalities: $\text{Pic}(X) = H^2(X, \mathbf{Z}) = H^2(X_0, \mathbf{Z}) = \text{Pic}(X_0)$.

Then $X$ is analytically isomorphic to $X_0$.

The proof of (5.1) will be prepared by several lemmata. We denote by $l$ a general line in $\mathbf{P}_2$ meeting the discriminant locus $\Delta$ of the conic bundle transversally. Then $S = S_1 = \Phi^{-1}(l)$ is a smooth surface.

Lemma 5.2. $S$ is the blow-up of a ruled surface $\mathbf{P}(O_{\mathbf{P}_1} \oplus O_{\mathbf{P}_1}(k))$ in three points.

Proof. Clearly $S$ is the blow-up of a ruled surface $\mathbf{P}(O \oplus O(k))$ in say $d$ points (with $d = \deg \Delta$). Now $K_S^2 = (K_X + L_2)^2.L_2 = (L_1 + L_2)^2.L_2 = 5$ (c.f. 1.4).

Hence $d = 3$. □

Lemma 5.3.

(1) $-K_S = L_1 + L_2$
(2) $\chi(S, L_1) = 3$
(3) $H^2(S, L_1) = 0, h^0(S, L_1) \geq 3$
(4) $L_1 | S$ is generated by global sections.
(5) $h^0(S, L_1) = 3$, $H^0(S, L - L_2) = 0$ and $\nu : H^0(S, L_1) \to H^0(F, L_1)$ is an isomorphism, ($F$ a fiber of the conic bundle).

Proof. (1) follows from $K_X = -L_1 - 2L_2$ by adjunction.
(2) is clear by Riemann-Roch.
(3) $H^2(S, L_1) = H^0(S, -2L_1 - L_2) = 0$, since $L_1 | F = \mathcal{O}(2)$, where $F$ is a general fiber of $\Phi | S$. So by (2) : $h^0(S, L_1) \geq 3$.
(4) Here we use the results and notations of Sect. 7. By (7.2) the instability of the conic bundle $X$ fulfills $n(X) \leq \deg \Delta - 2$, hence $n(X) \leq 1$. Thus $n(X) = 1$. Consequently $S$ is $F_0$ or $F_1$ blown up in 3 points. In other words, $S$ is $\mathbb{P}_2$ blown up in 4 points. No 3 of them can be collinear, otherwise we would have a section $C$ with $C^2 = -2$. So $S$ is a del Pezzo surface and it follows easily that $L_1 | S = -K_S - L_2$ is nef. The global generatedness can be deduced either directly or by computing Fujita's $\Delta$-genus: $\Delta(S, L_1) \leq 0$ and by applying Fujita’s fundamental results. Note that always $\Delta \geq 0$, hence $\Delta(S, L_1) = 0$, which gives already the first claim of (5).
(5) Use the exact sequence
$$0 \to H^0(S, L_1 - L_2) \to H^0(S, L_1) \to H^0(F, L_1)$$
with $F$ a general fiber of $\Phi$. Then (4) together with $L_1 | F = \mathcal{O}(2)$ gives the claim. \(\Box\)

Lemma 5.4.

(1) $H^2(S, L_1 + \mu L_2) = 0$ for all $\mu \in \mathbb{Z}$.
(2) $H^1(S, L_1 + \mu L_2) = 0$ for all $\mu \geq -1$.
(3) $H^1(X, L_1 + \mu L_2) = 0$ for all $\mu \geq -2$.

Proof. (1) $H^2(S, L_1 + \mu L_2) = H^0(S, -2L_1 - (\mu + 1)L_2) = 0$ for all $\mu$, since $L_1 | F$ is positive for a general fiber $F$ of $\Phi$.
(2) Now let $\mu \geq -1$. From the exact sequence
$$0 \to (L_1 + \mu L_2)|S \to (L_1 + (\mu + 1)L_2)|S \to L_1 | F \to 0$$
we see that it is sufficient to show surjectivity of $H^0(S, L_1) \to H^0(L_1 | F)$. But this was already proved in 5.3 (5).
(3) Now use the exact sequence on $X$

$$0 \to L_1 + \mu L_2 \to L_1 + (\mu + 1)L_2 \to (L_1 + (\mu + 1)L_2)|_S \to 0.$$ 

By (1) and (2) we get for $\mu \geq -2$

$$H^1(X, L_1 + \mu L_2) \cong H^1(X, L_1 + (\mu + 1)L_2).$$

Since $H^1(X, L_1 + \mu L_2) \cong H^1(P_2, \Phi_* L_1) \otimes O_{P_2}(\mu) = 0$ for $\mu \gg 0$, we conclude.

**Proof of Proposition 5.1.** By Riemann-Roch and our assumptions: $\chi(X, L_1) = \chi(X_0, L_1) = 3$, so from 5.4 (3) we obtain $h^0(X, L_1) \geq 3$. Since $h^0(S, L_1 - L_2) = 0$, we conclude:

$$H^0(X, L_1 - L_2) = 0,$$

hence the restriction $H^0(X, L_1) \to H^0(S, L_1)$ is injective. Since $h^0(S, L_1) = 3$, we conclude $h^0(X, L_1) = 3$, so $r$ is an isomorphism. But this implies that $L_1$ is nef: assume that there is a curve $C \subset X$ with $(L_1, C) < 0$. Then for generic $l \subset P_2 : C \cap S_l = \emptyset$, since otherwise we would find $s \in H^0(X, L_1)$ such that $s|C \neq 0$ (use 5.3 (4) and the fact that $r$ is an isomorphism). Thus $\Phi(C) \cap l = \emptyset$ which is absurd. Now $L_1$ being nef, $-K_X = L_1 + 2L_2$ is ample as sum of two nef line bundles generating $Pic(X)$. So $X$ is Fano and consequently $X \cong X_0$ by Iskovskikh’s classification. □

6. Moishezon twistor spaces are not topologically projective

For $X$ a compact complex manifold, let $w_2(X) \in H^2(X, \mathbb{Z}/2\mathbb{Z})$ be its second Stiefel-Whitney class, whose vanishing means that $K_X$ is divisible by two in $Pic(X)$.

**Theorem 6.1.** Let $X$ be a projective threefold. Then: $b_1(X) = b_3(X) = w_2(X) = 0$ iff $X$ is one of the following:

i) Fano with $b_2 = 1$, of index $r = 2$ or 4 (in this last case, $X = P_3$),

ii) a $P_1$-bundle $P(V)$ over a surface $S$ with $b_1(S) = 0$, with $V$ a 2-bundle over $S$ such that $(\det V + K_S)$ is divisible by 2 in $Pic(S)$.

iii) obtained from the above manifolds by blowing-up finitely many points.
Remarks. 1. It is obvious that the conditions $b_1 = b_3 = w_2 = 0$ are necessary to belong to the above classes.

2. If one only assumes that $X$ has at most terminal singularities, and that $b_1 = b_3 = 0$, it is still true that $X$ is uniruled.

Proof. We have: $h^{1,0} = h^{3,0} = 0$, hence: $\chi(O_X) = 1 + h^{2,0} \geq 1$. Thus: $K_X$ is not nef (2.2). Let $\varphi : X \to Y$ be the contraction of an extremal ray in $X$. By Mori’s list and because $K_X = 2L, L \in Pic(X)$, we see that if $\varphi$ is a modification, it has to be the contraction of a smooth divisor $E$ of $X$, $E$ isomorphic to $P_2$, with normal bundle $E|_E \cong O_E(1)$ (because in all other cases, a curve $C \subset E$ exists such that: $-K_X.C = 1$, contradicting $w_2 = 0$). Thus: $Y$ is smooth and satisfies the same conditions: $b_1 = b_3 = w_2 = 0$ as $X$. We can thus assume that $\dim(Y) \leq 2$.

Assume first that $Y = S$ is a surface; then $Y$ is smooth, and $\varphi$ can’t be a conic bundle, otherwise a curve $C$ exists, which is contained in a fiber of $\varphi$ such that $(-K_X.C) = 1$, again contradicting $w_2 = 0$. Hence $\varphi$ is a $P_1$-bundle, and $b_1(S) = b_1(X) = 0$. Moreover, $K_X = O_{P(V)}(-2) + \varphi^*(detV + K_S)$, if $X = P(V)$ for $V$ a rank 2 bundle over $S$, so we are in case (ii). Assume now that $Y = C$ is a curve. Let $F$ be a smooth fiber of $\varphi$; then $F$ is a minimal Del Pezzo surface, otherwise, an expectional curve of the first kind $C_0$ on $F$ would satisfy: $1 = (-K_F.C_0) = (-K_X.C_0)$, contradicting: $w_2 = 0$. Thus $F$ is either $P_2$ or $P_1 \times P_1$. The case $F = P_2$ is again excluded, since: $-K_X|_F = -K_F = O_F(3)$ in this case. The case $F = P_1 \times P_1$ is also excluded by the proposition below.

The last possible case is: $\dim Y = 0$, so $X$ is Fano with $b_2(X) = 1$, and $r = 2, 4$ since $w_2 = 0$. □

Proposition 6.2. There is no quadric bundle $\varphi : X \to C \cong P_1(C)$ with $2 = b_2(X); b_3(X) = w_2(X) = 0$.

Proof. If $\varphi$ were smooth, we would have $b_2(X) = 3$. The set $\Delta$ of singular fibers of $\varphi$, which are isomorphic to the quadric cone in $P_3$ after [Mo] is thus nonempty.
Since:
\[ \chi(X) = \chi(C) \cdot \chi(F) + \sum_{c \in \Delta} (\chi(X_c) - \chi(F)) \]
where \( \chi \) is the topological Euler-Poincaré characteristic, \( F = P_1 \times P_1 \), and \( X_c := \varphi^{-1}(c) \), we get from
\[ \chi(X_c) = 3, \chi(F) = 4, \chi(X) = 6, \]
that \( \delta \) consists of exactly two points. □

On the other hand, we can embed \( X \) in a \( P_3 \)-bundle \( P := P(E^*) \), where \( E^* \) is a 4-bundle on \( C \) normalised in such a way that \( X \subset |2L| \), with \( L = O_P(1) \).

Let \( c_1 \in \mathbb{Z} \) be the degree of \( E \). We have a quadrilinear symmetric map
\[ \Psi : S^2(E) \to S^2(\text{det}E) \]
which sends any quadratic form \( B \) on \( E \) to its discriminant. \( X \) is the zero locus of some \( s \in H^0(P, 2L) \), and let \( \sigma := \Psi \circ s \in H^0(C, S^2(\text{det}E)) = H^0(C, \mathcal{L}) \), where \( \mathcal{L} \) has degree \( 2c_1 \). Then we conclude \( c_1 = 1 \) since \( \{ \sigma = 0 \} = \Delta \). We now compute:
\[ K_X = (K_p + 2L)|_X = (-4L + \varphi^*(c_1 - 2))|_X, \]
and so \( w_2(X) \neq 0 \) since \( c_1 \) is odd. (Here: \( \text{Pic}(C) \) is identified with \( Z \) in the usual way).

**Corollary 6.3.** Let \( M^4 \) be a compact connected anti-self dual Riemannian fourfold, and let \( \tau : Z \to M^4 \) be its twistor space ([AHS]).

Assume that \( Z \) is Moishezon, but not projective. Then there is no projective threefold \( Z_0 \) which is homeomorphic to \( Z \) if \( n \geq 3 \) is even, with \( n = b_2(Z) - 1 \).

Probably this remains true if \( n \) is odd, too. This answers a question (3.15) asked in [C2].

**Remarks.** Recall that \( \tau : Z \to M^4 \) is a differentiable (non holomorphic) submersion whose fibers are holomorphic rational curves on \( Z \) with normal bundle \( O(1) \oplus O(1) \), and that \( w_2(Z) = 0 \). Recall that if \( Z \) is Moishezon, it is “almost Fano”, i.e. the Kodaira dimension of \( K_Z^{-1} \) is 3. ([P],[V]).

It is shown in [C] that \( M^4 \) is homeomorphic to either \( S^4 \) or the connected sum \( \# nP_2(C) \) of \( n \) copies of \( P_2(C) \) if \( Z \) is Moishezon. It is shown in [H] that if \( Z \) is projective, it is either \( P_3(C) \) or \( P(Tp_2(C)) \), with \( M^4 \) respectively \( S^4 \) or \( P_2(C) \) with metrics conformal to the usual ones. Examples with arbitrary \( n \) are known to exist ([P2] : \( n = 2 \); [K] : \( n = 3 \); [L] all \( n \)). It is shown
in [C2], [L2] that small generic deformations of Kurke-Lebrun’s examples are not in the class $C$, thus showing that Kodaira-Spencer stability theorem is not true in the class of compact manifolds bimeromorphic to Kähler ones. The above corollary thus exhibits another difference between these $Z$ and projective manifolds.

**Proof.** Let $M = M^4$, thus $M$ is topologically $nP_2(C)$, with $n \geq 2$. We describe $H^2(X, \mathbb{Z})$ together with its bilinear intersection form. Let $(\alpha_1, ..., \alpha_n)$ be an orthogonal basis of $H^2(M, \mathbb{Z})$ (ie : $\alpha_i \alpha_j = 0$ if $i \neq j$, $\alpha_i^2 = 1$). We identify $\alpha_i$ and $\tau^* \alpha_i$. Let $\bar{c} = \frac{1}{2}c_1(Z)$. A $\mathbb{Z}$-basis of $H^2(Z, \mathbb{Z})$ is then : $(c, \alpha_1, ..., \alpha_n)$ where : $c = \frac{1}{2}(\bar{c} + \alpha_1 + ... + \alpha_n)$, which is integral (see [P3]).

The intersection form is defined by :

\[
\begin{align*}
\bar{c}^3 &= 2(4 - n); \bar{c}^2 \cdot \alpha_i = 0; \bar{c} \cdot \alpha_i^2 = -2 & \text{for all } i, \\
c^3 &= 1 - n; c^2 \cdot \alpha_i = -1; c \cdot \alpha_i^2 = -1 & \text{for all } i.
\end{align*}
\]

We now assume that $Z_0$ is a projective threefold homeomorphic to $Z$.

**Lemma 6.4.** $Z_0$ is not blow-up in a point of any smooth projective threefold $Z_1$.

**Proof.** Otherwise there would exist $E$ and $L \neq 0$ in $Pic(Z_0)$ such that : $E^3 = 1, E^2L = E.L^2 = 0$ (just take the class $E$ of the exceptional divisor of the blow-up, and the class $L$ of the lifting of any ample line bundle on $Z_1$).

However, a direct computation shows that the equations :

\[(\epsilon c + \epsilon_1 \alpha_1 + ... + \epsilon_n \alpha_n)^3 = 1 = \epsilon[\epsilon^2(1 - n) - 3(\sum \epsilon_i^2 + \epsilon(\sum \epsilon_i))]\]

have no integer solutions $(\epsilon, \epsilon_i), (\lambda, \lambda_i)$ if $n \geq 3$. □

**Lemma 6.5.** $Z_0$ is not a $P_1$-bundle over any algebraic surface $S$.

**Proof.** Let $\varphi_0 : Z_0 \rightarrow S$ be any such $P_1$-bundle structure. Then $(\varphi_0)^*(H^2(S, \mathbb{Z}))$ generates a sublattice of rank $n$ in $H^2(Z_0, \mathbb{Z})$ (which has rank $(n + 1)$), and consisting of classes $L$ such that : $L^3 = 0$. Now, if

\[L = \lambda c_1 + \lambda_1 \alpha_1 + ... + \lambda_n \alpha_n,\]
one has:

\[ L^3 = \lambda[\lambda^2(1-n) - 3(\sum \lambda_i^2 + \lambda\lambda_i)] = \lambda Q(\lambda, \lambda_i), \]

where \( Q \) is a definite negative quadratic form on \( \mathbb{R}^{n+1} \). Thus \( \varphi_0^* (H^2(S, \mathbb{Z})) = \tau^* H^2(M^4, \mathbb{Z}) \). But this shows that the intersection form on \( S \) would be definite of rank \( n \geq 2 \), which is impossible if \( n \) is even by Hodge index theorem (which forces \( h^{1,1}(S) = 1 \)). □

(6.4) and (6.5) imply now together with theorem (6.1) that \( Z_0 \) has \( b_2 = 1 \), contradiction.

7. A BOUND FOR THE DEGREE OF INSTABILITY OF A CONIC BUNDLE

Definition and Construction 7.1 (1) Let \( S \) be a smooth rational surface with a surjective holomorphic map \( \phi : S \to \mathbb{P}_1 \). Let \( C \subset S \) be a section of \( \phi \). \( C \) is said to be minimal if its selfintersection number \( C^2 \) is minimal with respect to all sections of \( \phi \). We call

\[ n(\phi) = -C^2, \]

where \( C \) is minimal, the degree of \( \phi \). Loosely speaking, when it is clear which map \( \phi \) is meant, we put \( n(S) = n(\phi) \).

(2) Let \( \Phi : X \to \mathbb{P}_2 \) be a proper conic bundle, i.e. the discriminant locus \( \Delta \subset \mathbb{P}_2 \) is not empty. Let \( d \) be the degree of \( \Delta \) which number we also call the degree of the conic bundle \( \Phi \). Let \( G = \mathbb{P}_2 \) be the variety of lines in \( \mathbb{P}_2 \). Let \( G^* \) be the Zariski open set in \( G \) consisting of those lines which meet \( \Delta \) in \( d \) distinct points tranversely. Then for \( l \in G^* \), the surface \( S_l = \Phi^{-1}(l) \) is a smooth surface and in fact a Hirzebruch surface \( F_k = \mathbb{P} (\mathcal{O} \oplus \mathcal{O}(-e)) \) blown up in \( d \) points. We denote by \( n(l) = n(\Phi|S_l) \) its degree of instability. Finally let \( n(X) = n(\Phi) \) be the minimum of all \( n(l), l \in G^* \). We call \( n(X) \), or better \( n(\Phi) \), the degree of instability of the conic bundle \( X \).

Our main result in this section is

**Theorem 7.2.** Assume that the conic bundle \( \Phi : X \to \mathbb{P}_2 \) is standard (i.e. \( Pic(X) = \mathbb{Z}K_X + \Phi^*(Pic(\mathbb{P}_2)) \)) and assume moreover that the degree of \( \Phi \) is \( d \). Then

\[ n(X) \leq d - 2, \]
in particular \( n(X) \) is finite.

First let us show the following

**Proposition 7.3.** Let \( \pi : S_0 \to P_1 \) be a ruled surface, i.e. a \( P_1 \)-bundle over \( P_1 \). Let \( \sigma : S \to S_0 \) be the blow-up of \( b \geq 3 \) distinct points on \( S_0 \). Let \( n \) be the degree of instability of \( S \to P_1 \). Assume that \( n \geq b - 1 \). Then there exists a unique minimal section of \( S \).

**Proof.** Write \( S_0 = P(O(\mathcal{O}(-\nu))) \) with \( \nu \geq 0 \). Then \( \nu \) is the degree of instability of \( S_0 \). Let \( C_0 \) be a minimal section of \( S_0 \); so \( C_0^2 = -\nu \). If \( \nu > 0 \), then \( C_0 \) is unique.

Assume first \( \nu \geq 2 \). Then we claim that the strict transform \( \overline{C}_0 \) of \( C_0 \) in \( S \) is the unique minimal section of \( S \). In fact, take a section \( C \) of \( S_0 \) such that the strict transform \( \overline{C} \) is minimal and assume of course that \( C \neq C_0 \), if also \( C_0 \) is minimal. Since

\[
C^2 \geq \nu
\]

by the elementary theory of ruled surfaces, we have for the strict transform

\[
\overline{C} = -n \geq \nu - b,
\]

hence \( n \leq b - \nu \leq n + 1 - \nu \) by our assumption. This contradicts \( \nu \geq 2 \) and settles the proposition in this case.

In case \( \nu \geq 1 \) we see by the same construction, that we must have

\[
C^2 = 1, \overline{C}^2 = \nu - b = 1 - b,
\]

i.e. all \( b \) points have to be on \( C \), if \( C \neq C_0 \). Observe that here we must have \( C \cap C_0 = \emptyset \), hence none of the points to be blown up is on \( C_0 \). Now let \( \overline{C}' \) be another minimal section. Then by the same reasoning as for \( C \) all points to be blown up are on \( C' \), too. But this contradicts \( C.C' = C^2 = 1 \).

It remains to settle the case \( \nu = 0 \). But this is an obvious exercise. \( \Box \)

Coming back to our conic bundle \( \Phi : X \to P_2 \) and to the proof of (7.2), we assume that \( n(X) \geq d - 1 \). Then by (7.3) there exists for every \( l \in G^* \) a unique minimal section \( C_l \) of \( \Phi_l : S_l \to l \) (observe \( d \geq 3 \)). We want to show that the curves \( C_l \) form an algebraic family.
Proposition 7.4. There exists a unique component $T$ of the Chow scheme of curves in $X$ and a bimeromorphic map $\Phi_* : T \to G$ together with Zariski open sets $T^* \subset T$, $G^{**} \subset G^*$ such that for all $t \in T^*$:

$$\{t\} = C_l \text{ with } l = \Phi_*(t),$$

where $\{t\}$ denotes the curve parametrised by $t$.

Before giving the proof of (7.4) let us first show how (7.2) is proved by means of (7.4). Assume as before that $n(X) \geq d - 1$. Fix $a \in \mathbb{P}_2 \setminus \Delta$ and let

$$P_a = \{l \in G| a \in l\}$$

be the pencil of lines through $a$. Let $D$ be the Zariski closure of $\bigcup C_l$, where $l$ runs over $P_a \cap G^*$.

By (7.4) $D$ is a prime divisor in $X$ such that $\Phi|D : D \to \mathbb{P}_2$ is bimeromorphic. But this divisor is not a linear combination of $K_X$ and $\Phi^*(\mathcal{O}(1))$: intersect with a general fiber of $\Phi$ to obtain the contradiction. Hence $\Phi$ is not a standard conic bundle, contradicting our assumption.

It remains to give the

**Proof of 7.4.** (1) First we compute $(-K_X.C_l)$ for $l \in G^*$. We have an exact sequence, namely the normal bundle sequence for the embeddings $C_l \subset S_l \subset X$:

$$0 \to \mathcal{O}(-n(l)) \to N_{C_l|X} \to \mathcal{O}(1) \to 0.$$ 

Here $N = N_{C_l|X}$ is the normal bundle of $C_l$ in $X$. We conclude $c_1(N) = 1 - n(l)$, hence

$$(-K_X.C_l) = 3 - n(l) \quad (\ast).$$

(2) Thus the curves $C_l$ form a bounded family and therefore there exists a component $T$ of the Chow scheme containing all $C_l$ for $l$ in some nonempty Zariski open subset $U$ of $G^*$. We have

$$\dim T \leq h^0(N) \leq 2,$$

thus $\dim T = 2$.

(3) For $t \in T$ generic, we let

$$\Phi_*(t) = \Phi(C),$$
where $C$ is the section determined by $t$. Clearly $\Phi_*$ extends to a meromorphic map $T \to G$.

By construction there exists a Zariski open set $G^{**} \subset G^+$ such that $C_l \subset \Phi^{-1}_*(l)$ for $l \in G^{**}$. We have even $C_l = \Phi^{-1}_*(l)$ ; otherwise we would have some $t \in T$ such that the curve $B_t$ corresponding to $t$ is contained in $S_l$. But $B_t^2 = -n(l)$ by $(\ast)$, and because of the fact that $(-K_X.B_t)$ does not depend on $t$. Hence $\Phi_*$ is bimeromorphic. □

Note that $C_l^2 = -n(X)$ for all $l \in G^{**}$. 

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