THE K-ENERGY ON HYPERSURFACES AND STABILITY

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INTRODUCTION

The notion of stability for a polarized projective variety was introduced by D. Mumford for the study of the moduli problem of projective varieties. The stability has been verified by Mumford for smooth algebraic curves, D. Gieseker for algebraic surfaces and Viehweg for algebraic manifolds, which are polarized by m-pluri-canonical bundles for m sufficiently large ([Md], [Gi], [Vi]). However, it still seems to be a challenging problem to check the stability for a given polarized variety, even if the variety is a singular hypersurface in some projective space. The purpose of this paper is to give a sufficient and intrinsic condition for a hypersurface to be stable or semistable. The condition is given in terms of the properness or lower boundedness of a generalized K-energy, which was introduced by T. Mubachi for Kähler manifolds. In particular, we will prove that any hypersurface is semistable if it has only orbifold singularities of codimension at least two and admits a Kähler-Einstein orbifold metric.

We denote by $R_{n,d}$ the space of all homogeneous polynomials on $\mathbb{C}^{n+2}$ of degree $d$, and $B$ the projective space $PR_{n,d}$. Any point $[f]$ in $B$ determinates a unique hypersurface $\Sigma_f$ in $\mathbb{CP}^{n+1}$ of degree $d$. The special linear group $G = SL(n + 2, \mathbb{C})$ induces an action on the vector space $R_{n,d}$ by assigning $f$ to $f \circ \sigma^{-1}$ for any $\sigma$ in $G$. Then we say that $\Sigma_f$ is stable if the orbit $Gf$ is closed and the stabilier of $f$ in $G$ is finite; we say that $\Sigma_f$ is semistable if the zero in $R_{n,d}$ is not contained in the closure of the orbit $Gf$. It is well-known that any smooth hypersurface $\Sigma_f$ is stable. However, if $\Sigma_f$ has only one isolated

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singularity, then it may not be semistable. The simplest examples are those cubic surfaces in $\mathbb{CP}^3$ with one singularity of type other than $A_1$ or $A_2$.

The K-energy is a functional on the space of admissible Kähler metrics in the Kähler class given by the polarization. It is in fact a Donaldson functional on a "virtual" holomorphic bundle and is defined in terms of the Bott-Chern class associated to the invariant polynomial $\text{Ch}_{n+1}$ defining the $(n+1)$-th Chern Character (cf. section 3). Here is our first theorem,

**Theorem 0.1.** Let $\Sigma_f$ be a hypersurface in $\mathbb{CP}^{n+1}$, then $\Sigma_f$ is stable if the K-energy is proper (see Definition 5.1), and $\Sigma_f$ is semistable if the K-energy is bounded from below.

In fact, the proof here yields a stronger result which can be described as follows: let $g_{FS}$ be the Fubini-Study metric on $\mathbb{CP}^{n+1}$, for any $\sigma \in G$, $\sigma^*g_{FS}$ restricts to an admissible Kähler metric on $\Sigma_f$, we denote by $Gg_{FS}$ the set of such admissible Kähler metrics, then $\Sigma_f$ is stable if the K-energy is proper on $Gg_{FS}$, and $\Sigma_f$ is semistable if the K-energy is bounded from below on $Gg_{FS}$.

An easy corollary of Theorem 0.1 is the following

**Theorem 0.2.** Let $\Sigma_f$ be a normal hypersurface in $\mathbb{CP}^{n+1}$ of degree $d$. Then

1. if $3 \leq d \leq n + 1$ and $\Sigma_f$ has only orbifold singularities and admits a Kähler-Einstein orbifold metric, then $\Sigma_f$ is semistable;
2. if $d \geq n + 2$ and $\Sigma_f$ has only log-terminal singularities, then $\Sigma_f$ is stable.

The definition of a log-terminal singularity will be given in section 5. After some preparations, we give the proof for Theorem 0.1 in section 4 and for Theorem 0.2 in section 5. In section 6, we will briefly discuss some generalizations of above theorems without proof. In particular, Theorem 0.1, 0.2 will still be true for complete intersections in projective spaces. The details of these generalizations will appear in a forthcoming paper [T1], where we will deal with subvarieties in $\mathbb{CP}^N$ with canonical or anti-canonical polarization and higher codimensions.

It was S.T.Yau who brought the stability to my attention more than six years ago. I would like to thank him for sharing his insight that there must be a
connection between the existence of Kähler-Einstein metrics and the stability, analogous to the celebrated theorem that the existence of a Hermitian-Yang-Mills metric is equivalent to the stability on an irreducible holomorphic vector bundle (cf. [Do], [UY]).

1. SINGULAR HERMITIAN METRICS ON RELATIVE CANONICAL BUNDLES

As defined in last section, $R_{n,d}$ is the space of all homogeneous polynomials on $\mathbb{C}^{n+2}$ of degree $d$, and $B = PR_{n,d}$. We put

$$X = B \times \mathbb{C}P^{n+1}$$

and

$$\Sigma = \{([f]; x) \in X | f(x) = 0\}$$

It is easy to show that $\Sigma$ is a smooth hypersurface in $X$. We also define a subvariety

$$\Sigma_s = \{([f]; x) \in X | \frac{\partial f}{\partial z_i}(x) = 0, i = 0, 1, \ldots, n + 1\}$$

where $z_0, \ldots, z_{n+1}$ are homogeneous coordinates of $\mathbb{C}P^{n+1}$. Clearly, $\Sigma_s \subset \Sigma$.

**Lemma 1.1.** $\Sigma_s$ is smooth and irreducible, $\dim_{\mathbb{C}} \Sigma_s = \dim_{\mathbb{C}} \Sigma - (n + 1)$.

**Proof.** Let $([f]; x) \subset \Sigma_s$. The group $G = SL(n + 2, \mathbb{C})$ acts on $X$ as follows: $\forall \sigma \in G$,

$$\sigma([f]; x) = ([f \circ \sigma^{-1}]; \sigma(x)) .$$

Clearly, $G$ preserves both $\Sigma_s$ and $\Sigma$. Therefore, by a linear transformation, we may assume that $x = [1, 0, \ldots, 0]$. Let us write

$$f = \sum_{i_0 + \cdots + i_{n+1} = d} a_{i_0 \cdots i_{n+1}} z_0^{i_0} \cdots z_{n+1}^{i_{n+1}}$$

Since $\frac{\partial f}{\partial z_j}(x) = 0$ for $0 \leq j \leq n + 1$, we have

$$a_{d0\cdots0} = a_{d-110\cdots0} = \cdots = a_{d-10\cdots10\cdots0} = \cdots = a_{d-1\cdots1} = 0 .$$
Suppose that $a_{i_0 \cdots i_{n+1}} \neq 0$, so we may take $\{a_{i_0 \cdots i_{n+1}}\}_{(i_0 \cdots i_{n+1}) \neq (i'_0 \cdots i'_{n+1})}$ to be local coordinates of $\mathcal{X}$ near $([f]; x)$. Then
\[ d\left(\frac{\partial f}{\partial x_i}\right)(x) = da_{i_1 \cdots i_{n+1}} + \cdots \]
Therefore, $\Sigma$ is smooth at $([f]; x)$. Moreover,
\[ \dim_{\mathbb{C}} \Sigma = \dim_{\mathbb{C}} \mathcal{X} - (n + 2) \]
The irreducibility of $\Sigma$ follows from $H^0(\Sigma, \mathbb{C}) = \mathbb{C}$, which can be easily proved by using the Lefschetz theorem. \(\square\)

**Corollary 1.1.** $B_s = \pi_1(\Sigma_s)$ is irreducible, where $\pi_i: \mathcal{X} \to B$ or $\mathbb{C}P^{n+1}$ is the projection onto its $i$-th factor.

In fact, $B_s$ is the hypersurface in $B$ classifying all singular hypersurfaces in $\mathbb{C}P^{n+1}$.

Define $B_{ss}$ to be the subvariety of $B_s$ consisting of all polynomials $[f]$ of the form $f_1^2 f_2$ with $\deg(f_1)$ positive. One can easily compute
\[ \dim_{\mathbb{C}} B_{ss} = \left( n + d - 1 \right) + n + 1 \]
Therefore, $\text{codim}_{\mathbb{C}} B_{ss} \geq 2$.

Let $\mathcal{K}$ be the relative canonical bundle $K_{\Sigma} \otimes \pi_1^* K_B^{-1}$ over $\Sigma$. We can define a canonical metric $|| \cdot ||_{\mathcal{K}}$ on $\mathcal{K}|_{\Sigma \setminus \Sigma_s}$ as follows: at each point $([f]; x) \in \Sigma \setminus \Sigma_s$, we choose a local coordinate system $(y_1, \cdots, y_b, z_1, \cdots, z_{n+1})$ of $\mathcal{X}$ at $([f]; x)$ such that $y_1, \cdots, y_b$ is a local coordinate system of $B$ at $[f]$ and $z_1, \cdots, z_n$ are tangent to $\pi_1^{-1}([f])$ and $\{z_{n+1} = 0\}$ defines $\pi_1^{-1}([f])$ locally, then $\mathcal{K}$ has a trivialization at $([f]; x)$ given by
\[ a \left( dz_1 \wedge \cdots \wedge dz_n \wedge dy_1 \wedge \cdots \wedge dy_b \right) \otimes \pi_1^* (dy_1 \wedge \cdots \wedge dy_b)^{-1} \to a \in \mathbb{C} \]
we define
\[ ||a(dz_1 \wedge \cdots \wedge dz_n \wedge dy_1 \wedge \cdots \wedge dy_b) \otimes \pi_1^* (dy_1 \wedge \cdots \wedge dy_b)^{-1} ||_{\mathcal{K}}^2 = |a|^2 \left( \det(g_{ij}) \right)^{-1} \]
where $\{g_{ij}\}_{1 \leq i, j \leq n}$ is the metric tensor of the restriction of Fubini-Study metric $\omega_{FS}|_{\pi_1^{-1}([f])}$ on $\mathbb{C}P^{n+1}$, i.e.,
\[ \omega_{FS}|_{\pi_1^{-1}([f])} = \sqrt{-1} \frac{1}{2\pi} g_{ij} \, dz_i \wedge d\bar{z}_j \]
We need to check that it is well-defined. To see this, let \( (y'_1, \ldots, y'_b; z'_1, \ldots z'_{n+1}) \) be another local coordinate system with the properties stated above, then we have transition functions
\[
y'_i = F_i(y), \quad 1 \leq i \leq b
\]
\[
z'_\alpha = H_\alpha(z_1, \ldots, z_{n+1}; y_1, \ldots, y_b), \quad 1 \leq \alpha \leq n + 1
\]
satisfying:
\[
z'_{n+1} = H_{n+1}(z_1, \ldots, z_n, 0; 0, \ldots, 0) \equiv 0
\]
therefore,
\[
dy'_1 \wedge \cdots \wedge dy'_b = \det(DF) dy_1 \wedge \cdots \wedge dy_b
\]
\[
dz'_1 \wedge \cdots \wedge dz'_{n} \big|_{\pi^{-1}(f)} = \det\left(\frac{\partial H_\alpha}{\partial z_\beta}\right)_{1 \leq \alpha, \beta \leq n} dz_1 \wedge \cdots \wedge dz_n \big|_{\pi^{-1}(f)}
\]
\[
\omega_{FS} \big|_{\pi^{-1}(f)} = \frac{\sqrt{-1}}{2\pi} \sum_{\alpha, \beta=1}^{n} g'_{\alpha \beta} dz'_\alpha \cdots dz'_\beta
\]
\[
= \frac{\sqrt{-1}}{2\pi} \sum_{\alpha, \beta=1}^{n} g'_{\alpha \beta} \frac{\partial H_\alpha}{\partial z_\gamma} \frac{\partial H_\beta}{\partial z_\delta} dz_\gamma \wedge d\bar{z}_\delta
\]
i.e.,
\[
g_{\gamma \delta} = \sum_{\alpha, \beta=1}^{n+1} g'_{\alpha \beta} \frac{\partial H_\alpha}{\partial z_\gamma} \frac{\partial H_\beta}{\partial z_\delta}
\]
\[
= \sum_{\alpha, \beta=1}^{n} g'_{\alpha \beta} \frac{\partial H_\alpha}{\partial z_\gamma} \frac{\partial H_\beta}{\partial z_\delta} \quad \text{(since } \frac{\partial H_{n+1}}{\partial z_\gamma} = 0 \text{ for } \gamma \leq n) \]
it follows
\[
\det(g_{\gamma \delta})_{1 \leq \gamma, \delta \leq n} = \det(g'_{\alpha \beta})_{1 \leq \alpha, \beta \leq n} \cdot \det\left(\frac{\partial H_\alpha}{\partial z_\gamma}\right)_{1 \leq \alpha, \gamma \leq n}
\]
Now \( \mathcal{K} \) has a new trivialization at \( ([f]; x) \), i.e.
\[
a'(dz'_1 \wedge \cdots \wedge dz'_n \wedge dy'_1 \wedge \cdots \wedge dy'_b) \otimes \pi_1^*(dy'_1 \wedge \cdots \wedge dy'_b)^{-1} \rightarrow a' \in \mathbb{C}
\]
The local section \( a \) in the first trivialization becomes
\[
a' = a \det\left(\frac{\partial H_\alpha}{\partial z_\gamma}\right)_{1 \leq \alpha, \gamma \leq n}^{-1}
\]
in the second trivialization, so
\[
|a'|^2 \det(g'_{\alpha \beta})_{1 \leq \alpha, \beta \leq n}^{-1} = |a|^2 \det(g_{\gamma \delta})_{1 \leq \gamma, \delta \leq n}^{-1}
\]
i.e., \( \| \cdot \|_\mathcal{K} \) is a well-defined hermitian norm on \( K|_{\Sigma \setminus \Sigma_s} \). The group \( G \) acts on the total space of \( \mathcal{K} \) and the subgroup \( SU(n + 2) \) acts by isometries of \( \| \cdot \|_\mathcal{K} \) on \( K|_{\Sigma \setminus \Sigma_s} \). Introduce a smooth function \( \psi \) on \( \Sigma \setminus \Sigma_s \) by

\[
\psi([f]; x) = \log \left( \frac{\sum_{\alpha=0}^{n+1} |\frac{\partial F}{\partial z_\alpha}|^2}{\sum_{d=i_0+\ldots+i_{n+1}} |a_{i_0\ldots i_{n+1}}|^2 \cdot (\sum_{\alpha=0}^{n+1} |z_\alpha|^2)^{d-1}} \right)
\]

where \( F \) is the defining polynomial of \( \Sigma \). Clearly, \( \psi \) diverges to \(-\infty\) as \( ([f]; x) \) tends to \( \Sigma_s \).

We define a new hermitian metric \( \| \cdot \| \) on \( \mathcal{K}|_{\Sigma \setminus \Sigma_s} \) by

\[
\| \cdot \|^2 = e^{\psi} \| \cdot \|_\mathcal{K}
\]

We will prove that this hermitian metric can be smoothly extended across \( \Sigma_s \) and compute its curvature form. Clearly, \( SU(n + 2) \) still acts on \( \mathcal{K}|_{\Sigma \setminus \Sigma_s} \) by isometries of \( \| \cdot \| \).

**Theorem 1.1.** The hermitian metric \( \| \cdot \| \) can be extended to be a smooth metric on \( \mathcal{K} \) over \( \Sigma \). Moreover, the curvature form of this metric is

\[
\pi^*_1 \omega_B + (d - n - 2) \pi_2^* \omega_{FS}
\]

where \( \omega_B \) is the Kähler form of the Fubini-Study metric on \( B = PR_{n,d} \).

**Proof.** Let \( ([f]; x) \) be any point in \( \Sigma_s \) and \( U \) be a neighborhood of \( ([f]; x) \) in \( \mathcal{K} \). We want to extend \( \| \cdot \| \) on \( \mathcal{K}|_{U \cap \Sigma \setminus \Sigma_s} \) to \( \mathcal{K}|_{U \cap \Sigma} \). By using the isometries from \( SU(n + 2) \), we may assume that

\[
x = [1, 0, \ldots, 0]
\]

Since \( \Sigma \) is smooth and \( \pi^*_1([f]) \) is singular at \( x \), there is a local coordinate system \( \{y_1, \ldots, y_b; z_1, \ldots, z_{n+1}\} \) of \( \mathcal{K} \) at \( ([f]; x) \) such that \( \{y_1, \ldots, y_{b-1}; z_1, \ldots, z_{n+1}\} \) restricts to a local coordinate sytem of \( \Sigma \) near \( ([f]; x) \), in particular,

\[
\frac{\partial F}{\partial y_b} \neq 0 \quad \text{in} \quad U,
\]
then $\mathcal{K}$ has a local holomorphic section

$$S_U = \frac{dz_1 \wedge \cdots \wedge dz_{n+1} \wedge dy_1 \wedge \cdots \wedge dy_{b-1}}{\frac{\partial F}{\partial y_b}} \otimes \pi_1^*(dy_1 \wedge \cdots \wedge dy_b)$$

At any point $([f]; y) \in U \cap \Sigma \setminus \Sigma_s$, for some $i, z_1, \ldots, z_{i}, \ldots, z_{n+1}$ give local coordinates of $\pi_1^{-1}([f])$ at $y$, for simplicity, we assume $i = n + 1$. Since $F \equiv 0$ on $\Sigma$

$$\frac{\partial F}{\partial z_1} dz_1 + \cdots + \frac{\partial F}{\partial z_{n+1}} dz_{n+1} + \cdots + \frac{\partial F}{\partial y_b} dy_b \equiv 0 \text{ on } \Sigma.$$

Multiplying both sides by $dz_1 \wedge \cdots \wedge dz_n$, we obtain

$$S_U = \pm \frac{dz_1 \wedge \cdots \wedge dz_n \wedge dy_1 \wedge \cdots \wedge dy_{b}}{\frac{\partial F}{\partial z_{n+1}}} \otimes \pi_1^*(dy_1 \wedge \cdots \wedge dy_b)^{-1}$$

Let $\{g_{ij}\}$ be the corresponding hermitian metrics of $\omega_{FS}$ in local coordinates $z_1, \ldots, z_{n+1}$, i.e.,

$$\omega_{FS} = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{n+1} g_{ij} dz_i \wedge d\bar{z}_j$$

Then

$$\omega_{FS}^n = n! \left( \frac{\sqrt{-1}}{2\pi} \right)^n \sum_{i,j=1}^{n+1} (-1)^{i+j+n+1} \det(g_{\alpha\beta})_{1 \leq \alpha, \beta \leq n+1}^{a \neq i, \beta \neq j} dz_1 \wedge \cdots \wedge d\bar{z}_i \wedge \cdots \wedge d\bar{z}_j \wedge \cdots \wedge d\bar{z}_n$$

$$= n! \left( \frac{\sqrt{-1}}{2\pi} \right)^n \left\{ \det(g_{\alpha\beta})_{1 \leq \alpha, \beta \leq n} - 2\text{Re}(\sum_{i=1}^{n} (-1)^{n+i+1} \det(g_{\alpha\beta})_{1 \leq \alpha \leq n+1 \atop \alpha \neq i, \beta \neq n} \frac{\partial z_{n+1}}{\partial z_i}) \right\}$$

$$\frac{\partial z_{n+1}}{\partial z_i} + \sum_{i,j=1}^{n} (-1)^{i+j} \det(g_{\alpha\beta})_{1 \leq \alpha, \beta \leq n+1}^{a \neq i, \beta \neq j} \frac{\partial z_{n+1}}{\partial z_i} \frac{\partial z_{n+1}}{\partial z_j}$$

$$dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n$$
(since \(\frac{\partial z_{n+1}}{\partial z_i} = -\frac{\partial F}{\partial z_i} / \frac{\partial F}{\partial z_{n+1}}\))

\[
= n!(\frac{-1}{2\pi})^n \left\{ \det(g_{\alpha\beta})_{1 \leq \alpha, \beta \leq n} \left( \frac{\partial F}{\partial z_{n+1}} \right)^2 \right. \\
+ \sum_{i,j=1}^{n} (-1)^{i+j} \det(g_{\alpha\beta})_{1 \leq \alpha, \beta \leq n} \frac{\partial F}{\partial z_i} \frac{\partial F}{\partial z_j} \\
\left. \cdot \left| \frac{\partial F}{\partial z_{n+1}} \right|^{-2} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n \right\}
\]

\[
= n!(\frac{-1}{2\pi})^n \left| \frac{\partial F}{\partial z_{n+1}} \right|^{-2} \det(g_{ij})_{1 \leq i,j \leq n+1} \sum_{i,j=1}^{n+1} g^{ij} \frac{\partial F}{\partial z_i} \frac{\partial F}{\partial z_j} dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n
\]

where \(\{g^{ij}\}\) is the inverse matrix of \(\{g_{ij}\}\). Therefore

\[
\|S_v\|_T^2 = \left\{ \det(g_{ij})_{1 \leq i,j \leq n+1} \sum_{i,j=1}^{n+1} g^{ij} \frac{\partial F}{\partial z_i} \frac{\partial F}{\partial z_j} \right\}^{-1}
\]

Now we may assume that \(z_1, \ldots, z_{n+1}\) are coordinates of \(\mathbb{C}^{n+1} \subset \mathbb{CP}^{n+1}\).

Then

\[
g_{ij} = \frac{1}{1 + \sum_{i=1}^{n+1} |z_i|^2} \left( \delta_{ij} - \frac{z_i z_j}{1 + \sum_{i=1}^{n+1} |z_i|^2} \right)
\]

\[
g^{ij} = (1 + |z|^2) (\delta_{ij} + z_j z_i)
\]

it follows

\[
\sum_{i,j=1}^{n+1} g^{ij} \frac{\partial F}{\partial z_i} \frac{\partial F}{\partial z_j} = (1 + |z|^2) \left( \sum_{i=1}^{n+1} \left| \frac{\partial F}{\partial z_i} \right|^2 + \sum_{i,j=1}^{n+1} z_i \frac{\partial F}{\partial z_i} \cdot z_j \frac{\partial F}{\partial z_j} \right)
\]

but \(\sum_{i=1}^{n+1} z_i \frac{\partial F}{\partial z_i} + \frac{\partial F}{\partial z_0} = \sum_{i=1}^{n+1} z_i \frac{\partial F}{\partial z_i} + \frac{\partial F}{\partial z_0} = dF \equiv 0\) on \(\Sigma\), so

\[
\sum_{i,j=1}^{n+1} g^{ij} \frac{\partial F}{\partial z_i} \frac{\partial F}{\partial z_j} = (1 + |z|^2) \sum_{i=1}^{n+1} \left| \frac{\partial F}{\partial z_i} \right|^2
\]

On the other hand,

\[
\det(g_{ij})_{1 \leq i,j \leq n+1} = \frac{1}{(1 + |z|^2)^{n+2}}
\]
Therefore,
\[ \| S_V \|^2 = e^\psi \frac{(1 + |z|^2)^{n+1}}{\sum_{i=0}^{n+1} |\frac{\partial F}{\partial z_i}|^2} \]
\[ = \left( \sum_{i_0 + \ldots + i_{n+1} = d} |a_{i_0 \ldots i_{n+1}}|^2 \right)^{-1} (1 + |z|^2)^{n+2-d} \]

This shows that the hermitian metric extends smoothly to a metric on \( K \) over \( \Sigma \). The curvature form of this metric follows easily from the above formula for \( \| S_V \|^2 \). \( \square \)

By the definition of the hermitian metric \( \| \cdot \| \), one can easily deduce the following from Theorem 1.1,

**Corollary 1.2.** The curvature form of the metric \( \| \cdot \|_K \) is given by
\[ R(\| \cdot \|_K) = \pi_1^* \omega_B + (d - n - 2) \pi_2^* \omega_{FS} + \frac{\sqrt{-1}}{2\pi} \partial \overline{\partial} \psi. \]

2. EXTENSION OF QUILLEN METRICS

In this section, we first introduce a hermitian metric on the tautological line bundle \( O_B(-1) \) over \( B \). This metric will be related to the intrinsic structure of the underlying variety through the K-energy in next section. We will also show that this metric is actually the Quillen metric on the determinant line bundle of certain holomorphic “virtual” bundle over \( \Sigma \). This explains how we arrived at this metric.

Define a current \( u \) as follows: for any smooth \((2b-2)\)-form \( \varphi \) on \( B \), where \( b \) is the complex dimension of \( B \), we have
\[ \int_B \varphi \wedge u = \int_{\Sigma} \pi_1^* \varphi \wedge \pi_2^* \omega_{FS}^{n+1} \]
It is easy to see that \( u \) is positive, closed and smooth in \( B \setminus B \). To determine the cohomology class \([u]\) in \( H^2(B, \mathbb{R}) \), we simply plug \( \omega^{b-1} \) into (2.1) and find
\[ \int_B \omega_B^{b-1} \wedge u = \int_{\Sigma} \pi_1^* \omega_B^{b-1} \wedge \pi_2^* \omega_{FS}^{n+1} \]
\[ = \int_{\Sigma} \pi_1^* \omega_B^{b-1} \wedge \pi_2^* \omega_{FS}^{n+1} \wedge (\pi_1^* \omega_B + d \cdot \pi_2^* \omega_{FS}) \]
\[ = \int_{\Sigma} \pi_1^* \omega_B^{b-1} \wedge \pi_2^* \omega_{FS}^{n+1} \wedge \pi_1^* \omega_B = 1 \]
Therefore, the cohomology class \([u]\) coincides with the Kähler class of the Fubini-Study metric, which is just the first Chern class of \(O_B(1)\) over \(B\).

**Proposition 2.1.** There is a \(1/\delta\)-Hölder continuous hermitian metric \(\| \cdot \|_u\) on \(O_B(-1)\) over \(B\) such that \(-u\) is its curvature form in the sense of distribution, namely, for any smooth \((2b-2)\)-form \(\varphi\) on \(B\), we have

\[
\int_B \varphi \land u = \int_B \varphi \land \left( \omega_B + \frac{-1}{2\pi} \partial \bar{\partial} \log \left( \frac{\| \cdot \|_u^2}{\| \cdot \|_B^2} \right) \right)
\]

where \(\| \cdot \|_B\) denotes the standard metric on \(O_B(-1)\) whose curvature form is \(\omega_B\).

This is a special case of Theorem 5.2 in [T1]. The readers can find the proof there. We note that such a metric \(\| \cdot \|_u\) is unique up to multiplications by positive constants.

Since \(\psi\) has only logarithmic singularity along \(\Sigma_s\), the integral

\[
\int_{\Sigma} \psi \pi_1^* \varphi \land \pi_2^* \omega_{FS}^n
\]

is finite for any smooth \(2b\)-form \(\varphi\) on \(B\). We can define a generalized function \(\psi_B\) by

\[
\int_B \psi_B \varphi = \int_{\Sigma} \psi \pi_1^* \varphi \land \pi_2^* \omega_{FS}^n
\]

where \(\varphi\) is a smooth \(2b\)-form on \(B\).

**Lemma 2.1.** The function \(\psi_B\) is smooth outside \(B_s\), continuous outside \(B_{ss}\) and bounded from above.

**Proof.** The smoothness of \(\psi_B\) outside \(B_s\) follows from the fact that the fibration \(\pi_1 : \Sigma \backslash \Sigma_s \mapsto B \backslash B_s\) is smooth. Since \(\psi\) is bounded from above, so is \(\psi_B\). Moreover,

\[
\psi_B(y) = \int_{\pi_1^{-1}(y)} \psi \omega_{FS}^n
\]
for any point \( y \) in \( B \setminus B_s \). Given any point \( y \) in \( B_s \setminus B_{ss} \), using the fact that the singular set of \( \pi_1^{-1}(y) \) is of complex dimension at most \( n - 1 \), it is easy to show

\[
\lim_{z \rightarrow y, z \in B \setminus B_s} \psi_B(z) = \int_{\pi_1^{-1}(y)} \psi \omega_F^{n_s} < \infty
\]

Clearly, it follows the continuity of \( \psi_B \) at \( y \). In fact, the value of \( \psi_B \) is given by the integral on the right in (2.5). The lemma is proved. □

By Proposition 2.1, the closed form \((d - n - 2)u + d(n + 1)\omega_B\) represents the same cohomology class as \((n + 2)(d - 1)\omega_B\) does.

**DEFINITION 2.1.** Fix \( d > 1 \), we define a singular hermitian metric on \( O_B(-1) \) by

\[
\left\| \begin{array}{c}
\psi_B \\
\end{array} \right\| = e^{\frac{(2n+1)(d-1)}{2(n+2)(n-1)}} \left\| \begin{array}{c}
\psi \\
\end{array} \right\|_B
\]

where \( \left\| \begin{array}{c}
\psi \\
\end{array} \right\|_u \) and \( \left\| \begin{array}{c}
\psi \\
\end{array} \right\|_B \) are given as in Proposition 2.1.

Later on we will see how this metric is related to the K-energy.

In the following we will show that this metric is nothing but the Quillen metric on certain determinant line bundle over \( B \setminus B_s \).

Consider a “virtual” holomorphic bundle \( E \) over \( \Sigma \),

\[
E = (n + 1)(K^{-1} - K) \otimes (L - L^{-1})^n - n(n + 2 - d)(L - L^{-1})^{n+1}
\]

where

\[
\mathcal{L} = \pi_2^* \mathcal{O}_{P^{n+1}}(1)
\]

We denote by \( \det(E, \pi_1) \) the determinant line bundle over \( B \) through the fibration \( \pi_1 : \Sigma \rightarrow B \). It is defined to be

\[
\prod_{j=0}^{n} \left( \det(K^{-1} \otimes L^{n-2j}, \pi_1) \otimes \det(K \otimes L^{n-2j}, \pi_1)^{-1} \right)^{p(j)} \otimes \prod_{j=0}^{n+1} \det(L^{n+1-2j}, \pi_1)^{-q(n+1)}
\]

where \( p = n + 1 \) and \( q = n(n + 2 - d) \). For any coherent sheaf \( \mathcal{F} \) over \( \Sigma \), the determinant line bundle \( \det(\mathcal{F}, \pi_1) \) is defined as follows (cf. [KH]): we first define a presheaf by assigning any open subset \( V \subset B \) to the cohomology
$H^i(\pi^{-1}(V), \mathcal{F}|_{\pi^{-1}(V)})$, then the direct image $R^i\pi_1_*\mathcal{F}$ is simply the sheaf generated by this presheaf, and define $\text{det}(\mathcal{F}, \pi_1)$ to be the double dual of the sheaf

\begin{equation}
\bigotimes_{i=0}^{\infty} \text{det}(R^i\pi_1_*\mathcal{F})^{(-1)^i}
\end{equation}

By the Grothendick-Riemann-Roch Theorem, one can compute the first Chern class

\begin{equation}
C_1(\text{det}(\mathcal{E}, \pi_1)) = -2^{n+1}(n+2)(d-1)\omega_B,
\end{equation}

therefore, the determinant line bundle $\text{det}(\mathcal{E}, \pi_1)$ coincides with $\mathcal{O}_B(-(n+2)(d-1))$.

Let us recall definition of the Quillen metric on $\text{det}(\mathcal{E}, \pi_1)$. Let $h$ be the hermitian metric on $\mathcal{L} = \pi^*_2\mathcal{O}_{\mathbb{C}P^{n+1}}(1)$ with curvature form $\pi^*_2\omega_{FS}$. Fix a smooth hypersurface $X$ in $\mathbb{C}P^{n+1}$, the Fubini-Study metric restricts to a Kähler metric $\omega_X$ on $X$ and consequently induces a hermitian metric $h_X$ on $\mathcal{K}|_X$. Let $\mathcal{F}$ be one of bundles $\mathcal{K}^\pm \otimes \mathcal{L}^{n-2j}$ or $\mathcal{L}^{n+1-2j}$. Then the fiber of the determinant line bundle $\text{det}(\mathcal{F}, \pi_1)$ over $X$ can be naturally identified with the space

\begin{equation}
\bigotimes_{i=1}^{n} \text{det}(H^i(X, \mathcal{F}|_X))^{(-1)^i}
\end{equation}

The metrics $h$ and $h_X$ yield a hermitian metric $h_\mathcal{F}$ on $\mathcal{F}$. Together with the Kähler metric $\omega_X$, this $h_\mathcal{F}$ defines an inner product $\langle \cdot, \cdot \rangle$ on $H^i(X, \mathcal{F}|_X)$, which is identified with the space of harmonic $\mathcal{F}$-valued $(0,i)$-forms,

\begin{equation}
\langle s_1, s_2 \rangle = \int_X (s_1(x), s_2(x))_\mathcal{F} \omega_X^n, \quad s_1, s_2 \in H^i(X, \mathcal{F}|_X)
\end{equation}

where $(\cdot, \cdot)_\mathcal{F}$ denotes the inner product on the space of $\mathcal{F}$-valued $(0,i)$-forms on $X$ induced by $h_\mathcal{F}$ and $\omega_X$. In a local coordinate system $\{z_1, \ldots, z_n\}$ of $X$ and a local holomorphic frame $e$ of $\mathcal{F}$, we can express

\begin{align}
\begin{aligned}
s_1 &= \sum \varphi_{\alpha_1 \cdots \alpha_i} dz_{\alpha_1} \wedge \cdots \wedge dz_{\alpha_i} \\
s_2 &= \sum \psi_{\beta_1 \cdots \beta_i} dz_{\beta_1} \wedge \cdots \wedge dz_{\beta_i}
\end{aligned}
\end{align}

then

\begin{equation}
\langle s_1, s_2 \rangle_\mathcal{F} = a g^{\alpha_1 \beta_1} \cdots g^{\alpha_i \beta_i} \varphi_{\beta_1} \cdots \beta_i \psi_{\alpha_1} \cdots \alpha_i
\end{equation}
where $a$ is a local positive function representing $h_X$ in the frame $e$ and $\{g^{\alpha\beta}\}$ is the inverse of the metric $\omega_X$ in the local coordinates. The inner product induces a $L^2$-metric on $\det H^i(X, \mathcal{F}|_X)$. Unfortunately, this $L^2$-metric does not vary continuously as $X$ varies in $B$. A discontinuous point in $B$ is where the dimension of $H^i(X, \mathcal{F}|_X)$ jumps for some $i$. The Quillen metric is introduced to eliminate this discontinuity. A Quillen metric is defined in terms of the $L^2$-metric and the Ray-Singer analytic torsion.

Let $P_X$ be the harmonic projection from the space $\Gamma^i(X, \mathcal{F}|_X)$ of $\mathcal{F}$-valued $(0,i)$-forms onto $H^i(X, \mathcal{F}|_X)$. We define an Zeta function

\begin{equation}
\zeta_X(s) = -\frac{1}{\Gamma(s)} \int_0^\infty u^{s-1} \sum_{i=0}^n (-1)^i \text{tr} \left( e^{-u} \Box_X^i - P_X \right) du , \quad \text{Re}(s) > n
\end{equation}

where

\begin{equation}
\Gamma(s) = \int_0^\infty u^{s-1} e^{-u} du
\end{equation}

and $\Box_X^i$ denotes the Laplacian on $\Gamma^i(X, \mathcal{F}|_X)$ induced by $h_X$ and $\omega_X$, i.e., if $\bar{\partial}^*$ denotes the adjoint operator of $\bar{\partial}$ on $\Gamma^i(X, \mathcal{F}|_X)$ with respect to $h$ and $\omega$, then

\begin{equation}
\Box_X^i = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*
\end{equation}

It is now well-known that $\zeta_X(s)$ extends to be a meromorphic function on $\mathbb{C}$ and the extension, still denoted by $\zeta_X$, is holomorphic at the origin.

**Definition 2.2.** (Ray and Singer [RS]). The analytic torsion $\tau(\mathcal{F}|_X)$ is defined to be $-\zeta'_X(0)$.

**Definition 2.3.** The Quillen metric $\| \cdot \|_{Q, \mathcal{F}}$ on the determinant line bundle $\det(\mathcal{F}, \pi_1)_X$ is the metric obtained by multiplying the $L^2$-metric by $e^{\tau(\mathcal{F}|_X)}$.

It can be proved that this Quillen metric depends on $X$ smoothly. Using (2.12) and $\| \cdot \|_{Q, \mathcal{F}}$ for $\mathcal{F}$ being either $K^\pm \otimes \mathcal{L}^{n-2j}$ or $\mathcal{L}^{n+1-2j}$, we can construct a Quillen metric $\| \cdot \|_{Q, \mathcal{E}}$ on $\det(\mathcal{E}, \pi_1)$.

**Theorem 2.2.** There is a positive constant $c$ such that the Quillen metric $\| \cdot \|_{Q, \mathcal{E}}$ is the same as $c \| \cdot \|_{Q}^{2n+1(n+2)(d-1)}$. 
This theorem can be proved by the same arguments as in the proof of Theorem 4.1 and using two transgression formulas of Bismut, Gillet and Soule in [BGS] (see also [BF]). We omit the proof, since it is not used in the proof of our theorems.

**Corollary 2.1.** The Quillen metric \( || \cdot ||_{Q,E} \) over \( B \setminus B_s \) extends to be a continuous metric over \( B \setminus B_{ss} \).

Clearly, it follows from Theorem 2.1 and Lemma 2.1.

3. The Generalized K-energy

In this section, we define the K-energy on singular varieties in \( \mathbb{C}P^N \), and compute its complex Hessian. The K-energy was first introduced by T. Mabuchi [Ma] on compact Kähler manifolds. Our K-energy here is nothing else but the limit of the K-energy of T. Mabuchi, in case the singular variety is a limit of smooth Kähler manifolds.

Let \( X \) be an irreducible subvariety in \( \mathbb{C}P^N \). We denote by \( S(X) \) the subvariety of all singular points in \( X \). We assume that the complex codimension of \( S(X) \) is at least two, or equivalently, \( X \) is normal. Let \( L \) be the restriction to \( X \) of the hyperplane line bundle over \( \mathbb{C}P^N \). A hermitian metric \( h \) on \( L \) is admissible if it extends to be a smooth metric over \( \mathbb{C}P^N \) with positive curvature, in particular, for any \( \sigma \) in \( G = SL(N + 1, \mathbb{C}) \), the restriction of \( \sigma^* h_{FS} \) to \( X \) is an admissible metric on \( L \), where \( h_{FS} \) is the standard metric on the hyperplane bundle whose curvature gives the Fubini-Study metric. We denote by \( \mathcal{M}_X^+ \) the set of all admissible metrics. Note that any metric \( h \) in \( \mathcal{M}_X^+ \) is of the form \( e^{-\varphi} h_{FS} \) for some function \( \varphi \).

**Lemma 3.1.** Let \( \{h_t\}_{0 \leq t \leq 1} \) be a loop in \( \mathcal{M}_X^+ \). Then

\[
(3.1) \quad \int_0^1 \int_X \varphi_t (s(\omega_t) - c) \omega^n_t \wedge dt = 0
\]

where \( \{h_t = e^{-\varphi_t} h_0\} \), \( \omega_t \) is the curvature \( R(h_t) \) of \( h_t \), \( s(\omega_t) \) denotes the scalar curvature of the Kähler metric \( \omega_t \) on \( X \) and \( c \) is any constant.

**Proof.** First we note that \( \psi, |\nabla \psi| \) and \( \Delta \psi \) are bounded for \( \psi = \varphi_t \) or \( \frac{\partial \varphi_t}{\partial t} \), where \( |\nabla \psi| \) is the pointwise norm of the gradient of \( \psi \) with respect to the
metric induced by \( \omega_t \). Let \( \pi : \tilde{X} \to X \) be a smooth resolution. Fix a Kähler form \( \tilde{\omega} \) on \( \tilde{X} \). If \( \text{Ric}(\omega) \) denotes the Ricci curvature of the metric induced by a Kähler form \( \omega \), then

\[ \text{Ric}(\tilde{\omega}) - \pi^* \text{Ric}(\omega_t) = \partial \bar{\partial} \log \left( \frac{\pi^* \omega^n_t}{\tilde{\omega}^n} \right) \]  

Using the assumption that \( S(X) \) is of at least codimension two and integrating by parts, one can easily prove

\[ \int_0^1 \int_{\tilde{X}} \pi^* \varphi_t (n \text{Ric}(\tilde{\omega}) - c \pi^* \omega_t) \wedge \omega_t \wedge dt \]

\[ = \left[ \log \left( \frac{\pi^* \omega^n_t}{\tilde{\omega}^n} \right) - \frac{\pi^* \omega^n_t}{\tilde{\omega}^n} \right]_0^1 \tilde{\omega}^n \]

\[ + \int_0^1 \int_{\tilde{X}} \pi^* \varphi_t (n \text{Ric}(\tilde{\omega}) - c \pi^* \omega_t) \wedge \pi^* \omega_t \wedge dt \]

\[ = \int_0^1 \int_{\tilde{X}} \pi^* \varphi_t (n \text{Ric}(\tilde{\omega}) - c \pi^* \omega_t) \wedge \pi^* \omega_t \wedge dt \]

Define \( h_{t,s} = e^{-s \varphi_t} h_0 \) and

\[ D(s) = \int_0^1 \int_{\tilde{X}} s \pi^* \varphi_t (n \text{Ric}(\tilde{\omega}) - c \pi^* \omega_t) \wedge \pi^* \omega_t \wedge dt \]

where \( \omega_{t,s} = \omega_0 + s \partial \bar{\partial} \varphi_t \). It suffices to show that \( D' \equiv 0 \). Write \( \varphi_{t,s} = s \varphi_t \), and we compute

\[ D'(s) = \int_0^1 \int_{\tilde{X}} \frac{\partial^2 \varphi_{t,s}}{\partial t \partial s} (n \text{Ric}(\tilde{\omega}) - c \pi^* \omega_t) \wedge \pi^* \omega_t \wedge dt \]

\[ + n \int_0^1 \int_{\tilde{X}} \frac{\partial \varphi_{t,s}}{\partial t} \partial \bar{\partial} \left( \frac{\partial \varphi_{t,s}}{\partial s} \right) \wedge (n \text{Ric}(\tilde{\omega}) - c \pi^* \omega_t) \wedge \omega_t \wedge dt \]

\[ = \int_0^1 \int_{\tilde{X}} \frac{\partial}{\partial t} \left( \frac{\partial \varphi_{t,s}}{\partial s} (n \text{Ric}(\tilde{\omega}) - c \pi^* \omega_t) \wedge \pi^* \omega_t \wedge dt \right) \]

The lemma is proved. \( \square \)
By this lemma, we can define the generalized K-energy $D(h_0, h_1)$ by
\begin{equation}
D(h_0, h_1) = - \int_0^1 \int_X \phi_t \left( s(\omega_t) - \mu \right) \omega_t^n \wedge dt
\end{equation}
where $h_0, h_1 \in \mathcal{M}_X^+$, $\{h_t\}$ is any path from $h_0$ to $h_1$, and $\mu$ is defined to be the ratio
\begin{equation}
\frac{\int_X s(\omega_t) \omega_t^n}{\int_X \omega_t^n}
\end{equation}
It is known that $\mu$ is independent of $t$ and any choice of admissible metrics in $\mathcal{M}_X^+$. Obviously, by Lemma 3.1, we have
\begin{align*}
D(h_0, h_1) &= -D(h_1, h_0) \\
D(h_0, h_2) &= D(h_0, h_1) + D(h_1, h_2)
\end{align*}
The following proposition can be proved by the same arguments in the proof of Lemma 3.1.

**Proposition 3.1.** Assume that there is smooth function $f_0$ on $X \setminus S(X)$ satisfying:
\begin{equation}
\text{Ric}(\omega_0) - \frac{\mu}{n} \omega_0 = \partial \bar{\partial} f_0 \quad \text{on } X \setminus S(X)
\end{equation}
Then we have
\begin{equation}
D(h_0, h_1) = \int_X \left( \log \left( \frac{\omega_t^n}{\omega_0^n} \right) - f_0 \right) \omega_t^n + \int_X f_0 \omega_0^n - \frac{\mu}{n} \left( I(h_0, h_1) - J(h_0, h_1) \right)
\end{equation}
where $I(h_0, h_1)$ and $J(h_0, h_1)$ are defined as follows: write $h_1 = e^{-\varphi} h_0$, then $\omega_1 = \omega_0 + \partial \bar{\partial} \varphi$, and
\begin{align}
I(h_0, h_1) &= \int_X \varphi (\omega_0^n - \omega_1^n) \\
J(h_0, h_1) &= \int_0^1 \frac{I(h_0, e^{-t\varphi} h_0)}{t} dt
\end{align}
In case $X$ is a complete intersection in $\mathbb{C}P^N$, the assumption (3.7) is automatically true.

Next, we will compute the complex Hessian of the K-energy. Let $Y$ be a complex manifold of dimension $m$ and $h : Y \hookrightarrow \mathcal{M}_X^+$ be a smooth map. Fix a hermitian metric $h_0$ in $\mathcal{M}_X^+$, the pull-back $h^* D(h_0, \cdot)$ is a real function on $Y$. 
We want to compute its complex Hessian. We define a hermitian metric \( h_{L,Y} \) on the bundle \( \pi^*_{L}L \) over \( Y \times X \), where \( \pi_j \) is the projection from \( Y \times X \) onto its \( j \)-th factor,

\[
(3.11) \quad h_{L,Y}|_{\{y\} \times X} = h(y)|_{X}
\]

Note that each \( h(y) \) is a metric on \( L \) with positive curvature. We denote by \( R(h_{L,Y}) \) the curvature form of \( h_{L,Y} \) on \( Y \times X \). It is easy to see that as a 2-form, \( R(h_{L,Y}) \) restricts to \( R(h_y) \) on \( \{y\} \times X \). The Kähler metrics defined by \( R(h_y) \), where \( y \in Y \), induce a hermitian metric \( k_Y \) on the relative canonical bundle \( \pi^*_YK_X \) over \( Y \times X \). We denote by \( \text{Ric}_{X|Y} \) the curvature of this hermitian metric. Then

\[
(3.12) \quad \text{Ric}_{X|Y} = \pi^*_Y \text{Ric}(\omega_0) - \partial \bar{\partial} \log \left( \frac{\omega_y^n}{\omega_0^n} \right)
\]

where \( \omega_y = R(h_y) \) and \( \omega_0 = R(h_0) \). It follows from admissibility of \( h_y \) that the second form in (3.12) is bounded. Moreover, \( \text{Ric}_{X|Y} \) restricts to \( \text{Ric}(\omega_y) \) on \( \{y\} \times X \).

**Proposition 3.2.** Let \( X \) be an irreducible, normal projective variety, and \( Y, h, \) etc. be as above. Then the complex Hessian \( D(h_0, h(\cdot)) \) is given weakly as follows: for any smooth \((2m-2)\)-form \( \phi \) with compact support in \( Y \),

\[
(3.13) \quad -\int_Y D(h_0, h(y)) \partial_Y \bar{\partial}_Y \phi = \int_{Y \times X} \left( \text{Ric}_{X|Y} - \frac{\mu}{n+1} \omega_{L,Y} \right) \wedge \omega_{L,Y}^n \wedge \pi^*_1 \phi
\]

**Proof.** We may assume that \( Y \) is an open ball in \( \mathbb{C}^m \). Then for any \( y \) in \( Y, \{h(ty)\}_{0 \leq t \leq 1} \) is a path from \( h(0) \) to \( h(y) \). We define \( \varphi_y = -h(y)h(y)^{-1}, \omega_y = R(h_y) \) as above, and \( \mu \) to be as in (3.6). Then by the definition, we have

\[
(3.14) \quad D(h_0, h(y)) = D(h_0, h(0)) + D(h(0), h(y))
\]

\[
= D(h_0, h(0)) - \int_0^1 \int_X \varphi_{ty} (n \text{Ric}(\omega_{ty}) - \mu \omega_{ty}) \wedge \omega_{ty}^{n-1} \wedge dt
\]

\[
= D(h_0, h(0)) + \int_0^1 \int_{\{y\} \times X} \varphi_{ty} (n \text{Ric}_{X|Y} - \mu \omega_{L,Y}) \wedge \omega_{L,Y}^{n-1} \wedge dt
\]

where \( \omega_{L,Y} = R(h_{L,Y}) \). Let \( \eta : \mathbb{R}^1 \mapsto \mathbb{R}^1 \) be a smooth map satisfying: \( 0 \leq \eta \leq 1, \eta(t) = 0 \) for \( t \leq 1 \) and \( \eta(t) = 1 \) for \( t \geq 2 \). We denote by \( \rho \) the distance
function from the singular set \( S(X) \). Define dilation maps \( \tau_t : Y \times X \mapsto Y \times X \) by putting \( \tau_t(y, x) = (ty, x) \). Then for any \((2m - 2)\)-form \( \phi \) with compact support in \( Y \), using (3.14) and (3.12), we have

(3.15)

\[
- \int_Y D(h_0, h(y)) \partial_Y \overline{\partial} Y \phi
= \lim_{\epsilon \to 0^+} \left( \int_Y \left( \int_0^1 \int_{\{ty\} \times X} \eta(\frac{\rho}{\epsilon}) \phi_{ty} \left(n \text{ Ric}_{X|Y} - \mu \omega_{L,Y} \right) \wedge \omega_{\nu}^{n-1} \wedge \partial_Y \overline{\partial} Y \phi \wedge dt \right) \right)
= \lim_{\epsilon \to 0^+} \left( \int_Y \int_{Y \times X} \tau_t^* \phi \wedge \partial \overline{\partial} \left( \eta(\frac{\rho}{\epsilon}) \phi_{ty} \tau_t^* \left(n \text{ Ric}_{X|Y} - \mu \omega_{L,Y} \right) \wedge \tau_t^* \omega_{\nu}^{n-1} \right) \wedge dt \right)
\]

\[
= \int_0^1 \int_{Y \times X} \tau_t^* \phi \wedge \partial \overline{\partial} \phi_{ty} \wedge \tau_t^* \left(n \text{ Ric}_{X|Y} - \mu \omega_{L,Y} \right) \wedge \tau_t^* \omega_{\nu}^{n-1} \wedge dt
+ \lim_{\epsilon \to 0^+} \int_0^1 \int_{Y \times X} \tau_t^* \phi \wedge \left( \phi_{ty} \partial \overline{\partial} \eta(\frac{\rho}{\epsilon}) + 2 \text{Re} \left( \partial \eta(\frac{\rho}{\epsilon}) \wedge \partial \phi_{ty} \right) \right) \wedge
\]

\[
\wedge \tau_t^* \left(n \text{ Ric}_{X|Y} - \mu \omega_{L,Y} \right) \wedge \tau_t^* \omega_{\nu}^{n-1} \wedge dt
\]

It follows from (3.12) and the remark after it that the integrand in last integral is bounded by \( C \rho^{-2} \) for some constant \( C \). Since the singular set is of codimension at least two, the last limit is zero. Therefore,

(3.16)

\[
- \int_Y D(h_0, h(y)) \partial_Y \overline{\partial} Y \phi
= \int_0^1 \int_{Y \times X} \tau_t^* \phi \wedge \partial \overline{\partial} - \phi_{ty} \wedge \tau_t^* \left(n \text{ Ric}_{X|Y} - \mu \omega_{L,Y} \right) \wedge \tau_t^* \omega_{\nu}^{n-1} \wedge dt
\]

\[
= \int_0^1 \int_{Y \times X} \tau_t^* \phi \wedge \left( \tau_t^* \left(n \text{ Ric}_{X|Y} \right) \wedge \frac{\partial}{\partial t} \tau_t^* \omega_{\nu}^{n-1} - \frac{\mu}{n+1} \frac{\partial}{\partial t} \tau_t^* \omega_{\nu}^{n+1} \right) \wedge dt
\]

\[
= \int_{Y \times X} \tau_t^* \phi \wedge \left( \text{ Ric}_{X|Y} \wedge \omega_{\nu}^{n-1} - \frac{\mu}{n+1} \omega_{\nu}^{n+1} \right)
- \int_0^1 \int_{Y \times X} \tau_t^* \phi \wedge \frac{\partial}{\partial t} \left( \tau_t^* \left(n \text{ Ric}_{X|Y} \right) \right) \wedge \tau_t^* \omega_{\nu}^{n-1} \wedge dt
\]

It remains to show that the last integral in (3.16) vanishes. By (3.12),

(3.17)

\[
\frac{\partial}{\partial t} \left( \tau_t^* \left(n \text{ Ric}_{X|Y} \right) \right) = -\partial \overline{\partial} \left( \Delta_t \varphi_y \right)
\]
where $\Delta_{ty}$ is the laplacian of the metric with Kähler form $\omega_{ty}$. Therefore,

$$
\int_0^1 \int_{Y \times X} \pi_1^* \phi \wedge \frac{\partial}{\partial t} (\tau_t^*(Ric_{X/Y})) \wedge \tau_t^* \omega_{L,Y}^n \wedge dt
$$

$$
= - \int_Y \partial_Y \bar{\partial}_Y \phi \cdot \int_X \int_0^1 \Delta_{ty} \rho_y \omega_{ty}^n \wedge dt = 0
$$

Then (3.13) follows. □

**Corollary 3.1.** Let $X$ be in $\mathbb{C}P^N$ and $G$ be the linear group $\text{SL}(N + 1, \mathbb{C})$ acting naturally on $\mathbb{C}P^N$. Define $GX$ to be the variety $\{(\sigma, x) \mid x \in \sigma(X)\}$ in $G \times \mathbb{C}P^N$. Then for any smooth $(2 \dim \mathbb{C}G - 2)$-form $\phi$ with compact support in $G$,

$$
(3.18)
$$

$$
- \int_G D(h_{FS}, \sigma^* h_{FS}) \partial_G \bar{\partial}_G \phi = \int_{GX} \left( Ric_{GX,G} - \frac{\mu}{n + 1} \sigma^* \omega_{FS} \right) \wedge (\sigma^* \omega_{FS})^n \wedge \pi_1^* \phi
$$

where $\omega_{FS}$ is the curvature form of $h_{FS}$, and $\text{Ric}_{GX,G}$ is the curvature of the metric on the relative canonical bundle $\mathcal{K}_{GX,G}$ induced by Kähler metrics $\sigma^* \omega_{FS}$.

**Proof.** Note that $GX$ is biholomorphic to $G \times X$ by identifying $(\sigma, x)$ with $(\sigma, \sigma^{-1}(x))$. Therefore, (3.18) follows from Proposition 3.2. □

**Remark 3.1.** Proposition 3.2 is a special case of a general theorem which can be roughly described as follows: Let $E$ be a holomorphic “virtual” bundle over $X$ and $\mathcal{M}_E$ denotes the space of all hermitian metrics on $E$. To avoid the complexity caused by singularity, we assume that $X$ is smooth. Let $\Phi$ be any symmetric $GL_\mathbb{C}$-invariant multi-linear function of $n + 1$ variables, such as, the function $\text{Ch}_{n+1}$ defining the $(n+1)$-th Chern character. Then it follows from [BC] that, modulo $\text{Im}(\partial)$ and $\text{Im}(\bar{\partial})$, the $(n,n)$-form

$$
(3.19) \quad \text{BC}(\Phi; h_0, h_1) = \int_0^1 \Phi(R(h_t), \cdots, R(h_t); h_t h_t^{-1}) dt
$$

is independent of the path $\{h_t\}$ from $h_0$ to $h_1$. This is the Bott-Chern class associated to the invariant function $\Phi$. Integrating this Bott-Chern class over $X$, we obtain the Donaldson functional

$$
(3.20) \quad D(\Phi; h_0, h_1) = \int_X \text{BC}(\Phi; h_0, h_1)
$$
The K-energy coincides, up to multiplication by constants, the Donaldson functional $D(Ch_{n+1}; \cdot, \cdot)$ restricted to $\mathcal{M}_X^+$, which is a subset in $\mathcal{M}_E$, in case

$$E = (n+1)(K_X^{-1} - K_X) \otimes (L - L^{-1})^n - \mu(L - L^{-1})^{n+1}$$

Let $Y$ be a complex manifold and $h : Y \to \mathcal{M}_E$ be a map as in Proposition 3.2. Then for any $(2m - 2)$-form $\psi$ on $Y$ with compact support,

$$-\int_Y D(\Phi; h_0, h(y)) \partial_Y \overline{\partial}_Y \psi = \int_{Y \times X} \Phi(R(h(y)), \cdots, R(h(y))) \wedge \pi_1^* \psi$$

4. The proof of Theorem 0.1

We will adopt the notations in sections 1 and 2. Let $X = \Sigma_f$ be a hypersurface of degree $d$ in $\mathbb{C}P^{n+1}$, and $G$ be the linear group $\text{SL}(n+2, \mathbb{C})$. As before, $h_{FS}$ denotes the standard metric on the hyperplane bundle over $\mathbb{C}P^{n+1}$. Its curvature $\omega_{FS}$ is the Kähler form of the Fubini-Study metric.

**Theorem 4.1.** Let $B = PR_{n,d}$ be the space of all hypersurfaces of degree $d$, and $\| \cdot \|_Q$ be the hermitian metric on $\mathcal{O}_B(-1)$ defined in (2.6). Then for any $\sigma$ in $G$, we have

$$D(h_{FS}, \sigma^* h_{FS}) = \frac{(n+2)(d-1)}{n+1} \log \left( \frac{\| \cdot \|_Q(f \circ \sigma^{-1})}{\| \cdot \|_Q(f)} \right)$$

**Proof.** Let us denote by $\tilde{D}$ the function on $G$ defined by the right side of (4.1). It is obviously continuous on $G$. By using the definition of $\| \cdot \|_Q$ in (2.6) and Corollary 1.2, we have that for any smooth $(2 \dim \mathbb{C}G - 2)$-form $\phi$ with compact support in $G$,

$$-\int_G \tilde{D} \cdot \partial_G \overline{\partial}_G \phi = \int_{G \Sigma_f} \left( \frac{n+2-d}{n+1} \pi_2^* \omega_{FS} - \pi_1^* \circ T_f^* \omega_B - \partial \overline{\partial} \psi \right) \wedge \pi_2^* \omega_{FS} \wedge \pi_1^* \phi$$

$$= \int_{G \Sigma_f} \left( -(T_f \times \text{id})^* R(\| \cdot \|_\mathcal{K}) - \frac{n(n+2-d)}{n+1} \pi_2^* \omega_{FS} \right) \wedge \pi_2^* \omega_{FS} \wedge \pi_1^* \phi$$

where $T_f : G \to B$ is defined by $T_f(\sigma) = [f \circ \sigma^{-1}]$. Notice that $(T_f \times \text{id})^* R(\| \cdot \|_\mathcal{K})$ restricts to $\text{Ric}_{G \times G}$ on $G \Sigma_f$ in (3.18). Therefore, the difference $D(h_{FS}, \sigma^* h_{FS}) - \tilde{D}(\sigma)$ is a pluriharmonic function on $G$. Since $G$ is
simply-connected, we can write
\[ D(h_{FS}, \sigma^* h_{FS}) - \bar{D}(\sigma) = (\log |w|^2) (\sigma) \]
for some holomorphic function \( w \) on \( G \). On the other hand, using the definitions of \( D(\cdot, \cdot) \) and \( \| \cdot \|_Q \), it is not hard to prove that \( w \) has at most polynomial growth at the infinity of \( G \), so \( w \) extends to be a meromorphic function on any compactification of \( G \). However, \( G \) has a normal compactification \( \overline{G} \) in \( \mathbb{C}P^{n^2+2n} \) with irreducible divisor at infinity. Therefore, \( w \) has to be a nonzero constant. Then the theorem follows. 

The tautological line bundle \( \mathcal{O}_B(-1) \) over \( B \) can be written
\[ (4.3) \quad \mathcal{O}_B(-1) = \{(x,v) | x \in B, v \in x\} \]
The hermitian metric \( \| \cdot \|_B \) on \( \mathcal{O}_B(-1) \) over \( B \) is defined as follows: Let \( \langle \cdot, \cdot \rangle \) be the standard inner product on \( \mathbb{R}^n \), then for any \( (x,v) \) in \( \mathcal{O}_B(-1) \),
\[ (4.4) \quad \| (x,v) \|_B = \sqrt{\langle v, v \rangle} \]
Let us recall a simple fact.

**Lemma 4.1.** Let \( \Sigma_f \) be a hypersurface in \( \mathbb{C}P^{n+1} \) defined by \( f \). We define a function \( F_0 \) on \( G \) by
\[ F_0(\sigma) = \log \| ([f \circ \sigma^{-1}], f \circ \sigma^{-1}) \|_B, \quad \sigma \in G \]
Then
i) \( X \) is stable if and only if \( F_0 \) is proper on \( G \);
ii) \( X \) is semistable if and only if \( F_0 \) is bounded from below on \( G \).

**Proof.** Let us just show (i) here. The proof for (ii) is identical and omitted. First we assume that \( f \) is stable. If the function \( F_0 \) is not proper on the orbit \( G \), then there is a sequence \( \{\sigma_i\} \) in \( G \) which has no limit point in \( G \), such that \( F_0(\sigma_i) \) are uniformly bounded. By the definition of \( F_0 \) and \( \| \cdot \|_B \), the square norms \( \langle \sigma_i(f), \sigma_i(f) \rangle \) are uniformly bounded, i.e., all points \( \sigma_i(f) \) lie in a compact subset of \( R_{n,d} \). Without loss of generality, we may assume that \( \lim_{i \to \infty} \sigma_i(f) \) exists in \( R_{n,d} \). By the definition of the stability, the orbit \( G \cdot f \) is closed, so we may assume that \( \lim_{i \to \infty} \sigma_i(f) = \sigma(f) \) for some \( \sigma \) in \( G \). Therefore, the stabilizer of \( f \) in \( G \) is not finite, which contradicts to the assumption that \( f \)
is stable. Conversely, if $F_0$ is proper on $G$, it is clear that the stabilizer of $f$ in $G$ is finite. Take any sequence $\sigma_i$ in $G$ such that $\sigma_i(f)$ has a limit in $R_{n,d}$, then $F_0(\sigma_i)$ is uniformly bounded independent of $i$. By the properness, it implies that $\sigma_i$ has a limit $\sigma$ in $G$. This shows that the orbit is closed, consequently, $f$ is stable. □

Let $\| \cdot \|_Q$ be the Quillen metric defined in (2.6). Then

\[(4.5) \quad \| \cdot \|_B = e^{-\frac{(n+1)}{2(n+2)(d-1)}\psi_B} \| \cdot \|_Q^{\frac{n+2-d}{n+1}}\]

where $\| \cdot \|_u$ is defined in Proposition 2.1. Then by Proposition 2.1 and Lemma 2.1, there is a positive constant $c$ such that $\| \cdot \|_B \geq c \| \cdot \|_Q$, therefore, we have

**Lemma 4.2.** Let $\Sigma_f$ be a hypersurface in $CP^{n+1}$ defined by $f$. Define a function $F_f$ on $G$ by

$$F_f(\sigma) = \log(\| f \circ \sigma^{-1} \|_Q)$$

Then

i) $f$ is stable if the function $F_f$ is proper on $G$ in the following sense:

for any $C > 0$, there is a constant $\delta = \delta(C) > 0$, so that $F_f(\sigma) \geq C$

whenever $\inf_{z \in \Sigma_f} \frac{|\sigma(z)|}{|z|} \leq \delta$.

ii) $f$ is semistable if the function $F_f$ is bounded from below.

Now Theorem 0.1 follows easily from Lemma 4.2 and Theorem 4.1.

5. THE LOWER BOUNDEDNESS OF THE K-ENERGY

In this section, we give some sufficient conditions for the K-energy to be bounded from below or proper on the space of admissible Kähler metrics.

We assume that $X$ is a normal, irreducible variety satisfying (3.7). Then by (3.8), we can have

\[(5.1) \quad D(h_0, h_1) = \int_X \log \left( \frac{\omega^n}{e^{f_0\omega^n}} \right) \omega^n + \int_X f_0\omega^n - \frac{\mu}{n} (I(h_0, h_1) - J(h_0, h_1))\]

**Lemma 5.1.** For any $h_0, h_1$ in $M_X^+$, 

$$\frac{n}{n+1}I(h_0, h_1) \geq I(h_0, h_1) - J(h_0, h_1) \geq \frac{1}{n+1}I(h_0, h_1).$$
It follows from a straightforward computation (cf. [T2]). We omitted the proof.

**DEFINITION 5.1.** We say the K-energy $D(\cdot, \cdot)$ proper if for any fixed $h_0$ and $c > 0$, there is a constant $C = C(h_0, c)$ such that $D(h_0, h) \leq c$ implies $I(h_0, h) \leq C$.

The above lemma, together with (5.1), implies that if $\mu \leq 0$, the K-energy is bounded from below or proper if the first integral in (5.1) is so. We recall that a variety $X$ has only log-terminal (resp. canonical) singularities if it satisfies the following two conditions: (i) for some integer $r > 1$, the Weil divisor $rK_X$ is Cartier; (ii) if $\pi: \tilde{X} \rightarrow X$ is a resolution and $\{E_i\}$ the family of all exceptional prime divisors of $\pi$, then $rK_{\tilde{X}} = \pi^*(rK_X) + r \sum \alpha_i E_i$ with $\alpha_i > -1$ (resp. $\alpha_i \geq 0$)(cf. [Re]). Suppose that $X$ is a normal variety satisfying (3.7). Then it is clear from (i) and (ii) that $X$ has only log-terminal singularities if and only if

$$K_{\tilde{X}} = \frac{\mu}{n} \pi^*(\mathcal{O}_{CP^n}(-1)) + \sum \alpha_i E_i$$

for some $\alpha_i > -1$. Let $S_i$ be a section of $[E_i]$ defining $E_i$, and $\|\cdot\|_i$ be a fixed hermitian metric on $[E_i]$, then (3.7) and (5.2) imply that there is a smooth volume form $d\tilde{V}$ on $\tilde{X}$ such that

$$\pi^*f_0 = \log \left( \frac{d\tilde{V}}{\omega_0^a} \right) - \sum \alpha_i \log \|S_i\|^2_i$$

consequently, we have

$$\int_X e^{f_0} \omega_0^a = \int_{\tilde{X}} \left( \prod \|S_i\|^{-2\alpha_i}_i \right) d\tilde{V} < \infty$$

**Theorem 5.2.** If $\mu \leq 0$ and $X$ is a normal variety with only log-terminal singularities and satisfying (3.7), the K-energy is proper in the sense of Definition 5.1.

**Proof.** Using the assumption that all singular points are log-terminal, one can show (cf. [T2]) that there are $C$, $\epsilon > 0$, such that for any $h = e^{-\varphi}h_0$ in $\mathcal{M}_X^+$,

$$\int_X e^{f_0-\epsilon(\varphi-\sup_X \varphi)} \omega_0^a = \int_{\tilde{X}} e^{-\epsilon(\varphi-\sup_X \varphi)} \left( \prod \|S_i\|^{-2\alpha_i}_i \right) d\tilde{V} \leq C$$
By the concavity of logarithmic function, we deduce from this

\[ (5.6) \quad \int_X \log \left( \frac{(\omega_0 + \partial \overline{\partial} \varphi)^n}{e^{\int_0 \omega_0^n}} \right) \geq c I(h_0, h) - C'' \]

where \( C'' \) is a constant independent of \( h \). Therefore, if \( \mu \leq 0 \), the K-energy in (3.8) is proper. □

Remark 5.1. In case \( \mu < 0 \), there should be a much weaker condition for the K-energy to be proper in the sense of Definition 5.1.

The following theorem is taken from [DT], whose proof follows an idea in [BM].

**Theorem 5.3.** If \( X \) is a Kähler-Einstein orbifold, then the K-energy is bounded from below.

**Remark 5.2.** It is natural to ask when the K-energy is proper. Let \( \alpha(X) \) be the holomorphic invariant defined in [T2]. It can be proved that the K-energy is proper if this invariant is greater than \( \frac{n}{n+1} \), where \( n \) is the complex dimension of \( X \). Using the fact that any smooth hypersurface of degree greater than 2 is stable, one can easily show that the K-energy is proper on the subset of Kähler metrics of the form \( \sigma^* \omega_{FS}|_X \), where \( X = \Sigma_f \), \( \deg(f) \geq 3 \), \( \sigma \in SL(n+2, \mathbb{C}) \) and \( \omega_{FS} \) denotes the Fubini-Study metric. We conjecture that the K-energy is proper if \( X \) has a Kähler-Einstein metric and no holomorphic vector fields.

It is obvious that Theorem 0.2 follows from Theorem 5.1, 5.2 and the fact that \( \mu = n + 2 - d \) in case of hypersurfaces.

6. Some Generalizations

In this section, we give some generalizations of Theorem 0.1 and 0.2. They can be proved by the same method with some modifications. The details of the proof will appear in a forthcoming paper [T1].

Let \( \pi : \mathcal{X} \to \mathcal{H} \) be a holomorphic fibration between projective varieties satisfying:

1. \( \mathcal{X} \) is a subvariety in \( \mathcal{H} \times \mathbb{CP}^N \), and \( \pi \) coincides with the projection \( \pi_1 \) onto the first factor;
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(2) There is an Zariski open subset $\mathcal{H}_0$ in $\mathcal{H}$ such that $\pi^{-1}(h)$ is a smooth $n$-fold for each $h$ in $\mathcal{H}_0$;
(3) the linear group $G = SL(N + 1, \mathbb{C})$ acts on both $\mathcal{H}_0$ and $\pi^{-1}(\mathcal{H}_0)$, and the action commutes with $\pi_1, \pi_2$;
(4) either the hyperplane bundle on $\mathbb{C}P^n$ restricts to $rK^\pm$ on each $\pi^{-1}(h)$ for some nonzero rational number $r$, or the first Chern class of $\pi^{-1}(h)$ vanishes, where $h$ is in $\mathcal{H}_0$.

The typical example of such a fibration is a universal family of subvarieties in $\mathbb{C}P^n$ over a Hilbert scheme.

**Theorem 6.1.** Let $\pi : \mathcal{X} \to \mathcal{H}$ be as above. We denote by $\mathcal{L}$ the pull-back of the hyperplane bundle over $\mathbb{C}P^n$ by $\pi_2$. Put

\[
\mathcal{E} = (n + 1)(K^{-1}_\mathcal{K}) \otimes (\mathcal{L} - \mathcal{L})^n - n\mu(\mathcal{L} - \mathcal{L})^{n+1}
\]

where $\mathcal{K} = K_{\mathcal{X}} \otimes K^{-1}_\mathcal{H}$ be the relative canonical bundle and $K^{-1}_{\pi^{-1}(h)} = \mu L\big|_{\pi^{-1}(h)}$ for some $h$ in $\mathcal{H}_0$. Assume that the inverse of the determinant line bundle $L_\mathcal{H} = \det(\mathcal{E}, \pi)$ is ample on $\mathcal{H}$. Then for any $h$ in $\mathcal{H}_0$, we have

(i) $X = \pi^{-1}(h)$ is stable with respect to $G$ and $L_\mathcal{H}$ if the K-energy on $X$ is proper;
(ii) $X$ is semistable with respect to $G$ and $L_\mathcal{H}$ if the K-energy on $X$ is bounded from below.

In particular, such an $X$ is semistable if it admits a Kähler-Einstein metric.

An example of such a fibration in Theorem 6.1 is provided by complete intersections in $\mathbb{C}P^{n+k}$. Let $R_{n+k,d}$ be the space of all homogeneous polynomials on $\mathbb{C}^{n+k+1}$ of degree $d$. Put

\[
B = B_1 \times \cdots \times B_k, \quad B_i = PR_{n+k,d_i}
\]

where $d_i$ are integers greater than one. Then $B$ classifies all complete intersections in $\mathbb{C}P^{n+k}$ defined by $k$ homogeneous polynomials of degree $d_1, \ldots, d_k$. Some of these intersections may be of dimension higher than $n$. Let $\mathcal{X}$ be the universal family of those intersections, and $\mathcal{H} = B$. Then one can show that
the determinant line bundle $L_B$ in Theorem 6.1 is isomorphic to

\[2^{n+1} d_1 \cdots d_k \bigotimes_{i=1}^k \mathcal{O}_{B_i} \left( \frac{n+k+1 - \sum_{j=1}^{k} d_j}{d_i} - (n+1) \right)\]

Since each $d_i \geq 2$, $(n+1)d_i$ is greater than $n + k + 1 - \sum_{j=1}^{k} d_j$. This follows that $L_{B}^{-1}$ is ample on $B$. Therefore, we have

**Theorem 6.2.** Let $X$ be a normal complete intersection in $\mathbb{C}P^{n+k}$. Then

$X$ is stable if the $K$-energy is proper, and $X$ is semistable if the $K$-energy is bounded from below. In particular, $X$ is semistable if it admits a Kähler-Einstein orbifold metric.

There are more applications of Theorem 6.1 in addition to Theorem 6.2 on complete intersections.

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