Quasi-convergence of the Ricci flow

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We study a collection of Riemannian metrics which collapse under the Ricci flow, and show that the quasi-convergence equivalence class of an arbitrary metric in this collection contains a 1-parameter family of locally homogeneous metrics.

1. Introduction and statement of main theorem.

In [1], Hamilton and Isenberg studied the Ricci flow of a family of solv-geometry metrics on twisted torus bundles. This family contains no Einstein metrics, so the (normalized) Ricci flow cannot converge. Hamilton–Isenberg introduced the concept of quasi-convergence to describe its behavior, writing

"...the Ricci flow of all metrics in this family asymptotically approaches the flow of a sub-family of locally homogeneous metrics..."

The intent of this paper is to make that statement more precise. In so doing, we answer a question of Hamilton, who asked whether an arbitrary metric in this class would converge to a unique locally homogeneous limit or would exhibit a more nuanced behavior.

Definition 1.1. If $g, h$ are evolving Riemannian metrics on a manifold $M^n$, we say $g$ quasi-converges to $h$ if for any $\varepsilon > 0$ there is a time $t_\varepsilon$ such that

$$\sup_{M^n \times [t_\varepsilon, \infty)} |g - h|_h < \varepsilon.$$ 

Quasi-convergence is an equivalence relation. Indeed, the standard fact that $|U(V, V)| \leq |U|_h |V|^2_h$ for any symmetric 2-tensor $U$ and vector field $V$ implies that $g$ quasi-converges to $h$ if and only if for all $t \geq t_\varepsilon$,

$$(1 - \varepsilon) h(V, V) \leq g(V, V) \leq (1 + \varepsilon) h(V, V).$$

We now state our result, using notation defined in [1] and to be reviewed in §2 below.
Theorem 1.2. If \( g \) is any solv-Gowdy metric on a twisted torus bundle \( M^3_{\Lambda} \), there is a locally homogeneous metric \( h \) in its quasi-convergence equivalence class \([g]\). Moreover, if \( h \) corresponds to the data \((\alpha(\theta), \Omega, F)\), the locally homogeneous metrics in \([g]\) are exactly those with the data \((\ell + \alpha(\theta), \Omega, F)\), \( \ell \in \mathbb{R} \).

Remark 1.3. Similar quasi-convergence of the Ricci flow to a 1-parameter family was conjectured for a class of \( T^3 \) metrics studied in [2].

The paper is organized as follows. §2 describes the bundles \( T^2 \to M^3_{\Lambda} \to S^1 \) and the solv-Gowdy metrics under study. It turns out that at large times, an arbitrary solv-Gowdy metric \( g \) behaves much like locally homogeneous metrics. §3 quantifies this observation and explicitly constructs a family \( h_\varepsilon \) of locally homogeneous metrics existing for all \( t > 0 \) which approximate \( g \) for times \( t \geq t_\varepsilon \). In §4, we show that this family enjoys a certain compactness property which allows us to prove the existence part of the main theorem. The heuristic here is that \( g \) resembles a single locally homogeneous metric closely enough that the metrics \( h_\varepsilon \) are not too far apart at \( t = 0 \). §5 completes the main theorem by explaining the very special sort of non-uniqueness which can occur: distinct locally homogeneous metrics define distinct equivalence classes unless they differ only by a dilation of the base circle.

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2. Review of solv-Gowdy geometries.

We begin by briefly recalling some notation and results of [1]. Readers familiar with that paper may skip this section.

To construct an arbitrary solv-Gowdy metric \( g \), take \( \Lambda \in SL(2, \mathbb{Z}) \) with eigenvalues \( \lambda_+ > 1 > \lambda_- \). In coordinates \( \theta, x, y \) on \( \mathbb{R}^3 \), chosen so that the \( x, y \) axes coincide with the eigenvectors of \( \Lambda \), define

\[
(2.1) \quad g = e^{2A} d\theta \otimes d\theta + e^{F+W} dx \otimes dx + e^{F-W} dy \otimes dy,
\]

where \( F \) is constant and \( A, W \) depend only on \( \theta \). Clearly, \( g \) descends to a metric on the product of the line and the torus \( T^2 \). Let \( \Lambda \) act on \( \mathbb{R} \times T^2 \) by \((\theta, x, y) \mapsto (\theta + 2\pi, \lambda_- x, \lambda_+ y)\). If

\[
(2.2) \quad A(\theta + 2\pi) = A(\theta)
\]
and

\( W(\theta + 2\pi) = W(\theta) + 2\log \lambda_+ \),

then \( \Lambda \) is an isometry, and \( g \) becomes a well defined metric on the mapping torus \( M^{3}_\Lambda \), regarded as a twisted \( T^2 \) bundle over \( S^1 \). Notice that \( A \) governs the length of the base circle, while \( F \) and \( W \) respectively describe the scale and skew of the fibers. We denote arc length by

\[
\tag{2.4}
s(\theta) = \int_0^\theta e^{A(u)} du
\]

and set

\[
\tag{2.5}
Z = \frac{\partial}{\partial s} W.
\]

Then we can write the Ricci tensor as

\[
\tag{2.6}
Rc = -\frac{1}{2} e^{2A} Z^2 d\theta \otimes d\theta - \frac{1}{2} e^{F+W} \frac{\partial Z}{\partial s} dx \otimes dx + \frac{1}{2} e^{F-W} \frac{\partial Z}{\partial s} dy \otimes dy.
\]

The locally homogeneous solv-Gowdy metrics are easily characterized.

**Lemma 2.1.** A solv-Gowdy metric \( g \) is locally homogeneous if and only if \( W \) depends linearly on arc length.

**Proof.** If \( g \) is locally homogeneous, then \( R = -\frac{1}{2} Z^2 \) is constant in space. Since \( Z \) is continuous, it follows that \( \partial^2 W/\partial s^2 = 0 \).

If \( Z \) is constant in space, let \( P_0 = (\theta_0, x_0, y_0) \), \( P_1 = (\theta_1, x_1, y_1) \) be points in \( M^{3}_\Lambda \). It will suffice to construct a diffeomorphism \( \Phi : U_0 \to U_1 \), where \( U_0, U_1 \) are neighborhoods of \( P_0, P_1 \) respectively, such that \( \Phi(P_0) = P_1 \) and \( \Phi^* g = g \). If \( \Phi \) is given in coordinates \( (\theta, x, y) \) by

\[
\Phi(\theta, x, y) = (\tau(\theta, x, y), \xi(\theta, x, y), \eta(\theta, x, y)),
\]

the pullback condition \( \Phi^* g = g \) is equivalent to the system

\[
\tag{2.7a}
e^{2A(\theta)} = \left( \frac{\partial \tau}{\partial \theta} \right)^2 e^{2A(\tau)} + \left( \frac{\partial \xi}{\partial \theta} \right)^2 e^{F+W(\tau)} + \left( \frac{\partial \eta}{\partial \theta} \right)^2 e^{F-W(\tau)}
\]

\[
\tag{2.7b}
e^{F+W(\theta)} = \left( \frac{\partial \tau}{\partial x} \right)^2 e^{2A(\tau)} + \left( \frac{\partial \xi}{\partial x} \right)^2 e^{F+W(\tau)} + \left( \frac{\partial \eta}{\partial x} \right)^2 e^{F-W(\tau)}
\]

\[
\tag{2.7c}
e^{F-W(\theta)} = \left( \frac{\partial \tau}{\partial y} \right)^2 e^{2A(\tau)} + \left( \frac{\partial \xi}{\partial y} \right)^2 e^{F+W(\tau)} + \left( \frac{\partial \eta}{\partial y} \right)^2 e^{F-W(\tau)}.
\]
Note that $s(\theta)$ is invertible, because $\frac{\partial s}{\partial \theta} = e^{A(\theta)} > 0$, and define

$$
\tau(\theta, x, y) = s^{-1}(s(\theta) + s(\theta_1) - s(\theta_0))
$$

$$
\xi(\theta, x, y) = x_1 + e^{-\frac{A}{2}(s(\theta_1)-s(\theta_0))} (x - x_0)
$$

$$
\eta(\theta, x, y) = y_1 + e^{\frac{A}{2}(s(\theta_1)-s(\theta_0))} (y - y_0).
$$

Clearly, $\Phi : P_0 \mapsto P_1$. Equation (2.7a) is satisfied, because

$$
\frac{\partial \tau}{\partial \theta} = \frac{\partial \theta}{\partial s}(\tau) \cdot \frac{\partial s}{\partial \theta}(\theta) = e^{-A(\tau)+A(\theta)}.
$$

To see that (2.7b) is satisfied, let $\omega$ denote $W$ regarded as a linear function of arc length, so that $W(\theta) = \omega(s(\theta))$. Then we can write

$$
\log \left( \left( \frac{\partial \xi}{\partial x} \right)^2 e^{W(\tau)} \right) = -Z \cdot (s(\theta_1) - s(\theta_0)) + \omega(s(\theta) + s(\theta_1) - s(\theta_0))
$$

$$
= \omega(s(\theta)) = W(\theta).
$$

Equation (2.7c) is verified in a similar fashion.

\textbf{Remark 2.2.} When studying a single locally homogeneous solv-Gowdy metric, one can always make $A$ constant in space by a reparameterization of $S^1$; but it will not be convenient for us to do so.

If an arbitrary solv-Gowdy metric $g$ evolves by the Ricci flow

(2.8) \[ \frac{\partial}{\partial t} g = -2 \operatorname{Rc}, \]

we shall abuse notation and allow the quantities introduced above to depend also on time. We find that $g$ remains a solv-Gowdy metric and that (2.8) is equivalent to the system

(2.9a) \[ \frac{\partial}{\partial t} A = \frac{1}{2} Z^2 \]

(2.9b) \[ \frac{\partial}{\partial t} W = \frac{\partial}{\partial s} Z \]

(2.9c) \[ \frac{\partial}{\partial t} F = 0, \]

whose solution exists for all $t \geq 0$. It is most convenient to study $Z$ and recover $A$ and $W$ by integration. $Z$ evolves by

(2.10) \[ \frac{\partial}{\partial t} Z = \frac{\partial^2}{\partial s^2} Z - \frac{1}{2} Z^3, \]
where the operator $\partial^2/\partial s^2$ plays the role of the Laplacian and evolves according to the commutator

$$\frac{\partial}{\partial t} \left[ \frac{\partial}{\partial s}, \frac{\partial}{\partial s} \right] = -\frac{1}{2} Z^2 \frac{\partial}{\partial s}. \tag{2.11}$$

For all $t \geq 0$, we identify $S^1$ with the circle $x = 0, y = 0$ and denote its length by

$$L(t) \doteq \int_{S^1} ds = \int_0^{2\pi} e^{A(\theta,t)} d\theta. \tag{2.12}$$

Notice that (2.3) implies the important integral condition

$$\int_{S^1} Z ds = 2 \log \lambda_+, \tag{2.13}$$

which is preserved by the flow.

If an evolving solv-Gowdy metric is locally homogeneous at $t = 0$, it remains so under the Ricci flow. For such metrics, $Z$ is the function of time alone

$$Z(t) = \frac{1}{\sqrt{t + 1/\zeta^2}}, \tag{2.14}$$

where $\zeta \doteq Z(0)$ is positive by (2.13). The sub-family of locally homogeneous solv-Gowdy metrics can thus be indexed by $(\alpha(\theta), \Omega, F)$, where

$$\alpha(\theta) \doteq A(\theta,0) \tag{2.15a}$$

$$\Omega \doteq W(0,0). \tag{2.15b}$$

We now summarize the estimates we shall use from [1]. Let $g$ be a solution to the Ricci flow whose initial data $g(\cdot,0)$ is a $C^2$ solv-Gowdy metric. Hamilton–Isenberg organize the proof of their main theorem into four steps. In Step 1, they show there is $C > 0$ depending on $Z(\cdot,0)$ such that for all $t > 0$,

$$\left| Z(\cdot,t) \right| \leq \frac{1}{\sqrt{t + C}} < \frac{1}{\sqrt{t}}. \tag{2.16}$$

By Step 2, there is a time $T > 0$ and constants $m \doteq Z_{\min}(T), M \doteq Z_{\max}(T)$ depending on $L(0), Z(\cdot,0)$ and satisfying $0 < m \leq M < 1/\sqrt{T}$ such that for all $t \geq T$,

$$\frac{1}{\sqrt{t - T + 1/m^2}} \leq Z(\cdot,t) \leq \frac{1}{\sqrt{t - T + 1/M^2}}. \tag{2.17}$$
By Step 1 again, there are $C, C' > 0$ depending on $L(0), Z(\cdot, 0)$ such that for all $t \geq T + 1,$

\begin{equation}
C \sqrt{t - T} \leq L(t) \leq C' \sqrt{t - T}.
\end{equation}

By Step 4, there is $C > 0$ depending on $L(0), Z(\cdot, 0)$ such that for all $t \geq T,$

\begin{equation}
\left| \frac{\partial}{\partial s} Z(\cdot, t) \right| \leq \frac{C}{(1 + m^2(t - T))^2}.
\end{equation}

3. Construction of approximating metrics.

As a first step in proving the existence part (Theorem 4.1) of our main theorem, we find times $t_\varepsilon$ and construct locally homogeneous metrics $h_\varepsilon$ with the following properties: $h_\varepsilon$ is in a sense the average of $g$ at $t_\varepsilon$; $h_\varepsilon$ remains $\varepsilon$-close to $g$ for all times $t \geq t_\varepsilon$; and most importantly, $h_\varepsilon$ exists for all $t \geq 0$.

**Proposition 3.1.** For any $\varepsilon > 0$, there is a time $t_\varepsilon > 0$ and a locally homogeneous solv-Gowdy metric $h_\varepsilon$ evolving by the Ricci flow for $0 < t < \infty$ such that

$$\text{sup}_{\mathcal{M}_h^3 \times (t_\varepsilon, \infty)} |g - h_\varepsilon|_{h_\varepsilon} < \varepsilon.$$  

Before proving this, we collect some technical observations.

**Lemma 3.2.** For any $\varepsilon > 0$, there is $t_\varepsilon > 0$ such that $Z$ satisfies the pinching estimate

\begin{equation}
Z_{\max}(t) - Z_{\min}(t) \leq \frac{\varepsilon}{L(t)},
\end{equation}

and the decay estimate

\begin{equation}
\frac{1}{\sqrt{t - t_\varepsilon + 1/m_\varepsilon^2}} \leq Z(\cdot, t) \leq \frac{1}{\sqrt{t - t_\varepsilon + 1/M_\varepsilon^2}},
\end{equation}

for all $t \geq t_\varepsilon,$ where $m_\varepsilon, M_\varepsilon$ are defined by

\begin{equation}
0 < m_\varepsilon \triangleq Z_{\min}(t_\varepsilon) \leq Z_{\max}(t_\varepsilon) \triangleq M_\varepsilon < \infty
\end{equation}

and satisfy

\begin{equation}
m_\varepsilon \leq M_\varepsilon \leq m_\varepsilon + \varepsilon \quad \text{and} \quad M_\varepsilon^2 \leq (1 + \varepsilon)m_\varepsilon^2.
\end{equation}
Moreover, we can choose \( t_\varepsilon \) so that
\[
\int_{t_\varepsilon}^{\infty} \left| \frac{\partial Z}{\partial s} \right| \, dt \leq \varepsilon.
\]

Proof. Let \( T, m, M \) be as in (2.17) and let \( C \) be the constant in (2.19). Let \( t_* = \max \{ T + C/ (m^4 \epsilon), T + 1 \} \) and suppose \( t \geq t_* \). Then (2.19) implies
\[
\int_{t_*}^{\infty} \left| \frac{\partial Z}{\partial s} \right| \, dt \leq \int_{0}^{\infty} \frac{C}{m^4 (t + t_* - T)^2} \, dt = \frac{C}{m^4 (t_* - T)} \leq \varepsilon,
\]
and (2.18) implies there is \( C' > 0 \) such that
\[
L(t) \leq C' \sqrt{t - T}.
\]
Hence for such times
\[
Z_{\text{max}}(t) - Z_{\text{min}}(t) \leq \int_{S^1} \left| \frac{\partial Z}{\partial s} \right| \, ds \leq CC' \frac{\sqrt{t - T}}{(1 + m^2 (t - T))^2}.
\]
Choose \( t_\varepsilon \geq t_* \) large enough that (3.1) holds for \( t \geq t_\varepsilon \), and that (3.4) holds for \( m_\varepsilon, M_\varepsilon \) defined by (3.3). This is possible, because
\[
\left( \frac{Z_{\text{max}}(t)}{Z_{\text{min}}(t)} \right)^2 \leq \frac{t - T + 1/m^2}{t - T + 1/M^2} \leq 1 + \frac{1}{m^2 (t - T)}.
\]
Then since \( \frac{\partial}{\partial t} Z = \frac{\partial^2}{\partial s^2} Z - \frac{1}{2} Z^3 \), we observe that
\[
\frac{d}{dt} Z_{\text{min}} \geq -\frac{1}{2} Z_{\text{min}}^3 \quad \text{and} \quad \frac{d}{dt} Z_{\text{max}} \leq -\frac{1}{2} Z_{\text{max}}^3.
\]
A routine use of the maximum principle (proved in [3]) now establishes (3.2) for all \( t \geq t_\varepsilon \). \( \square \)

Remark 3.3. The proof shows that for \( t \geq T + 1 \),
\[
Z_{\text{max}} - Z_{\text{min}} = O (t - T)^{-3/2},
\]
a result which also follows directly from (2.17).
Lemma 3.4. Let $\varepsilon > 0$ be given and let $t_\varepsilon, m_\varepsilon, M_\varepsilon$ be as in Lemma 3.2. Then there is a locally homogeneous solv-Gowdy metric

$$h_\varepsilon = e^{2A_\varepsilon} d\theta \otimes d\theta + e^{F_\varepsilon + W_\varepsilon} dx \otimes dx + e^{F_\varepsilon - W_\varepsilon} dy \otimes dy$$

evolving by the Ricci flow for $0 < t < \infty$ so that for $t \geq t_\varepsilon$,

$$\frac{1}{\sqrt{t - t_\varepsilon + 1/m_\varepsilon^2}} \leq Z_\varepsilon(t) \leq \frac{1}{\sqrt{t - t_\varepsilon + 1/M_\varepsilon^2}},$$

where $Z_\varepsilon = \frac{\partial W_\varepsilon}{\partial \theta} = e^{-A_\varepsilon} \frac{\partial W_\varepsilon}{\partial \theta}$. Moreover, $h_\varepsilon$ is constructed so that for all $\theta \in S^1$, $A_\varepsilon(\theta, t_\varepsilon) = A(\theta, t_\varepsilon)$ and $|W(\theta, t_\varepsilon) - W(\theta, t_\varepsilon)| \leq \varepsilon$.

Proof. Define

$$(3.5) \quad Z_\varepsilon(t) \doteq \frac{1}{\sqrt{t + (1/\zeta_\varepsilon^2) - t_\varepsilon}},$$

where

$$(3.6) \quad \zeta_\varepsilon \doteq \int_{S^1} Z ds / \int_{S^1} ds,$$

with the RHS evaluated at $t_\varepsilon$. Observe that $Z_\varepsilon$ is well defined for all $t \geq 0$, because $|Z(t)| < 1/\sqrt{t}$ by (2.16), whence

$$1/\zeta_\varepsilon^2 - t_\varepsilon \geq 1/Z_{\text{max}}^2(t_\varepsilon) - t_\varepsilon > 0.$$ 

Now recall that locally homogeneous solv-Gowdy metrics form a 3-parameter family and define

$$(3.7a) \quad \alpha_\varepsilon(\theta) \doteq A(\theta, t_\varepsilon) - \frac{1}{2} \int_{0}^{t_\varepsilon} Z_\varepsilon^2 dt$$

$$(3.7b) \quad \Omega_\varepsilon \doteq W(0, t_\varepsilon)$$

$$(3.7c) \quad F_\varepsilon \doteq F.$$ 

Notice that $h_\varepsilon$ is well defined; indeed, the identities

$$2 \log \lambda_+ = \int_{S^1} Z ds = \zeta_\varepsilon \int_{S^1} ds = \int_{S^1} \zeta_\varepsilon e^{A_\varepsilon} d\theta = \int_{S^1} Z_\varepsilon ds_\varepsilon$$

show that the integral condition (2.13) is satisfied at $t_\varepsilon$, hence for all time.
The first assertion of the lemma is verified by the elementary observation
\[ m_\varepsilon = Z_{\min}(t_\varepsilon) \leq \zeta_\varepsilon \leq Z_{\max}(t_\varepsilon) = M_\varepsilon, \]
which follows from (3.6). The second assertion is trivial; to prove the third, simply notice that
\[ |W(\theta, t_\varepsilon) - W_\varepsilon(\theta, t_\varepsilon)| \leq \int_{S^1} |Z - \zeta_\varepsilon| \, ds \leq (Z_{\max} - Z_{\min})(t_\varepsilon) \cdot L(t_\varepsilon) \leq \varepsilon. \]

**Proof of Proposition 3.1.** Without loss of generality, assume \( 0 < \varepsilon \leq 1/6 \).

Let \( t \geq t_\varepsilon \) and observe that
\[
|(A - A_\varepsilon)(\theta, t)| = \frac{1}{2} \left| \int_{t_\varepsilon}^{t} \left( Z^2 - Z_\varepsilon^2 \right)(\theta, \tau) \, d\tau \right| \\
\leq \frac{1}{2} \int_{t_\varepsilon}^{t} \left( \frac{1}{\tau - t_\varepsilon + 1/M_\varepsilon^2} - \frac{1}{\tau - t_\varepsilon + 1/m_\varepsilon^2} \right) \, d\tau \\
= \log \sqrt{\frac{1 + M_\varepsilon^2 (t - t_\varepsilon)}{1 + m_\varepsilon^2 (t - t_\varepsilon)}}.
\]

Then since \(|e^u - 1| \leq e^U - 1\) when \(|u| \leq U\), we have
\[
\left| e^{2A} - e^{2A_\varepsilon}(\theta, t) \right| = e^{2A_\varepsilon} \left| e^{2(A - A_\varepsilon)} - 1 \right| \leq e^{2A_\varepsilon} \frac{M_\varepsilon^2 - m_\varepsilon^2}{m_\varepsilon^2}
\]
and hence
\[
\left( (h_\varepsilon)^{\theta\theta} \right)^2 (g^{\theta\theta} - (h_\varepsilon)^{\theta\theta})^2 \leq \varepsilon^2.
\]

Because \( W_\varepsilon \) is constant in time, we have
\[
|(W - W_\varepsilon)(\theta, t)| \leq |W(\theta, t) - W(\theta, t_\varepsilon)| + |W(\theta, t_\varepsilon) - W_\varepsilon(\theta, t_\varepsilon)| \\
\leq \left| \int_{t_\varepsilon}^{t} \frac{\partial Z}{\partial s} \, d\tau \right| + \varepsilon \\
\leq 2\varepsilon,
\]
whence substituting \( \delta = 2\varepsilon \leq 1/3 \) in the crude estimate \( e^\delta \leq 1 + \delta + \frac{\delta^2}{2} \) (which holds for \( 0 \leq \delta \leq 1 \)) gives
\[
\left| e^{(F + W) - e^{F_\varepsilon + W_\varepsilon}}(\theta, t) \right| = e^{F_\varepsilon + W_\varepsilon} \left| e^{(W - W_\varepsilon)} - 1 \right| \leq 3\varepsilon e^{F_\varepsilon + W_\varepsilon}.
\]
and thus
\[(h_\varepsilon)^{xx} (g_{xx} - (h_\varepsilon)_{xx})^2 \leq 9\varepsilon^2.\]

The estimate for \[((h_\varepsilon)^{yy} (g_{yy} - (h_\varepsilon)_{yy})^2\text{ is entirely analogous. We have shown that}
\[|g - h_\varepsilon|^2 = (h_\varepsilon)^{ac} (h_\varepsilon)^{bd} (g_{ab} - (h_\varepsilon)_{ab}) (g_{cd} - (h_\varepsilon)_{cd}) \leq 19\varepsilon^2\]
for \(t \geq t_\varepsilon\), which is clearly equivalent to the desired result. \(\Box\)

4. Existence.

We have seen that for any \(\varepsilon > 0\), there is a natural choice \(h_\varepsilon\) of locally homogeneous metric approximating \(g\) for times \(t \geq t_\varepsilon\). In view of our non-uniqueness result (Theorem 5.1), it is remarkable that these choices are close enough to one another that we can prove the existence of a locally homogeneous metric in \([g]\).

**Theorem 4.1.** There is a locally homogeneous solv-Gowdy metric \(h_\infty\) evolving by the Ricci flow for \(0 \leq t < \infty\) such that for any \(\varepsilon > 0\) there is a time \(t_\varepsilon > 0\) with
\[\sup_{\mathcal{M}_A^j \times [t_\varepsilon, \infty)} |g - h_\infty|_{h_\infty} < \varepsilon.\]

Again, we first obtain some preliminary results.

**Lemma 4.2.** Let \(\{\varepsilon_j\}\) be a sequence with \(\varepsilon_j \searrow 0\). For each \(j\), let \(h_j\) denote the metric \(h_{\varepsilon_j}\) given by Proposition 3.1. Then there is a subsequence \(j_k\) and a locally homogeneous metric \(h_\infty\) with data \((\alpha_\infty (\theta), \Omega_\infty, F_\infty)\) such that
\[\alpha_{j_k} (\theta), \Omega_{j_k}, F_{j_k} \rightarrow (\alpha_\infty (\theta), \Omega_\infty, F_\infty)\]
uniformly in \(\theta\). (Here, and throughout the proof, a subscript such as \(j\) denotes quantities corresponding to the metric \(h_j \equiv h_{\varepsilon_j}\).)

**Proof.** The argument is constructed from four claims, as follows: Claim 4.3 bounds \(\partial_\theta A_j (\cdot, t_j)\), hence \(\partial_\theta A_j (\cdot, t_j)\) by construction, hence \(\partial_\theta A_j (\cdot, 0)\) by (4.1) and the local homogeneity of \(h_j\). Combining this with Claim 4.4 proves \(\{A_j (\cdot, 0)\}\) is bounded and equicontinuous. Since Claim 4.5 bounds
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\[ \frac{\partial}{\partial s_j} W_j (\cdot, 0), \] this lets us bound \[ \frac{\partial}{\partial \theta} W_j (\cdot, 0). \] Combining this with Claim 4.6 then proves \( \{W_j (\cdot, 0)\} \) is bounded and equicontinuous. Because \( F_j \equiv F \) by construction, this lets us extract a subsequence of the \( h_j \) whose initial data converge uniformly to the data of a locally homogeneous metric \( h_\infty \) existing for all \( t \geq 0 \).

Notice that if \( j < k \), we may (and shall) assume \( t_j \leq t_k \).

**Claim 4.3.** There is \( C < \infty \) such that

\[
\sup_{M_\theta^3 \times [T, \infty)} \left| \frac{\partial A}{\partial \theta} \right| < C.
\]

Compute

\[
\frac{\partial}{\partial t} \left( \frac{\partial A}{\partial \theta} \right) = \frac{\partial}{\partial \theta} \left( \frac{1}{2} Z^2 \right) = e^A Z \frac{\partial Z}{\partial s}.
\]

Since by (2.17),

\[
\frac{\partial}{\partial t} A \leq \frac{1}{2} \cdot \frac{1}{t - T + 1/M^2}
\]

for \( t \geq T \), there is \( C' > 0 \) such that

\[
A (\cdot, t) \leq \log C' + \log \sqrt{t - T + 1/M^2}
\]

for \( t \geq T \). Then by (2.19), we have

\[
\left| \frac{\partial}{\partial t} \left( \frac{\partial A}{\partial \theta} \right) \right| \leq C' \sqrt{t - T + 1/M^2} \frac{1}{\sqrt{t - T + 1/M^2}} \cdot \frac{C''}{(1 + m^2 (t - T))^2}
\]

for all \( t \geq T \). Since there is \( B > 0 \) depending only on the initial data such that \( -B \leq \partial A/\partial \theta \leq B \) at \( t = T \), the claim follows.

**Claim 4.4.** The sequence \( \{\alpha_j (\theta)\} \) is bounded for each \( \theta \in S^1 \).

Let \( \theta \in S^1 \) be arbitrary. For \( j < k \), consider

\[
\alpha_j (\theta) - \alpha_k (\theta) = A (\theta, t_j) - \frac{1}{2} \int_0^{t_j} Z_j^2 \, dt - A (\theta, t_k) + \frac{1}{2} \int_0^{t_k} Z_k^2 \, dt
\]

\[
= \frac{1}{2} \int_{t_j}^{t_k} (Z_k^2 - Z_j^2) \, dt + \frac{1}{2} \int_0^{t_j} (Z_k^2 - Z_j^2) \, dt.
\]
Since \(1/\zeta_k^2 - t_k \geq 1/M_j^2 - t_j\), we obtain a familiar estimate for the first integral:

\[
\frac{1}{2} \left| \int_{t_j}^{t_k} (Z_k^2 - Z_j^2) \, dt \right| \leq \log \sqrt{\frac{1 + M_j^2 (t_k - t_j)}{1 + m_j^2 (t_k - t_j)}} \leq \log \sqrt{1 + \varepsilon_j}.
\]

Write the second integral as

\[
\frac{1}{2} \int_0^{t_j} (Z_k^2 - Z_j^2) \, dt = \log \sqrt{\frac{1/\zeta_j^2 - t_j}{1/\zeta_k^2 - t_k}} + \log \sqrt{\frac{t_j + (1/\zeta_k^2 - t_k)}{1/\zeta_j^2}}
\]

where

\[
P_{jk} \triangleq (1 - \zeta_j^2 t_j) \left(1 + \frac{t_j}{1/\zeta_k^2 - t_k}\right) > 0.
\]

Since

\[
\frac{1/M^2 - T}{t_j + 1/M^2 - T} \leq 1 - \zeta_j^2 t_j \leq \frac{1/m^2 - T}{t_j + 1/m^2 - T}
\]

and

\[
\frac{t_j + 1/m^2 - T}{1/m^2 - T} \leq 1 + \frac{t_j}{1/\zeta_k^2 - t_k} \leq \frac{t_j + 1/M^2 - T}{1/M^2 - T},
\]

we conclude that

\[
\frac{1/M^2 - T}{1/m^2 - T} \leq P_{jk} \leq \frac{1/m^2 - T}{1/M^2 - T}.
\]

**Claim 4.5.** There are \(0 < Z_* \leq Z^* < \infty\) such that \(Z_j(0) \in [Z_*, Z^*]\) for all \(j\).

Note how

\[
1/Z_j^2(0) = 1/\zeta_j^2 - t_j \geq 1/Z_{\text{max}}^2(t_j) - t_j \geq 1/M^2 - T > 0
\]

by (2.16) and (2.17), and similarly

\[
1/Z_j^2(0) = 1/\zeta_j^2 - t_j \leq 1/Z_{\text{min}}^2(t_j) - t_j \leq 1/m^2 - T < \infty.
\]

**Claim 4.6.** There are \(\Omega_* \leq \Omega^*\) such that \(\Omega_j \in [\Omega_*, \Omega^*]\) for all \(j\).
Suppose \( j < k \). Then since \( \Omega_j \triangleq W(0, t_j) \), we have
\[
|\Omega_k - \Omega_j| = |W(0, t_k) - W(0, t_j)| \leq \int_{t_j}^{t_k} \left| \frac{\partial W}{\partial t} \right| dt = \int_{t_j}^{t_k} \left| \frac{\partial s}{\partial s} \right| dt \leq \varepsilon_j.
\]

**Lemma 4.7.** If \( h_\infty \) is a locally homogeneous metric with data
\[
(\alpha_\infty(\theta), \Omega_\infty, F')
\]
and \( \{h_j\} \) is a sequence of locally homogeneous metrics with data
\[
(\alpha_j(\theta), \Omega_j, F)
\]
converging to \( (\alpha_\infty(\theta), \Omega_\infty, F') \) uniformly in \( \theta \), then for any \( \varepsilon > 0 \) there is \( J_\varepsilon \) such that for each \( j \geq J_\varepsilon \)
\[
\sup_{M^3_\lambda \times [0, \infty)} |h_j - h_\infty|_{h_\infty} < \varepsilon.
\]

**Proof.** The integral condition
\[
\int_{S^1} Z_\infty(0) e^{\alpha_\infty(\theta)} d\theta = 2 \log \lambda_+ = \int_{S^1} Z_j(0) e^{\alpha_j(\theta)} d\theta
\]
shows that \( Z_j(0) \to Z_\infty(0) \). For \( \delta > 0 \) to be determined, choose \( J_\varepsilon \) large enough that
\[
\sup_{\theta \in S^1} |\alpha_\infty(\theta) - \alpha_j(\theta)| \leq \delta \quad \text{and} \quad \left| \frac{Z_\infty^2(0)}{Z_j^2(0)} - 1 \right| \leq \delta
\]
for all \( j \geq J_\varepsilon \), and consider
\[
(A_\infty - A_j)(\theta, t) = (\alpha_\infty - \alpha_j)(\theta) + \frac{1}{2} \int_0^t (Z_\infty^2 - Z_j^2) dt.
\]
For any \( \lambda, \mu > 0 \) we have the now-familiar inequality
\[
\log \left( 1 - \frac{\mu - \lambda}{\lambda} \right) \leq \int_0^t \left( \frac{1}{t + \lambda} - \frac{1}{t + \mu} \right) dt \leq \log \left( 1 + \frac{\mu - \lambda}{\lambda} \right).
\]
Since
\[
\frac{1}{2} \int_0^t (Z_\infty^2 - Z_j^2) dt = \frac{1}{2} \int_0^t \left( \frac{1}{t + 1/Z_\infty^2(0)} - \frac{1}{t + 1/Z_j^2(0)} \right) dt
\]
and
\[
\frac{1/Z_j^2(0) - 1/Z_\infty^2(0)}{1/Z_\infty^2(0)} \leq \delta,
\]
we get our first estimate:
\[
|(A_\infty - A_j)(\theta, t)| \leq \delta + \log \sqrt{1 + \delta}.
\]
Next observe that when \(0 < \delta \leq \log 2\) we have \(e^\delta \leq 1 + 2\delta\) and thus obtain our second estimate:
\[
|(W_\infty - W_j)(\theta, t)| = |W_\infty(\theta, 0) - W_j(\theta, 0)|
\]
\[
= \left| \int_0^\theta Z_\infty(0) \cdot e^{\alpha_\infty(u)} du - \int_0^\theta Z_j(0) \cdot e^{\alpha_j(u)} du \right|
\]
\[
\leq \int_0^\theta Z_\infty(0) \cdot e^{\alpha_\infty(u)} \left| 1 - e^{\alpha_j(u) - \alpha_\infty(u)} \right| du
\]
\[
+ \int_0^\theta Z_j(0) \cdot e^{\alpha_j(u)} \left| \frac{Z_\infty(0)}{Z_j(0)} - 1 \right| du
\]
\[
\leq 3\delta (2 \log \lambda_+).
\]
As in the proof of Theorem 3.1, it follows that we can make \(|h_\infty - h_j|_{h_\infty}\) as small as desired by choosing \(\delta = \delta(\varepsilon)\) appropriately.

**Proof of Theorem 4.1.** Note that \(|g - h_\infty|_{h_\infty}\) will be small if both \(|g - h_j|_{h_j}\) and \(|h_j - h_\infty|_{h_\infty}\) are. So take the subsequence of metrics \(h_{jk}\) and times \(t_{jk}\) given by Lemma 4.2 and pass to a further subsequence according to Lemma 4.7.

\(\square\)

5. Uniqueness.

Distinct locally homogeneous solv-Gowdy metrics belong to the same equivalence class if and only if they differ merely by a dilation of arc length. In that case, we shall see that they approach one another at the rate \(C/t\), where the constant depends on the initial difference in length of the base circle.

**Theorem 5.1.** Let \(h\) and \(h_*\) be locally homogeneous metrics corresponding to the data \((\alpha(\theta), \Omega, F)\) and \((\alpha_*(\theta), \Omega_*, F_*)\) respectively. If for some
constant $\ell$ we have $\alpha_* \equiv \alpha + \ell$ and $\Omega_* = \Omega$ and $F_* = F$, then $h$ and $h_*$ quasi-converge with

$$|h_* - h|_h = O\left(\frac{1}{t}\right).$$

In all other cases, there are $\delta > 0$ and $\theta \in S^1$ such that

$$|h_* - h|_h (\theta, t) \geq \delta$$

for all $t > 0$, so $h$ and $h_*$ do not quasi-converge.

Proof. We consider three cases.

Case 5.2. $\alpha_* \equiv \alpha + \ell$, $\Omega_* = \Omega$, $F_* = F$.

Writing

$$Z(t) = \frac{1}{\sqrt{t + 1/\zeta^2}}$$

and

$$Z_* (t) = \frac{1}{\sqrt{t + 1/\zeta_*^2}},$$

we observe that $\ell = \log (\zeta / \zeta_*)$, since by the integral condition (2.13) we have

$$\left(\frac{\zeta}{\zeta_*}\right) = \int_{S^1} e^{\alpha_*(\theta)} d\theta = \int_{S^1} e^{\alpha(\theta)} d\theta = e^\ell. \tag{5.1}$$

It follows that the function

$$\omega (\theta) \equiv \int_0^\theta \left(\zeta_* e^{\alpha_*(u)} - \zeta e^{\alpha(u)}\right) du \tag{5.2}$$

is identically zero. So for all $\theta \in S^1$ and $t \geq 0$ we have

$$(W_* - W) (\theta, t) = (W_* - W) (\theta, 0) = \Omega_* - \Omega + \omega (\theta) = 0.$$

Now notice that

$$(A_* - A) (\theta, t) = (\alpha_* - \alpha) (\theta) + \frac{1}{2} \int_0^t \left(Z_*^2 (\tau) - Z_*^2 (\tau)\right) d\tau = \ell + \phi (t),$$

where

$$\phi (t) \equiv \frac{1}{2} \log \frac{1 + \zeta_*^2 t}{1 + \zeta_*^2 t} \tag{5.3}.$$
It is clear by (5.1) that $A^* - A \to 0$ uniformly in $\theta$ as $t \to \infty$. In fact, this identifies the critical rate at which distinct locally homogeneous metrics $h, h^*$ approach each other, because

$$(e^{2A^*} - e^{2A}) (\theta, t) = e^{2A(\theta,t)} \left( e^{2(\ell + \phi(t))} - 1 \right)$$

and hence

$$|h_* - h|_h = \left| h^{\theta\theta} (h_* - h)_{\theta\theta} \right| = \left| e^{2(\ell + \phi(t))} - 1 \right| = \frac{1/\zeta^2 - 1/\zeta^2}{t + 1/\zeta^2}.$$ 

**Case 5.3.** $\alpha_* \equiv \alpha + \ell, \Omega_* = \Omega, F_* \neq F$.

Notice that $W_* - W \equiv 0$ and $A_* - A \to 0$ as above. Without loss of generality, suppose $F_* - F = \delta > 0$. Then for all $\theta \in S^1$ and $t \geq 0$ we have

$$e^{F_* + W_*} - e^{F + W} = e^{F + W} \left( e^{F_* - F - 1} \right) > \delta e^{F + W}$$

and hence

$$|h_* - h|_h \geq |h^{xx} (h_* - h)_{xx}| > \delta > 0.$$ 

**Case 5.4.** Either $\alpha_* \neq \alpha + \ell$ or $\Omega_* \neq \Omega$.

Observe that we can always find $\theta$ with

$$(W_* - W) (\theta, 0) = \Omega_* - \Omega + \omega (\theta) \neq 0,$$

since $\omega$ cannot be identically zero if $\alpha_* \neq \alpha + \ell$. Without loss of generality, assume $(W_* - W) (\theta, 0) = \delta > 0$. Then if $F_* \geq F$, we have

$$e^{F_* + W_* (\theta, t)} - e^{F+W (\theta, t)} = e^{F+W (\theta, t)} \left( e^{F_* - F - 1} \right) \geq e^{F+W (\theta, t)} \left( e^{\delta} - 1 \right)$$

for all $t \geq 0$ and hence

$$|h_* - h|_h (\theta, t) \geq |h^{xx} (h_* - h)_{xx}| (\theta, t) > \delta > 0.$$ 

On the other hand, if $F \geq F_*$ we obtain

$$e^{F_* - W_* (\theta, t)} - e^{F-W (\theta, t)} = e^{F-W (\theta, t)} \left( e^{F_* - F - 1} \right) \leq e^{F-W (\theta, t)} \left( e^{-\delta} - 1 \right)$$

for all $t \geq 0$ and thus

$$|h_* - h|_h (\theta, t) \geq |h^{yy} (h_* - h)_{yy}| (\theta, t) > \frac{\delta}{1 + \delta} > 0.$$

$\square$
References.

