Positive Mass Theorem for Hypersurface in 5-Dimensional Lorentzian Manifolds

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1. Introduction.

It is well-known that Positive Mass Theorem has a fundamental importance in Einstein's general relativity. The positive mass theorem for 5-dimensional Lorentzian manifolds is therefore interesting in the context of Kaluza-Klein theory which provides a 5-dimensional general relativity containing both Einstein's 4-dimensional theory of gravity and Maxwell's theory of electromagnetism. This idea of Kaluza-Klein was enthusiastically received by unified-field theorists and was extended to higher dimensions to include the strong and weak forces (i.e., 11-dimensional supergravity theories and 10-dimensional superstrings). We refer to review article [OW] for higher-dimensional unified theories from the general relativity side. Mathematically, the existence of Spin$^c$ structures on orientable 4-manifolds provides a unified treatment on gravity and electromagnetism. In this paper we adapt Witten's method and the analytic arguments of Parker and Taubes to such a Spin$^c$ structure. This yields a Positive Mass Theorem (Theorem 1.6 below) for hypersurfaces in 5-dimensional Lorentzian manifolds.

Let $N$ be a 5-dimensional Lorentzian manifold with Lorentzian metric $\tilde{g}$ of signature $(-1, 1, 1, 1, 1)$, which satisfies the Einstein equations

$$\tilde{R}_{\alpha\beta} - \frac{\tilde{R}}{2} \tilde{g}_{\alpha\beta} = T_{\alpha\beta},$$

where $\tilde{R}_{\alpha\beta}$, $\tilde{R}$ are the Ricci and scalar curvatures of $\tilde{g}$ respectively, $T_{\alpha\beta}$ is a symmetric tensor field which is interpreted physically as the energy-momentum tensor of matter.

**Definition 1.1.** A spacelike hypersurface $M$ of $N$ is called asymptotically flat of order $\tau$ if there is a compact set $K \subset M$ such that $M - K$ is the disjoint union of a finite number of subsets $M_1, \cdots, M_k$ — called the "ends"
of $M$ — each diffeomorphic to the complement of a contractible compact set in $R^4$. Under the diffeomorphism the metric of $M_l \subset M$ is of the form

$$g_{ij} = \delta_{ij} + a_{ij}$$

in the standard coordinates $\{x^i\}$ on $R^4$, where $a_{ij}$ satisfies

$$a_{ij} = O(r^{-\tau}), \quad \partial_k a_{ij} = O(r^{-\tau-1}), \quad \partial_l \partial_k a_{ij} = O(r^{-\tau-2}).$$

Furthermore, the second fundamental form of $M$ satisfies

$$h_{ij} = O(r^{-\tau-1}), \quad \partial_k h_{ij} = O(r^{-\tau-2}).$$

A $U(1)$ line bundle $L$ over $M$ is called asymptotically flat of order $\tau$ if there is a trivialization of $L$ over the end and a $u(1)$-value 1-form $A$ such that on end $M_l$, the connection on $L$ can be written as

$$dA_j = \partial_j + A_j, i,$$

where $A_j$ is real, and satisfies

$$A_j = O(r^{-\tau-1}), \quad \partial_k A_j = O(r^{-\tau-2}).$$

We will often identify the end $M_l \subset M$ with the corresponding set $M_l \subset R^4$.

The curvature $F_A = dA$ of such a connection on $L$ may be interpreted physically as the electromagnetic field. For spacelike asymptotically flat hypersurface $M$ and asymptotically flat line bundle $L$, we can define the total energy, the total linear momentum and the total electromagnetic momentum. They are defined in each asymptotic end $M_l$ as limits over the sphere $S_{R,l}$ of radius $R$ in $M_l \subset R^4$.

**Definition 1.2.** Total energy of end $M_l$ is defined as

$$E_l = \lim_{R \to \infty} C_4^{-1} \int_{S_{R,l}} (\partial_j g_{ij} - \partial_i g_{ij}) d\Omega^i,$$

total linear momentum of end $M_l$ is defined as

$$p_{lk} = \lim_{R \to \infty} C_4^{-1} \int_{S_{R,l}} 2(h_{ik} - \delta_{ik} h_{jj}) d\Omega^i,$$

total electromagnetic momentum of end $M_l$ is defined as

$$q_{li} = \lim_{R \to \infty} C_4^{-1} \left( \int_{S_{R,l}} 2A_j d\Omega^i - \int_{S_{R,l}} 2A_i d\Omega^j \right),$$

where $C_4 = 12\omega_3$ and $\omega_3$ is the volume of unit sphere $S^3$ with standard metric.
Definition 1.3. The current matrix of electromagnetic field on end $M_i$ is defined by

$$\Omega_i = (\omega_{ij}),$$

where

$$\begin{align*}
\omega_{11} &= 2^{-1}(-q_{12}^2 - q_{13}^2 - q_{14}^2 + q_{13}^2 + q_{14}^2 + q_{13}^2 + q_{14}^2), \\
\omega_{22} &= 2^{-1}(-q_{12}^2 + q_{13}^2 + q_{14}^2 + q_{13}^2 - q_{14}^2 - q_{13}^2), \\
\omega_{33} &= 2^{-1}(q_{12}^2 - q_{13}^2 + q_{14}^2 - q_{13}^2 + q_{14}^2 - q_{13}^2), \\
\omega_{44} &= 2^{-1}(q_{12}^2 + q_{13}^2 - q_{14}^2 - q_{13}^2 - q_{14}^2 + q_{13}^2), \\
\omega_{12} &= \sum_{k \neq \{i,j\}} q_{i1k}q_{1kj}, \quad 1 \leq i, j \leq 4, \ i \neq j.
\end{align*}$$

When the asymptotic order $\tau > 1$, these quantities are finite, independent on the choice of asymptotic coordinates. Since $q_{ij} = -q_{ji}$, $\Omega_i$ is real symmetric. Moreover, $\Omega_i$ is traceless.

The following Positive Mass Conjecture was proved first by R. Schoen and S.T. Yau [SY1, SY2, SY3], then by E. Witten [W, PT].

Theorem 1.4 (Schoen-Yau, Witten). Let $N$ be a 4-dimensional Lorentzian manifold with Lorentzian metric $\bar{g}$ of signature $(-1,1,1,1)$, which satisfies the Einstein equations (1.1), $M \subset N$ be a spacelike asymptotically flat hypersurface of order $\tau > \frac{1}{2}$. If $M$ satisfies the dominant energy condition

$$T_{00} \geq \sqrt{\sum_i T_{0i}^2}, \quad \text{and} \quad T_{00} \geq |T_{\alpha\beta}|,$$

then, for each end $M_i$, we have

$$E_i \geq \sqrt{\sum_i p_{li}^2}.$$

If $E_{l_0} = 0$ for some $l_0$, then $M$ has only one end and $N$ is flat over $M$.

One key point in Witten's argument is to prove that there is a positive definite Hermitian metric on $\text{Spin}(3,1)$ spinors. This fact was verified by T. Parker and C. Taubes [PT] in terms of representation theory of spin group $SL(2,C)$, and was extended to $\text{Spin}(4,1)$ spinors by the author in terms of representation theory of spin group $SU(1,1)$. Consequently, Positive Mass
Conjecture can be proved for spin spacelike hypersurface in 5-dimensional Lorentzian manifolds [Z1]. It should be true for all spin group Spin(n, 1), an issue we will address elsewhere.

Now since \( N \) is 5-dimensional and \( M \) is an orientable hypersurface in \( N \), \( M \) has a \( \text{Spin}^c \) structure. It means that there is a \( U(1) \) line bundle \( L \) on \( N \) such that \( S \otimes L^{1/2} \) is globally-defined over \( M \), where \( S \) is (locally) spinor bundle of \( N \), which is not globally-defined on \( N \) except that \( N \) is spin. Denote \( W = S \otimes L^{1/2} \). \( W \) is called the complex Witten-Dirac spinor bundle, and \( L \) is called \( \text{Spin}^c \) structure. Let \( A \) be a \( U(1) \) connection 1-form on \( L \), and denote \( F^M_A \) as the curvature of \( L \) restricted on \( M \). The corresponding connection on \( L^{1/2} \) is \( \tilde{d}_A = d + \frac{1}{2} A \). Let \( \nabla \) be the metric connection on \( S \). Then the globally-defined connection \( \nabla_A \) and the metric on \( W \) are defined as follows: write \( \phi = s_1 \otimes \sigma_1, \psi = s_2 \otimes \sigma_2 \) locally, where \( s_1, s_2 \in S, \sigma_1^2, \sigma_2^2 \in L \), then

\[
\nabla_A \phi = \nabla s_1 \otimes \sigma_1 + s_1 \otimes \tilde{d}_A \sigma_1,
\]

\[
\langle \phi, \psi \rangle_W = \langle s_1, s_2 \rangle_S \cdot \langle \sigma_1, \sigma_2 \rangle_L.
\]

Obviously, \( \nabla_A \) is compatible with the metric \( \langle , \rangle_W \). At each \( p \in M \), we fix an orthonormal frame \( \{e_\alpha| \alpha = 0, 1, 2, 3, 4 \} \) with \( e_0 \) normal to \( M \) and \( e_1, e_2, e_3, e_4 \) tangent to \( M \). (Here, and henceforth, repeated indices are summed with Latin indices running from 1 to 4 and Greek indices running from 0 to 4.) Denote \( \{e^\alpha| \alpha = 0, 1, 2, 3, 4 \} \) as its dual frame.

**Definition 1.5.** The above \( M \) satisfies the charged dominant energy condition if

\[
(1.10) \quad T_{00} \geq \sqrt{\sum_i T_{0i}^2} + \sqrt{\sum_{i,j} F_{AiJ}^2}, \text{ and } T_{00} \geq |T_{\alpha\beta}| + |F_{A\alpha\beta}|.
\]

**Theorem 1.6.** Let \( N \) be a 5-dimensional Lorentzian manifold with Lorentzian metric \( \tilde{g} \) of signature \((-1,1,1,1,1)\), which satisfies the Einstein equations (1.1), \( M \subset N \) be a spacelike asymptotically flat hypersurface of order \( \tau > 1 \). Let \( L \) be the \( \text{Spin}^c \) structure of complex Witten-Dirac spinor bundle of \( M \) with \( U(1) \) connection \( A \), which is also asymptotically flat of order \( \tau > 1 \). If \( M \) satisfies the charged dominant energy condition (1.10), then, for each end \( M_i \), we have

\[
E_i \geq \left\{ \begin{array}{ll}
\sqrt{|Q_l|^2 + 2|q_{12}q_{34} + q_{13}q_{42} + q_{14}q_{23}|} & \text{if } |P_l| = 0, \\
|P_l| + \sqrt{2^{-1}|Q_l|^2 + \tilde{P}_l^2} & \text{if } |P_l| \neq 0,
\end{array} \right.
\]
where \( |P_i| = \sqrt{\sum_i p_i^2}, \ |Q_i| = \sqrt{\sum_{i<j} q_{ij}^2} \) and \( \bar{P}_i = |P_i|^{-1}(p_{i1}, p_{i2}, p_{i3}, p_{i4})^t \) if \( |P_i| \neq 0 \). If \( E_{l_0} = 0 \) for some \( l_0 \), then \( M \) has only one end and \( N, L \) are flat over \( M \). Moreover, \( p_{lok} = 0, q_{loij} = 0 \).

We also prove an analogous theorem for 4-dimensional Lorentzian manifolds in the appendix. Namely,

**Theorem 1.7.** Let \( N \) be a 4-dimensional Lorentzian manifold with Lorentzian metric \( \tilde{g} \) of signature \((-1,1,1,1)\), which satisfies the Einstein equations \((1.1)\), \( M \subset N \) be a spacelike asymptotically flat hypersurface of order \( \tau > \frac{1}{2} \).

Let \( L \) be the \( \text{Spin}^c(3,1) \) structure of \( N \) with \( U(1) \) connection \( A \), which is also asymptotically flat of order \( \tau > \frac{1}{2} \) over \( M \). If \( M \) satisfies the charged dominant energy condition \((1.10)\), then, for each end \( M_1 \), we have

\[
E_i \geq \sqrt{|P_i|^2 + |Q_i|^2 + 2|p_{i1}q_{l23} + p_{i2}q_{l31} + p_{i3}q_{l12}|},
\]

where \( |P_i| = \sqrt{\sum_i p_i^2}, \ |Q_i| = \sqrt{\sum_{i<j} q_{ij}^2} \). If \( E_{l_0} = 0 \) for some \( l_0 \), then \( M \) has only one end and \( N, L \) are flat over \( M \). Moreover, \( p_{lok} = 0, q_{loij} = 0 \).

### 2. Spinors.

Let \( N \) be a 5-dimensional Lorentzian manifold, and \( M \) be a spacelike hypersurface in \( N \). Denote \( H \) as the field of quaternions. The hyper-unitary group \( HU(1,1) = \text{Spin}^0(4,1) \) is the double covering group of connected Lorentz group \( SO(4,1) \) (see [Ha], p272). A \( \text{Spin}^c \) structure on \( N \) is a globally defined \( \text{Spin}^0(1,1) \) bundle \( W \) over \( M \) locally of the form \( W = S \otimes L^\frac{1}{2} \).

For any \( X \in \text{End}(W) \), denote \( X^* \) the adjoint of \( X \) under \( HU(1,1) \times \mathbb{Z}_2 U(1) \) Hermitian structure. Denote

\[
\mathcal{R} = \{ X \in \text{End}(W), X = X^*, \text{Trace}(X) = 0 \}.
\]

There is an invariant metric on \( \mathcal{R} \) defined for \( X, Y \in \mathcal{R} \) by,

\[
\langle X, Y \rangle = -\frac{1}{2} \text{Re(Trace}(XY))).
\]

Moreover, for any \( X \in T^*N \) with coordinate \( (x_0, x_1, x_2, x_3, x_4) \), we have a canonical identification of \( X \) to an element in \( \mathcal{R} \), i.e.,

\[
X \mapsto \left( \begin{array}{cc}
  x_0 & x \\
 -x & -x_0
  \end{array} \right),
\]

\[(2.1)\]
where \( x = x_1 + x_2I + x_3J + x_4K \). As in [Z1] one can prove that this defines an isometry \( T^*N \equiv \mathbb{R} \).

The spinor bundle \( W \) has a \( HU(1,1) \times \mathbb{Z}_2 U(1) \) invariant Hermitian metric defined by

\[
(\phi, \psi) = \bar{\xi}_1 \cdot \eta_1 - \bar{\xi}_2 \cdot \eta_2
\]

for \( \phi = (\xi_1, \xi_2)^t \in W, \psi = (\eta_1, \eta_2)^t \in W \). This metric is not positive definite.

The Clifford multiplication is the map \( T^*N \otimes W \rightarrow W \) that sends \( X \otimes \phi \) to \( X\phi \), where \( X\phi \) means that spinor \( \phi \) is multiplied by the corresponding matrix (2.1) of covector \( X \). Obviously, \( XY + YX = -2g(X,Y) \cdot Id \). The choice of a timelike covector \( e^0 \) gives another Hermitian metric on \( W \) by

\[
(\phi, \psi) = (e^0 \phi, \psi) = \bar{\xi}_1 \cdot \eta_1 + \bar{\xi}_2 \cdot \eta_2
\]

for \( \phi = (\xi_1, \xi_2)^t \in W, \psi = (\eta_1, \eta_2)^t \in W \). This new metric is positive definite and \( Sp(1) \times Sp(1) \times \mathbb{Z}_2 U(1) \) invariant. Furthermore, for any \( X \in T^*_p M \), spinors \( \phi, \psi \in W \), we have

\[
(2.2)
(X\phi, \psi) = (\phi, X\psi), \quad (x\phi, \psi) = - (\phi, x\psi), \quad (e^0 \phi, \psi) = (\phi, e^0 \psi).
\]

The proofs of above facts are similar to those in [Z1]. By (2.1), we get a canonical representation of the coframe

\[
(2.3)
e^0 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e^1 \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

\[
e^2 \mapsto \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad e^3 \mapsto \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}, \quad e^4 \mapsto \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix}.
\]

Now we derive the Pauli representation. We identify \( H \cong C^2 \) as follows:

For any \( x_H = x_1 + x_2I + x_3J + x_4K = (x_1 + x_2I) + J(x_3 - x_4I) \in H \), we identify it to \( x_C = (x_1 + x_2I, x_3 - x_4I)^t \in C^2 \). Since \( I \cdot x_H = I(x_1 + x_2I) + J(-I)(x_3 - x_4I), \ J \cdot x_H = J(x_1 + x_2I) - (x_3 - x_4I), \) and \( K \cdot x_H = J(-I)(x_1 + x_2I) - I(x_3 - x_4I) \). We can obtain the following canonical Pauli representation

\[
(2.4)
I \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad J \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad K \mapsto \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}.
\]

For any \( x_H, y_H \in H \), we have \( \Re(\bar{x}_H y_H) = \Re(\bar{x}_C y_C) \). This fact implies that, for any \( \phi, \psi \in W, \Re(\phi, \psi)_H = \Re(\phi, \psi)_C \), where \( (\ , \ )_H \) is quaternions
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Hermitian metric on $W$ and $\langle , \rangle_C$ is the corresponding complex Hermitian metric on $W$ while $W$ is viewed as a complex rank-4 bundle.

Obviously, $W = W^+ \oplus W^-$ over $M$, where $W^\pm = \{ \phi \in W : \ast \phi = \pm \phi \}$ ($\ast = -e^1 e^2 e^3 e^4$). The 'half spinor bundles' $W^\pm$ are orthogonal w.r.t. metrics $\langle , \rangle$ and $\langle , \rangle$. Moreover, since $e^0 \ast = \ast e^0$, $e^0$ preserves $W^\pm$. Now the space of 2-forms of $M$ splits as the self-dual part $\Lambda^+$ and the anti-self-dual part $\Lambda^-$, $\Lambda^\pm = \text{span}\{e^I_\pm, e^J_\pm, e^K_\pm\}$, where

\begin{align*}
e^I_\pm &= e^1 \wedge e^2 \pm e^3 \wedge e^4, \\
e^J_\pm &= e^1 \wedge e^3 \pm e^4 \wedge e^2, \\
e^K_\pm &= e^1 \wedge e^4 \pm e^2 \wedge e^3.
\end{align*}

Define the Clifford multiplication of 2-form on $W$ by: $(e^i \wedge e^j) = e^i e^j \ (i \neq j)$.

A straightforward computation shows $\Lambda^\pm W^\mp = 0$. Furthermore,

\begin{align*}(2.5) \quad e^I_+ &= -e^I_+ \quad e^I_- &= -e^I_- \\
(2.6) \quad e^J_+ &= -e^J_+ \\
(2.7) \quad e^J_- &= -e^J_- \\
(2.8) \quad e^K_+ &= -e^K_+ \\
(2.9) \quad e^I_+ e^J_+ &= 2e^K_+ \\
(2.10) \quad e^I_- e^J_- &= 2e^K_- 
\end{align*}

3. Hypersurface $\text{Spin}^c$ Dirac operator.

Let $N$ be a 5-dimensional Lorentzian manifold, and $M$ be a spacelike hypersurface in $N$. Fix a point $p \in M$ and an orthonormal basis $\{e_\alpha\}$ of $T_pN$ with $e_0$ normal and $e_1, e_2, e_3, e_4$ tangent to $M$. Extend $e_1, e_2, e_3, e_4$ to an orthonormal frame in a neighbourhood of $p$ in $M$ such that $(\nabla_i e_j)_p = 0$. Extend this to a local orthonormal frame $\{e_\alpha\}$ for $N$ with $(\nabla_0 e_j)_p = 0$. Let $\{e^\alpha\}$ be the dual frame. Then $(\nabla_i e^j)_p = -h_{ij} e^0, \ (\nabla_i e^0)_p = -h_{ij} e^j$, where $h_{ij} = \langle \nabla_i e_0, e_j \rangle, \ 1 \leq i, j \leq 4$, are the components of the second fundamental form at $p$. The metric connection $\nabla$ and $\nabla$, together with a $U(1)$ connection $A$ on $L$, induce two connections on $W$. These induced connections on $W$, which we denote by $\nabla_A, \nabla_A$ respectively, are related by

\begin{align*}(3.1) \quad &\nabla_A = \nabla_A + \frac{1}{2} h_{ij} e^0 e^j.
\end{align*}
By definition, $\tilde{\nabla}_A$ is compatible with the metric $(\cdot, \cdot)$, i.e.,
\[d\left((\phi, \psi) \ast e_i\right) = \left(\left(\tilde{\nabla}_A e_i, \phi \ast e_i\right) + \left(\phi, \tilde{\nabla}_A e_i\right)\right) \ast 1.\]

Using (2.2) and (3.1), we can prove that $\nabla_A$ is also compatible with the metrics $(\cdot, \cdot)$ and $(\cdot, \cdot)$, i.e.,
\[d\left((\phi, \psi) \ast e_i\right) = \left(\left(\nabla_A e_i, \phi \ast e_i\right) + \left(\phi, \nabla_A e_i\right)\right) \ast 1.\]

In a local orthonormal coframe $\{e^i\}$ of $M$, Spin$^c$ Dirac operator $D_A$ and the hypersurface Spin$^c$ Dirac operator $\tilde{D}_A$ are defined by
\[D_A = e^i \nabla_{A_i}, \quad \tilde{D}_A = e^i \tilde{\nabla}_{A_i},\]
respectively. Obviously, $D_A$ is self-adjoint with respect to the metric $\langle \cdot, \cdot \rangle$. We also have the following standard Weitzenböck formula:
\[D_A^2 = \nabla^* \nabla_A + \frac{R}{4} + \frac{1}{2} F_A^M,\]
where $R$ is the scalar curvature of $M$, and $F_A^M$ is the restriction on $M$ of the curvature of $L$. From (3.1), we have
\[\tilde{D}_A = D_A + \frac{H}{2} e^0,\]
where $H = \sum h_{ii}$ is the mean curvature of $M$. Moreover,
\[d\left(\langle e^i \phi, \psi \rangle \ast e^i\right) = \left(\langle D_A e_i, \phi \ast e_i\rangle - \langle \phi, D_A e_i\rangle\right) \ast 1 = \left(\langle \tilde{D}_A e_i, \phi \ast e_i\rangle - \langle \phi, \tilde{D}_A e_i\rangle\right) \ast 1.\]

and
\[d\left(\langle \phi, \tilde{\nabla}_{A_i} \psi \rangle \ast e^i\right) = \left(\langle \tilde{\nabla}_{A_i} e_i, \phi \ast e_i\rangle - \langle \phi, (-\tilde{\nabla}_{A_i} + h_{ij} e^0 e^i) e^i \rangle \tilde{\nabla}_{A_i} e_i\rangle\right) \ast 1.\]

It follows that the adjoints under the metric $(\cdot, \cdot)$ are $D_A^* = D_A$, $\tilde{D}_A^* = \tilde{D}_A$, $\tilde{\nabla}_{A_i}^* = -\tilde{\nabla}_{A_i} + h_{ij} e^0 e^j$. With the information, we can easily derive (as in [Z1]) the following two Weitzenböck formulas,

\[
\tilde{D}_A^2 = \nabla^* \nabla_A + \frac{1}{4} (R + H^2) - \frac{1}{2} \nabla_i H e^0 e^i + \frac{1}{2} F_A^M
\]

\[
= \tilde{\nabla}_A^* \tilde{\nabla}_A + \frac{1}{2} (T_{00} + T_{0i} e^0 e^i + F_A^M).
\]
The integral form of Weitzenböck formula (3.3) is
\[
(3.4) \quad \int_M \left| \nabla_A \phi \right|^2 + \left< \phi, \tilde{R} \phi \right> - \left| \tilde{D}_A \phi \right|^2 = \frac{1}{2} \int_{\partial M} \left< \phi, [\epsilon^i, \epsilon^j] \nabla_A j \phi \right> \ast e^i,
\]
where $\tilde{R} = \frac{1}{2} (T_{00} + T_{0t} e^0 e^t + F^M_A)$, and $[\epsilon^i, \epsilon^j] = e^i e^j - e^j e^i$.

Now recall that $M$ and $L$ are asymptotically flat of order $\tau > 1$ with asymptotic coordinates $\{dx^i\}$ on the end. Orthonormalizing $\{dx^i\}$ yields an orthonormal coframe
\[
e^i = dx^i + \frac{1}{2} a_{ik} dx^k + O(r^{-\tau-1}).
\]
Denote $e^0$ as $dx^0$. Then, on each end,
\[
\nabla_A j = \partial_j - \frac{1}{4} \Gamma^k_{jl} dx^k dx^l + \frac{1}{2} A_j i + O(r^{-2\tau-1}),
\]
\[
\tilde{D}_A = dx^j \partial_j - \frac{1}{4} \Gamma^k_{jl} dx^j dx^k dx^l + \frac{H}{2} dx^0 + \frac{1}{2} dx^j A_j i + O(r^{-2\tau-1}),
\]
where $\Gamma^k_{jl} = \frac{1}{2} (\partial_j g_{kl} + \partial_l g_{kj} - \partial_k g_{jl}) = O(r^{-\tau-1})$. Therefore $\tilde{D}_A$ gives the maps for the weighted Hölder spaces
\[
C^{2,\alpha}(W) \times C^{1,\alpha}_{-\tau-1}(W) \times C^{0,\alpha}_{-\tau-2}(W)
\]
defined by connection $\nabla_A$ on $W$. Here we are using the weighted spaces defined in the papers of Bartnik [B] and Lee-Parker [LP]. For constant spinor $\phi_0$, $\partial_i \phi_0 = 0$, we have $\tilde{D}_A \phi_0 \in C^{1,\alpha}_{-\tau-1}(W)$, and $\tilde{D}_A^2 \phi_0 \in C^{0,\alpha}_{-\tau-2}(W)$.

The following lemma can be easily proved in the spirit of [PT].

**Lemma 3.1.** Suppose $M$, $L$ are asymptotically flat of order $\tau > 1$ and $\phi$, $\{\phi_i\} \in W$ are $C^1$ spinors which satisfy $\nabla_A \phi = 0$, $\nabla_A \phi_i = 0$ for each $i$,

(i) If $\lim_{x \to \infty} \phi(x) = 0$, where the limit is taken along $M$ in one asymptotic end, then $\phi = 0$.

(ii) If $\{\phi_i\}$ are linearly independent in some end, then they are linearly independent everywhere on $M$.

**Proof.** By the assumption, we have $\nabla_A i \phi = -\frac{1}{2} h_{ij} e^0 e^j \phi$. Then
\[
\left| d|\phi|^2 \right| = 2 \left| \Re \left< \nabla_A \phi, \phi \right> \right| \leq C |h| |\phi|^2.
\]
Therefore the lemma can be proved in the same way as Lemma 4.1, [Z1]. □
Lemma 3.2. If $M$, $L$ are asymptotically flat of order $\tau > 1$ and the charged dominant energy condition (1.10) holds on $M$, then the map
\[ \tilde{D}_A^2 : C^2_{-\tau}(W) \to C^0_{-\tau-2}(W) \]
is an isomorphism.

Proof. First note that the lower order term in (3.2)
\[ \left( \frac{1}{4}(R + H^2) - \frac{1}{2}\nabla_i H e^0 e^i + \frac{1}{2}F_A^M \right) \]
lies in $C^0_{-\tau-2}(W)$. Consequently, Theorem 9.2(d) of [LP] shows that $\tilde{D}_A^2$ is an isomorphism provided it is injective. To show injectivity, suppose that $\phi \in C^2_{-\tau}(W)$ satisfies $\tilde{D}_A^2 \phi = \tilde{\nabla}_A^* \tilde{\nabla}_A \phi + R \phi = 0$. Integrating over the region $M_r \subset M$ inside radius $r$ in asymptotic coordinates, we have
\[ \int_{M_r} |\tilde{\nabla}_A \phi|^2 + \langle R \phi, \phi \rangle = \int_{\partial M_r} \langle \phi, \tilde{\nabla}_A \phi \rangle * e^i. \]
But
\[ \langle \phi, \tilde{\nabla}_A \phi \rangle = \left( \phi, \left( \nabla_{Ai} \phi + \frac{1}{2}h_{ij} e^0 e^j . \phi \right) \right) = O(r^{-2\tau-1}), \]
and $\text{Vol}(\partial M_r) = O(r^3)$ by (1.2), (1.3). Hence the right hand side of the above integral vanishes in the limit as $r \to \infty$. Therefore $\tilde{\nabla}_A \phi = 0$ on $M$. Hence $\phi = 0$ by Lemma 3.1 (i), and the proof of the lemma is complete. \(\square\)

Theorem 3.3. If $M$, $L$ are asymptotically flat of order $\tau > 1$ and the charged dominant energy condition (1.10) holds on $M$, then for any constant spinor $\phi_0$ on ends, the following boundary value problem has a unique solution $\phi \in C^2_{-\tau}(W)$,
\[ \begin{cases} \tilde{D}_A \phi & = \ 0, \\ \lim_{r \to \infty} \phi & = \ \phi_0. \end{cases} \]

Proof. Since $\tilde{D}_A^2 \phi_0 \in C^0_{-\tau-2}(W)$, Lemma 3.2 show that there is unique $\phi_1 \in C^2_{-\tau}(W)$ such that $\tilde{D}_A^2 \phi_1 = -\tilde{D}_A^2 \phi_0$. Then $\phi = \phi_1 + \phi_0$ satisfies $\tilde{D}_A^2 \phi = 0$. Let $\psi = \tilde{D}_A \phi \in C^1_{-\tau-1}(W)$, then
\[ \int_{M_r} |\tilde{\nabla}_A \psi|^2 + \langle R \psi, \psi \rangle = \int_{\partial M_r} \langle \psi, \tilde{\nabla}_A \psi \rangle * e^i = \int_{\partial M_r} O(r^{-2\tau-3}) \to 0 \]
as $r \to \infty$. Therefore $\tilde{\nabla}_A \psi = 0$ on $M$. Hence $\psi = 0$ by Lemma 3.1 (i) and $\phi$ is the unique solution of (3.5). \(\square\)

In this section, we will prove Positive Mass Theorem.

Proof of Theorem 1.6. Fix a constant spinor \( \phi_0 \neq 0 \) on \( M_l \) and \( \phi_0 = 0 \) on the other ends. Let \( \phi = \phi_0 + \phi_1 \) be the solution of (3.5) with \( \phi_1 \in C^2_{-\tau}(W) \).

As in [Z1] we have

\[
\int_M \left| \vec{\nabla} A \phi \right|^2 + \left\langle \phi, \vec{R} \phi \right\rangle = \frac{1}{2} \int_{\partial M_{\infty}} \left\langle \phi_0, [dx^i, dx^j] \vec{\nabla} J \phi_0 \right\rangle * dx^i + \frac{1}{4} \int_{\partial M_{\infty}} \left\langle \phi_0, [dx^i, dx^j] J_i \phi_0 \right\rangle * dx^i
\]

or

\[
\int_M \left| \vec{\nabla} A \phi \right|^2 + \left\langle \phi, \vec{R} \phi \right\rangle = \frac{C_4}{4} \sum_i \left( \left\langle \phi_0, E_l \phi_0 \right\rangle + \left\langle \phi_0, p_k dx^0 dx^k \phi_0 \right\rangle + \sum_{i<j} \left\langle \phi_0, dx^i dx^j q_{ij} \phi_0 \right\rangle \right).
\]

We next simplify these terms algebraically. For this we temporarily drop the subscript on \( \phi_0 \), writing \( \phi_0 = (\phi^+, \phi^-) \in W^+ \oplus W^- \). Similarly, we drop the subscript \( l \) from \( E_l, P_l, Q_l, \Omega_l, p_{li} \) and \( q_{lij} \). When \( |P| \neq 0 \), we choose \( \phi^- \) so that \( p_k dx^0 dx^k \phi^+ = -|P| \phi^- \). Then

\[
\left\langle \phi_0, p_k dx^0 dx^k \phi_0 \right\rangle = \left\langle \phi^+, p_k dx^0 dx^k \phi^- \right\rangle + \left\langle \phi^-, p_k dx^0 dx^k \phi^+ \right\rangle
\]

or

\[
\int_M \left| \vec{\nabla} A \phi \right|^2 + \left\langle \phi, \vec{R} \phi \right\rangle = \frac{C_4}{4} \sum_i \left( \left\langle \phi_0, E_l \phi_0 \right\rangle + \left\langle \phi_0, p_k dx^0 dx^k \phi_0 \right\rangle + \sum_{i<j} \left\langle \phi_0, dx^i dx^j q_{ij} \phi_0 \right\rangle \right).
\]

Denote the self-dual part of total electromagnetic momentum of end \( M_l \) by

\[
q_1^+ = 2^{-1}(q_{12} + q_{34}), \quad q_2^+ = 2^{-1}(q_{13} + q_{42}), \quad q_3^+ = 2^{-1}(q_{14} + q_{23}),
\]

and anti-self-dual part of total electromagnetic momentum of end \( M_l \) by

\[
q_1^- = 2^{-1}(q_{12} - q_{34}), \quad q_2^- = 2^{-1}(q_{13} - q_{42}), \quad q_3^- = 2^{-1}(q_{14} - q_{23}).
\]

Let \( q^+ = e^+_l q_1^+ + e^+_2 q_2^+ + e^+_3 q_3^+ \), \( q^- = e^-_l q_1^- + e^-_2 q_2^- + e^-_3 q_3^- \), then

\[
\sum_{i<j} \left\langle \phi_0, dx^i dx^j q_{ij} \phi_0 \right\rangle = \left\langle \phi^+, q^+ i \phi^+ \right\rangle + \left\langle \phi^-, q^- i \phi^- \right\rangle
\]

or

\[
\int_M \left| \vec{\nabla} A \phi \right|^2 + \left\langle \phi, \vec{R} \phi \right\rangle = \left\langle \phi^+, (q^+ - |P|^{-2} p_k p_j dx^k q^- dx^j) i \phi^+ \right\rangle.
\]
Using (2.6), (2.7), (2.8), (2.9) and (2.10), we obtain

\[
p_k p_j dx^k e^I_{-} dx^j = \left(- p_k p_1 dx^k dx^1 - p_k p_2 dx^k dx^2 + p_k p_3 dx^k dx^3 + p_k p_4 dx^k dx^4 \right) e^I_{+}
\]

\[
= \left(p_1^2 + p_2^2 - p_3^2 - p_4^2 + 2p_1 p_3 dx^1 dx^3 - 2p_4 p_2 dx^4 dx^2 + 2p_1 p_2 dx^1 dx^4 + 2p_2 p_3 dx^2 dx^3 \right) e^I_{+}
\]

\[
= \left[p_1^2 + p_2^2 - p_3^2 - p_4^2 + (p_1 p_3 - p_4 p_2) e^I_{+} + (p_1 p_4 + p_2 p_3) e^K_{+}\right] e^I_{+}
\]

\[
= \left(p_1^2 + p_2^2 - p_3^2 - p_4^2\right) e^I_{+} + 2 \left(p_1 p_4 + p_2 p_3\right) e^I_{+}
\]

\[
+ 2 \left(p_2 p_4 - p_1 p_3\right) e^K_{+}.
\]

Similarly, one finds that

\[
p_k p_j dx^k e^I_{-} dx^j = 2 \left(p_2 p_3 - p_1 p_4\right) e^I_{+} + \left(p_1^2 - p_2^2 + p_3^2 - p_4^2\right) e^I_{+}
\]

\[
+ 2 \left(p_1 p_2 + p_3 p_4\right) e^K_{+},
\]

\[
p_k p_j dx^k e^K_{-} dx^j = 2 \left(p_1 p_3 + p_2 p_4\right) e^I_{+} + 2 \left(p_3 p_4 - p_1 p_2\right) e^I_{+}
\]

\[
+ \left(p_1^2 - p_2^2 - p_3^2 + p_4^2\right) e^K_{+}.
\]

Denote

\[
c_1 = q_1^+ - |P|^{-2} \left(\left(p_1^2 + p_2^2 - p_3^2 - p_4^2\right) q_1^- + 2 (p_2 p_3 - p_1 p_4) q_2^- + 2(p_1 p_3 + p_2 p_4) q_3^-\right),
\]

\[
c_2 = q_2^+ - |P|^{-2} \left(2(p_1 p_4 + p_2 p_3) q_1^- + \left(p_1^2 - p_2^2 + p_3^2 - p_4^2\right) q_2^- + 2(p_3 p_4 - p_1 p_2) q_3^-\right),
\]

\[
c_3 = q_3^+ - |P|^{-2} \left(2(p_2 p_4 - p_1 p_3) q_1^- + 2(p_1 p_2 + p_3 p_4) q_2^- + \left(p_1^2 - p_2^2 - p_3^2 + p_4^2\right) q_3^-\right).
\]
When $|P| = 0$, we set
\[
\begin{align*}
c_1 &= q_i^+ - |S|^{-2} \left( (s_1^2 + s_2^2 - s_3^2 - s_4^2)q_i^- + 2(s_1s_3 - s_1s_4)q_2^- \\ &+ 2(s_1s_3 + s_2s_4)q_3^- \right), \\
c_2 &= q_i^+ - |S|^{-2} \left( 2(s_1s_4 + s_2s_3)q_1^- + (s_1^2 - s_2^2 + s_3^2 - s_4^2)q_2^- \\ &+ 2(s_3s_4 - s_1s_2)q_3^- \right), \\
c_3 &= q_i^+ - |S|^{-2} \left( 2(s_2s_4 - s_1s_3)q_1^- + 2(s_1s_2 + s_3s_4)q_2^- \\ &+ (s_1^2 - s_2^2 - s_3^2 + s_4^2)q_3^- \right),
\end{align*}
\]

where $s_1, s_2, s_3, s_4$ are arbitrary real numbers such that $|S| = \sqrt{\sum_i s_i^2} \neq 0$.

We choose $\phi^-$ by $s_kdx^0dx^k\phi^+ = -|S|\phi^-$. Then we can repeat the above calculation, replacing $p_k$ by $s_k$. Therefore,
\[
\begin{align*}
\sum_{i < j} \langle \phi_0, dx^i dx^j q_{ij} i\phi_0 \rangle &= \langle \phi^+, (e_+^I c_1 + e_+^J c_2 + e_+^K c_3) i\phi^+ \rangle \\
&= 2\langle \phi^+, (Ic_1 + Jc_2 + Kc_3) i\phi^+ \rangle.
\end{align*}
\]

By the Pauli representation (2.4), we have
\[
\Re \sum_{i < j} \langle \phi_0, dx^i dx^j q_{ij} i\phi_0 \rangle = 2\Re \langle \phi^+, C\phi^+ \rangle,
\]

where
\[
C = \begin{pmatrix} c_1 & -ic_2 + c_3 \\ ic_2 + c_3 & c_1 \end{pmatrix},
\]

which has real eigenvalues $\lambda = \pm |C|, |C| = \sqrt{\sum_i c_i^2}$. Now we take $\phi^+$ to be the eigenspinor of eigenvalue $-|C|$ with $|\phi^+|^2 = \frac{1}{2}$. We obtain
\[
E - |P| - |C| = 4C_4^{-1} \int_M \left| \tilde{\nabla}_A \phi \right|^2 + \langle \phi, \tilde{R}\phi \rangle \geq 0.
\]

Next we compute $|C|$. Denote
\[
t_k = \begin{cases} |S|^{-1}s_k & \text{if } |P| = 0, \\ |P|^{-1}p_k & \text{if } |P| \neq 0. \end{cases}
\]
Obviously, $\sum_k t_k^2 = 1$. A straightforward computation gives
\[
\left( (t_1^2 + t_2^2 - t_3^2 - t_4^2) + 2(t_2t_3 - t_1t_4)q_1^2 + 2(t_1t_3 + t_2t_4)q_2^2 \right)^2
\]
\[
+ \left( 2(t_1t_4 + t_2t_3)q_1^2 + (t_1^2 - t_2^2 + t_3^2 - t_4^2)q_2^2 + 2(t_3t_4 - t_1t_2)q_3^2 \right)^2
\]
\[
+ \left( 2(t_2t_4 - t_1t_3)q_1^2 + (t_1^2 - t_2^2 + t_3^2 - t_4^2)q_2^2 + 2(t_3t_4 - t_1t_2)q_3^2 \right)^2
\]
\[
= (q_1^2 + q_2^2 + q_3^2)^2.
\]

Therefore
\[
|C|^2 = (q_1^2 + q_2^2 + q_3^2)^2 + (q_4^2)^2 + (q_1^2 + q_2^2 + q_3^2)^2
\]
\[- 2(t_1^2 + t_2^2 - t_3^2 - t_4^2)q_1^2 - 2(t_2t_3 - t_1t_4)q_1^2 - 4(t_1t_3 + t_2t_4)q_1^2 q_2^2 - 4(t_3t_4 - t_1t_2)q_3^2
\]
\[- 4(t_1t_4 + t_2t_3)q_1^2 q_2^2 - 4(t_3t_4 - t_1t_2)q_3^2 q_2^2 - 2(t_1^2 - t_2^2 + t_3^2 - t_4^2)q_2^2
\]
\[- 4(t_2t_4 - t_1t_3)q_1^2 q_2^2 - 4(t_1t_2 + t_3t_4)q_3^2 q_2^2 - 2(t_1^2 - t_2^2 + t_3^2 - t_4^2)q_3^2
\]
\[
= \frac{1}{2} |Q|^2 + \tilde{T} \Omega \tilde{T},
\]

where $\tilde{T} = (t_1, t_2, t_3, t_4)^t$. Now we show when $|P| = 0$, there is another choice of constant spinor $\phi_0$ such that the third term in (4.1) has sharper value. First, by mean value inequality and (4.4),
\[
|C|^2 \leq 2((q_1^2)^2 + (q_2^2)^2 + (q_3^2)^2 + (q_4^2)^2) = |Q|^2.
\]

On the other hand,
\[
\text{Re} \sum_{i<j} \left< \phi_0, dx^ix^jdq_{ij}i\phi_0 \right> = \text{Re} \left< \phi^+, q^+i\phi^+ \right> + \text{Re} \left< \phi^-, q^-i\phi^- \right>
\]
\[
= 2\text{Re} \left< \phi^+, Q^+\phi^+ \right> - 2\text{Re} \left< \phi^-, Q^-\phi^- \right>,
\]

where
\[
Q^+ = \begin{pmatrix} -q_1^+ & -iq_2^+ + q_3^+ \\ iq_2^+ + q_3^+ & q_1^+ \end{pmatrix}, \quad Q^- = \begin{pmatrix} -q_1^- & -iq_2^- + q_3^- \\ iq_2^- + q_3^- & q_1^- \end{pmatrix}.
\]

When $|P| = 0$, we can choose $\phi^+, \phi^-$ freely. So we choose $\phi^+$ to be the eigenspinor of eigenvalue $-|Q^+|$ of $Q^+$, and choose $\phi^-$ to be the eigenspinor of eigenvalue $|Q^-|$ of $Q^-$ such that $|\phi^+|^2 + |\phi^-|^2 = 1$, $|Q^+|^2 = \sum_i (q_i^+)^2$, $|Q^-|^2 = \sum_i (q_i^-)^2$. Then,
\[
-\text{Re} \sum_{i<j} \left< \phi_0, dx^ix^jdq_{ij}i\phi_0 \right> = 2|Q^+|^2|\phi^+|^2 + 2|Q^-|^2|\phi^-|^2.
\]
We choose $\phi^- = 0$ if $|Q^+| \geq |Q^-|$, and $\phi^+ = 0$ if $|Q^+| \leq |Q^-|$. Thus

$$-\Re \sum_{i<j} \langle \phi_0, dx^i dx^j q_{ij} i \phi_0 \rangle = 2 \max \left\{ |Q^+|, |Q^-| \right\}$$

$$= \sqrt{|Q|^2 + 2|q_{12}q_{34} + q_{13}q_{42} + q_{14}q_{23}|}.$$ 

By (4.5), we know to get a sharper result by choose constant spinor in this way when $|P| = 0$. The proof of the first part of Theorem 1.6 is complete.

Now suppose $E_i = 0$. Then $p_{1k} = 0$, $1 \leq k \leq 4$, $c_{1j} = 0$, $1 \leq j \leq 3$ and $q_{1ij} = 0$, $1 \leq i, j \leq 4$. Take constant spinor $\{\psi_\mu : \mu = 1, 2, 3, 4\}$ which form a basis of $W$ on $M_1$ and $\psi_\mu = 0$ on all other ends $M_i$, where we take $W$ as complex bundle. Let $\psi_\mu$ be the solutions of $D_A \psi_\mu = 0$ constructed from this data by Theorem 3.3. The vanishing of $E_1$ then implies $\tilde{\nabla}_A \psi_\mu = 0$ and $\psi_\mu \rightarrow 0$ uniformly on each end except $M_1$. But this contradicts Lemma 3.1 (i) unless $M_1$ is the only end of $M$. By Lemma 3.1 (ii), $\{\psi_\mu : \mu = 1, 2, 3, 4\}$ are linearly independent everywhere on $M$, so in a local frame $\{e_i\}$ of $M$,

$$-\frac{1}{4} \tilde{R}_{\alpha\beta ij} e^\alpha e^\beta \psi_\mu + \frac{1}{2} F_{Ai\mu} \psi_\mu = (\tilde{\nabla}_A \tilde{\nabla}_B - \tilde{\nabla}_B \tilde{\nabla}_A - \tilde{\nabla}_{A(e_i, e_j)} ) \psi_\mu = 0.$$ 

In terms of (2.3), (2.4), we obtain

$$\left( \begin{array}{c} \tilde{R}_{ij1}^+ - F_{Ai\mu} \\ \tilde{R}_{ij2}^+ \\ \tilde{R}_{ij3}^+ \\ \tilde{R}_{ij01}^+ - \tilde{R}_{ij02}^+ \\ \tilde{R}_{ij03}^+ + \tilde{R}_{ij04}^+ \\ \tilde{R}_{ij01}^- + \tilde{R}_{ij02}^- \\ \tilde{R}_{ij03}^- + \tilde{R}_{ij04}^- \end{array} \right) \psi_\mu = 0,$$

where

$$\tilde{R}_{ij1}^\pm = \tilde{R}_{ij12} \pm \tilde{R}_{ij34}, \quad \tilde{R}_{ij2}^\pm = \tilde{R}_{ij13} \pm \tilde{R}_{ij42}, \quad \tilde{R}_{ij3}^\pm = \tilde{R}_{ij14} \pm \tilde{R}_{ij23}.$$ 

This immediately implies that, over $M$, $\tilde{R}_{ij}\alpha\beta = 0$, $F_{Ai\mu} = 0$. Therefore $T_{00} = 0$ by the Einstein equations, and $\tilde{R}_{0i0j} = 0$, $F_{Ai0} = 0$ by the charged dominant energy condition. Thus the proof of Theorem 1.6 is complete. □

5. Appendix: Analogue on 4-Lorentzian Manifolds.

In this appendix, we assume $N$ is a 4-dimensional Lorentzian manifold with Lorentzian metric $\tilde{g}$ of signature $(-1,1,1,1)$, which satisfies the Einstein equations (1.1), and $M$ is a spacelike hypersurface in $N$ which is asymptotically flat of order $\tau > \frac{1}{2}$. Let $L$ be a $U(1)$ line bundle which is a $Spin^c(3,1)$
structure of $N$. We assume $L$ is also asymptotically flat of order $\tau > \frac{1}{2}$ over $M$. The total energy, the total linear momentum and the total electromagnetic momentum of each end of $M$ can be defined same as (1.7), (1.8), (1.9) except that we integrate over the sphere in 3-dimensional asymptotically flat ends.

**Proof of Theorem 1.7.** Let $V$ be (locally) $SL(2, C)$ bundle. The complex spinor bundle $W$ of $N$ is equal to $(\tilde{V} \otimes L^{\frac{1}{2}}) \oplus (V^* \otimes L^{\frac{1}{2}})$. Note that $L^{\frac{1}{2}}$ is globally-defined over $M$ since every orientable 3-manifold is spin. Now the Clifford multiplication can be defined as follows: For any $X \in T^*N$ with coordinate $(x_0, x_1, x_2, x_3)$, we identify it to the corresponding elements $X \in Hom(\tilde{V} \otimes L^{\frac{1}{2}} \rightarrow V^* \otimes L^{\frac{1}{2}})$ and $X^\sigma \in Hom(V^* \otimes L^{\frac{1}{2}} \rightarrow \tilde{V} \otimes L^{\frac{1}{2}})$,

$$X \mapsto \begin{pmatrix} x_0 - x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 + x_1 \end{pmatrix}, \quad X^\sigma \mapsto \begin{pmatrix} x_0 + x_1 & -x_2 - ix_3 \\ -x_2 + ix_3 & x_0 - x_1 \end{pmatrix}.$$ 

Then the Clifford multiplication $T^*N \otimes W \rightarrow W$ is defined by $X \otimes (\xi, \eta)^t = (X^\eta, X^\sigma \xi)^t$. We refer to [PT, Z2] for details. Now let $\phi_0 = (\xi_0, \eta_0)^t$,

$$p_{lk}dx^0dx^k\phi_0 + \sum_{i<j} dx^i dx^j q_{ij} i\phi_0 = (C_{l1}\xi_0, C_{l2}\eta_0)^t,$$

where

$$C_{l\xi} = \begin{pmatrix} p_{l1} - q_{l23} & -p_{l2} + q_{l31} - i(p_{l3} - q_{l12}) \\ -p_{l2} + q_{l31} + i(p_{l3} - q_{l12}) & -p_{l1} + q_{l23} \end{pmatrix},$$

$$C_{l\eta} = \begin{pmatrix} -p_{l1} + q_{l23} & p_{l2} + q_{l31} + i(p_{l3} + q_{l12}) \\ p_{l2} + q_{l31} - i(p_{l3} + q_{l12}) & p_{l1} + q_{l23} \end{pmatrix}.$$ 

Note $C_{l\xi}$ has eigenvalues $\pm \lambda_{l\xi}$,

$$\lambda_{l\xi} = \sqrt{(p_{l1} - q_{l23})^2 + (p_{l2} - q_{l31})^2 + (p_{l3} - q_{l12})^2},$$

and $C_{l\eta}$ has eigenvalues $\pm \lambda_{l\eta}$,

$$\lambda_{l\eta} = \sqrt{(p_{l1} + q_{l23})^2 + (p_{l2} + q_{l31})^2 + (p_{l3} + q_{l12})^2}.$$ 

We choose spinor $\phi_0 = (\xi_0, \eta_0)$ such that $\xi_0$ is the eigenspinor of eigenvalue $-\lambda_{l\xi}$ and $\eta_0$ is the eigenspinor of eigenvalue $-\lambda_{l\eta}$. Moreover, $|\xi_0|^2 + |\eta_0|^2 = 1$. Then

$$\langle \phi_0, p_{lk}dx^0dx^k\phi_0 \rangle + \sum_{i<j} \langle \phi_0, dx^i dx^j q_{ij} i\phi_0 \rangle = -\lambda_{l\xi}|\xi_0|^2 - \lambda_{l\eta}|\eta_0|^2.$$
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We choose $\eta_0 = 0$ if $\lambda_l \geq \lambda_{l_0}$, and $\xi_0 = 0$ if $\lambda_l \leq \lambda_{l_0}$. Thus, if $M$ satisfies the charged dominant energy condition, then

$$E_l \geq \sqrt{|P_l|^2 + |Q_l|^2 + 2|p_{l1}q_{l2} + p_{l2}q_{l1}| + p_{l3}q_{l12}}.$$ 

If $E_{l_0} = 0$ for some $l_0$, then $M$ has only one end, $p_{l_0k} = 0$, $q_{l_0ij} = 0$, and $R_{\alpha\beta\gamma\delta} = 0$, $F_{A\alpha\beta} = 0$ over $M$. Thus the proof of Theorem 1.7 is complete. □

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