The classification of exceptional Dehn surgeries on 2-bridge knots

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We will classify all exceptional Dehn surgeries on 2-bridge knots according to whether they produce reducible, toroidal, or Seifert fibered manifolds.

1. Introduction.

A nontrivial Dehn surgery on a hyperbolic knot $K$ in $S^3$ is exceptional if the resulting manifold is either reducible, or toroidal, or a Seifert fibered manifold whose orbifold is a sphere with at most three exceptional fibers, called a small Seifert fibered space. Thus an exceptional Dehn surgery is non-hyperbolic, and using a version of Thurston’s orbifold theorem proved by Boileau and Porti [BP], it can be shown that a non-exceptional surgery on a 2-bridge knot is also hyperbolic, see Remark 6.3. There has been much research into determining how many exceptional surgeries a knot can admit. See the survey articles [Go, Lu] for details.

The purpose of this paper is to classify all exceptional Dehn surgeries on 2-bridge knots. Hatcher and Thurston [HT] have shown that there is no reducible surgery on hyperbolic 2-bridge knots. We will determine all toroidal surgeries and small Seifert fibered surgeries on these knots, which will then complete the classification.

We use $[b_1, \ldots, b_n]$ to denote the partial fraction decomposition $1/(b_1 - 1/(b_2 - \ldots - 1/b_n)\ldots)$. Recall that a 2-bridge knot $K$ is non-hyperbolic if and only if $K = K_{1/q}$ for some $q$, in which case $K$ is a $(2, q)$ torus knot, and surgery on $K$ is well understood. $K$ is a twist knot if it is equivalent to some $K_{p/q}$ with $p/q = [b, \pm 2]$ for some integer $b$. Since $[b, \pm 2] = [b \mp 1, \mp 2]$, we may assume that $b$ is even. Let $K(\gamma)$ be the manifold obtained by $\gamma$ surgery on $K$. We always assume that $\gamma \neq \infty$, that is, the surgery is nontrivial. With respect to the standard meridian-longitude pair on $\partial N(K)$, each slope $\gamma$ is identified with a rational number, see Rolfsen’s book [R]. The following is the main theorem of this paper. Note that part (4) of the theorem is the case of surgery on the Figure 8 knot, and is due to Thurston [Th]. It is included in the theorem for the sake of completeness.
Theorem 1.1. Let $K$ be a hyperbolic 2-bridge knot.

1. If $K \neq K_{[b_1, b_2]}$ for any $b_1, b_2$, then $K$ admits no exceptional surgery.

2. If $K = K_{[b_1, b_2]}$ with $|b_1|, |b_2| > 2$, then $K(\gamma)$ is exceptional for exactly one $\gamma$, which yields a toroidal manifold. When both $b_1$ and $b_2$ are even, $\gamma = 0$. If $b_1$ is odd and $b_2$ is even, $\gamma = 2b_2$.

3. If $K = K_{[2n, \pm 2]}$ and $|n| > 1$, $K(\gamma)$ is exceptional for exactly five $\gamma$: $K(\gamma)$ is toroidal for $\gamma = 0, \pm 4$, and is small Seifert fibered for $\gamma = \mp 1, \pm 2, \pm 3$.

4. If $K = K_{[2n, -2]}$ is the Figure 8 knot, $K(\gamma)$ is exceptional for only nine $\gamma$: $K(\gamma)$ is toroidal for $\gamma = 0, 4, -4$, and is Seifert fibered for $\gamma = -1, -2, -3, 1, 2, 3$.

We will use a result of Hatcher and Thurston [HT] to determine all toroidal surgeries on a hyperbolic 2-bridge knot $K$, see Lemma 2.2 below. In general it is more difficult to determine small Seifert fibered surgeries, due to the fact that there is no essential surfaces in such a manifold, unless its Euler number is 0. Here we will use essential lamination theory developed by Gabai and Oertel [GO]. The readers are referred to [GO] for definitions and basic properties concerning essential branched surfaces and essential laminations, which play a central role in the proof of the theorem. We will use Brittenham's criterion [Br], which says that if $M$ is a small Seifert fibered space containing an essential branched surface $\mathcal{F}$, then each component of $M - \text{Int } N(\mathcal{F})$ is an $I$-bundle over some compact surface $G$. The idea of the proof is to construct essential laminations in surgered manifolds whose complementary regions are not $I$-bundles. We will apply some techniques developed by Delman [De1, De2] and Roberts [Ro] to construct essential laminations in the surgered manifolds.

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2. Reducible, toroidal, or Seifert fibered surgeries.

Reducible surgeries and toroidal surgeries on 2-bridge knots are completely determined by Lemmas 2.1 and 2.2. Certain surgeries on twist knots are shown to be Seifert fibered in Corollary 2.4.

**Lemma 2.1 (Hatcher-Thurston).** Let $K$ be a 2-bridge knot. Then $K(r)$ is reducible if and only if $K$ is a $(2,q)$ torus knot, and $r = 2q$.

*Proof.* See [HT, Theorem 2]. □

**Lemma 2.2.** Let $K$ be a hyperbolic 2-bridge knot.

1. If $K(\gamma)$ is toroidal for some $\gamma$, then $K = K_{[b_1,b_2]}$ for some $b_1, b_2$.

2. If $|b_i| > 2$ for $i = 1,2$, there is exactly one such $\gamma$. When both $b_i$ are even, $\gamma = 0$. When $b_1$ is odd and $b_2$ is even, $\gamma = 2b_2$.

3. If $K = K_{[2n,2]}$ and $|n| > 1$, $K(\gamma)$ is toroidal if and only if $\gamma = 0$ or $-4$. For $K = K_{[2n,-2]}$, $\gamma = 0$ or $4$.

4. If $K = K_{[2,-2]}$, then $K(\gamma)$ is toroidal if and only if $\gamma = 0$, $4$, or $-4$.

*Proof.* We refer the reader to [HT] for notations. If $K(\gamma)$ is toroidal, there is an essential punctured torus $T$ in the knot exterior. By Theorem 1 of [HT], $T$ is carried by some $\Sigma_{[b_1,\ldots,b_k]}$, where $[b_1,\ldots,b_k]$ is an expansion of $p/q$. By the proof of Theorem 2 of [HT], we have $0 = 2 - 2g = n(2-k)$, where $g$ is the genus of $T$, and $n$ is the intersection number between $\partial T$ and a meridian of $K$. Therefore $k = 2$. This proves (1). The rest follows by determining all the possible expansions of type $[b_1,b_2]$ for $p/q$. The boundary slopes of the surfaces can be calculated using Proposition 2 of [HT]. By the proof of [Pr, Corollary 2.1], an incompressible punctured torus $T$ in the exterior of a 2-bridge knot will become an essential torus after surgery along the slope of $\partial T$. □

**Lemma 2.3.** Let $L = k_1 \cup k_2$ be the Whitehead link, which is the 2-bridge link associated to the rational number $p'/q' = [2,2,-2]$. Let $L(\gamma_1,\gamma_2)$ be the manifold obtained by $\gamma_i$ surgery on $k_i$. If $\gamma_1 = -1/n$ and $\gamma_2 = -1, -2$ or $-3$, then $L(\gamma_1,\gamma_2)$ is a small Seifert fibered space.
Proof. By definition $L(\infty, \gamma_2)$ is the manifold obtained from $S^3$ by $\gamma_2$ surgery on $k_2$. After $-1$ surgery on $k_2$, the knot $k_1$ becomes a trefoil knot in $L(\infty, -1) = S^3$. Since the exterior of a torus knot is a Seifert fibered space with orbifold a disk with two cones, it is easy to see that all surgeries but one yield Seifert fibered spaces, each having an orbifold a disk with at most three cone points. For this trefoil, the exceptional surgery has coefficient $-6$, yielding a reducible manifold. Thus $L(-1/n, -1)$ is a small Seifert fibered space for any $n$.

After $-2$ surgery on $k_2$, the knot $k_1$ becomes a knot in $\mathbb{R}P^3 = L(\infty, -2)$. The link $L$ is drawn in Figure 1(a), where the curve $C$ is a curve on $\partial N(k_2)$ of slope $-2$, so it bounds a disk in $L(\infty, -2)$. Thus a band sum of $k_1$ and $C$ forms a knot $k_1'$ isotopic to $k_1$ in $L(\infty, -2)$. The link $L' = k_1' \cup k_2$ is shown in Figure 1(b). Using Kirby Calculus one can show that $L(-1/n, -2) = L'(-2 - 1/n, -2)$. The exterior of $k_1'$ in $S^3$ is a Seifert fibered space with orbifold a disk with two cones, in which $k_2$ is a singular fiber of index 3. Thus after $-2$ surgery on $k_2$, the manifold $L(\infty, -2) - \text{Int} N(k_1')$ is still Seifert fibered, with orbifold a disk with two cones. The fiber slope on $\partial N(k_1')$ is 6. It follows that all but the 6 surgery on $k_1'$ in $L(\infty, -2)$ yield small Seifert fibered manifolds. In particular, $L(-1/n, -2) = L'(-2 - 1/n, -2)$ are small Seifert fibered manifolds for all $n$.

The proof for $\gamma_2 = -3$ is similar. One can show that the band sum of $k_1$ and the curve $C$ of slope $-3$ on $\partial N(k_2)$ is isotopic to the curve $k_1'$ shown in Figure 1(c), which is a $(3, -2)$ torus knot. By the same argument as above one can show that $L(-1/n, -3) = L(-3 - 1/n, -3)$ are small Seifert fibered manifolds for all $n$. \(\square\)
Recall that a 2-bridge knot $K$ is a twist knot if $K = K_{p/q}$, and $p/q = [2n, \pm 2]$ for some $n$.

**Corollary 2.4.** If $K = K_{p/q}$ is a twist knot with $p/q = [2n, \pm 2]$, then $K(\gamma)$ is a small Seifert fibered space for $\gamma = \mp 1$, $\mp 2$ and $\mp 3$.

**Proof.** Consider the case $p/q = [2n, 2]$. The proof for $p/q = [2n, -2]$ is similar. Let $L = k_1 \cup k_2$ be a 2-bridge link associated to the rational number $p'/q' = [2, 2, -2]$. Notice that after $-1/n$ surgery on $k_1$, the knot $k_2$ becomes the knot $K = K_{[2n,2]}$ in $S^3 = L(-1/n, \infty)$. Therefore by Lemma 6, $K(\gamma) = L(-1/n, \gamma)$ are small Seifert fibered spaces for $\gamma = -1, -2$ and $-3$. □

### 3. Delman’s construction, and the proof of Theorem 1.1(1).

For each rational number $p/q$, there is associated a diagram $D(p/q)$, which is the minimal subdiagram of the Hatcher-Thurston diagram [HT, Figure 4] that contains all minimal paths from 1/0 to $p/q$. See [HT, Figure 5] and [De1]. $D(p/q)$ can be constructed as follows. Let $p/q = [a_1, \ldots, a_k]$ be a continued fraction expansion of $p/q$. To each $a_i$ is associated a “fan” $F_{a_i}$ consisting of $a_i$ simplices, see Figure 2(a) and 2(b) for the fans $F_4$ and $F_{-4}$. The edges labeled $e_1$ are called initial edges, and the ones labeled $e_2$ are called terminal edges. The diagram $D(p/q)$ can be constructed by gluing the $F_{a_i}$ together in such a way that the terminal edge of $F_{a_i}$ is glued to the initial edge of $F_{a_{i+1}}$. Moreover, if $a_ia_{i+1} < 0$ then $F_{a_i}$ and $F_{a_{i+1}}$ have one edge in common, and if $a_ia_{i+1} > 0$ then they have a 2-simplex in common. See Figure 2(c) for the diagram of $[2, -2, -4, 2]$. Notice that the fans $F_{-2}$ and $F_{-4}$ in the figure share a common triangle.
To each vertex $v_i$ of $D(p/q)$ is associated a rational number $r_i/s_i$. It has one of the three possible parities: odd/odd, odd/even, or even/odd, denoted by $o/o$, $o/e$, and $e/o$, respectively. Note that the three vertices of any simplex in $D(p/q)$ have mutually different parities.

We consider $D(p/q)$ as a graph on a disk $D$, with all vertices on $\partial D$, containing $\partial D$ as a subgraph. The boundary of $D$ forms two paths from the vertex $1/0$ to the vertex $p/q$. The one containing the vertex $0/1$ is called the top path, and the one containing the vertex $1/1$ is called the bottom path. Edges on the top path are called top edges. Similarly for bottom edges.

Let $\Delta_1, \Delta_2$ be two simplices in $D(p/q)$ with an edge in common. Assume that the two vertices which are not on the common edge are of parity $o/o$. Then the arcs indicated in Figure 3(a) and (b) are called channels. A path $\alpha$ in $D(p/q)$ is a union of arcs, each of which is either an edge of $D(p/q)$ or a channel.

Let $v$ be a vertex on a path $\alpha$ in $D(p/q)$. Let $e_1, e_2$ be the edges of $\alpha$ incident to $v$. Then the corner number of $v$ in $\alpha$, denoted by $c(v; \alpha)$ or simply $c(v)$, is defined as the number of simplices in $D(p/q)$ between the edges $e_1$ and $e_2$. A path $\alpha$ from $1/0$ to $p/q$ is an allowable path if it has at least one channel, and $c(v) \geq 2$ for all $v$ in $\alpha$.

Now assume that $K = K_{p/q}$ is a 2-bridge knot. Then $q$ is an odd number. Recall that $K_{p/q} = K_{p'/q}$ if $p' \equiv p^{\pm 1} \mod q$, and $K_{-p/q}$ is the mirror image of $K_{p/q}$. We may assume without loss of generality that $p$ is even, and $1 < p < q$. This is because $K_{(q-p)/q}$ is equivalent to the mirror image of $K_{p/q}$, so the result of $\gamma$ surgery on the first is the same as that of $-\gamma$ surgery on the second. Note that $q - p$ and $p$ have different parity, since $q$ is odd. The following result is due to Delman. See [De1] and [De2, Proposition 3.1].
Theorem 3.1 (Delman). Given an allowable path $\alpha$ of $D(p/q)$, there is an essential branched surface $F$ in $S^3 - K$ which remains essential after all nontrivial surgeries on the knot $K$. \hfill \qed

Lemma 3.2. If there is an allowable path $\alpha$ in $D(p/q)$ such that $c(v) > 2$ for some vertex $v$ in $\alpha$, then $K(\gamma)$ is not a small Seifert fibered space for any $\gamma$. \hfill \qed

Proof. It was shown in [Br, Corollary 4] that if $F$ is an essential branched surface in a small Seifert fibered space $M$, then each component of $M - \text{Int} N(F)$ is an $I$-bundle over a compact surface $G$, such that the vertical surface $\partial v N(F)$ (also called cusps) is the $I$-bundle over $\partial G$. It has been shown in [De1] that for each vertex $v$ of $\alpha$ there is a component $W_v$ of $S^3 - \text{Int} N(F)$ such that $W_v$ is a solid torus whose meridian disk intersects the cusps $c(v)$ times. In particular, if $c(v) > 2$ then $W_v$ is not an $I$-bundle as above. Since $F$ is an essential branched surface in $K(\gamma)$, it follows that $K(\gamma)$ is not a small Seifert fibered space. \hfill \qed

Lemma 3.3. Suppose $p$ is even, $q$ is odd, and $1 < p < q - 1$. If $p/q$ does not have partial fraction decomposition of type $[r_1, r_2]$, then $D(p/q)$ has an allowable path $\alpha$ such that some vertex $v$ on $\alpha$ has $c(v) > 2$. \hfill \qed

Proof. Let $[a_1, \ldots, a_n]$ be the partial fraction decomposition of $p/q$ such that all $a_i$ are even. Then $a_1 \geq 2$. If $a_i = 2$ for all $i$, then $p/q = (q - 1)/q$, contradicting our assumption. Thus either some $a_i < 0$, or some $a_i > 4$. We separate the two cases.

CASE 1. Some $a_i < 0$.

Let $a_i$ be the first negative number. Then $a_{i-1} > 0$, so there is a sign change. By [De2] there is a channel $\alpha_0$ in $F_{a_{i-1}} \cup F_{a_i}$ starting at a bottom edge and ending at a top edge, where $F_{a_i}$ is the fan in $D(p/q)$ corresponding to $a_i$. Let $\alpha_1$ be the part of the bottom path of $D(p/q)$ from the vertex $1/0$ to the initial point of $\alpha_0$, and let $\alpha_2$ be the part of the top path from the end point of $\alpha_0$ to the vertex $p/q$. Then $\alpha = \alpha_1 \cup \alpha_0 \cup \alpha_2$ is an allowable path in $D(p/q)$. We need to show that if $c(v) = 2$ for all vertices $v$ on this path, then $p/q = [r_1, r_2]$ for some $r_1, r_2$.

Consider the vertices on $\alpha_1$. Since $c(v_i) = 2$ for all $v_i$, each vertex $v_i$ is incident to exactly one non boundary edge $e_i$ of $D(p/q)$, which must have the other end on a vertex $v_i'$ in the top path. If some of these $v_i'$ are different,
then since all faces of $D(p/q)$ are triangles, it is clear that some $v_j$ on $\alpha_1$ would have at least two non boundary edges, which would be a contradiction. Similarly, each vertex on $\alpha_2$ has a unique non boundary edge, leading to a common vertex on the bottom path, so the diagram $D(p/q)$ looks exactly as in Figure 4(a). It is the union of two fans $F_{r_1}$ and $F_{r_2}$ with $r_1 > 0$, and $r_2 < 0$. Therefore, $p/q = [r_1, r_2]$.

**CASE 2. Some $a_i \geq 4$.**

In this case there is a channel $\alpha_0$ with both ends on the bottom path. Construct an allowable path $\alpha = \alpha_1 \cup \alpha_0 \cup \alpha_2$ with $\alpha_1, \alpha_2$ in the bottom path. Similar to Case 1, it can be shown that each vertex on $\alpha_i$ has a unique non boundary edge leading to a common vertex $v_i'$ on the top path, so $D(p/q)$ looks like that in Figure 4(b). In this case $p/q = [r_1, r_2]$, with both $r_i > 0$.

**Corollary 3.4.** Let $K$ be a 2-bridge knot. If $K \neq K_{[b_1, b_2]}$ for any $b_1, b_2$, then $K(\gamma)$ is non-exceptional for all $\gamma$.

**Proof.** $K$ is not a $(2, q)$ torus knot, otherwise $K = K_{[2q, \pm 1]} = K_{[2q, \mp 1]}$. Hence by Lemmas 2.1 and 2.2, $K(\gamma)$ is irreducible and atoroidal. By Lemmas 3.3 and 3.2, $K(\gamma)$ is not a small Seifert fiber space. Therefore, $K(\gamma)$ is non-exceptional. □
4. Surgery on twisted Whitehead links, proof of Theorem 1.1(2).

A twisted Whitehead link is a two bridge link $L$ associated to a rational number $[2, r, -2]$ for some $r \neq 0$. See Figure 5 for a twisted Whitehead link with $r = -6$. When $r = \pm 2$, $L$ is a Whitehead link. It has been determined exactly which 2-bridge link complements contain persistent laminations [Wu]. The next lemma shows that if $|r| > 2$ then there is a persistent lamination with some desired property.

Recall that a slope $\gamma$ of a knot $K$ is an integral slope if it intersects the meridian of $K$ exactly once.

**Lemma 4.1.** Let $L = k_1 \cup k_2$ be a twisted Whitehead link associated to the rational number $p/q = [2, r, -2]$. Let $L(\gamma_1, \gamma_2)$ be the manifold obtained by $\gamma_i$ surgery on $k_i$. If $|r| > 2$, and one of the $\gamma_i$ is not an integral slope, then $L(\gamma_1, \gamma_2)$ is not a small Seifert fiber space.

**Proof.** By considering the mirror image of $L$ if necessary we may assume that $r < 0$. If $r$ is even, then $[2, r, -2]$ is a partial fraction decomposition with even coefficients, and $r \leq -4$. There is an allowable path in $D(p/q)$ with two channels, as shown in Figure 6(a), where $r = -4$. If $r$ is odd, then $p/q = [2, r + 1, 2]$, in which case $D(p/q)$ also has an allowable path with two channels. See Figure 6(b) for the case $r = -3$.

Let $\mathcal{F}$ be the essential branched surface in the link exterior associated to the above allowable path in $D(p/q)$, as constructed in [De2]. There is one solid torus component $V_i$ in $S^3 - \text{Int} N(\mathcal{F})$ for each $k_i$, containing $k_i$ as a central curve. From the construction of $\mathcal{F}$ one can see that each channel contributes two cusps, one on each $\partial V_i$. Actually from [De2, Figure 3.5] we see that the two cusps corresponding to a channel are around two points of $L$ on a level sphere with same orientation. Since each $k_i$ intersects the
sphere at two points with different orientations, those two cusps must be around different components of $L$. One is referred to [Wu] for more details about surgery on 2-bridge links.

As the allowable path above has two channels, each $V_i$ has two meridional cusps. Thus $F$ remains an essential branched surface after surgery on $L$. Moreover, since one of the $\gamma_i$ is non-integral, after surgery $V_i$ becomes a solid torus whose meridional disk intersects the cusps at least four times. By [Br, Corollary 4], the surgered manifold is not a small Seifert fiber space. □

**Corollary 4.2.** Let $K = K_{[b_1,b_2]}$ be a two bridge knot with $|b_i| > 2$ for $i = 1, 2$. Then $K(\gamma)$ is exceptional for only one $\gamma$, which yields toroidal manifold. When both $b_1$ and $b_2$ are even, $\gamma = 0$. If $b_1$ is odd and $b_2$ is even, $\gamma = 2b_2$.

**Proof.** By Lemmas 2.1 and 2.2, $K(\gamma)$ is irreducible, and it is toroidal for exactly one $\gamma$ as described in the corollary. So it remains to show that $K(\gamma)$ is never a small Seifert fiber space.

Since $K$ is a knot, at least one of the $b_i$ is an even number. We may assume without loss of generality that $b_1 = 2n$ for some integer $n$, because $K_{[b_1,b_2]}$ is equivalent to $K_{[b_2,b_1]}$, by turning the standard diagram for the first knot upside down.

Let $L = k_1 \cup k_2$ be a 2-bridge link associated to the rational number $p/q = [2, b_2, -2]$. Notice that after $-1/n$ surgery on $k_1$, the other component $k_2$ becomes the knot $K = K_{[2n,b_2]}$. Therefore, doing $\gamma$ surgery on $K$ is the same as doing $-1/n$ surgery on $k_1$, then doing some $\gamma'$ surgery on $k_2$. Since
$-1/n$ is non integral, and $|b_2| > 2$, the result follows from Lemma 4.1. □

**Corollary 4.3.** Let $K = K_{[b, \pm 2]}$ with $|b| > 2$. If $\gamma$ is a non integral slope, then $K(\gamma)$ is not a small Seifert fiber space.

**Proof.** As above, $K(\gamma) = L(\gamma, \pm 1)$, where $L = L_{[2, b, -2]}$. Since $\gamma$ is non integral, the result follows from Lemma 4.1. The result also follows from [Br]. □

## 5. Roberts' construction of essential branched surfaces.

In [Ro] Roberts constructed branched surfaces in certain knot complements, which can be extended to essential branched surfaces in $K(\gamma)$ for all $\gamma$ in an infinite interval. We will describe her results and construction in this section, and apply them to surgery on twist knots in the next section.

Let $E(K) = S^3 - \text{Int} N(K)$ be the exterior of a knot $K$ in $S^3$, let $R'$ be a (possibly non orientable) compact surface in $S^3$ with $\partial R' = K$. Let $R = R' \cap E(K)$.

Let $S$ be a surface in $E(K)$ which has interior disjoint from $R$, and has a single boundary curve $\partial S = a_1 \cup b_1 \cup \ldots \cup a_n \cup b_n$, where $b_i$ are mutually disjoint arcs on $R$, and $a_i$ are arcs on $T = \partial N(K)$. By specifying a cusp at each $b_i$, the union of $R$ and $S$ becomes a branched surface $B = \langle R, S \rangle$ in $E(K)$. The cusps will be assigned in such a way that each $a_i$ on $T$ is one of the four types indicated in Figure 7. Note that $\partial B$ is a train track on $T$, and each component of $T - \partial B$ is a digon, i.e a disk with two cusps.
Remark. The pictures on Figure 7 are mirror images of that in [Ro, Figure 22]. Thus for example, type $P_1$ here is of type $N_1$ in [Ro]. Apparently we are using different coordinate systems. This paper adopts the convention that the meridian-longitude pair $(m, l)$ on $T$ is chosen so that when $K$ is endowed with the same orientation as that of $l$, the linking number $lk(m, K) = 1$, measured using the right hand rule. See [R]. With this convention, types $P_1, P_2$ in Figure 7 will have positive contributions to any slope $\gamma$ carried in the train track.

Let $N(B)$ be a regular neighborhood of the branched surface $B$ in $E(K)$ with the natural $I$-bundle structure. A surface of contact is a properly embedded compact surface $P$ in $N(B)$, transverse to the $I$-fibers, with $\partial P \subset \partial_v N(B) \cup T$, such that the intersection of $\partial P$ with each component of the vertical surfaces $\partial_v N(B)$ is either empty or a single arc.

Let $p_1, p_2, n_1, n_2$ be the numbers of $a_i$ of type $P_1, P_2, N_1, N_2$ respectively. Let $r$ be the slope of $\partial R$ on $T$. Let

$$J = \{ r + (p_1 - n_1) \frac{x}{x+1} + (p_2 - n_2)x \mid x > 0 \}$$

Then Roberts' theorem [Ro, Theorem 2.3] can be stated as

**Theorem 5.1 (Roberts).** If $B = \langle R, S \rangle$ constructed above is an essential branched surface in $E(K)$, and has no planar surface of contact, then $B$ extends to an essential branched surface $B_\gamma$ in $K(\gamma)$ for all slope $\gamma \in J$.

The construction of the extended branched surface is as follows. Let $T \times I$ be a small neighborhood of $T$ in $E(K)$ with $T = T \times 0$, such that $B \cap (T \times I) = \partial B \times I$. Add the digons $T \times 1 - B$ to $B$, and branched so that the cusps on the two edges of each digon lie on different sides. The definition of $J$ guarantees that the train track $\partial B$ on $T$ can be split to produce a curve $C$ of slope $\gamma$ on $T$. Split $\partial B \times I$ accordingly and, after Dehn filling, cap off $C$ by a meridian disk in the Dehn filling solid torus, so that one obtains a branched surface $B_\gamma$ in $K(\gamma)$. It was shown in [Ro] that $B'$ carries an essential lamination in $K(\gamma)$.

Denote by $E(B)$ the exterior of $B$ in $E(K)$, i.e $E(B) = E(K) - \text{Int } N(B)$.

**Corollary 5.2.** For any $\gamma \in J$, the manifold $E(B)$ is homeomorphic to a component $W$ of the exterior of $B_\gamma$ in $K(\gamma)$, with horizontal surface of $B$ identified to the horizontal surface of $B_\gamma$ on $\partial W$. 
Proof. Examine the above construction. After adding the digons of $T \times 1 - B$ to $B$, the branched surface is topologically homeomorphic to $B \cup (T \times 1)$, which cuts off a region isotopic to $E(B)$. Clearly this region is not affected by the later changes, and its horizontal surface is the restriction of that on $E(B)$.


As noticed earlier, any twist knot can be written as $K_{[2n, \pm 2]}$ for some $n$, because if $b$ is odd then $K_{[b, 2]} = K_{[b-1, -2]}$. Since $K_{[2n, -2]}$ is the mirror image of $K_{[-2n, 2]}$, we need only consider knots of type $K_{[2n, 2]}$. We can also assume that $n \neq 0, 1$, otherwise the knot is a trivial knot or a trefoil knot.

Lemma 6.1. Let $K = K_{[2n, 2]}$ be a twist knot with $n \neq 0, 1$. Then $K(\gamma)$ is not a small Seifert fiber space for all $\gamma < -4$.

Proof. A knot $K = K_{[2n, 2]}$ has two spanning surfaces as indicated in Figure 8, where $2n = 4$. The first surface is a punctured Klein bottle, and the second one is a punctured torus.

Let $R$ be the punctured Klein bottle of Figure 8(1) in the knot exterior. Add a disk $S$ to $R$ such that the boundary of $S$ is the circle indicated in Figure 8(1). The boundary of $S$ consists of four arcs $a_1 \cup b_1 \cup a_2 \cup b_2$, where $b_i$ lies on $R$, and $a_i$ on the torus $T = \partial N(K)$. The arrows at the arcs $b_i$ indicate the side of the cusp. This determines the branched surface $B = \langle R, S \rangle$, and hence the type of $a_i$ on $T$. One of the $a_i$ (the one on top) is of type $P_1$, and the other one of type $N_2$. Using the notation in Section 5, we have $p_1 = n_2 = 1$, and $p_2 = n_1 = 0$. By calculating the linking number between
$K$ and $\partial R$, one can see that $\partial R$ has slope $r = -4$ on $T$. Therefore,

$$J = \{ r + (p_1 - n_1) \frac{x}{x + 1} + (p_2 - n_2)x \mid x > 0 \} = \{ -4 + \frac{x}{x + 1} - x \mid x > 0 \}$$

$$= \{ -4 - \frac{x^2}{x + 1} \mid x > 0 \} = (-\infty, -4)$$

We need to show that $B$ is an essential branched surface in $E(K)$. It is clear that $N(B)$ is topologically a solid torus. By examining the cusp on $\partial N(B)$, one can see that the exterior of $B$ is the same as that of the surface in Figure 9, with cusp the boundary of the surface $F$. Note that $\partial F$ runs along the meridian of the solid torus $N(F)$ 2$n$ - 1 times. Hence $E(B)$, the exterior of $B$, is a solid torus with a single cusp running along the longitude 2$n$ - 1 times. Since $n \neq 0, 1$, it is easy to see that $E(B)$ satisfies all conditions for $B$ to be essential, i.e, $E(B)$ is indecomposable, and the horizontal surface on $\partial E(B)$ is essential. One also needs to check that the branched surface $B$ satisfies all the intrinsic essentiality properties, i.e, it has no disk or half disk of contact, it contains no Reeb branched surfaces, and it fully carries a lamination. This is straightforward.

To apply Roberts' theorem, we also need to show that $B = (R, S)$ has no planar surface of contact. Assume that $P$ is a compact surface embedded in $N(B)$, transverse to the $I$-fibers, with $\partial P \subset \partial_v N(B) \cup T$. Cutting along the two arcs $b_1, b_2$ along which $S$ is glued to $R$, the branched surface $B$ becomes two surfaces $S$ and $R' = R - b_1 \cup b_2$. Let $u, v$ be the number of times that $P$ intersects an $I$-fiber of $S$ and $R'$ respectively. Then the gluing at $b_i$ gives the equation $v = u + v + w$, where $w$ is the number of times $\partial P$ passes the cusp at $b_i$. So $u = w = 0$, and $P$ lies in $N(R)$. Since $R$ is nonplanar, $P$ can not be planar.

It now follows from Theorem 5.1 that $B$ extends to an essential branched surface $B_\gamma$ for all slopes $\gamma < -4$. By Corollary 5.2, $E(B)$ can be considered
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as a component of the exterior of $B_\gamma$. Since $E(B)$ is a solid torus with a
cusp running along the longitude $|2n - 1| \geq 3$ times, it is not an $I$-bundle.
Therefore, by the result of Brittenham [Br], $K(\gamma)$ is not a small Seifert fiber
space for all $\gamma < -4$.

□

Lemma 6.2. Let $K = K_{[2n,2]}$ be a twist knot with $|n| > 2$. Then $K(\gamma)$ is
not a small Seifert fiber space for all $\gamma > 0$.

Proof. The proof is very similar to that of Lemma 6.1, only instead of using
the non orientable surface in Figure 8(1), we use the orientable surface $R$
in Figure 8(2). As a Seifert surface of $K$, the boundary of $R$ has slope 0. Also,
the arcs $a_1, a_2$ are now of type $P_2$ and $N_1$, so $p_1 = n_2 = 0$, $p_2 = n_1 = 1$, and

$$J = \left\{ x + \left( p_1 - n_1 \right) \frac{x}{x + 1} + \left( p_2 - n_2 \right) x \mid x > 0 \right\} = \left\{ 0 - \frac{x}{x + 1} + x \mid x > 0 \right\}$$

$$= \left\{ \frac{2}{x + 1} \mid x > 0 \right\} = (0, \infty)$$

The exterior of the branched surface $B = \langle R, S \rangle$ is the same as that of a
band with $2n$ twists. Since $|n| > 2$, it is not an $I$-bundle. Therefore, one
can use the argument in the proof of Lemma 6.1 to obtain the conclusion.

□

Proof of Theorem 1.1. Parts (1) and (2) are exactly Corollaries 3.4 and 4.2.
For part (3), consider the knot $K_{[2n,2]}$. By Lemma 2.2 and Corollary 2.4,
surgeries with $\gamma = 0, 4$ are the only toroidal ones, and $\gamma = -1, -2, -3$ are
Seifert fibered. By Corollary 4.3, Lemmas 6.1 and 6.2, there are no other
small Seifert fibered surgeries. Since no surgeries on $K_{[2n,2]}$ are reducible
(Lemma 2.1), all surgeries with $\gamma \neq 0, -1, -2, -3, -4$ are non-exceptional.
Since $K_{[2n,-2]}$ is the mirror image of $K_{[-2n,2]}$, the result follows.

Part (4) follows from the well known result of Thurston about surgery
on the Figure 8 knot [Th], which is stronger as the non-exceptional surgeries
are shown to be hyperbolic. The result as stated can also be proved using
the techniques in this paper: As toroidal and reducible surgeries are known,
we only need to deal with small Seifert fibered surgeries. By [Br] all non-
integral surgeries on $K = K_{[-2,2]}$ are not small Seifert fibered. By Lemma
6.1 $K(\gamma)$ are not small Seifert fibered for $\gamma < -4$. Since $K$ is amphicheiral,
this is also true for $\gamma > 4$.

□
Remark 6.3. Boileau and Porti [BP] proved a version of Thurston’s Orbifold Theorem, showing that the geometrization conjecture is true for manifolds which admit a finite group action whose fixed point set is a non-empty 1-manifold. Using this result one can show that non-exceptional surgeries on 2-bridge knots are also hyperbolic. The proof is as follows. A \( p/q \) 2-bridge knot can be obtained by taking two arcs of slope \( p/q \) on the “pillowcase”, then joining the ends with two trivial arcs. From this picture it is easy to see that \( K \) is a strongly invertible knot, i.e. there is an involution \( \varphi \) of \( S^3 \) such that \( \varphi(K) = K \), and the fixed point set of \( \varphi \) is a circle \( S \) intersecting \( K \) at two points. The map \( \varphi \) restricts to an involution of \( E(K) = S^3 - \text{Int} \, N(K) \), which can be extended to an involution \( \bar{\varphi} \) of the surgered manifold. Since \( \bar{\varphi} \) has fixed point set a nonempty 1-manifold, the result follows from [BP].

References.


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