Convergence of the J-flow on Kähler Surfaces

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Donaldson defined a parabolic flow of potentials on Kähler manifolds which arises from considering the action of a group of symplectomorphisms on the space of smooth maps between manifolds. One can define a moment map for this action, and then consider the gradient flow of the square of its norm. Chen discovered the same flow from a different viewpoint and called it the J-flow, since it corresponds to the gradient flow of his J-functional, which is related to Mabuchi’s K-energy. In this paper, we show that in the case of Kähler surfaces with two Kähler forms satisfying a certain inequality, the J-flow converges to a zero of the moment map.

1. Introduction.

In [Do], Donaldson described how a number of geometric situations fit into a general framework of diffeomorphism groups and moment maps. In the Kähler setting, he used this framework to define a natural parabolic flow, as follows. Suppose that \((M, \omega)\) is a compact Kähler manifold of dimension \(n\) and let \(\chi_0\) be another Kähler form on \(M\), in a different Kähler class. Consider the infinite-dimensional manifold \(\mathcal{M}\) of diffeomorphisms \(f : M \to M\), homotopic to the identity. \(\mathcal{M}\) carries a natural symplectic form \(\Omega\) defined by

\[
\Omega_f(v, w) = \int_M \omega(v, w) \frac{\chi_0^n}{n!},
\]

for sections \(v, w\) of \(f^*(TM)\). The group \(\mathcal{G}\) of exact \(\chi_0\)-symplectomorphisms of \(M\) acts on \(\mathcal{M}\) by composition on the right, preserving \(\Omega\). We can identify the Lie algebra of \(\mathcal{G}\) with the space of functions on \(M\) of integral zero with respect to the volume form induced by \(\chi_0\). A moment map \(\mu : \mathcal{M} \to \text{Lie}(\mathcal{G})^*\) for the group action is given by

\[
\mu(f) = \frac{f^*(\omega) \wedge \chi_0^{n-1}}{\chi_0^n} - \frac{\int_M \omega \wedge \chi_0^{n-1}}{\int_M \chi_0^n},
\]

where we are using the \(L^2\) inner product to identify \(\text{Lie}(\mathcal{G})\) with its dual. It is natural to look for solutions of

\[
\mu(f) = 0 \quad (\text{mod} \ \mathcal{G}).
\]
These points form the symplectic quotient. Under certain conditions, one would hope that the gradient flow $f_t$ of the function $\|\mu\|^2$ on $\mathcal{M}$ would converge to give a solution of (1.1). The gradient flow can be rewritten as a flow of Kähler forms $(f_t^*\chi_0)^{-1}$ on $M$. This defines a parabolic flow on the space of Kähler potentials and is the object of study of this paper.

At around the same time, Chen [C1] independently discovered the same flow as the gradient flow of his $J$-functional. He later called it the $J$-flow [C2]. He showed in [C1] that the $J$-functional is related to the Mabuchi K-energy [Ma], which plays a key role in the study of Kähler geometry and stability in the sense of geometric invariant theory (see [Y2], [T2], [T3] and [PS] for example).

Explicitly, the $J$-flow is defined as follows. Let $c$ be the constant given by

$$c = \frac{\int_M \omega \wedge \chi_0^{n-1}}{\int_M \chi_0^n},$$

and let $\mathcal{H}$ be the space of Kähler potentials

$$\mathcal{H} = \{ \phi \in C^\infty(M) \mid \chi_\phi = \chi_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \phi > 0 \}.$$

The $J$-flow is the flow on $\mathcal{H}$ given by

$$\frac{\partial \phi_t}{\partial t} = c - \frac{\omega \wedge \chi_\phi^{n-1}}{\chi_\phi^n},$$

$$\phi_0 = 0.$$  \hspace{1cm} (1.2)

A critical point of the $J$-flow gives a Kähler metric $\chi$ satisfying

$$\omega \wedge \chi^{n-1} = c\chi^n.$$  \hspace{1cm} (1.3)

Donaldson [Do] asked whether one can find a solution to (1.3) in the class $[\chi_0]$ under certain assumptions. He noted that a necessary condition is that $[nc\chi_0 - \omega]$ be a Kähler class, and conjectured that this condition be sufficient. Chen [C1] confirmed this conjecture in the case $n = 2$, without using the $J$-flow, by observing that (1.3) reduces to a Monge-Ampère equation which can be solved by the well-known result of Yau [Y1]. The conjecture is still open for $n > 2$.

Chen [C1] shows that Donaldson’s conjecture would imply a result on the lower bound of the Mabuchi K-energy for compact Kähler manifolds $M$ with negative first Chern class. Namely, if $-\omega \in c_1(M)$ with $\omega > 0$, then for Kähler classes $[\chi_0]$ satisfying

$$nc[\chi_0] - [\omega] > 0,$$
the Mabuchi K-energy would have a lower bound in the class $[\chi_0]$. Solutions of the $J$-flow exist for a short time by general theory, since the flow is parabolic. In [C2], Chen showed that the flow always exists for all time for any smooth initial data. He also showed that if the bisectional curvature of $\omega$ is non-negative then the $J$-flow converges to a critical metric.

In general, the behaviour of the flow is not known. In this paper, we deal with the case $n = 2$ with no curvature restrictions. Our main result is as follows.

**Main Theorem** Suppose that $(M, \omega)$ has dimension $n = 2$ and that 
\[ nc\chi_0 - \omega > 0. \]
Then the $J$-flow (1.2) converges in $C^\infty$ to a smooth critical metric.

The outline of the paper is as follows. In section 2 we state some preliminary facts about the flow and introduce notation. In section 3, the maximum principle is used to derive an estimate on the second derivatives of $\phi$ in terms of $\phi$ itself. In section 4, a $C^0$ estimate for $\phi$ is given. The argument uses the second order estimate, a Moser iteration argument applied to the exponential of $-\phi$ and the result of Tian [T1] (see also [TY]) on the existence of constants $\alpha > 0$ and $C$ such that 
\[ \int_M e^{-\alpha \phi} \chi_0^n \leq C, \]
for all $\phi$ in $\mathcal{H}$ with $\sup_M \phi = 0$. In section 5, the proof of the main theorem is completed.

### 2. Preliminaries and notation.

From now on, assume that $\omega$ has been scaled so that $c = 1/n$. We will work in local coordinates, and write 
\[ \omega = \frac{\sqrt{-1}}{2} g_{\overline{z}^i z^j} dz^i \wedge d\overline{z}^j, \quad \chi_0 = \frac{\sqrt{-1}}{2} \chi_{0 \overline{z}^i z^j} dz^i \wedge d\overline{z}^j, \]
and 
\[ \chi = \frac{\sqrt{-1}}{2} \chi_{\overline{z}^i z^j} dz^i \wedge d\overline{z}^j = \frac{\sqrt{-1}}{2} (\chi_{0 \overline{z}^i z^j} + \partial_i \overline{\partial_j} \phi) dz^i \wedge d\overline{z}^j, \]
where $\chi = \chi_\phi$ (suppressing the $t$-subscript.) The operators $\Lambda_\omega$ and $\Lambda_\chi$ act on $(1, 1)$ forms $\alpha = \frac{\sqrt{-1}}{2} \alpha_{\overline{z}^i z^j} dz^i \wedge d\overline{z}^j$ by 
\[ \Lambda_\omega \alpha = g_{\overline{z}^i z^j} \alpha_{\overline{z}^i z^j}, \quad \text{and} \quad \Lambda_\chi \alpha = \chi_{\overline{z}^i z^j} \alpha_{\overline{z}^i z^j}. \]
The $J$-flow (1.2) can be written
\[
\frac{\partial \phi}{\partial t} = \frac{1}{n}(1 - \Lambda \chi \omega)
\]
\[
\phi|_{t=0} = 0.
\]  
(2.1)

Differentiating with respect to $t$ gives
\[
\frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial t} \right) = \tilde{\triangle} \left( \frac{\partial \phi}{\partial t} \right),
\]  
(2.2)

where the operator $\tilde{\triangle}$ acts on functions $f$ by
\[
\tilde{\triangle} f = \frac{1}{n} \chi^k_j \chi^l_i g_{ij} \partial_k \partial_l f.
\]

For convenience, write
\[
h^{kl} = \chi^k_j \chi^l_i g_{ij}.
\]

The tensor $h^{kl}$ is positive definite and its inverse defines a Hermitian metric on $M$. The operator $\tilde{\triangle}$ is, up to a constant factor, the Laplacian associated to this Hermitian metric.

By the maximum principle for parabolic equations, (2.2) implies that
\[
\inf_M (\Lambda \chi_0 \omega) \leq \Lambda \chi \omega \leq \sup_M (\Lambda \chi_0 \omega),
\]  
(2.3)

which gives a lower bound for $\chi$,
\[
\chi \geq \frac{1}{\sup_M (\Lambda \chi_0 \omega)} \omega.
\]  
(2.4)

The $J$-functional [C1] is defined by
\[
J_{\omega, \chi_0}(\phi) = \int_0^1 \int_M \frac{\partial \phi_t}{\partial t} \frac{\omega \wedge \chi_0^{n-1}}{(n-1)!} dt,
\]

where $\{\phi_t\}$ is a path in $\mathcal{H}$ between 0 and $\phi$. The functional is independent of the choice of path. We will need the following formula for the functional in the case $n = 2$. Taking the path $\phi_t = t\phi$, we see that
\[
J_{\omega, \chi_0}(\phi) = \frac{1}{2} \int_M \phi \omega \wedge (\chi_0 + \chi).
\]  
(2.5)
Chen also makes use of the $I$-functional,

$$I_{\omega,\chi_0}(\phi) = \int_0^1 \int_M \frac{\partial \phi_t}{\partial t} \frac{\chi_0^n}{n!} dt.$$  

This is a well-known functional in Kähler geometry (see [Ma]). Notice that $I(\phi) = 0$ along the flow. For $n = 2$, this functional is given by

$$I_{\omega,\chi_0}(\phi) = \frac{1}{6} \int_M \phi (\chi_0^2 + \chi \wedge \chi_0 + \chi^2). \quad (2.6)$$

In the course of the paper, $C_0, C_1, \ldots$ will denote constants depending only on the initial data $\omega$ and $\chi_0$. Curvature expressions such as $R_{i\overline{j}k}$ will always refer to the metric $g_{i\overline{j}}$.

### 3. Second order estimate.

We use the maximum principle to obtain an estimate on the second derivative of $\phi$ in terms of $\phi$. We choose to calculate the evolution of $(\log \Lambda_{\omega \chi} - A\phi)$ for some constant $A$ (compare to [Y1], [Au] or [Si] for the analogous estimate for the well-known Monge-Ampère equation, and [Ca] for the Kähler-Ricci flow.)

**Theorem 3.1.** Suppose that $(M, \omega)$ has dimension $n = 2$ and that

$$\chi_0 - \omega > 0. \quad (3.1)$$

Let $\phi = \phi_t$ be a solution of the $J$-flow (2.1) on $[0, \infty)$. Then there exist constants $A > 0$ and $C > 0$ depending only on the initial data such that for any time $t \geq 0$, $\chi = \chi_{\phi_t}$ satisfies

$$\Lambda_{\omega \chi} \leq C e^{A (\phi - \inf_{M \times [0,t]} \phi)}. \quad (3.2)$$

**Proof.** We will calculate

$$(\tilde{\Delta} - \frac{\partial}{\partial t}) (\log (\Lambda_{\omega \chi}) - A\phi).$$

Using normal coordinates for $\omega$, first calculate

$$\tilde{\Delta} (\Lambda_{\omega \chi}) = \frac{1}{n} h^{k\overline{l}} \partial_k \partial_{\overline{l}} (g^{\overline{j}} \chi_{ij})$$

$$= \frac{1}{n} h^{k\overline{l}} R_{k\overline{l}j} \chi_{ij} + \frac{1}{n} h^{k\overline{l}} g^{\overline{j}} \partial_k \partial_{\overline{l}} \chi_{ij}.$$
And
\[
\frac{\partial}{\partial t}(\Lambda \omega \chi) = \frac{\partial}{\partial t}(g^{ij} \partial_i \partial_j \phi) \\
= -\frac{1}{n} g^{ij} \partial_i \partial_j (\chi^k g_{k\ell}) \\
= \frac{1}{n} (g^{ij} \partial_i (\chi^\ell \partial_j \chi_{\ell \pi} \chi^{\pi}) g_{k\ell} + g^{ij} \chi^k R_{ij\ell}) \\
= \frac{1}{n} (g^{ij} h^{pi} \partial_i \partial_j \chi_{\ell \pi} - g^{ij} h^{pi} \chi^r \partial_i \chi_{\ell \pi} \partial_j \chi_{r \pi} \\
- g^{ij} h^{pi} \chi^r \partial_i \chi_{r \pi} \partial_j \chi_{\ell \pi} + \chi^k R_{k\ell}).
\]

Now
\[
\Delta \log(\Lambda \omega \chi) = \frac{\hat{\Delta}(\Lambda \omega \chi)}{\Lambda \omega \chi} - \frac{|\hat{\nabla}(\Lambda \omega \chi)|^2}{(\Lambda \omega \chi)^2},
\]
where
\[
|\hat{\nabla}(\Lambda \omega \chi)|^2 = \frac{1}{n} h^{ij} \partial_i (\Lambda \omega \chi) \partial_j (\Lambda \omega \chi).
\]

Note that by the Kähler property of \( \chi \), we have
\[
\partial_i \partial_j \chi_{\ell \ell} = \partial_i \partial_j \chi_{\ell \ell}.
\]

Then
\[
(\hat{\Delta} - \frac{\partial}{\partial t}) \log(\Lambda \omega \chi) \\
= \frac{1}{n \Lambda \omega \chi} (h^{ij} R_{ij\ell} \chi^{\ell \ell} - n |\hat{\nabla}(\Lambda \omega \chi)|^2 \Lambda \omega \chi) + g^{ij} h^{pi} \chi^r \partial_i \chi_{r \pi} \partial_j \chi_{\ell \pi} \\
+ g^{ij} h^{pi} \chi^r \partial_i \chi_{r \pi} \partial_j \chi_{\ell \pi} - \chi^k R_{k\ell}).
\]

We need the following lemma to deal with the second term on the right hand side.

**Lemma 3.2.**
\[
n |\hat{\nabla}(\Lambda \omega \chi)|^2 \leq (\Lambda \omega \chi) g^{ij} h^{pi} \chi^r \partial_i \chi_{r \pi} \partial_j \chi_{\ell \pi}.
\]
Proof. Using normal coordinates for \( \omega \) in which \( \chi \) is diagonal, and making use of the Cauchy-Schwartz inequality, we obtain

\[
n|\nabla (\Lambda \omega \chi)|^2 = \sum_{i,j,k} \chi^{ij} \chi^{k\ell} \partial_k \chi_{j\ell} \chi_{i\ell} \leq \sum_{i,j} \left( \sum_k (\chi^{ij})^2 |\partial_k \chi_{j\ell}|^2 \right)^{1/2} \left( \sum_k (\chi^{k\ell})^2 |\partial_k \chi_{i\ell}|^2 \right)^{1/2}
\]

\[
= \left( \sum_i \left( \sum_k (\chi^{ij})^2 |\partial_k \chi_{i\ell}|^2 \right)^{1/2} \right)^2
\]

\[
= \left( \sum_i \chi_{ii} \left( \sum_k (\chi^{ij})^2 |\partial_k \chi_{i\ell}|^2 \right)^{1/2} \right)^2
\]

\[
\leq \sum_i \chi_{ii} \sum_{i,k} (\chi^{ij})^2 |\partial_k \chi_{i\ell}|^2
\]

\[
= (\Lambda \omega \chi) \sum_{i,k} (\chi^{ij})^2 \chi_{i\ell} \partial_k \chi_{i\ell}
\]

\[
= (\Lambda \omega \chi) \sum_{i,k} (\chi^{ij})^2 \chi_{i\ell} \partial_k \chi_{i\ell}
\]

\[
\leq (\Lambda \omega \chi) \sum_{i,j,k} (\chi^{ij})^2 \partial_j \chi_{j\ell} \partial_k \chi_{i\ell}
\]

\[
= (\Lambda \omega \chi) g^{\ell \eta} h^{\nu \pi} \partial_\eta \chi_{\nu \pi} \partial_\ell \chi_{\eta \pi}.
\]

Let \( C_0 \) be a constant satisfying

\[
R_{\kappa \ell} \geq -C_0 g_{\kappa \eta} g^{\ell \eta}.
\]

Then,

\[
(\tilde{\Delta} - \frac{\partial}{\partial t}) \log(\Lambda \omega \chi) \geq \frac{1}{n \Lambda \omega \chi} (-C_0 h^{\ell \eta} g_{\kappa \eta} g^{\ell \eta} \chi_{\kappa \ell} - \chi^{k\ell} R_{k\ell})
\]

\[
= \frac{1}{n} (-C_0 h^{\ell \eta} g_{\kappa \eta} - \frac{1}{\Lambda \omega \chi} \chi^{k\ell} R_{k\ell}).
\]
Now calculate
\[
(\tilde{\Delta} - \frac{\partial}{\partial t})\phi = \frac{1}{n}(h^{kl} \partial_k \partial_l \phi + \chi^{ij} g_{ij} - 1)
= \frac{1}{n}(\chi^{ij} \chi^{kl} \chi_{kl} - h^{kl} \chi_{0kl} + \chi^{ij} g_{ij} - 1)
= \frac{1}{n}(2\chi^{ij} g_{ij} - h^{kl} \chi_{0kl} - 1).
\]

At this point we must choose our value of \( A \). From our assumption (3.1), we can choose \( 0 < \epsilon < 1/3 \) to be sufficiently small so that
\[
\chi_0 \geq (1 + 3\epsilon) \omega.
\]

Let \( A \) be given by
\[
A = \frac{C_0}{\epsilon}.
\]

Fix a time \( t > 0 \). There is a point \((x_0, t_0)\) in \( M \times [0, \tilde{t}] \) at which the maximum of \((\log(\Lambda_\omega \chi) - A \phi)\) is achieved. We may assume that \( t_0 > 0 \). At this point, we have
\[
0 \geq \frac{1}{n} \left( -C_0 h^{kl} g_{kl} - \frac{1}{\Lambda_\omega \chi} \chi^{kl} R_{kl} - 2A \chi^{ij} g_{ij} + Ah^{kl} \chi_{0kl} + A \right)
\geq \frac{1}{n} \left( -C_0 h^{kl} g_{kl} - \frac{1}{\Lambda_\omega \chi} \chi^{kl} R_{kl} - 2A \chi^{ij} g_{ij} + (1 - \epsilon) Ah^{kl} \chi_{0kl} \right)
+ \epsilon Ah^{kl} g_{kl} + A
= \frac{1}{n} \left( -\frac{1}{\Lambda_\omega \chi} \chi^{kl} R_{kl} - 2A \chi^{ij} g_{ij} + (1 - \epsilon) Ah^{kl} \chi_{0kl} + A \right).
\]

From the lower bound (2.4) on \( \chi_{kl} \), the term \( \chi^{kl} R_{kl} \) is bounded above and hence at \((x_0, t_0)\), we have
\[
1 + (1 - \epsilon) h^{kl} \chi_{0kl} - 2\chi^{ij} g_{ij} \leq \frac{C_1}{\Lambda_\omega \chi}.
\]

From (3.3), we get
\[
1 + (1 + \epsilon) h^{kl} g_{kl} - 2\chi^{ij} g_{ij} \leq \frac{C_1}{\Lambda_\omega \chi}.
\] (3.4)

We will compute in normal coordinates at \( x_0 \) for \( \omega \) in which \( \chi \) is diagonal and has eigenvalues \( \lambda_1, \lambda_2 \). From (2.4), \( \lambda_1 \) and \( \lambda_2 \) are bounded below by
a positive constant. We want to show that they are also bounded above. First, observe that for \( n = 2 \),

\[
\frac{1}{\Lambda \chi \omega} = \frac{\det \chi}{(\det \omega)(\Lambda \omega \chi)},
\]

and by (2.3), this is bounded along the flow.

Multiplying (3.4) by \((\det \chi / \det \omega)\) gives,

\[
\lambda_1 \lambda_2 + (1 + \epsilon) \left( \frac{\lambda_2}{\lambda_1} + \frac{\lambda_1}{\lambda_2} \right) - 2(\lambda_1 + \lambda_2) \leq C_2.
\]

From (2.3), we may suppose that one of the eigenvalues, say \( \lambda_2 \), is bounded from above. Rewrite the inequality as

\[
\lambda_1(\lambda_2 + (1 + \epsilon) \frac{1}{\lambda_2} - 2) + (1 + \epsilon) \frac{\lambda_2}{\lambda_1} - 2\lambda_2 \leq C_2.
\]

Then, since the function \( f : (0, \infty) \to \mathbb{R} \) defined by

\[
f(x) = x + (1 + \epsilon) \frac{1}{x} - 2,
\]

is bounded below by a small positive constant depending on \( \epsilon \), we see that \( \lambda_1 \) must also be bounded above. Hence at the point \((x_0, t_0)\), there exists \( C \) depending only on the initial data such that

\[
\Lambda \omega \chi \leq C.
\]

Then, on \( M \times [0, t] \),

\[
\log(\Lambda \omega \chi) - A\phi \leq \log C - A \inf_{M \times [0, t]} \phi.
\]

Exponentiating gives

\[
\Lambda \omega \chi \leq Ce^{A(\phi - \inf_{M \times [0, t]} \phi)},
\]

completing the proof of the theorem.


We prove an estimate on the \( C^0 \) norm of \( \phi \) using a Moser iteration method applied to the exponential of the solution rather than a power of the solution (compare to [Y1]) and the estimate of Theorem 3.1.
Theorem 4.1. Suppose that \((M, \omega)\) has dimension \(n = 2\) and that
\[ \chi_0 - \omega > 0. \]

Let \(\phi_t\) be a solution of the \(J\)-flow (2.1) on \([0, \infty)\). Then there exists a constant \(\tilde{C} \) depending only on the initial data such that
\[ \| \phi_t \|_{C^0(M)} \leq \tilde{C}. \]

Proof. Suppose first that \(\inf_M \phi_t\) is bounded from below uniformly in time. We will show that this implies the above estimate. Since the functional \(J_{\omega, \chi_0}\) decreases along the flow, there exists a constant \(C_0\) such that
\[ \int_M \phi_t \omega \wedge (\chi_0 + \chi_{\phi_t}) \leq C_0, \]
using (2.5). Let \(C_1\) be a positive constant satisfying
\[ \omega^2 \leq C_1 \omega \wedge \chi_0. \]

Then
\[ \int_M \phi_t \omega^2 = \int_M (\phi_t - \inf_M \phi_t) \omega^2 + \int_M \inf_M \phi_t \omega^2 \]
\[ \leq C_1 \int_M (\phi_t - \inf_M \phi_t) \omega \wedge \chi_0 + \inf_M \phi_t \int_M \omega^2 \]
\[ \leq C_1 C_0 - C_1 \int_M \phi_t \omega \wedge \chi_{\phi_t} + \inf_M \phi_t \left( \int_M \omega^2 - C_1 \int_M \omega \wedge \chi_0 \right) \]
\[ = C_1 C_0 - C_1 \int_M (\phi_t - \inf_M \phi_t) \omega \wedge \chi_{\phi_t} \]
\[ + \inf_M \phi_t \left( \int_M \omega^2 - 2C_1 \int_M \omega \wedge \chi_0 \right) \]
\[ \leq C_1 C_0 + \inf_M \phi_t \left( \int_M \omega^2 - 2C_1 \int_M \omega \wedge \chi_0 \right). \]

This gives an upper bound for \(\int_M \phi_t \omega^2\) depending on the lower bound for \(\inf_M \phi_t\). Since \(\triangle \omega \phi_t > -\Lambda \omega \chi_0\) along the flow, it follows from the existence of a lower bound on the Green’s function of \(\omega\) that \(\sup_M \phi_t\) is bounded from above, giving us the required estimate.

Now suppose that no such lower bound for \(\inf_M \phi_t\) exists. Then we can assume that there is a sequence of times \(t_i \to \infty\) such that

(i) \(\inf_M \phi_{t_i} = \inf_{t \in [0, t_i]} \inf_M \phi_t\)
(ii) \( \inf_M \phi_{t_i} \to -\infty \).

We will seek a contradiction. For a fixed \( i \), write
\[
\psi_{t_i} = \phi_{t_i} - \sup_M \phi_{t_i}.
\]

Notice that \( \sup_M \phi_{t_i} \) is bounded from below by zero from (2.6) and the fact that \( I(\phi_t) = 0 \). Hence
\[
\|\psi_{t_i}\|_{C^0} \to \infty.
\]

The following proposition is the key result of this section.

**Proposition 4.2.** Let \( M \) be a compact complex surface with two Kähler metrics \( \chi_0 \) and \( \omega \). Suppose that \( \psi \in C^\infty(M) \) satisfies the conditions
\[
\chi_\psi = \chi_0 + \frac{\sqrt{-1}}{2} \partial \bar{\partial} \psi > 0, \quad \sup_M \psi = 0,
\]
and
\[
\Lambda_\omega \chi_\psi \leq Ce^\frac{A}{2}(\psi - \inf_M \psi).
\]
Then there exists a constant \( C' \) depending only on \( M, \omega, \chi_0 \) and the constants \( A \) and \( C \) such that
\[
\|\psi\|_{C^0} \leq C'.
\]

We apply this proposition to \( \psi = \psi_{t_i} \) and obtain a contradiction since
\[
\Lambda_\omega \chi_{\psi_{t_i}} = \Lambda_\omega \chi_{\phi_{t_i}} \leq Ce^\frac{A}{2}(\phi_{t_i} - \inf_{t \in [0, t_i]} \inf_M \phi_t) = Ce^\frac{A}{2}(\psi_{t_i} - \inf_M \psi_{t_i}),
\]
where we have used Theorem 3.1 and condition (i) above. It remains to prove the proposition.

**Proof of Proposition 4.2** Let \( \delta \) be a small positive constant, to be determined later. Set \( B = A/(1 - \delta) \) and let \( u = e^{-B\psi} \).

Now, for \( \beta = n/(n - 1) = 2 \), the Sobolev inequality for functions \( f \) on \((M, \omega)\) is
\[
\|f\|_{2,\beta}^2 \leq C_2(\|\nabla f\|_2^2 + \|f\|_2^2),
\]
for \( C_2 \) depending on \( \omega \). We will apply this to \( u^{p/2} \) for \( p \geq 1 \). This gives
\[
\left(\int_M e^{-Bp\beta \omega^2} \frac{\omega^2}{2}\right)^{1/\beta} \leq C_2 \left(\int_M \|
abla e^{-Bp\psi/2} \omega^2 \|_2^2 + \int_M e^{-Bp\omega^2} \frac{\omega^2}{2}\right). \quad (4.1)
\]
Now calculate
\[
\int_M \left| \nabla e^{-Bp\psi/2} \right|^2 \frac{\omega^2}{2} = \sqrt{-1} \int_M \partial e^{-Bp\psi/2} \wedge \overline{\partial e^{-Bp\psi/2}} \wedge \omega
\]
\[
= \frac{B p^2}{4} \sqrt{-1} \int_M e^{-Bp\psi} \partial \psi \wedge \overline{\partial \psi} \wedge \omega
\]
\[
= -\frac{B p}{4} \sqrt{-1} \int_M \partial(e^{-Bp\psi}) \wedge \overline{\partial \psi} \wedge \omega
\]
\[
= \frac{B}{2} \int_M e^{-Bp\psi} \frac{\sqrt{-1}}{2} \partial \psi \wedge \omega
\]
\[
= \frac{B}{2} \int_M e^{-Bp\psi} (\chi \psi - \chi_0) \wedge \omega
\]
\[
= \frac{B}{2} \int_M e^{-Bp\psi} (\Lambda \omega \chi \psi - \Lambda \omega \chi_0) \frac{\omega^2}{2}
\]
\[
\leq C B p \int_M e^{-Bp\psi} e^{\Lambda(p-\inf_M \psi)} \frac{\omega^2}{2}
\]
\[
= C B p e^{\Lambda \inf_M \psi} \int_M e^{-(p-(1-\delta)) B \psi} \frac{\omega^2}{2},
\]
where we have used the estimate
\[
\Lambda \omega \chi \psi \leq C e^{\Lambda(p-\inf_M \psi)}.
\]

Then in (4.1),
\[
\left( \int_M u^{p \beta} \frac{\omega^2}{2} \right)^{1/\beta} \leq C_3 p e^{-A \inf_M \psi} \int_M u^{p-(1-\delta)} \frac{\omega^2}{2}.
\]
Raising to the power $1/p$ and writing $\gamma = 1 - \delta$ gives
\[
\|u\|_{p \beta} \leq C_3^{1/p} p^{1/p} e^{-(A/p) \inf_M \psi} \|u\|^{(p-\gamma)/p}_{p-\gamma}.
\]
Take the logarithm of both sides to get
\[
\log \|u\|_{p \beta} \leq \frac{1}{p} \log C_3 + \frac{1}{p} \log p + \frac{1}{p} \sup_M (-A \psi) + \frac{(p - \gamma)}{p} \log \|u\|_{p-\gamma}.
\]
We now apply the iteration. First, replace $p$ with $p \beta + \gamma$ to get
\[
\log \|u\|_{p \beta^2 + \gamma \beta} \leq \frac{1 + \frac{\beta}{p \beta + \gamma}}{p \beta + \gamma} \log C_3 + \frac{1}{p \beta + \gamma} (\beta \log p + \log(p \beta + \gamma))
\]
\[
+ \frac{1 + \beta}{p \beta + \gamma} \sup_M (-A \psi) + \frac{\beta(p - \gamma)}{p \beta + \gamma} \log \|u\|_{p-\gamma}.
\]
Repeat this procedure, replacing $p$ with $p\beta + \gamma$ to obtain for any positive integer $k$,

\[
\log \|u\|_{p\beta^{k+1} + \gamma(\beta + \beta^2 + \ldots + \beta^k)} \leq \frac{1 + \beta + \beta^2 + \ldots + \beta^k}{p\beta^k + \gamma(1 + \beta + \beta^2 + \ldots + \beta^{k-1})} \log C_3 \\
+ \frac{1}{p\beta^k + \gamma(1 + \beta + \beta^2 + \ldots + \beta^{k-1})} \left( \beta^k \log p + \beta^{k-1} \log(p\beta + \gamma) + \ldots \\
+ \log(p\beta^k + \gamma(1 + \beta + \ldots + \beta^{k-1})) \right) \\
+ \frac{1 + \beta + \beta^2 + \ldots + \beta^k}{p\beta^k + \gamma(1 + \beta + \beta^2 + \ldots + \beta^{k-1})} \sup_{M} (-A \psi) \\
+ \frac{\beta^k(p - \gamma)}{p\beta^k + \gamma(1 + \beta + \beta^2 + \ldots + \beta^{k-1})} \log \|u\|_{p^{-\gamma}}.
\] (4.2)

Now set $p = 1 + \delta$. Then, since $\beta = 2$ we have

\[
p\beta^k + \gamma(1 + \beta + \beta^2 + \ldots + \beta^{k-1}) = 1 + \beta + \beta^2 + \ldots + \beta^k + \delta.
\]

Notice that the second term on the right hand side of (4.2) is bounded by

\[
\log p + \frac{1}{\beta} \log \beta^2 + \ldots + \frac{1}{\beta^k} \log(\beta^{k+1}) \leq \log p + \log \beta\left( \sum_{i=1}^{k} \frac{i + 1}{\beta^i} \right) \leq C_4.
\]

Then

\[
\log \|u\|_{p\beta^{k+1} + \gamma(\beta + \beta^2 + \ldots + \beta^k)} \leq \log C_3 + C_4 + \sup_{M} (-A \psi) + 2\delta \max(\log \|u\|_{2\delta}, 0).
\]

Using the fact that $A = (1 - \delta)B$ and $-B \psi = \log u$, and letting $k$ tend to infinity,

\[
\log \|u\|_{C^0} \leq C_5 + 2 \max(\log \|u\|_{2\delta}, 0).
\]

Hence we get the following inequality for $\psi$,

\[
\|\psi\|_{C^0} \leq C_6 + C_7 \max \left( \log \left( \int_M e^{-2\delta B \psi^2} \frac{\omega^2}{2} \right)^{1/2\delta} , 0 \right). \tag{4.3}
\]

We can now finish the estimate. First, define

\[
P(M, \chi_0) = \{ \Phi \in C^2(M) \mid \chi_0 + \sqrt{-1} \frac{1}{2} \partial \bar{\partial} \Phi \geq 0, \sup_M \Phi = 0 \}.
\]
Then Proposition 2.1 of \[T1\] (see section 4.4, \[Ho\]) states that there exist constants \(\alpha > 0\) and \(C_8\) depending only on \((M, \chi_0)\) such that

\[
\int_M e^{-\alpha \Phi \chi_0} \frac{n}{n!} \leq C_8 \quad \text{for all } \Phi \in P(M, \chi_0).
\]

Define \(\delta\) to be

\[
\delta = \min \left\{ \frac{\alpha}{4A}, \frac{1}{2} \right\} > 0.
\]

Then the required estimate follows from (4.3), since \(\psi\) belongs to \(P(M, \chi_0)\).

5. Convergence of the flow.

In this section we complete the proof of the main theorem. We assume, using the result of \[C2\], that a solution \(\phi = \phi_t\) for the \(J\)-flow exists for all time. From Theorem 3.1 and Theorem 4.1 we have uniform estimates on \(\phi\) and the derivatives \(\partial_i \partial_j \phi\), using the fact that

\[
\chi_{ij} = \chi_{0} \bar{\chi}_0 + \partial_i \partial_j \phi > 0.
\]

Since the operator

\[
\frac{1}{n} (1 - \Lambda \chi \omega),
\]

is concave in the \(\chi_{ij}\), it is well known that, by the work of Evans [E1, E2] and Krylov [Kr] (see also [Tr]), one can deduce a uniform Hölder estimate on the second derivatives \(\partial_i \partial_j \phi\). By differentiating the equation (2.1) and applying standard Schauder estimates for parabolic equations (see [LSU] for example), one can obtain uniform estimates on all of the derivatives of \(\phi\). It then follows that there is a sequence of times \(t_j \to \infty\) such that \(\phi_{t_j}\) converges in \(C^\infty\) to some smooth function \(\phi_\infty\). In order to show that we have convergence without having to pass to a subsequence, we will use a modification of the argument in [Ca].

Notice that \(\partial \phi / \partial t\) satisfies the heat equation

\[
\frac{\partial}{\partial t} \left( \frac{\partial \phi}{\partial t} \right) = \Delta \left( \frac{\partial \phi}{\partial t} \right).
\]

Since we have uniform bounds for \(\chi_{ij}\) from above and away from zero, and bounds on \(\frac{\partial}{\partial t} \chi_{ij}\) and all the covariant derivatives of \(\chi_{ij}\) and \(\frac{\partial}{\partial t} \chi_{ij}\), it follows from the Harnack inequality of Li and Yau [LY] and the argument in [Ca]
that there exist positive constants $C_0$ and $\eta$, which are independent of $t$, such that
\[
\sup_M \left( \frac{\partial \phi}{\partial t} \right) - \inf_M \left( \frac{\partial \phi}{\partial t} \right) \leq C_0 e^{-\eta t}.
\]
Since
\[
\int_M \frac{\partial \phi}{\partial t} \chi^2 = 0,
\]
$\partial \phi/\partial t$ must take on the value zero somewhere on $M$ for each $t$, and so
\[
\left| \frac{\partial \phi}{\partial t} \right| \leq C_0 e^{-\eta t}.
\]
Hence for any $0 < s < s'$, and any $x \in M$,
\[
|\phi(x, s') - \phi(x, s)| = \left| \int_s^{s'} \frac{\partial \phi}{\partial t}(x, t) dt \right|
\leq \int_s^{s'} |\frac{\partial \phi}{\partial t}(x, t)| dt
\leq C_0 \int_s^{s'} e^{-\eta t} dt
= C_0 \frac{1}{\eta} (e^{-\eta s} - e^{-\eta s'}),
\]
which tends to zero as $s$ and $s'$ tend to infinity. Hence $\phi_t$ converges in the $C_0$ norm to $\phi_\infty$. It must converge also in the $C^\infty$ topology, since otherwise there would exist an integer $N$, an $\epsilon > 0$ and a sequence $t_j \rightarrow \infty$ with
\[
\|\phi_{t_j} - \phi_\infty\|_{C^N} \geq \epsilon.
\]
Since $\phi$ is bounded in all the $C^k$ norms, one could pass to a subsequence of the $\phi_{t_j}$ which would converge to some $\phi'_\infty \neq \phi_\infty$, giving the contradiction. This completes the proof.

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