Changing sign solutions of a conformally invariant fourth-order equation in the Euclidean space

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We prove the existence of infinitely many solutions for the critical equation $\Delta^2 u = |u|^{2^* - 2} u$ in $\mathbb{R}^n$, where $\Delta^2$ denotes the bilaplacian for the euclidean metric. These solutions are non-equivalent in the sense that we cannot pass from one to another by translation and rescaling. Moreover, infinitely many of them must change sign.

Fourth-order equations of critical Sobolev growth have been an intensive target of investigations in the last years, particularly because of the applications of the fourth-order Paneitz operator to conformal geometry and also because of the parallel that exists between fourth-order equations of critical growth and their second-order analogues. References for the Paneitz operator are Branson [2] and Paneitz [7]. We consider in this paper the following fourth-order equation

(1) \[ \Delta^2 u = |u|^{2^* - 2} u \]

on $\mathbb{R}^n$, $n \geq 5$, where $2^* = 2n/(n - 4)$ is the critical exponent for the Sobolev embedding of $H^2_2$-spaces (consisting of functions in $L^2$ with two derivatives in $L^2$) into $L^p$-spaces, and $\Delta^2 = \Delta_\xi^2$ is the bilaplacian operator with respect to the Euclidean metric $\xi$. In [6], Lin proved that the only smooth positive solutions of (1) are the functions given by

(2) \[ u_{\lambda,a}(x) = \alpha_n \left( \frac{\lambda}{1 + \lambda^2 |x - a|^2} \right)^{(n-4)/2}, \]

where $\alpha_n = (n(n - 4)(n^2 - 4))^{(n-4)/8}, \lambda > 0$ and $a \in \mathbb{R}^n$. The result extends to non-trivial non-negative solutions of (1) when they belong to the Beppo-Levi space $D_2^2(\mathbb{R}^n)$. Following standard terminology, we say that two
solutions $u$ and $v$ of an equation such as (1) are equivalent if they are related by an equation such as

$$v(x) = \lambda^{-(n-4)/2}u \left(\frac{x-a}{\lambda}\right)$$

for some $\lambda > 0$ and $a \in \mathbb{R}^n$. Thanks to the above mentioned result of Lin [6], two smooth positive solutions of (1) are always equivalent. Indeed,

$$u_{\lambda,a}(x) = \lambda^{(n-4)/2}u_{1,0}(\lambda(x-a)).$$

Moreover, it is easily checked that equivalent solutions have the same energy in the sense that

$$\int_{\mathbb{R}^n} (\Delta v)^2 \, dx = \int_{\mathbb{R}^n} (\Delta u)^2 \, dx$$

if $u$ and $v$ are related by (3). The energy of the $u_{\lambda,a}$’s in (2) is precisely the quantum of energy of a bubble in the blow-up study of positive solutions of Paneitz-type equations. We refer to Hebey and Robert [4] for more details.

The purpose of this paper is to prove the following theorem. Such a theorem is the analogue of Ding’s result [3] when passing from the second-order critical equation $\Delta u = |u|^{4/(n-2)}u$ to the fourth-order critical Equation (1) we consider in this paper.

**Theorem.** There exists a sequence $(u_k)^\infty_{k=1}$ of solutions of (1) whose energy tends to $+\infty$ as $k \to +\infty$, namely such that

$$\int_{\mathbb{R}^n} (\Delta u_k)^2 \, dx \to +\infty$$

as $k \to +\infty$. In particular, there exist infinitely many non-equivalent solutions of equation (1). These solutions $u_k$ necessarily change sign when $k$ is large.

We prove the theorem in the rest of the paper following Ding’s approach [3] when proving the existence of infinitely many non-equivalent solutions of the second order critical equation $\Delta u = |u|^{4/(n-2)}u$. Specific technical difficulties are attached to the fourth-order case.

**Proof of the theorem.** The Paneitz operator $P^n_h$ on the unit $n$-sphere $(S^n, h)$ reads as

$$P^n_h u = \Delta_h^2 u + c_n \Delta_h u + d_n u,$$

where $c_n = (n^2 - 2n - 4)/2$ and $d_n = (n(n - 4)(n^2 - 4))/16$ (see Paneitz [7] and Branson [2] for the definition of $P^n_h$). We let $\Phi : S^n \setminus \{N\} \to \mathbb{R}^n$ be the
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stereographic projection of north pole \( N \) in \( S^n \). Then, as is well known,

\[
(\Phi^{-1})^* h = \phi^{4/(n-4)} \xi,
\]

where

\[
\phi(x) = 4^{(n-4)/4} (1 + |x|^2)^{-(n-4)/2}.
\]

We let \( u \in C^2(\mathbb{R}^n) \) be a solution of (1) and let \( \hat{u} : S^n \to \mathbb{R} \) be given by

\[
\hat{u} = (u \phi^{-1}) \circ \Phi.
\]

By the conformal properties of \( P^n_h \)

\[
\phi^{2^* - 1} (P^n_h \hat{u}) \circ \Phi^{-1} = P^n_\xi u = \Delta^2 \xi u = |u|^{2^* - 2} u = \phi^{2^* - 1} (|\hat{u}|^{2^* - 2} \hat{u}) \circ \Phi^{-1}.
\]

Therefore, \( \hat{u} \) is a solution of

\[
P^n_h \hat{u} = |\hat{u}|^{2^* - 2} \hat{u}.
\]

Moreover, it is easily checked that

\[
\int_{\mathbb{R}^n} |u|^{2^*} \, dx = \int_{S^n} |\hat{u}|^{2^*} \, dv_h.
\]

Conversely, if \( \hat{u} \) is a solution of (7), then \( u : \mathbb{R}^n \to \mathbb{R} \) given by \( u = (\hat{u} \circ \Phi^{-1}) \phi \) is a solution of (1) satisfying (8). As a remark, if \( \hat{u} \in H^2_2(S^n) \) is a solution of (7), then \( \hat{u} \in L^p(S^n) \) for all \( p \), and \( \hat{u} \) is actually in \( C^4(S^n) \). We claim now that

\[
\int_{\mathbb{R}^n} (\Delta \xi u)^2 \, dx < +\infty
\]

In order to prove (9), we let \( \tilde{\xi} \) be the Riemannian metric on \( \mathbb{R}^n \) given by \( \tilde{\xi} = \phi^{4/(n-4)} \xi \). Then, if \( g \) is a Riemannian metric on \( \mathbb{R}^n \), we let \( L_g \) be the conformal Laplacian with respect to \( g \) given by

\[
L_g u = \Delta_g u + \frac{n-2}{4(n-1)} S_g u,
\]
where $S_{g}$ is the scalar curvature of $g$. By the conformal properties of $L_{g}$,

$$
\Delta_{\xi} u = L_{\xi} u
= \phi^{(n+2)/(n-4)} L_{\xi} \left( u \phi^{-(n-2)/(n-4)} \right)
= \phi^{(n+2)/(n-4)} \left( \Delta_{\xi} (u \phi^{-(n-2)/(n-4)}) + \frac{n(n-2)}{4} u \phi^{-(n-2)/(n-4)} \right).
$$

Therefore, we have that

$$
\int_{\mathbb{R}^{n}} (\Delta_{\xi} u)^{2} \, dv_{\xi} = \int_{\mathbb{R}^{n}} \phi^{4/(n-4)} \left( \Delta_{\xi} (u \phi^{-(n-2)/(n-4)}) \right. \\
\left. + \frac{n(n-2)}{4} u \phi^{-(n-2)/(n-4)} \right)^{2} \, dv_{\xi},
$$

and we can also write that

$$
\Delta_{\xi} (u \phi^{-(n-2)/(n-4)}) = \Delta_{\xi} \left( (\hat{u} \circ \Phi^{-1}) \phi^{-2/(n-4)} \right)
= \Delta_{\xi} (\hat{u} \circ \Phi^{-1}) \phi^{-2/(n-4)} + \Delta_{\xi} (\phi^{-2/(n-4)}) (\hat{u} \circ \Phi^{-1}) \\
- 2 \langle \nabla (\hat{u} \circ \Phi^{-1}); \nabla \phi^{-2/(n-4)} \rangle_{\xi},
$$

where $\langle \cdot; \cdot \rangle_{\xi}$ is the scalar product with respect to $\xi$. It follows that

$$
\int_{\mathbb{R}^{n}} (\Delta_{\xi} u)^{2} \, dv_{\xi} \leq 4 \int_{\mathbb{S}^{n}} (\Delta_{h} \hat{u})^{2} \, dv_{h} + C_{1} (I_{1} + I_{2} + I_{3})
\leq C_{2} + C_{1} (I_{1} + I_{2} + I_{3}),
$$

where $C_{1}, C_{2} > 0$ are positive constants, and

$$
I_{1} = \int_{\mathbb{S}^{n}} \left( \Delta_{h} (\phi^{-2/(n-4)} \circ \Phi) \right)^{2} (\phi^{4/(n-4)} \circ \Phi) \, dv_{h},
$$

$$
I_{2} = \int_{\mathbb{S}^{n}} \left( \phi^{4/(n-4)} \circ \Phi \right) \left| \nabla (\phi^{-2/(n-4)} \circ \Phi) \right|_{h}^{2} \, dv_{h},
$$

$$
I_{3} = \int_{\mathbb{S}^{n}} (\phi^{-2} \circ \Phi) (u \circ \Phi)^{2} \, dv_{h}.
$$
Thanks once again to the conformal invariance of the conformal Laplacian, we can write that

\[ I_1 = \int_{\mathbb{R}^n} \left( \Delta \xi (\phi^{-2/(n-4)}) \right)^2 \phi^{4/(n-4)} dv_{\xi} \]
\[ = \int_{\mathbb{R}^n} \phi^{(2n+4)/(n-4)} \left( \phi^{-(n+2)/(n-4)} \Delta \xi \phi - \frac{n(n-2)}{4} \phi^{-2/(n-4)} \right)^2 dx \]
\[ \leq C_3 \int_{\mathbb{R}^n} (\Delta \phi)^2 dx + C_4 \int_{\mathbb{R}^n} \phi^{2^*} dx < +\infty, \]

where \( C_3, C_4 > 0 \) are positive constants. In a similar way, we can write that

\[ I_2 = \int_{\mathbb{R}^n} \phi^{4/(n-4)} \left| \nabla \phi^{2^*/(n-4)} \right|_{\xi}^2 dv_{\xi} \]
\[ = \int_{\mathbb{R}^n} \phi^{2^*} \left| \nabla \phi^{2^*/(n-4)} \right|_{\xi}^2 dx < +\infty. \]

At last, by (6), we also have that

\[ |I_3| \leq C_5 \int_{\mathbb{R}^n} dv_{\xi} \]
\[ = C_5 \int_{\mathbb{R}^n} \phi^{2^*} dx < +\infty, \]

where \( C_5 > 0 \) is a positive constant. Hence, (9) is true. In a similar way, we claim that we also have that

\[ \int_{\mathbb{R}^n} |\nabla u|^{2^*} dx < +\infty, \]

where \( 2^* = 2n/(n-2) \) is the critical Sobolev exponent for the embedding of \( H^2_1 \)-spaces (consisting of functions in \( L^2 \) with one derivative in \( L^2 \)) into \( L^p \)-spaces. Another possible equation for \( 2^* \) is \( 2^* = 2 \times 1^5 \). In order to prove (10), we note that, by (6),

\[ |\nabla (\tilde{u} \circ \Phi^{-1})|_{\xi} = \phi^{2^*/(n-4)} \left| \nabla \tilde{u} \right|_h. \]

Then, we write that

\[ \int_{\mathbb{R}^n} |\nabla u|^{2^*} dx \leq C_6 \int_{\mathbb{R}^n} |\nabla (\tilde{u} \circ \Phi^{-1})|_{\xi}^{2^*} \phi^{2^*} dx + C_7 \int_{\mathbb{R}^n} |\nabla \phi|^{2^*} dx \]
\[ \leq C_8 \int_{\mathbb{R}^n} \phi^{2^*} dx + C_6 \int_{\mathbb{R}^n} |\nabla \phi|_{\xi}^{2^*} dx < +\infty, \]

where \( C_6, C_7, C_8 > 0 \) are positive constants. This proves (10).
Now we consider $\eta \in C^\infty_c(\mathbb{R}^n)$ be such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $B_0(1)$ and $\eta \equiv 0$ in $\mathbb{R}^n \setminus B_0(2)$, where $B_0(r)$ stands for the open Euclidean ball of centre $0$ and radius $r$ in $\mathbb{R}^n$. For $R > 0$, we set

$$
\eta_R(x) = \eta \left( \frac{x}{R} \right)
$$

and let $u$ be a solution of (1). Multiplying (1) by $\eta_R u$ and integrating by parts over $\mathbb{R}^n$, we get that

$$
(11) \quad \int_{\mathbb{R}^n} \eta_R |u|^{2t} \, dx = \int_{\mathbb{R}^n} \Delta \xi (\eta_R u) \Delta u \, dx = I_1(R) + I_2(R) - 2I_3(R),
$$

where

$$
I_1(R) = \int_{B_0(2R)} \eta_R (\Delta \xi u)^2 \, dx,
$$

$$
I_2(R) = \int_{A_R} (\Delta \xi \eta_R) u (\Delta \xi u) \, dx,
$$

$$
I_3(R) = \int_{A_R} \langle \nabla \eta_R; \nabla u \rangle \xi (\Delta \xi u) \, dx,
$$

and where $A_R$ is the annulus $A_R = B_0(2R) \setminus B_0(R)$. Clearly, thanks to (9), we have that

$$
I_1(R) \longrightarrow \int_{\mathbb{R}^n} (\Delta \xi u)^2 \, dx,
$$

as $R \to +\infty$. We also have that

$$
\int_{\mathbb{R}^n} \eta_R |u|^{2t} \, dx \longrightarrow \int_{\mathbb{R}^n} |u|^{2t} \, dx,
$$

as $R \to +\infty$. Independently, letting $V(R) = \text{Vol}_\xi(A_R)$, by help of Hölder’s inequality, and noting that $V(R) \leq CR^n$, we can write that

$$
|I_2(R)| = \left| \int_{A_R} (\Delta \xi \eta_R) u (\Delta \xi u) \, dx \right| 
\leq CR^{-2} \|u\|_{2t} \left( \int_{A_R} (\Delta \xi u)^{2n/(n+4)} \, dx \right)^{(n+4)/2n}
$$
\[ \leq CR^{-2}\|u\|_{2n} \left( \int_{A_R} (\Delta \xi u)^2 \, dx \right)^{1/2} V(R)^{2/n} \]
\[ \leq C\|u\|_{2n} \left( \int_{A_R} (\Delta \xi u)^2 \, dx \right)^{1/2}. \]

Hence, \( I_2(R) \to 0 \) as \( R \to +\infty \). In a similar way, by (10), we can write that

\[ |I_3(R)| \leq CR^{-1}\|\nabla u\|_{2n} \left( \int_{A_R} |\Delta \xi u|^{2n/(n+2)} \, dx \right)^{(n+2)/2n} \]
\[ \leq CR^{-1}\|\nabla u\|_{2n} \left( \int_{A_R} (\Delta \xi u)^2 \, dx \right)^{1/2} V(R)^{1/n} \]
\[ \leq C\|\nabla u\|_{2n} \left( \int_{A_R} (\Delta u)^2 \, dx \right)^{1/2}. \]

Hence, we also have that \( I_3(R) \to 0 \) as \( R \to +\infty \). Passing to the limit as \( R \to +\infty \) in (11), we get that if \( \hat{u} \) is a solution of (7), then \( u: \mathbb{R}^n \to \mathbb{R} \) given by \( u = (\hat{u} \circ \Phi^{-1})\phi \) is a solution of (1) such that

\[ \int_{\mathbb{R}^n} (\Delta \xi u)^2 \, dx = \int_{\mathbb{R}^n} |u|^{2^*} \, dx \]
\[ = \int_{S^n} |\hat{u}|^{2^*} \, dv_h < +\infty. \]

In view of this result, and in order to prove the theorem, it suffices to prove that there exists a sequence \((\hat{u}_k)_k\) of solution of (7) such that

\[ \int_{S^n} |\hat{u}_k|^{2^*} \, dv_h \to +\infty, \]
as \( k \to +\infty \). Let \( J \) be the functional associated to (7) given by

\[ J(u) = \frac{1}{2} \int_{S^n} ((\Delta h u)^2 + c_n |\nabla u|^2 + d_n u^2) \, dv_h - \frac{1}{2^{2^*}} \int_{S^n} |u|^{2^*} \, dv_h. \]

Let also \( G \) be a closed subgroup of the isometry group \( \text{Isom}_h(S^n) \) of \((S^n, h)\).

For \( q = 1, 2, \) and \( p > 1 \), we let

\[ H^p_{q,G}(S^n) = \{ u \in H^p_q(S^n) \text{ s.t. } u(g \cdot x) = u(x) \text{ for all } g \in G \text{ and a.a. } x \in S^n \}. \]
where $H^q_0(S^n)$ is the Sobolev space of functions in $L^p$ with $q$ derivatives in $L^p$. We denote by $O^x_G = \{ g \cdot x | g \in G \}$ the orbit of $x$ under $G$ and let

$$k = \min_{x \in S^n} \dim O^x_G.$$  

The composition of a continuous embedding and of a compact embedding is compact. Moreover, we know from the general result in Hebey and Vaugon [5] that if $k \geq 1$, then the embedding $H^q_1(S^n) \subset L^q(S^n)$ is continuous for all $1 < q < p_G^*$, and compact for all $1 < q < p_G^{**}$, where $p_G^* = +\infty$ if $n - k \leq p$, and $p_G^{**} = (n - k)p/(n - k - p)$ if $n - k > p$. Noting that $(2^*)_G > 2^*$ when $k \geq 1$, the sequence $H^2_1(S^n) \subset H^1_1(S^n) \subset L^{(2^*)}_G(S^n)$ then gives that the embedding $H^2_1(S^n) \subset L^{2^*}_G(S^n)$ is compact when $k \geq 1$.

In what follows, we let $G$ be such that $k \geq 1$ and such that $H^2_1(S^n)$ is infinite dimensional. For instance, as in Ding [3], we can let $G = O(n_1) \times O(n_2)$, where $n_1, n_2$ are such that $n_1 + n_2 = n + 1$ and $n_1, n_2 \geq 2$. In this example, $k = \min(n_1, n_2) - 1$. We claim now that there exists a sequence $(\hat{u}_m)_m$ of critical points of $J$ restricted to $H^2_1(S^n)$ such that

$$\int_{S^n} \hat{u}_m^{2\gamma} dv_h \longrightarrow +\infty,$$

as $m \to +\infty$. In order to prove this claim, we first let $\| \cdot \|$ be the norm on $H^2_1(S^n)$ be given by

$$\| u \|^2 = \int_{S^n} ((\Delta_h u)^2 + c_n |\nabla u|_h^2 + d_n u^2) \ dv_h.$$  

For $J$ as above, it is easily seen that $J$ is even, that $J(0) = 0$ and that

(A1) there exist $\rho, \alpha > 0$ such that $J > 0$ in $B_0(\rho) \setminus \{0\}$ and $J \geq \alpha$ on $S_0(\rho)$, and

(A2) $J$ satisfies the Palais–Smale condition, where $B_0(\rho)$ is the ball of centre $0$ and radius $\rho$ in $H^2_1(S^n)$, and $S_0(\rho)$ is the sphere of centre $0$ and radius $\rho$ in $H^2_1(S^n)$. We can also prove that for any finite dimensional subspace $E \subset H^2_1(S^n)$,

(A3) $E \cap \{ J \geq 0 \}$ is bounded.

Indeed, since $E$ is finite dimensional, there exists $C > 0$ such that for any $u \in E$, $\| u \| \leq C\| u \|_{2\gamma}$. Let $E = \text{span}\{ f_1, \ldots, f_N \}$, where the $f_i$’s are an
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orthonormal basis for $E$, and $u = \sum_{i=1}^{N} \alpha_i f_i$ be such that $\|u\| = 1$. Then, for $R > 0$,

$$J(Ru) = \frac{R^2}{2} - \frac{R^{2^*}}{2^*} \|u\|^{2^*}_{2^*}$$

$$\leq \frac{R^2}{2} \left( 1 - \frac{2R^{2^*} - 2}{2^*C^{2^*}} \right)$$

and (A3) follows. Now, by (A1)–(A3) we can apply Theorem 2.13 of Ambrosetti and Rabinowitz [1] and we get the existence of an increasing sequence $(\alpha_m)_m$ of critical values for $J$ given by

$$\alpha_m = \sup_{h \in \Gamma^*} \inf_{u \in S \cap E_m} J(h(u)),$$

where $S = S_0(1)$, $E_m = \text{span}\{f_1, \ldots, f_m\}$, $E_m^\perp$ is the orthogonal complement of $E_m$, $(f_i)_{i \geq 1}$ is an orthonormal basis of $H^2_{2,G}(S^n)$ and $\Gamma^*$ is the space of odd homeomorphisms of $H^2_{2,G}(S^n)$ onto $H^2_{2,G}(S^n)$ such that $J(h(B)) \geq 0$, where $B$ is the ball of centre 0 and radius 1 in $H^2_{2,G}(S^n)$. Then, in order to prove that there exists a sequence $(\hat{u}_m)_m$ of critical points of $J$ restricted to $H^2_{2,G}(S^n)$ such that (12) is true, it suffices to prove that

$$\alpha_m \rightarrow +\infty$$

as $m \rightarrow +\infty$. We define

$$T = \{ u \in H^2_{2,G}(S^n) \text{ s.t. } 2^\ast \|u\|^2 = 2\|u\|^{2^\ast}_{2^\ast} \}$$

and let

$$\beta_m = \inf_{u \in T \cap E_m} \|u\|.$$

Then,

$$\beta_m \rightarrow +\infty$$

as $m \rightarrow +\infty$. Indeed, if it is not the case, there exists $(u_m)_m$ such that $u_m \in E_m^\perp$ for all $m$, $u_m \in T$ for all $m$, the $u_m$’s are bounded in $H^2_{2,G}(S^n)$ and $u_m \rightharpoonup 0$ in $H^2_{2,G}(S^n)$ since $u_m \in E_m^\perp$. The compactness of the embedding $H^2_{2,G}(S^n) \subset L^{2^\ast}(S^n)$ then implies that (up to a subsequence) $u_m \rightarrow 0$ in $L^{2^\ast}(S^n)$. It follows that $u_m \rightarrow 0$ in $H^2_{2}(S^n)$ since $u_m \in T$ for all $m$. On the other hand, by the Sobolev inequality corresponding to the embedding.
$H^2_2(S^n) \subset L^2(S^n)$, and still since $u_m \in T$ for all $m$, there exists $C > 0$ such that $\|u_m\| \geq C$ for all $m$. A contradiction, and (15), is proved. For $u \in E_m^\perp$, we let
\[
h_m(u) = \frac{1}{2}\beta_m u.
\]
Following Ambrosetti and Rabinowitz [1], it is easily seen that $h_m$ extends to $\tilde{h}_m \in \Gamma^*$. Given $u \in H^2_2(S^n) \setminus \{0\}$, we let $\beta(u) \in \mathbb{R}$ be such that $\beta(u)u \in T$. Then, if $u \in S \cap E_m^\perp$,
\[
J(h_m(u)) = \frac{1}{2} \left( \frac{\beta_m}{2} \right)^2 \left( 1 - \left( \frac{\beta_m}{2\beta(u)} \right)^{2t-2} \right)
\]
and we get with (13) and (15) that (14) holds. In particular, there exists a sequence $(\tilde{u}_m)_m$ of critical points of $J$ restricted to $H^2_{2,G}(S^n)$ such that (12) holds. The $\tilde{u}_m$'s are solutions of (7) when restricted to $H^2_{2,G}(S^n)$ in the sense that for any $m$ and any $\varphi \in H^2_{2,G}(S^n)$,
\[
\int_{S^n} ((\Delta_h \tilde{u}_m)(\Delta_h \varphi) + c_n (\nabla \tilde{u}_m, \nabla \varphi)_h + d_n \tilde{u}_m \varphi) \, dv_h
= \int_{S^n} |\tilde{u}_m|^{2t-2} \tilde{u}_m \varphi \, dv_h
\]
Let $\varphi$ be any smooth function on $S^n$ or any function in $H^2_2(S^n)$. Let also $\varphi_G$ be given by the equation
\[
\varphi_G(x) = \int_G \varphi(\sigma(x)) \, d\mu(\sigma),
\]
where $d\mu$ is the Haar measure on $G$. Clearly, $\varphi_G$ is smooth and $G$-invariant if $\varphi$ is smooth or $\varphi_G \in H^2_{2,G}(S^n)$ if $\varphi \in H^2_2(S^n)$. Then we can write that
\[
\int_{S^n} ((\Delta_h \tilde{u}_m)(\Delta_h \varphi_G) + c_n (\nabla \tilde{u}_m, \nabla \varphi_G)_h + d_n \tilde{u}_m \varphi_G) \, dv_h
\]
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\[
\begin{aligned}
&= \int_G \left( \int_{S^n} \left( (\Delta_h \hat{u}_m)(\Delta_h (\varphi \circ \sigma)) \\
&\quad + c_n(\nabla \hat{u}_m, \nabla (\varphi \circ \sigma))_h + d_n \hat{u}_m (\varphi \circ \sigma) \right) dv_h \right) d\mu(\sigma) \\
&= |G| \int_{S^n} ((\Delta_h \hat{u}_m)(\Delta_h \varphi) + c_n(\nabla \hat{u}_m, \nabla \varphi)_h + d_n \hat{u}_m \varphi) \, dv_h,
\end{aligned}
\]

where \( |G| \) is the volume of \( G \) with respect to \( d\mu \), and that

\[
\begin{aligned}
&\int_{S^n} |\hat{u}_m|^{2^* - 2} \hat{u}_m \varphi_G \, dv_h \\
&= \int_G \left( \int_{S^n} |\hat{u}_m|^{2^* - 2} \hat{u}_m (\varphi \circ \sigma) dv_h \right) d\mu(\sigma) \\
&= |G| \int_{S^n} |\hat{u}_m|^{2^* - 2} \hat{u}_m \varphi \, dv_h.
\end{aligned}
\]

It follows that

\[
\int_{S^n} ((\Delta_h \hat{u}_m)(\Delta_h \varphi) + c_n(\nabla \hat{u}_m, \nabla \varphi)_h + d_n \hat{u}_m \varphi) \, dv_h = \int_{S^n} |\hat{u}_m|^{2^* - 2} \hat{u}_m \varphi \, dv_h,
\]

for all \( \varphi \in H^2_2(S^n) \) and all \( m \). In particular, for any \( m \), \( \hat{u}_m \) is a solution of (7). The \( u_m \)'s associated to the \( \hat{u}_m \)'s have to change sign for \( m \gg 1 \) according to the remark on equivalent solutions as given earlier and the fact that

\[
\int_{\mathbb{R}} (\Delta u_m)^2 \, dx = \int_{S^n} |\hat{u}_m|^{2^*} \, dv_h \rightarrow +\infty.
\]

This ends the proof of the theorem. \( \square \)

References


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