Examples of hypersurfaces flowing by curvature in a Riemannian manifold

ROBERT GULLIVER AND GUOYI XU

This paper gives some examples of hypersurfaces $\varphi_t(M^n)$ evolving in time with speed determined by functions of the normal curvatures in an $(n + 1)$-dimensional hyperbolic manifold; we emphasize the case of flow by harmonic mean curvature. The examples converge to a totally geodesic submanifold of any dimension from 1 to $n$, and include cases which exist for infinite time. Convergence to a point was studied by Andrews, and only occurs in finite time. For dimension $n = 2$, the destiny of any harmonic mean curvature flow is strongly influenced by the genus of the surface $M^2$.

1. Background

Unless otherwise mentioned, all Riemannian manifolds in this article are connected and complete. Let $M^n$ be a smooth, connected, orientable compact manifold of dimension $n \geq 2$, without boundary, and let $(N^{n+1}, g^N)$ be a smooth connected Riemannian manifold. $\sigma^N$ is any sectional curvature of $N^{n+1}$, $\mathcal{R}$ is the Riemann tensor of $N^{n+1}$ and $\nabla^N$ is the Levi-Civita connection corresponding to $g^N$. For a hyperbolic manifold, $\sigma^N \equiv -1$. When an index, such as $i$, is repeated in one term of an expression, summation $1 \leq i \leq n$ is indicated.

Suppose $\varphi_0 : M^n \to N^{n+1}$ is a smooth immersion of an oriented manifold $M^n$ into $N^{n+1}$; write $\vec{v}$ for the induced normal vector to $\varphi_0(M)$. The second fundamental form of $M$ is a covariant tensor, which we represent at each point by a matrix $A$, where the entry $A_{ij} = h_{ij} = \langle \nabla^N_{\partial_x^i} \vec{v}, \partial_{x_j} \rangle g^N$. The Weingarten tensor is given by the matrix $\mathcal{W}$, whose entry $\omega^k_i = h_{ij}g^{jk}$, and $\{g^{jk}\}$ is the pointwise inverse matrix of $\{g_{jk}\}$.

We seek a solution $\varphi : M^n \times [0, T) \to N^{n+1}$ to an equation

\[
\frac{\partial}{\partial t} \varphi(x, t) = -f(\lambda(\mathcal{W}(x, t)))\vec{v}(x, t),
\]

\[
\varphi(x, 0) = \varphi_0(x),
\]

(1.1)
where \( F(x, t) = f(\lambda(W(x, t))) \) and \( f \) is a smooth symmetric function, where \( \vec{v}(x, t) \) is the outward normal vector to \( \varphi(M^n, t) \). \( W(x, t) \) is the Weingarten matrix of \( \varphi(M^n, t) \) in \( N^{n+1} \) and \( \lambda(W) \) is the set of eigenvalues \( (\lambda_1, \ldots, \lambda_n) \) of \( W \). Define \( \varphi_t(x) = \varphi(x, t) \), then \( (\lambda_1, \ldots, \lambda_n) \) are the principal curvatures of the hypersurface \( M_t \triangleq \varphi_t(M) \subset N \).

For example, (1.1) becomes mean curvature flow when \( f(\lambda) = \sum_i \lambda_i \) (see [4, 8]).

Consider the solution \( \varphi : M^n \times [0, T) \rightarrow N^{n+1} \) of the following equations:

\[
\frac{\partial}{\partial t} \varphi(x, t) = -\left( \sum_i \lambda_i^{-1} \right)^{-1} \vec{v}(x, t),
\]

\[
\varphi(x, 0) = \varphi_0(x).
\]

Such a solution \( \varphi(x, t) \) is harmonic mean curvature flow; \( f(\lambda) = (\sum_i \lambda_i^{-1})^{-1} \) is the harmonic mean of the numbers \( \lambda_1, \ldots, \lambda_n \).

It has been noted that the mean curvature flow of hypersurfaces in a Riemannian \( (n + 1) \)-dimensional manifold, \( n \geq 2 \), does not have all the desirable properties satisfied for \( n = 1 \) [3]. For some purposes, harmonic mean curvature flow (1.2) may be the preferred way to extend curve-shortening flow to \( n \geq 2 \).

Andrews proved the following theorem in [2]:

**Theorem 1.1.** Let \( M^n \) and \( \varphi_0 \) be assumed as at the beginning of this paper, and that the Riemannian manifold \( (N^{n+1}, g^N) \) satisfies the following conditions:

\[-K_1 \leq \sigma^N \leq K_2, \quad |\nabla^N R^N|_{g^N} \leq L\]

for some non-negative constants \( K_1, K_2 \) and \( L \).

Assume every principal curvature \( \lambda_i \) of \( \varphi_0 \) satisfies the following condition:

\[\lambda_i > \sqrt{K_1} \]

Then there exists a unique smooth solution to (1.2) on a maximal time interval \([0, T), T < \infty\), and the immersion \( \varphi_t \) converges uniformly to a round point \( p \) in \( N^{n+1} \) as \( t \) approaches \( T \).

Also, we have the following theorem, to appear in [7]:

**Theorem 1.2.** Let \( M^n \) be a smooth, connected, orientable compact manifold of dimension \( n \geq 2 \), without boundary. Assume \( N^{n+1} \) is a non-positively
Examples of hypersurfaces in a Riemannian manifold

curved, simply connected smooth manifold, and suppose \( \varphi_0 : M^n \to N^{n+1} \) is a smooth immersion of \( M^n \). Assume every principal curvature of \( \varphi_0(M) \) is positive. Then there exists a unique smooth solution to (1.2) on a maximal time interval \([0, T)\), \( T < \infty \), and the immersion \( \varphi_t \) converges uniformly to a round point \( p \) in \( N^{n+1} \) as \( t \) approaches \( T \).

In the rest of this paper, except for Section 6, and unless otherwise mentioned, we consider harmonic mean curvature flow and let \( f(\lambda) = (\sum_i \lambda^{-1}_i)^{-1} \). We provide two specific examples of harmonic mean curvature flow for infinite time: in Section 2, with dimension reduction in the limit and in Section 3, with the limit manifold of the same dimension as \( M \). Note, these examples in Sections 2 and 3 provide barriers for harmonic mean curvature flow in Riemannian manifolds; further applications will be addressed in [7]. We discuss the limit behavior of the harmonic mean curvature flow at infinite time in section 4. Then we treat the special consequences of the Gauss–Bonnet theorem for two-dimensional surfaces in Section 5, and turn to examples of more general flows by functions of normal curvatures in Section 6.

2. The dimension-reduction example

In this section, we give an example where \( \varphi_t \) converges to \( \varphi_\infty \) in the \( C^\infty \) topology but the dimension of \( M_\infty = \varphi_\infty(M) \) is less than the dimension of \( M_t \), i.e., there is dimension reduction.

**Theorem 2.1.** Let \( N^3 \) be a hyperbolic manifold containing an embedded closed geodesic \( M_\infty \). Then there is a flow \( \varphi_t : M^2 \to N^3 \) by harmonic mean curvature, where \( M^2 \) is a torus, which converges to \( M_\infty \) as \( t \to +\infty \). The flow consists of immersions \( \varphi_t \), which become embedded for \( t \) sufficiently large.

For example, we may let the ambient manifold \( N \) be \( H^3/\mathbb{Z} \), where \( H^3 \) is hyperbolic space, represented as the Poincaré half-space \((\mathbb{R}^3)^+ = \{(x, y, z) | (x, y, z) \in \mathbb{R}^3, z > 0\}\) with the metric \( g^N_{ij} = \frac{1}{z^2} \delta_{ij} \) (\( \delta_{ij} = \delta_i^j \) = Kronecker delta), and the \( \mathbb{Z} \) action \( f : \mathbb{Z} \times H^3 \to H^3 \) is defined as

\[
f(k)(x, y, z) = 2^k(x, y, z).
\]

Recall that \( f(k) \) is an isometry of \( H^3 \) for each \( k \in \mathbb{Z} \).

Now we let \( N \) be the quotient manifold of \( H^3 \) under the \( \mathbb{Z} \)-action, with fundamental domain \( \{(x, y, z) | 1 \leq \sqrt{x^2 + y^2 + z^2} \leq 2\} \). Then \( M_\infty \) = the positive z-axis, modulo \( f(1) \), is a closed geodesic in \( N \).
**Proof.** Let \( \psi_0 : \mathbb{S}^1 \to N \) be an embedding as the given closed geodesic curve \( M_\infty \) in \( N \). We choose a unit vector field \( w(x) \) in \( (T_x\psi_0)^\perp \). Then for \( r > 0 \), we define

\[
\psi(x, \theta, r) = \psi_r(x, \theta) : \mathbb{S}^1 \times \mathbb{S}^1 \to N^3
\]

by

\[
\psi(x, \theta, r) = \psi_r(x, \theta) = \gamma(x, \theta, r),
\]

where \( \gamma(x, \theta, \cdot) \) is the unit-speed geodesic in \( N \) with \( \gamma(x, \theta, 0) = \psi_0(x) \) and \( \frac{d}{dr}\gamma(x, \theta, r) = \vec{N}(x, \theta) \) at \( r = 0 \). Here \( \vec{N}(x, \theta) \) is the unit tangent vector in \( T_{\psi_0(x)}N^3 \), which is perpendicular to \( T_x\psi_0 \) and makes the angle \( \theta \) with \( w(x) \). Then \( \psi_r(\mathbb{S}^1 \times \mathbb{S}^1) \) has two principal curvatures:

\[
\lambda_1(r) \equiv \tanh r, \quad \lambda_2(r) \equiv \coth r.
\]

In fact, for \( i = 1, 2 \), \( \lambda_i(r) \) is the logarithmic derivative of the length of a Jacobi field, and hence satisfies the Ricatti equation \( \lambda_i'(r) + (\lambda_i(r))^2 = 1 \).

We have constructed a one-parameter family of immersions \( \psi_r : M \to N \), \(-\infty < r < \infty\), with two principal curvatures: \( \lambda_1(r) \equiv \tanh r \) and \( \lambda_2(r) \equiv \coth r \). It may be observed that \( \psi_r \) is an embedding for \( r \) sufficiently small.

Now consider the harmonic mean curvature flow \( \varphi_t = \psi_{r(t)} : M \to N \), with initial conditions \( \varphi_0 = \psi_{r_0}, \quad r(0) = r_0 \), where \( r_0 \) is some fixed positive constant. The speed must satisfy

\[
\frac{\partial r}{\partial t} = \left\langle \frac{\partial \gamma}{\partial r}, \frac{\partial r}{\partial t}, \vec{v} \right\rangle = \left\langle \frac{\partial \gamma(x, r)}{\partial t}, \vec{v} \right\rangle = \left\langle \frac{\partial \psi(x, r)}{\partial t}, \vec{v} \right\rangle = \left\langle -F\vec{v}, \vec{v} \right\rangle = -F(\lambda_1, \lambda_2)
\]

\[
= -\frac{1}{\lambda_1^{-1} + \lambda_2^{-1}} = -\frac{\sinh r \cosh r}{(\sinh r)^2 + (\cosh r)^2}.
\]

In the first equation, we use the fact \( \frac{\partial \gamma}{\partial r} = \vec{v} \); in the third equation, we use the definition of \( \psi_r \), where \( \vec{v} = \vec{N}(x, \theta) \) is the outward normal vector of \( \psi_r(M) \) at \( (x, \theta) \in \mathbb{S}^1 \times \mathbb{S}^1 \).

Solving, we find

\[
r(t) = \frac{1}{2} \sinh^{-1} \left( e^{-t} \sinh 2r_0 \right).
\]

Note that \( r(t) \to 0 \) as \( t \to \infty \). \( \square \)
3. The no-dimension-reduction example

In this section, we give an example in which $M_t$ converges to $M_\infty$ in the $C^\infty$ topology and the dimension of $M_\infty$ is the same as the dimension of $M_t$, i.e., there is no dimension reduction.

**Theorem 3.1.** There is a compact surface $M^2$ of genus 2, a hyperbolic manifold $N^3$ diffeomorphic to $M \times \mathbb{R}$, a totally geodesic embedding $\psi_0 : M \rightarrow N$ and a flow by harmonic mean curvature $\varphi_t : M \rightarrow N$ such that as $t \rightarrow +\infty$, $\varphi_t(M) \rightarrow \psi_0(M)$ smoothly.

**Proof.** Let $\Omega$ be a regular geodesic octagon in the hyperbolic plane $H^2$, with angles $\pi/2$, and thus area $4\pi$. Label the edges as $\beta_1, \alpha_1', -\beta_1', -\alpha_1, \beta_2, \alpha_2', -\beta_2', -\alpha_2$, in that order, where the signs indicate orientation. Let $A_1$ be the orientation-preserving isometry of $H^2$, which maps the oriented geodesic segments $\alpha_1$ to $\alpha_1'$; $A_2$ maps $\alpha_2$ to $\alpha_2'$; $B_1$ maps $\beta_1$ to $\beta_1'$ and $B_2$ maps $\beta_2$ to $\beta_2'$. The group $G$ of isometries of $H^2$ generated by $A_1, A_2$ and $B_1$ also includes $B_2$. $G$ is isomorphic to the fundamental group of the compact surface of genus 2. (See [Katok [6], pp. 95–98] for the arithmetic properties of the group $G$.)

Let $\psi_0 : H^2 \rightarrow H^3$ be an embedding as a totally geodesic surface in $H^3$. The isometries in $G$ extend in a well-known fashion to isometries of $H^3$, leaving the distance from $\psi_0(H^2)$ invariant.

Choose a unit normal vector field $\vec{N}$ to $\psi_0(H^2)$. Define $\psi(\cdot, r) : H^2 \rightarrow H^3$ by $\psi(x, r) = \psi_r(x) = \gamma(x, r)$ and $\psi(x, 0) = \psi_0(x)$, where $\gamma(x, \cdot)$ is the unit-speed geodesic in $H^3$ with $\gamma(x, 0) = x$ and $\frac{\partial}{\partial r} \gamma(x, 0) = \vec{N}(x)$.

Then $\psi_r(H^2)$ is totally umbilic, with normal curvatures $\lambda(r) \equiv \tanh r$. In fact, $\lambda(r)$ satisfies the Ricatti equation $\lambda'(r) + (\lambda(r))^2 = 1$, with the initial condition $\lambda(0) = 0$.

Now let the group $G$ act by isometries on $H^2$ and on $H^3$. The quotient $H^2/G = M^2$ is a compact surface of genus 2, with fundamental domain $\Omega$, and the quotient $H^3/G = N^3$ is a non-compact hyperbolic manifold diffeomorphic to $M \times \mathbb{R}$. The group $G$ acting on $N$ preserves each of the hypersurfaces $\psi_r(H^2)$. We have constructed a one-parameter family of totally umbilic embeddings $\psi_r : M \rightarrow N, -\infty < r < \infty$, with normal curvatures $\equiv \tanh r$.

Now consider the harmonic mean curvature flow $\varphi_t : M \rightarrow N$, with initial conditions $\varphi_0 = \psi_{r_0}$, where $r_0$ is some fixed positive constant. The speed
must satisfy
\[ \frac{\partial r}{\partial t} = \langle \frac{\partial \gamma}{\partial r} \frac{\partial r}{\partial t}, \vec{v} \rangle = \langle \frac{\partial \varphi(x,t)}{\partial r}, \vec{v} \rangle = \langle \frac{\partial \psi(x,r)}{\partial r}, \vec{v} \rangle = \langle \frac{\partial \psi(x,r)}{\partial t}, \vec{v} \rangle = \langle -F \vec{v}, \vec{v} \rangle = -F(\lambda_1, \lambda_2) = -\frac{1}{\lambda_1^{-1} + \lambda_2^{-1}} = -\frac{1}{2} \tanh r. \]

In the first equation, we use the fact \( \frac{\partial \gamma}{\partial r} = \vec{N}(x) = \vec{v}. \) In the third equation, we use the definition of \( \psi_r \), where \( \vec{v} \) is the outward normal vector of \( \psi_r \).

Solving, we find
\[ r(t) = \sinh^{-1}(e^{-t/2} \sinh r_0). \]

Note that \( r(t) \to 0 \) as \( t \to \infty \).

**4. The limit behavior of harmonic mean curvature flow at infinite time**

In this section, we will give a sufficient condition where the harmonic mean curvature flow will exist forever and discuss the limit behavior. Let \( \varphi_t : M \to N \) be an immersion of \( M^n \) into a hyperbolic manifold \( N^{n+1} \).

**Definition 4.1.** We define the following notation:
\[ \tilde{F}^{kl} = \frac{\partial F}{\partial h_{kl}}, \quad \tilde{F}^{kl, pq} = \frac{\partial^2 F}{\partial h_{kl} \partial h_{pq}}, \quad \tilde{H}^k_i = \frac{\partial H}{\partial \omega^k_i}, \]
\[ \tilde{H}^{s,i}_{r,k} = \frac{\partial^2 H}{\partial \omega^k_i \partial \omega^s_r}, \quad \mathcal{R}_{ij} = \mathcal{R}_{i0j0}, \]

where 0 appearing as a tensor index represents the normal vector \( \vec{v} \) of \( \varphi(M) \) in \( N \). For any \( W : M \to \mathbb{R} \), we define:
\[ \mathcal{L}(W) = \tilde{F}^{kl} \nabla_k \nabla_l W. \]

Recall from Andrews [2] that \( \mathcal{L} \) is elliptic as long as \( \varphi_t(M) \) remains locally strictly convex.

**Theorem 4.1.** If \( N^{n+1} \) is a hyperbolic manifold, \( F(x) < \frac{1}{n} \) for any \( x \in M \), then \( \varphi_t(M) \) remains locally convex and \( F(x,t) < \frac{1}{n} \) for any \( x \in M \), \( t \in \)
$[0, +\infty), \lim_{t \to \infty} F(x, t) = 0$, and the harmonic mean curvature flow exists for all $t$ in $[0, +\infty)$.

Proof. By Andrews [2], using a curvature coordinate system at one point, we have the following formula:

$$\frac{\partial F}{\partial t} = \mathcal{L}(F) + F < \hat{F}, (\Psi^2) > + F < \hat{F}^i, (\mathcal{R}_i) >$$

(4.1)

$$= \mathcal{L}(F) + \sum_i F \frac{\partial f}{\partial \lambda_i} (\lambda_i^2 + \mathcal{R}_i)$$

$$\leq \mathcal{L}(F) + F^3 (n - \sum_i \lambda_i^{-2}) \leq \mathcal{L}(F) + F^3 \left( n - \frac{1}{n} F^{-2} \right).$$

Consider the ordinary differential equation (ODE)

$$\frac{\partial \tilde{F}}{\partial t} = \tilde{F}^3 \left( n - \frac{1}{n} \tilde{F}^{-2} \right),$$

$$\tilde{F}(0) = \max_{x \in M} F(x, 0).$$

Solving the above ODE, we get $\tilde{F}(t)^{-2} - n^2 = (\tilde{F}(0)^{-2} - n^2) e^{2t/n}$. Because $0 < \tilde{F}(0) = \max_{x \in M^n} F(x, 0) < \frac{1}{n}$, we get $\lim_{t \to \infty} \tilde{F}(t) = 0$.

By the maximum principle, $F(x, t) \leq \tilde{F}(t) < \frac{1}{n}$, for all $x \in M, t \in [0, +\infty)$, and therefore $\lim_{t \to \infty} F(x, t) = 0$.

On the other hand, we have the following estimate by the above evolution equation of $F$:

$$\frac{\partial F}{\partial t} \geq \mathcal{L}(F) + F^3 \left( - \sum_i \lambda_i^{-2} \right) \geq \mathcal{L}(F) - F.$$

Now consider the ODE

$$\frac{\partial \hat{F}}{\partial t} = -\hat{F},$$

$$\hat{F}(0) = \min_{x \in M} F(x, 0).$$

Then by the maximum principle again, we get for all $x \in M, t \in [0, +\infty)$

$$F(x, t) \geq \hat{F}(t) = \min_{x \in M} F(x, 0) e^{-t} > 0.$$

In particular, $\varphi_t(M)$ remains convex for all $t$. 
Finally, we have the following estimate of $H$. By Andrews [2]

\[
\frac{\partial}{\partial t} \omega_i^r = \hat{F}^{kl}_{,\omega} \nabla_k \nabla_i \omega_i^r + \hat{F}^{kl,pq} (\nabla_i h_{kl}) (\nabla_j h_{pq}) g^{jr} + \hat{F}^{kl} (h_{ml} \omega_m^r) \omega_i^r + \hat{F}^{st} \mathcal{R}_{st} h_{ij} g^{jr} + 2 \hat{F}^{pm} g^{tr} \omega_q^r \mathcal{R}_{pict} - \hat{F}^{pq} (g^{tr} \omega_s^r \mathcal{R}_{psqt}) + \hat{F}^{pq} g^{tr} (\nabla_i \mathcal{R}_{tpq} - \nabla_p \mathcal{R}_{qit}) - \hat{F}^{pq} g^{tr} (\nabla_i \mathcal{R}_{tpq} - \nabla_p \mathcal{R}_{qit}).
\]

Now referring to the last five terms above, we define

\[
(\text{I}) = \dot{H}_i^r \hat{F}^{kl} (h_{ml} \omega_k^m) \omega_i^r, \quad (\text{II}) = \dot{H}_r^i \hat{F}^{st} \mathcal{R}_{st} h_{ij} g^{jr},
\]

\[
(\text{III}) = 2 \dot{H}_i^r \hat{F}^{pm} g^{tr} \omega_m^r \mathcal{R}_{pict}, \quad (\text{IV}) = - \dot{H}_i^r (\hat{F}^{pq} g^{tr} \omega_s^r \mathcal{R}_{psqt} + \hat{F}^{pq} g^{ts} \omega_r^r \mathcal{R}_{pict}),
\]

\[
(\text{V}) = \dot{H}_i^r \hat{F}^{pq} g^{tr} (\nabla_i \mathcal{R}_{tpq} - \nabla_p \mathcal{R}_{qit}),
\]

then

\[
\frac{\partial}{\partial t} H = \dot{H}_r^i \left( \frac{\partial}{\partial t} \omega_i^r \right) = \dot{H}_r^i (\hat{F}^{kl} \nabla_k \nabla_i \omega_i^r) + \dot{H}_r^i \hat{F}^{kl,pq} (\nabla_i h_{kl}) (\nabla_j h_{pq}) g^{jr} + (\text{I}) + \cdots + (\text{V}).
\]

Note

\[
\hat{F}^{kl} \nabla_k \nabla_i H = \hat{F}^{kl} \nabla_k (\dot{H}_r^i \nabla_j \omega_i^r) = \hat{F}^{kl} \dot{H}_r^i (\nabla_k \omega_i^r) (\nabla_j \omega_i^r) + \hat{F}^{kl} \dot{H}_r^i \nabla_k \nabla_i \omega_i^r.
\]

Define

\[
(\text{J}) = \dot{H}_r^i \hat{F}^{kl,pq} (\nabla_i h_{kl}) (\nabla_j h_{pq}) g^{jr} - \hat{F}^{kl} \dot{H}_r^i (\nabla_k \omega_i^r) (\nabla_j \omega_i^r),
\]

we get

\[
\frac{\partial}{\partial t} H = \mathcal{L}(H) + (\text{J}) + (\text{I}) + \cdots + (\text{V}).
\]

It is straightforward to get

\[
(\text{I}) + (\text{II}) = H [< \dot{F}, (\mathcal{W}^2) > + \hat{F}^{ij} \mathcal{R}_{ij0}] \leq n F^2 H \leq \frac{1}{n} H
\]

and

\[
(\text{V}) = \frac{\partial f}{\partial \lambda_i} (\nabla_j \mathcal{R}_{ji0} - \nabla_i \mathcal{R}_{jj0}) = 0.
\]

Choose a curvature coordinate system around one point; then we could do the following calculation:

\[
(\text{J}) = \hat{F}^{kl,pq} (\nabla_i h_{kl}) (\nabla_i h_{pq}).
\]
But, by Lemma 2.22 in [1], we know $F$ is concave from the fact that $f$ is concave. So, we get $(J) \leq 0$.

Now

\[
(III) + (IV) = 2\dot{H}_r^i \dot{F}^{pm}_{r\omega^m} g^{tr} \omega^q_m \mathcal{R}_{psqr} - \dot{H}^i_r(\dot{F}^{pq} g^{tr} \omega^q_r \mathcal{R}_{psqr} + \dot{F}^{pq} g^{ts} \omega^s_r \mathcal{R}_{psqr})
\]

\[
= 2\delta^i_r \frac{\partial f}{\partial \lambda_p} \delta^m_p \delta^r_q \lambda_q \delta^m_q \mathcal{R}_{psqr}
- \delta^i_r \left( \frac{\partial f}{\partial \lambda_p} \delta^q_p \delta^r_q \lambda_q \delta^q_q \mathcal{R}_{psqr} + \frac{\partial f}{\partial \lambda_p} \delta^q_p \lambda_q \delta^r_q \mathcal{R}_{psqr} \right)
\]

\[
= 2\mathcal{R}_{prpr} \frac{\partial f}{\partial \lambda_p} (\lambda_p - \lambda_r)
= 2 \sum_{p < r} \mathcal{R}_{prpr} \left( \frac{\partial f}{\partial \lambda_p} - \frac{\partial f}{\partial \lambda_r} \right) (\lambda_p - \lambda_r)
\]

\[
= \left( \sum_k \lambda_k^{-1} \right)^{-2} \cdot \sum_{i,j} (-\mathcal{R}_{ijij}) \cdot (\lambda_i - \lambda_j)^2 (\lambda_i + \lambda_j) \cdot \lambda_i^{-2} \lambda_j^{-2}
\]

\[
\leq \sum_{i,j} (\lambda_i + \lambda_j) \cdot \left( \frac{\lambda_i^{-1} - \lambda_j^{-1}}{\sum_k \lambda_k^{-1}} \right)^2 \leq \sum_{i,j} (\lambda_i + \lambda_j) = 2nH.
\]

We have the following inequality for $H$ by the above estimates:

\[
\frac{\partial H}{\partial t} \leq \mathcal{L}(H) + \left( 2n + \frac{1}{n} \right) H.
\]

Now consider the ODE

\[
\frac{\partial \hat{H}}{\partial t} = \left( 2n + \frac{1}{n} \right) \hat{H},
\]

\[
\hat{H}(0) = \max_{x \in M} H(x, 0).
\]

Then by the maximum principle again, we get for all $x \in M, t \in [0, +\infty)$:

\[
H(x, t) \leq \hat{H}(t) = \max_{x \in M} H(x, 0) e^{(2n+\frac{1}{n})t} < +\infty.
\]

This shows that the harmonic mean curvature flow exists on $[0, +\infty)$. □

In the rest of this section, we do not assume the ambient manifold $N^{n+1}$ is a hyperbolic manifold.

**Proposition 4.1.** Assume $N^{n+1}$ is a smooth $n + 1 \geq 3$ dimensional manifold which is convex at infinity, the maximal existence time of the harmonic mean curvature flow $\varphi : M \times [0, T) \rightarrow N$ is $T = +\infty$, and as $t \rightarrow +\infty$, $M_t =$
ϕ(M, t) converges to a smooth n-dimensional submanifold \(M_\infty\) of \(N\) in the \(C^\infty\)-topology; then

\[
\max_{x \in M, t \in [0, +\infty)} \{|F(x, t)|, |\nabla F(x, t)|, |\nabla^2 F(x, t)|\} \leq C,
\]

where \(C\) is a constant depending on \(M_0\), \(N^{n+1}\) and \(M_\infty\).

**Proof.** Straightforward from the assumptions. □

**Proposition 4.2.** Assume \(N\) and \(M_t \to M_\infty\) are as in the hypotheses of Proposition 4.1. Then

\[
\lim_{t \to \infty} \int_{M_t} F^2 \, d\mu_t = 0.
\]

**Proof.** By Theorem 1.1 in [5], we have the formula \(\frac{\partial}{\partial t} (\int_{M_t} d\mu_t) = -\int_{M_t} FH \, d\mu_t\). Because \(\int_{M_t} d\mu_t \to \mu(M_\infty)\) as \(t \to \infty\), we could find an \(\epsilon\)-dense set \(\{t_k\}_{k=1}^\infty\) for any positive constant \(\epsilon > 0\) such that

\[
\lim_{k \to \infty} t_k = \infty
\]

and

\[
\lim_{k \to \infty} \int_{M_{t_k}} FH \, d\mu_{t_k} = 0.
\]

Then using the inequality \(H \geq n^2 F\), we get \(\lim_{k \to \infty} \int_{M_{t_k}} F^2 \, d\mu_{t_k} = 0\).

Now to get our conclusion we only need to show \(\frac{\partial}{\partial t} \int_{M_t} F^2 \, d\mu_t\) is uniformly bounded. First, we know from Proposition 4.1 that \(|F|, |\nabla F|\) and \(|\nabla^2 F|\) are uniformly bounded. So, we have

\[
\frac{\partial}{\partial t} \left( \int_{M_t} F^2 \, d\mu_t \right) = \int 2FF_t + F^2(-FH) \, d\mu_t
\]

\[
= \int 2F \left( \mathcal{L}(F) + \sum_{i=1}^n F \left( \frac{\partial f}{\partial \lambda_i} \right) (\lambda_i^2 + \mathcal{R}_{ii}) \right) - F^3 H \, d\mu_t
\]
Examples of hypersurfaces in a Riemannian manifold

( where we use equation (4.1) )

\[
\begin{align*}
\int 2nF^4 + 2F^4 \left( \sum_{i=1}^{n} \lambda_i^{-2} R_{ii} \right) + 2F \mathcal{L}(F) - F^3 H d\mu_t \\
\leq \int 2F^4 K_2 \left( \sum_{i=1}^{n} \lambda_i^{-2} \right) d\mu_t + \int 2F \mathcal{L}(F) d\mu_t \\
\leq C \int F^2 d\mu_t + 2 \int F \mathcal{L}(F) d\mu_t,
\end{align*}
\]

where the first inequality uses the following facts:

\( M_t \) is always contained in some compact set of \( N^{n+1} \), since \( N^{n+1} \) is convex at infinity, so its sectional curvature is bounded above by some constant \( K_2 \); and \( HF^{-1} = (\sum_{i=1}^{n} \lambda_i)(\sum_{i=1}^{n} \lambda_i^{-1}) \geq n^2 - 2n \).

Next, since we know the volume of \( M_t \) is always non-increasing and \( |F| \) is uniformly bounded, we get

\[ C \int_{M_t} F^2 \, d\mu_t \leq C_1, \]

where \( C_1 \) is some constant depending only on \( M_0, N \) and \( M_\infty \).

Since \( |\nabla^2 F| \) is uniformly bounded, we get

\[ 2 \int F \mathcal{L}(F) \, d\mu_t \leq 2n^2 \int F \nabla^2 F \, d\mu_t \leq C_2, \]

where \( C_2 \) is some constant depending on \( M_0, N \) and \( M_\infty \).

By all the above, we get

\[ \frac{\partial}{\partial t} \left( \int F^2 \, d\mu_t \right) \leq C_3, \]

where \( C_3 \) is another constant depending on \( M_0, N \) and \( M_\infty \).

Therefore,

\[ \lim_{t \to \infty} \int_{M_t} F^2 \, d\mu_t = 0. \]

\[ \square \]

Corollary 4.1. Assume \( N \) and \( M_t \to M_\infty \) are as assumed for Proposition 4.1. Then we have

\[ \lim_{t \to \infty} \left( \max_{x \in M} F(x, t) \right) = 0. \]
Proof. By Proposition 4.2, we have

$$0 = \lim_{t \to \infty} \int_{M_t} F^2 \, d\mu_t = \int_{M_\infty} \lim_{t \to \infty} F^2(x, t) \, d\mu_\infty,$$

so the corollary follows. \qed

By the above results, assume $N$ and $M_t \to M_\infty$ are as in the hypotheses of Proposition 4.1, we know that $F \equiv 0$ on the limit surface $M_\infty$, if $M_\infty$ is the smooth limit of the harmonic mean curvature flow, which implies that $\det \mathbb{W} = 0$ on $M_\infty$.

5. Classification of harmonic mean curvature flow on surfaces

In this section, we consider harmonic mean curvature flow for $n = 2$, where $M^2$ is an orientable surface, $N^3$ is a hyperbolic manifold and the harmonic mean $f(\lambda) = \frac{\lambda_1 \lambda_2}{\lambda_1 + \lambda_2}$. As before, we assume that $\varphi_0(M)$ is locally strictly convex.

In the following, we always assume $F(x, 0) < \frac{1}{2}$, i.e., $\lambda_1^{-1} + \lambda_2^{-1} > 2$, which will guarantee, that the harmonic mean curvature flow exists forever by Theorem 4.1. Note that, for example, $f(\lambda_1, \lambda_2) < \frac{1}{2}$ for the examples of Theorems 2.1 and 3.1, and that the horospheres have $f(\lambda_1, \lambda_2) \equiv \frac{1}{2}$.

We define $C_0 = 2\pi \chi(M_0) = \int_{M_t} (K - 1) \, d\mu_t$, where the second equation is true for any $M_t$ because of the Gauss–Bonnet theorem, where $\chi(M_0)$ is the Euler number of $M_0$; $K(x, t) = \lambda_1(x, t) \lambda_2(x, t)$, $\lambda_1(x, t)$ and $\lambda_2(x, t)$ are the principal curvatures at the point $x$ on $M_t$ in the ambient hyperbolic manifold $N^3$, and the Gauss equation, which implies the Gauss curvature $= K - 1$.

First, define $V(t) = \int_{M_t} 1 \, d\mu_t$, the area of $M_t$. Then using the formula

$$\frac{\partial}{\partial t} d\mu_t = -FH d\mu_t,$$

we get

$$\frac{d}{dt} V(t) = \int_{M_t} \frac{\partial}{\partial t} d\mu_t = \int_{M_t} (-FH) d\mu_t = \int_{M_t} (-K) d\mu_t$$

$$= -\int_{M_t} (K - 1) d\mu_t - \int_{M_t} 1 d\mu_t = -C_0 - V(t).$$
Solving the above ODE, we get
\[ V(t) = (V(0) + C_0)e^{-t} - C_0. \]

This shows that the area of \( M_t \) is determined by its genus and the area \( V(0) \) of the initial surface \( M_0 \).

There are three cases: \( C_0 < 0 \) and \( C_0 = 0 \) and \( C_0 > 0 \), corresponding to the surfaces with genus \( g > 1 \) (case I), \( g = 1 \) (case II) and \( g = 0 \) (case III), respectively.

(I) Let us first consider the case \( C_0 = 2\pi\chi(M_0) < 0 \). In this case, we have
\[
\lim_{t \to \infty} V(t) = -C_0 > 0,
\]
which means the limit surface has non-zero volume. We conjecture that in a hyperbolic manifold \( N^3 \), the limit surface will be the totally geodesic surface, if there is one in the homotopy class of \( M_0 \). This behavior is seen in Theorem 3.1.

(II) When \( C_0 = 2\pi\chi(M_0) = 0 \), we have
\[
\lim_{t \to \infty} V(t) = -C_0 = 0,
\]
which means the limit surface has zero volume. In fact, we could prove the following:

**Proposition 5.1.** If \( N^3 \) is a hyperbolic manifold, \( F(x,0) < \frac{1}{2} \) for all \( x \in M \) and the genus of \( M = 0 \), then
\[
\lim_{t \to \infty} (\max_{x \in M_t} H(x,t)) = +\infty.
\]

**Proof.** Because \( \int_{M_t} (K - 1) \, d\mu_t = C_0 = 0 \), we have \( \max_{x \in M_t} K(x,t) \geq 1 \). We also have \( \lim_{t \to \infty} (\max_{x \in M_t} F(x,t)) = 0 \), using the assumption \( F(x,0) < \frac{1}{2} \), by Theorem 4.1. Then for any \( x \in M_t, \, t > 0 \), we have the following:
\[
K(x,t) = H(x,t)F(x,t) \leq F(x,t)(\max_{x \in M_t} H(x,t)).
\]

Taking the maximum on the both sides of the above inequality, we have
\[
1 \leq \max_{x \in M_t} K(x,t) \leq (\max_{x \in M_t} F(x,t))(\max_{x \in M_t} H(x,t)).
\]

So
\[
\max_{x \in M_t} H(x,t) \geq \frac{1}{\max_{x \in M_t} F(x,t)}.
\]
Taking the limit on both sides, we get

\[
\lim_{t \to \infty} \left( \max_{x \in M_t} H(x, t) \right) \geq \frac{1}{\lim_{t \to \infty} \left( \max_{x \in M_t} F(x, t) \right)} = +\infty.
\]

\[\square\]

The above proposition means that there exists at least one blow-up point on the limit set; the example of Theorem 2.1 blows up at every point.

(III) Finally, when \(C_0 = 2\pi \chi(M_0) > 0\), we have an interesting geometric result. In this case, because

\[V(t) = (V(0) + C_0)e^{-t} - C_0,
\]

there exists some \(T_0, 0 < T_0 < +\infty\), such that \(V(T_0) = 0\). That means the harmonic mean curvature flow stops in finite time. But we have already proved that the flow will exist forever if \(F < \frac{1}{2}\). So under the assumption \(F < \frac{1}{2}\), this surface will not exist.

**Remark 5.1.** Observe that the non-existence of the initial surfaces in Case (III) above may also be proven by lifting the simply connected surface \(M_0\) to the universal cover \(H^3\) of \(N^3\) and applying the comparison principle with shrinking spheres centered at a point: the sphere of radius \(r\) has \(F = \frac{1}{2} \coth r > \frac{1}{2}\).

### 6. General geometric flows

In this section, we give examples for a general geometric flow (1.1) in a hyperbolic manifold \(N^{n+1}\), which will exist forever or for a computable finite time, and converge to a given totally geodesic submanifold \(P^k\) of any codimension. In this section, we always assume the existence of a totally geodesic submanifold \(P^k\) in \(N^{n+1}\).

First, by similar methods to those of Sections 2 and 3, we may prove a theorem for general dimensions and codimensions.

**Theorem 6.1.** Assume \(P^k\) is a compact totally geodesic submanifold of the hyperbolic manifold \(N^{n+1}\), where \(1 \leq k \leq n\). Let \(M\) be diffeomorphic to the unit sphere bundle of the normal bundle \(\perp P\) when \(k < n\); we choose \(M\) to be one of the two connected components of the unit sphere bundle of the normal bundle \(\perp P\) when \(k = n\). Then, we have a flow by harmonic mean curvature \(\varphi_t : M \to N\) such that as \(t \to +\infty\), \(\varphi_t(M) \to P\).
Proof. We only sketch the proof. We find the second fundamental form matrix of $\psi_r(M)$ with respect to a basis of curvature directions in the following:

$$\mathcal{W} = \begin{pmatrix} I_k \tanh r & 0_{k \times (n-k)} \\ 0_{(n-k) \times k} & I_{n-k} \coth r \end{pmatrix}. $$

Then, we find

$$(6.1) \quad \frac{\partial r}{\partial t} = -F = -\frac{\tanh r}{k + (n-k)(\tanh r)^2}.$$ 

Solving this ODE, we get

$$(\sinh r(t))^k (\cosh r(t))^{n-k} = Ce^{-t},$$

where $C = (\sinh r_0)^k (\cosh r_0)^{n-k}$ is a fixed positive constant. This shows that $\varphi_t := \psi_{r(t)}$ is a solution of harmonic mean curvature flow. Note that $r(t) \to 0$ as $t \to +\infty$. \qed

Now let $M^n$ be diffeomorphic to (one connected component of) the unit sphere normal bundle of $P_k$ in $N_{n+1}$, and let $\psi_r : M \to N$ define the hypersurface at distance $r > 0$ from $P_k$. We consider flow by an arbitrary symmetric function of the normal curvatures.

**Theorem 6.2.** For the symmetric function $f(\lambda_1, \ldots, \lambda_n)$, define

$$h(r) = f(\tanh r, \ldots, \coth r),$$

where $\tanh r$ is repeated $k$ times and $\coth r$ is repeated $n-k$ times. Choose $r_0 > 0$ and define

$$T_0 = \int_0^{r_0} \frac{1}{h(r)} dr, \quad 0 < T_0 \leq +\infty.$$ 

Then we may construct a flow

$$(6.2) \quad \frac{\partial}{\partial t} \varphi(\cdot, t) = f(\lambda(\mathcal{W}(x,t))) \vec{v}(x,t)$$

with initial condition $\varphi(\cdot, 0) = \psi_{r_0}$, which exists for time $0 \leq t \leq T_0 \leq \infty$, and $\varphi(\cdot, t)$ converges to the totally geodesic $k$-dimensional submanifold $P_k$ as $t \to T_0$. 

Examples of hypersurfaces in a Riemannian manifold 715
**Proof.** The hypersurface defined by $\varphi(\cdot, t) := \psi_{r(t)}$ flows by (6.2) if

$$\frac{\partial r}{\partial t} = -F(x, t) \equiv -h(r)$$

$$\Rightarrow \int_{r(0)}^{r(T_0)} \frac{1}{h(r)} dr = \int_{0}^{T_0} -1 dt$$

$$\Rightarrow T_0 = \int_{0}^{r_0} \frac{1}{h(r)} dr.$$

The conclusion now follows from the proof of Theorem 6.1, replacing equation (6.1) with equation (6.3). □

**Remark 6.1.** Note that the flow (6.2) is parabolic if $\frac{\partial f}{\partial \lambda_i} > 0$ ($1 \leq i \leq n$); parabolic for backwards time if $\frac{\partial f}{\partial \lambda_i} < 0$ ($1 \leq i \leq n$) and is a first-order partial differential equation (PDE) if $f$ is constant.

The following corollary is a generalization of both mean curvature flow ($m = 1, \ell = 0$) and of harmonic mean curvature flow ($m = n, \ell = n - 1$).

**Corollary 6.1.** Assume $P^k$ is a compact totally geodesic submanifold of $N^{n+1}$, where $1 \leq k \leq n$. Let $M$ be diffeomorphic to the unit sphere bundle of the normal bundle $\perp P$ when $k < n$; $M$ is one of the two components of the unit sphere bundle of $\perp P$ when $k = n$.

For integers $0 \leq m, \ell \leq n$, let $S_m$ and $S_\ell$ be the elementary symmetric functions of degree $m, \ell$, respectively, of the principal curvatures $\lambda_1, \ldots, \lambda_n$ of $M_t$. We have a flow by curvature function

$$F(x, t) = \frac{S_m(\lambda_1, \ldots, \lambda_n)}{S_\ell(\lambda_1, \ldots, \lambda_n)},$$

for time $0 \leq t < \infty$, such that $\varphi(t) : M \to N$ and $\varphi_t(M) \to P$ as $t \to +\infty$; assuming that the integers $m, \ell$ satisfy $|m - (n - k)| < |\ell - (n - k)|$.

**Remark 6.2.** Theorem 6.2 may also be applied to prove a partial converse of Corollary 6.1: assuming $P^k$ and $N^{n+1}$ are as in Corollary 6.1, if the opposite condition $|m - (n - k)| \geq |\ell - (n - k)|$ holds, then the same construction yields a flow of hypersurfaces by the curvature function $F = \frac{S_m}{S_\ell}$, which converges to the totally geodesic submanifold $P^k$ in finite time $T_0$. 

Proof. In the following, we fix an arbitrary positive constant \( r(0) = r_0 \). First, we have

\[
S_m = \sum_{0 \leq p, q \leq k} C_k^p (\tanh r)^p C_{n-k}^q (\coth r)^q = \sum_{0 \leq p, q \leq k} C_k^p C_{n-k}^q (\coth r)^{q-p},
\]

where \( C_k^p \) is the combinatorial coefficient \( \frac{k!}{p!(k-p)!} \).

Since \( \coth r \geq 1 \), it is easy to see

\[
S_m \sim \begin{cases} 
(\coth r)^m & \text{if } m \leq n-k, \\
(\coth r)^{2(n-k)-m} & \text{if } m > n-k,
\end{cases}
\]

where the notation \( S_m \sim (\coth r)^j \) means that there exist positive constants \( C_1 \) and \( C_2 \) such that \( C_1 (\coth r)^j \leq S_m \leq C_2 (\coth r)^j \). Here \( C_1 \) and \( C_2 \) will depend only on \( m, n, k, \ell \) and \( r_0 \).

Similarly, we have

\[
S_\ell \sim \begin{cases} 
(\coth r)^\ell & \text{if } \ell \leq n-k, \\
(\coth r)^{2(n-k)-\ell} & \text{if } \ell > n-k.
\end{cases}
\]

Therefore,

\[
F = \frac{S_m}{S_\ell} \sim \begin{cases} 
(\coth r)^{m-\ell} & \text{if } m, \ell \leq n-k, \\
(\coth r)^{\ell-m} & \text{if } m, \ell > n-k, \\
(\coth r)^{2(n-k)-m-\ell} & \text{if } \ell \leq n-k < m, \\
(\coth r)^{m+\ell-2(n-k)} & \text{if } m \leq n-k < \ell.
\end{cases}
\]

By Theorem 6.2, we obtain that the flow exists forever if and only if the power of \( \coth r \) is negative in the asymptotic estimate for \( F \) above. That is, if and only if \( m \) and \( \ell \) satisfy one of the following conditions:

\[
\begin{align*}
& m < \ell & \text{if } m, \ell \leq n-k, \\
& \ell < m & \text{if } m, \ell > n-k, \\
& 2(n-k) < m + \ell & \text{if } \ell \leq n-k < m, \\
& m + \ell < 2(n-k) & \text{if } m \leq n-k < \ell.
\end{align*}
\]

It is straightforward to see the above inequalities are equivalent to the inequality \(|m-(n-k)| < |\ell-(n-k)|\), which is our conclusion. \( \square \)
Remark 6.3. In particular, the cases $k = n$, $m = 1$ and $\ell = 0$ are the first examples we are aware of in the literature of a locally convex compact hypersurface flowing by mean curvature and converging smoothly to a submanifold in infinite time. In addition, the cases $k = n - 1$, $m = 0$ and $\ell = 1$ give examples of (backwards parabolic) inverse mean curvature flow existing forever and converging to a totally geodesic hypersurface. After reversing time to obtain parabolicity, this example of $-\frac{1}{H}$ flow is properly divergent as $t \to \infty$.

Acknowledgements

We would like to thank Gerhard Huisken for interesting discussions, and in particular for the observation that there are no examples in the literature for convergence of a compact hypersurface flowing by harmonic mean curvature in infinite time to a set of positive dimension. Also, we would like to thank the referee for pointing out a gap in the earlier version of this paper.

References


School of Mathematics
University of Minnesota
Minneapolis
MN 55455
USA

E-mail address: gulliver@math.umn.edu, guoyixu@math.umn.edu

Received April 08, 2009